# Hydrostatic Stokes equations with non-smooth Neumann data.

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## 1 Introduction

The model of the Primitive Equations for Ocean and Atmosphere has been extensively studied by several authors, ([10, 9, 2, 7, 5, 6]), who have established existence and uniqueness results for the stationary and non-stationary models.

Let us recall that the Primitive Equations are a variation of the Navier-Stokes system, where some simplifications have been made (based on the analysis of physical scales, because the domains of study have a depth scale negligible in comparison to horizontal scales). Concretely, rigid-lid hypothesis and hydrostatic pressure are imposed ([7]). These simplifications reduce the dimension of the system from a numerical point of view. However, does not make easier the mathematical analysis. For instance, this system is no longer parabolic for the vertical velocity, which depends upon derivatives for the horizontal velocity, loosing an order of regularity.

As far as we know, all the results concerning strong solutions for the Primitive Equations are based on Ziane's results for the stationary and linear case, see [12].

In [6], F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido used Ziane's results to obtain existence of strong solution for the (non-stationary nonlinear) Primitive Equations, global in time for small data or local in time for any data. However, the fact of using a stationary problem as an auxiliary result to prove existence (and uniqueness) of strong solution for the non-stationary problem forced to impose some regularity hypothesis on the data (more precisely, on the temporal derivative for the Neumann boundary condition) that we consider they are not optimal.

In Conca's work [3], the author defines a **very weak solution** for the Stokes problem. He analyses what kind of regularity can be obtained for a Stokes system when Dirichlet boundary

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data are only of $L^2(\partial\Omega)$ type. We recall that a weak solution has **regularity** $H^1(\Omega)$, and this implies that we have to impose Dirichlet data in $H^{1/2}(\partial\Omega)$. With his definition, Conca finds existence of solution for the Stokes system in the case of a boundary data 1/2 degree minus regular.

In the present paper, we pretend to study a very weak solution concept for the hydrostatic Stokes problem, with Dirichlet-Neumann boundary conditions.

By sections, we present the following main contributions of this paper:

> **In Section 2** we set up the formulation of the stationary hydrostatic Stokes problem (1), and we define the dual problem associated (2). Using a **mixed formulation**, we obtain, first, a weak solution for the dual problem (where the "hydrostatic divergence condition" does not vanish). Afterwards, we extend Ziane's results to obtain strong solution for the dual problem. Finally, we define the **very weak solution** for the hydrostatic Stokes problem and, using the strong regularity for the dual problem, we prove existence and uniqueness of very weak solution.
>
> **In Section 3** we give a differential interpretation of the very weak solution and a sense for the boundary conditions, Dirichlet at the bottom and Neumann at the surface.
>
> **In Section 4** we apply the results obtained in section 2 to the non-stationary hydrostatic Stokes Problem, and we weaken hypothesis upon time derivative for the wind stress tensor (imposed in [6]), obtaining that what we estimate is an optimal regularity result. The extension to the nonlinear non-stationary problem (non-stationary Primitive Equations) is a simple exercise, rewriting the argument made in [6].

# 2 Existence of very weak solution

## 2.1 Formulation of the problem

We consider an open, bounded and Lipschitz-continuous domain $\Omega \subseteq \mathbb{R}^3$ given by

$$\Omega = \{(\mathbf{x}, z) \in \mathbb{R}^3;\, \mathbf{x} = (x, y) \in S,\ -h(\mathbf{x}) < z < 0\},$$

where $S$ is an open bounded domain of $\mathbb{R}^2$ and $h : S \to \mathbb{R}_+$ is the depth function. The boundary $\partial\Omega$ can be written as $\partial\Omega = \Gamma_s \cup \Gamma_b \cup \Gamma_l$ where:

$$\Gamma_s = \{(\mathbf{x}, 0)/\, \mathbf{x} \in S\},$$

$$\Gamma_b = \{(\mathbf{x}, -h(\mathbf{x}))/\, \mathbf{x} \in S\},$$

and

$$\Gamma_l = \{(\mathbf{x}, z)/\, \mathbf{x} \in \partial S,\ -h(\mathbf{x}) < z < 0\}.$$

In all of this work, we impose the fundamental hypothesis:

$$h \geq h_{\min} > 0 \quad \text{in } S.$$

Concretely, the domain has the form:

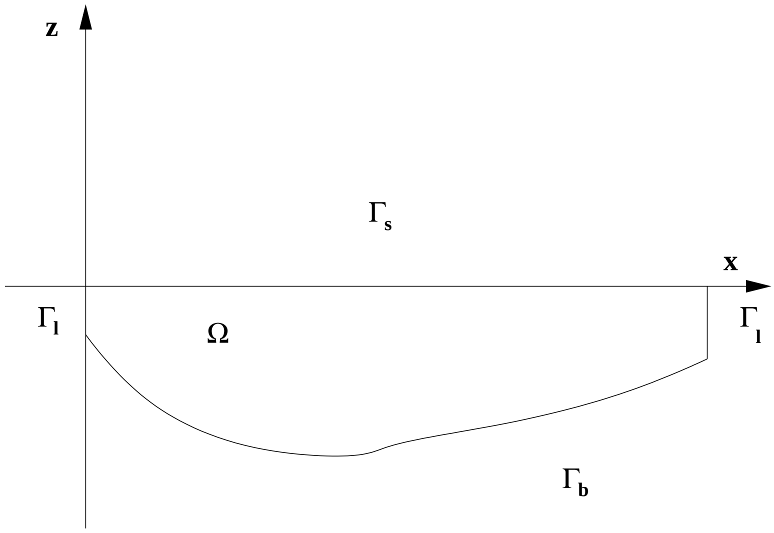


Figure 1: Domain to study.

We start from the hydrostatic Stokes problem:

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} - \nu_3\partial^2_{zz}\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\langle\mathbf{u}\rangle & = & 0 & \text{in } S, \\ \nu_3\partial_z\mathbf{u} & = & \tau & \text{on } \Gamma_s, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Gamma_b\cup\Gamma_l, \end{array}\right. \tag{1}$$

where $\langle\mathbf{u}\rangle(\mathbf{x}) = \displaystyle\int_{-h(\mathbf{x})}^{0}\mathbf{u}(\mathbf{x},z)dz$. The unknowns are $\mathbf{u}$ the horizontal components for the velocity, and a potential $p$ representing the surface pressure stress (and the centripetal forces). The data for (1) are the external forces, $\mathbf{f}:\Omega\to\mathbb{R}^2$ , the wind tension stress on the surface, $\tau:\Gamma_s\to\mathbb{R}^2$, and the eddy horizontal and vertical viscosities, $\nu>0$ and $\nu_3>0$ respectively. $\Delta$, $\nabla$ and $\nabla\cdot$ denote the 2-dimensional operator: $\partial^2_{xx}+\partial^2_{yy}$, $(\partial_x,\partial_y)$ and the horizontal divergence operator, respectively.

We define the following dual problem:

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{\Phi} - \nu_3\partial^2_{zz}\mathbf{\Phi} + \nabla\pi & = & \mathbf{g} & \text{in } \Omega, \\ \nabla\cdot\langle\mathbf{\Phi}\rangle & = & -\varphi & \text{in } S, \\ \nu_3\partial_z\mathbf{\Phi} & = & \mathbf{0} & \text{on } \Gamma_s, \\ \mathbf{\Phi} & = & \mathbf{0} & \text{on } \Gamma_b\cup\Gamma_l, \end{array}\right. \tag{2}$$

where $(\mathbf{\Phi},\pi)$ are the unknowns and $(\mathbf{g},\varphi)$ the data.

## 2.2 Weak and strong regularity for the dual problem.

The following functional spaces will be used:

$$H = \overline{\mathcal{V}}^{L^2} = \{\mathbf{v} \in L^2(\Omega)^2;\ \nabla \cdot \langle \mathbf{v} \rangle = 0 \ \text{ in } S, \langle \mathbf{v} \rangle \cdot \mathbf{n}_{\partial S} = 0\},$$

$$V = \overline{\mathcal{V}}^{H^1} = \{\mathbf{v} \in H^1(\Omega)^2;\ \nabla \cdot \langle \mathbf{v} \rangle = 0 \ \text{ in } S, \mathbf{v}|_{\Gamma_b \cup \Gamma_l} = \mathbf{0}\},$$

where

$$\mathcal{V} = \{\varphi \in C^\infty_{b,l}(\Omega)^2;\ \nabla \cdot \langle \varphi \rangle = 0 \ \text{ in } S\},$$

and

$$C^\infty_{b,l}(\Omega) = \{\varphi \in C^\infty(\Omega)^2;\ supp(\varphi) \text{ is a compact set } \subseteq \overline{\Omega} \backslash (\Gamma_b \cup \Gamma_l)\}.$$

We denote $H^1_{b,l}(\Omega)$ the space of function of $H^1(\Omega)$ vanishing on $\Gamma_b \cup \Gamma_l$ (i.e. , $H^1_{b,l}(\Omega) = \overline{C^\infty_{b,l}(\Omega)}^{H^1}$).

Using a mixed formulation for the problem (2) and extending Ziane's results about strong regularity for the problem (2) (that only were presented for $\varphi \equiv \mathbf{0}$), we will prove the following result:

**Theorem 2.1** *Suppose that $h \in C^3(S)$ and $\partial S \in C^3$. If $\mathbf{g} \in L^2(\Omega)^2$ and $\varphi \in \mathcal{H}$, being*

$$\mathcal{H} = \{\varphi /\ \varphi \in H^1(S),\ \int_S \varphi d\mathbf{x} = 0\},$$

*then there exists a unique solution of* (2) *with $\mathbf{\Phi} \in H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2$, $\pi \in H^1(S)$. Moreover, it verifies the following estimate:*

$$\|\mathbf{\Phi}\|^2_{H^2(\Omega)} + \|\pi\|^2_{H^1(S)} \leq C \left\{ \|\mathbf{g}\|^2_{L^2(\Omega)} + \|\varphi\|^2_{H^1(S)} \right\}. \tag{3}$$

First of all, we will obtain a weak solution of (2) using a mixed formulation for the problem. We set:

$$X = H^1_{b,l}(\Omega)^2, \qquad M = L^2_0(S),$$

and introduce the notation:

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega + \nu_3 \int_\Omega \partial_z \mathbf{u} \cdot \partial_z \mathbf{v} \, d\Omega, && \forall \mathbf{u}, \mathbf{v} \in X, \\
b(\mathbf{u}, p) &= -\int_S p(\nabla \cdot \langle \mathbf{u} \rangle) d\mathbf{x}, && \forall \mathbf{u} \in X, \forall p \in M, \\
\langle L, \mathbf{v} \rangle &= \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\Omega, && \forall \mathbf{v} \in X, \\
\langle R, q \rangle &= \int_S \varphi \, q \, d\mathbf{x} && \forall q \in M.
\end{aligned}$$

It is easy to verify that $a(\cdot,\cdot)$ is a bilinear, symmetric and continuous form on $X \times X$ and elliptic on the solenoidal space $V = \{\mathbf{v},\ \mathbf{v} \in X,\ b(\mathbf{v}, p) = 0,\ \forall p \in L^2_0(S)\}$; $b(\cdot,\cdot)$ is a continuous bilinear form on $X \times M$, $L$ is a linear form on $X$ and $R$ is a linear form on $M$.

Then, we consider the (abstract) mixed problem: Find $(\mathbf{\Phi}, \pi) \in X \times M$ such that:

$$\left\{ \begin{aligned} a(\mathbf{\Phi}, \mathbf{v}) + b(\mathbf{v}, \pi) &= \langle L, \mathbf{v} \rangle \quad \forall \mathbf{v} \in X, \\ b(\mathbf{\Phi}, q) &= \langle R, q \rangle \quad \forall q \in M. \end{aligned} \right. \tag{4}$$

**Proposition 2.2 (Existence and uniqueness of solution for** (4)**)** *([4]) Suppose that:*

- $a(\cdot,\cdot)$ *is a continuous bilinear $V$-elliptic form, i.e. there exists $a_0 > 0$ such that:*

$$a(\mathbf{v},\mathbf{v}) \geq a_0 \|\mathbf{v}\|_X \quad \forall \mathbf{v} \in V,$$

- $b(\cdot,\cdot)$ *is a bilinear form satisfying the inf-sup condition, i.e. there exists $\beta_0 > 0$ such that:*

$$\inf_{p \in M \setminus \{0\}} \sup_{\mathbf{v} \in X \setminus \{0\}} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_X \|p\|_M} \geq \beta_0.$$

*Then, for each pair $(L,R) \in X' \times M'$ the mixed problem* (4) *has a unique solution $(\mathbf{\Phi},\pi) \in X \times M$. Moreover, in this case, the following mapping is an isomorphism:*

$$(L,R) \in X' \times M' \longrightarrow (\mathbf{\Phi},\pi) \in X \times M$$

Therefore, in order to prove existence and uniqueness of weak solution of (2), $\mathbf{\Phi} \in H^1_{b,l}(\Omega)^2$ and $\pi \in L^2_0(S)$, we only have to prove that the inf-sup condition hold. To this aim, we use the following result:

**Lemma 2.3** *([4]) The 3-dimensional divergence operator, $\nabla_3 \cdot : W^\perp \longrightarrow L^2_0(\Omega)$ is an isomorphism, where $W = \{\mathbf{v} \in H^1_0(\Omega)^3,\ \nabla_3 \cdot \mathbf{v} = 0\}$, $W^\perp$ is the ortogonal space of $W$ respect to the $H^1_0(\Omega)$-norm, and $L^2_0(\Omega) = \{g \in L^2(\Omega),\ \int_\Omega g\, d\Omega = 0\}$.*

**Lemma 2.4** *The following inf-sup condition is verified:*

$$\sup_{\mathbf{v} \in H^1_0(\Omega)^2} \frac{\displaystyle\int_S p\, \nabla \cdot \langle \mathbf{v} \rangle \, d\mathbf{x}}{\|\mathbf{v}\|_{H^1_0(\Omega)}} \geq \sqrt{h_{\min}}\, \|p\|_{L^2(S)} \quad \forall p \in L^2_0(S).$$

**Proof:** We take $p \in L^2_0(S)$. Easily, we can deduce $\dfrac{1}{h(\mathbf{x})} p \in L^2_0(\Omega)$ (using that $h \geq h_{min} \geq 0$) and $\partial_z p = 0$. Indeed, we have:

$$\left\| \frac{1}{h(\mathbf{x})} p \right\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{h(\mathbf{x})}} \|p\|_{L^2(S)}.$$

Then, applying Lemma 2.3, there exists a function $\mathbf{U} = (\mathbf{u}, u_3) \in W^\perp \subset H^1_0(\Omega)^3$ such that $\nabla_3 \cdot \mathbf{U} = \dfrac{1}{h(\mathbf{x})} p$ and (in particular)

$$\|\nabla_3 \mathbf{U}\|_{L^2(\Omega)} \leq C \|\nabla_3 \cdot \mathbf{U}\|_{L^2(\Omega)} \leq C \left\| \frac{1}{h(\mathbf{x})} p \right\|_{L^2(\Omega)} \leq C \frac{1}{\sqrt{h_{\min}}} \|p\|_{L^2(S)}. \tag{5}$$

Rewriting the bilinear form $b(\cdot,\cdot)$, one has $\forall \mathbf{v} \in H^1_0(\Omega)^2$

$$\begin{aligned} b(\mathbf{v},p) &= -\int_S p(\nabla \cdot \langle \mathbf{v} \rangle)\, d\mathbf{x} = \int_S \nabla p\, \langle \mathbf{v} \rangle \, d\mathbf{x} = \int_\Omega \nabla p \cdot \mathbf{v}\, d\Omega \\ &= (\text{as } \partial_z p = 0) \int_\Omega \nabla_3 p \cdot (\mathbf{v}, v_3)\, d\Omega = -\int_\Omega p \nabla_3 \cdot (\mathbf{v}, v_3)\, d\Omega \end{aligned}$$

where $v_3$ is any function belonging to $H_0^1(\Omega)$. Then,

$$|b(\mathbf{u},p)| = \left| - \int_\Omega p\, \nabla_3 \cdot (\mathbf{u}, u_3) d\Omega \right| = \int_\Omega \frac{1}{h(\mathbf{x})} p^2 d\Omega = \int_S p^2 d\mathbf{x} = \|p\|^2_{L^2(S)}. \tag{6}$$

Therefore, using (5) and (6)

$$\frac{|b(\mathbf{u},p)|}{\|\nabla \mathbf{u}\|_{L^2(\Omega)} \|p\|_{L^2(S)}} = \frac{\|p\|_{L^2(S)}}{\|\nabla \mathbf{u}\|_{L^2(\Omega)}} \geq \sqrt{h_{\min}} \, \frac{\|\nabla_3 \mathbf{U}\|_{L^2(\Omega)}}{\|\nabla \mathbf{u}\|_{L^2(\Omega)}} \geq \sqrt{h_{\min}},$$

hence

$$\sup_{\mathbf{v} \in H_0^1(\Omega) \setminus \{0\}} \frac{|b(\mathbf{v},p)|}{\|\mathbf{v}\|_{H_0^1(\Omega)}} \geq \sqrt{h_{\min}} \, \|p\|_{L_0^2(S)}, \quad \forall p \in L_0^2(S).$$

■

Applying the Proposition 2.2 in our context, we can deduce the following result:

**Theorem 2.5** *Suppose $\Omega$ Lipschitz-continuous. If $\mathbf{g} \in (H_{b,l}^1(\Omega)^2)'$ and $\varphi \in (L_0^2(S))'$, then there exists a unique weak solution of* (2), $(\mathbf{\Phi}, \pi) \in H_{b,l}^1(\Omega)^2 \times L_0^2(S)$. *Moreover, there exists a constant $C > 0$ such that:*

$$\|\mathbf{\Phi}\|_{H^1(\Omega)} + \|\pi\|_{L^2(S)} \leq C \left\{ \|\mathbf{g}\|_{(H_{b,l}^1(\Omega))'} + \|\varphi\|_{(L_0^2(S))'} \right\}. \tag{7}$$

**Remark 2.1** *Taking into account that the space $L^2(S)^2/\mathbb{R}$ is isomorphic to $(L_0^2(S))'$, see [4], we obtain* (7) *replacing $\|\varphi\|_{(L_0^2(S))'}$ by $\|\varphi\|_{L^2(S)/\mathbb{R}}$.*

To prove Theorem 2.1, we will use Cattabriga's classic results for the Stokes problem in $S$, and the results for elliptic problems in $\Omega$ (see [12] and references therein cited), that we recall here:

**Proposition 2.6 (Regularity for the elliptic problem in $\Omega$, with mixed Neumann-Dirichlet boundary conditions)** *Assume $h \in C^3(S)$ and $\partial S \in C^3$. Let $\mathbf{u}$ the unique solution for:*

$$\begin{cases} -\nu \Delta \mathbf{u} - \nu_3 \partial^2_{zz} \mathbf{u} &= \mathbf{d} \qquad \text{in } \Omega, \\ \nu_3 \partial_z \mathbf{u} &= \tau \qquad \text{on } \Gamma_s, \\ \mathbf{u} &= \psi_l \ (\text{resp. } \psi_b) \quad \text{on } \Gamma_l \ (\text{resp. } \Gamma_b), \end{cases}$$

*Suppose $\mathbf{d} \in L^2(\Omega)^2$, $\tau \in H_0^{s-3/2}(\Gamma_s)^2$, $\psi_l \in H_0^{s-1/2}(\Gamma_l)^2$, $\psi_b \in H_0^{s-1/2}(\Gamma_b)^2$ with $3/2 \leq s < 2$. Then, $\mathbf{u} \in H^s(\Omega)^2$. Moreover, if $\psi_b \in H_0^{3/2+\varepsilon}(\Gamma_b)^2$, $\tau \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$ and $\psi_l \in H_0^{3/2+\varepsilon}(\Gamma_l)^2$, with $0 < \varepsilon < 1/2$, then $\mathbf{u} \in H^2(\Omega)^2$.*

**Proposition 2.7 (Regularity for the Stokes problem in $S$)** *Suppose $S \subseteq \mathbb{R}^2$ an open set such that $\partial S \in C^3$. Let $(\mathbf{u}, p)$ be the solution for the Dirichlet-Stokes problem:*

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \mathbf{a} \quad \text{in } S, \\ \nabla \cdot \mathbf{u} &= b \quad \text{in } S, \\ \mathbf{u} &= \mathbf{c} \quad \text{on } \partial S. \end{cases}$$

*If $\mathbf{a} \in L^\alpha(S)^2$, $b \in W^{s-1,\alpha}(S)$ and $\mathbf{c} \in (W^{s-\frac{1}{\alpha},\alpha}(\partial S))^2$ such that $\displaystyle\int_S b\, d\mathbf{x} = \int_{\partial S} \mathbf{c} \cdot \mathbf{n}_{\partial S}$, with $1 < \alpha < +\infty$ and $1 \leq s \leq 2$ be such that $s - \frac{1}{\alpha}$ is not an integer, then:*

$$\mathbf{u} \in (W^{s,\alpha}(S))^2 \quad \text{and} \quad p \in W^{s-1,\alpha}(S).$$

**Proof of Theorem** 2.1**:** From Theorem 2.5, we have that $\pi \in L^2_0(S)$, and thus $\nabla\pi \in H^{-1}(S)$. Therefore, we might consider the auxiliary function $\mathbf{v}$ as the unique solution of the problem:

$$\left\{\begin{array}{rcll} \nu\Delta\mathbf{v} &=& \nabla\pi & \text{in } S, \\ \mathbf{v} &=& \mathbf{0} & \text{on } \partial S. \end{array}\right. \tag{8}$$

Lax-Milgram's Theorem implies that $\mathbf{v} \in H^1_0(S)^2$. Now, we look at the elliptic problem verified by $\mathbf{w} = \mathbf{\Phi} - \mathbf{v}$ (without pressure and divergence nulle restriction):

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{w} - \nu_3\partial^2_{zz}\mathbf{w} &=& \mathbf{g} & \text{in } \Omega, \\ \nu_3\partial_z\mathbf{w} &=& \mathbf{0} & \text{on } \Gamma_s, \\ \mathbf{w} &=& \mathbf{0} & \text{on } \Gamma_l, \\ \mathbf{w} &=& -\mathbf{v} & \text{on } \Gamma_b. \end{array}\right. \tag{9}$$

Due to the regularity of the data and Proposition 2.6 (for $s = 3/2$) we deduce that $\mathbf{w} \in H^{3/2}(\Omega)^2$. Then, using Lemma $B$.1, $\langle\mathbf{w}\rangle \in H^{3/2}(S)^2$. As $\mathbf{v}$ is independent from $z$, we obtain:

$$-\nabla\cdot(h\,\mathbf{v}) = \nabla\cdot\langle\mathbf{w}\rangle - \nabla\cdot\langle\mathbf{\Phi}\rangle = \nabla\cdot\langle\mathbf{w}\rangle + \varphi \in H^{1/2}(S).$$

Now, as

$$\nabla\cdot\mathbf{v} = \frac{1}{h(\mathbf{x})}\nabla\cdot(h\,\mathbf{v}) - \frac{\nabla h\cdot\mathbf{v}}{h},$$

then $\nabla\cdot\mathbf{v} \in H^{1/2}(S)$. In particular, $\nabla\cdot\mathbf{v} \in H^{1/2-\varepsilon}(S)$ for all $\varepsilon > 0$. Therefore, if we consider the Stokes problem in $S$ that satisfies $(\mathbf{v}, \pi)$:

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{v} + \nabla\pi &=& \mathbf{0} & \text{in } S, \\ \nabla\cdot\mathbf{v} &\in& H^{1/2-\varepsilon}(S), & \text{in } S, \\ \mathbf{v} &=& \mathbf{0} & \text{on } \partial S, \end{array}\right. \tag{10}$$

from Proposition 2.7 (for $s = 1/2 - \varepsilon$ and $\alpha = 2$), we obtain that $\mathbf{v} \in H^{3/2-\varepsilon}(S)^2$ and $\pi \in H^{1/2-\varepsilon}(S)$, $\forall\varepsilon > 0$.

Returning to system (9) and using again Proposition 2.6 (now for $s = 2 - \varepsilon$), it verifies $\mathbf{w} \in H^{2-\varepsilon}(\Omega)$. In particular,

$$\frac{\partial\mathbf{\Phi}}{\partial z} = \frac{\partial\mathbf{w}}{\partial z} \in H^{1-\varepsilon}(\Omega)^2. \tag{11}$$

Now, we integrate $(2)_1$ in the $z$-variable, obtaining the problem:

$$\left\{\begin{array}{rcll} -\nu\Delta\langle\mathbf{\Phi}\rangle + h(\mathbf{x})\nabla\pi &=& \mathbf{G} & \text{in } S, \\ \nabla\cdot\langle\mathbf{\Phi}\rangle &=& -\varphi & \text{in } S, \\ \langle\mathbf{\Phi}\rangle &=& \mathbf{0} & \text{on } \partial S, \end{array}\right.$$

where $\mathbf{G} = \langle\mathbf{g}\rangle + \nu_3\partial_z\mathbf{w}(x,0) - \nu_3\partial_z\mathbf{w}(x,-h(\mathbf{x})) + \nabla h(\mathbf{x})\cdot(\nabla\mathbf{\Phi})(\mathbf{x},-h(\mathbf{x}))$. We will see that $\mathbf{G} \in L^2(S)^2$. From $\mathbf{g} \in L^2(\Omega)^2$ and (11), we have that $\langle\mathbf{g}\rangle + \nu_3\partial_z\mathbf{w}(\mathbf{x},0) - \nu_3\partial_z\mathbf{w}(\mathbf{x},-h(\mathbf{x})) \in$

$L^2(S)^2$. Therefore, we focus our attention on the term $\nabla h(\mathbf{x}) \cdot (\nabla \mathbf{\Phi})(\mathbf{x}, -h(\mathbf{x}))$ coming from $\langle \Delta \mathbf{\Phi} \rangle = \Delta \langle \mathbf{\Phi} \rangle - \nabla h \cdot (\nabla \mathbf{\Phi})|_{\Gamma_b}$. Taking into account that $\mathbf{\Phi}|_{\Gamma_b} = \mathbf{0}$, and deriving with respect to the $\mathbf{x}$-variables, we obtain:

$$(\nabla \mathbf{\Phi})|_{\Gamma_b} = (\partial_z \mathbf{\Phi})|_{\Gamma_b} \nabla h(\mathbf{x}).$$

Then,

$$\nabla h \cdot (\nabla \mathbf{\Phi})|_{\Gamma_b} = (\partial_z \mathbf{\Phi})|_{\Gamma_b} |\nabla h(\mathbf{x})|^2.$$

Therefore, it suffices to analyse the regularity of $(\partial_z \mathbf{\Phi})|_{\Gamma_b} |\nabla h(\mathbf{x})|^2$.
Since $h \in H^2(S)$, $|\nabla h|^2 \in L^p(S)$ for all $p > 1$. ¿From (11), $(\partial_z \mathbf{\Phi})|_{\Gamma_b} \in H^{1/2-\varepsilon}(\Gamma_b) \hookrightarrow L^{\frac{4}{1+2\varepsilon}}(\Gamma_b)$. In particular, $(\partial_z \mathbf{\Phi})|_{\Gamma_b} \in L^q(S)^2$ for a certain $q > 2$. Then $|\nabla h(\mathbf{x})|^2 (\partial_z \mathbf{\Phi})|_{\Gamma_b} \in L^2(S)^2$.

Notice that, using the equality $h \nabla \pi = \nabla(h \pi) - \pi \nabla h$, the previous problem can be rewritten as the Stokes problem:

$$\left\{ \begin{array}{rcll} -\nu \Delta \langle \mathbf{\Phi} \rangle + \nabla(h\pi) & = & \mathbf{G} + \pi \nabla h & \text{in } S, \\ \nabla \cdot \langle \mathbf{\Phi} \rangle & = & -\varphi & \text{in } S, \\ \langle \mathbf{\Phi} \rangle & = & \mathbf{0} & \text{on } \partial S, \end{array} \right.$$

with $\mathbf{G} + \pi \nabla h \in L^2(S)^2$ (using that $\pi \in H^{1/2-\varepsilon}(S)$), $\varphi \in H^1(S)$ such that $\displaystyle\int_S \varphi \, d\mathbf{x} = 0$. By Proposition 2.7, $\langle \mathbf{\Phi} \rangle \in H^2(S)$ and $h\pi \in H^1(S)$. In particular, $\pi \in H^1(S)$, using again that $h \geq h_{\min} > 0$.

Now, we go back to the dual problem (2). Moving the pressure term to the right hand side, we consider the corresponding elliptic problem. Then, using the new regularity hypothesis for $\pi$ and applying Proposition 2.6, one guarantees that $\mathbf{\Phi} \in H^2(\Omega)^2$. Finally, inequality (3) can be deduced by construction, thanks to the continuous dependence of the auxiliary problems (8), (9) and (10). ■

### 2.3 Very weak regularity for the primal problem.

Suppose the following regularity hypothesis for the data:

$$(H) \qquad \mathbf{f} \in (H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2)', \quad \tau \in H^{-3/2}(\Gamma_s).$$

Denote by $\langle \cdot, \cdot \rangle_S$ the duality in $(H^1(S))', H^1(S)$, by $\langle \cdot, \cdot \rangle_\Omega$ the duality $(H^2(\Omega) \cap H^1_{b,l}(\Omega))', H^2(\Omega) \cap H^1_{b,l}(\Omega)$ and by $\langle \cdot, \cdot \rangle_{\Gamma_s}$ the duality $H^{-3/2}(\Gamma_s), H^{3/2}_0(\Gamma_s)$.

Let $l : L^2(\Omega)^2 \times \mathcal{H} \to \mathbb{R}$ defined by:

$$l(\mathbf{g}, \varphi) = \langle \mathbf{f}, \mathbf{\Phi} \rangle_\Omega + \langle \tau, \mathbf{\Phi} \rangle_{\Gamma_s},$$

with $(\mathbf{\Phi}, \pi)$ the solution for the dual problem (2) with data $(\mathbf{g}, \varphi)$. It is easy to prove that $l$ is a linear and continuous operator from $L^2(\Omega)^2 \times \mathcal{H}$ into $\mathbb{R}$. Indeed, from (3) one has

$$\|l\|_{(L^2(\Omega) \times \mathcal{H})'} \leq C \left\{ \|\mathbf{f}\|_{(H^2(\Omega) \cap H^1_{b,l}(\Omega))'} + \|\tau\|_{H^{-3/2}(\Gamma_s)} \right\}.$$

**Definition 2.8** *A pair* $(\mathbf{u}, p)$ *is said a* **very weak solution of** (1) *if and only if the following conditions are verified:*

$$\left\{\begin{array}{c} \mathbf{u} \in L^2(\Omega)^2, \qquad p \in \mathcal{H}', \\ \displaystyle\int_\Omega \mathbf{u} \cdot \mathbf{g}\, d\Omega + \langle p, \varphi\rangle_S \;=\; l(\mathbf{g}, \varphi), \quad \forall \mathbf{g} \in L^2(\Omega)^2, \, \forall \varphi \in \mathcal{H} \end{array}\right. \tag{12}$$

Applying the classical Riesz' identification, one has easily the following:

**Lemma 2.9** *Assuming* $(H)$, *there exists a unique very weak solution* $(\mathbf{u}, p)$. *Moreover, it verifies:*

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{\mathcal{H}'} \le C \left\{ \|f\|_{(H^2(\Omega)\cap H^1_{b,l})'} + \|\tau\|_{H^{-3/2}(\Gamma_s)} \right\}. \tag{13}$$

In order to rewrite the bilinear form of (12) in terms of $(\mathbf{\Phi}, \pi)$, one needs the following result

**Proposition 2.10** *The space* $(H^1(S))'/\mathbb{R}$ *is isomorphic to* $\mathcal{H}'$.

**Proof:** In Appendix A.

Using the dual problem (2), we rewrite the very weak definition as:

$$\left\{\begin{array}{c} \mathbf{u} \in L^2(\Omega)^2, \qquad p \in (H^1(S))'/\mathbb{R}, \\ \displaystyle\int_\Omega \mathbf{u} \cdot \left(-\nu\Delta\mathbf{\Phi} - \nu_3 \partial^2_{zz}\mathbf{\Phi} + \nabla\pi\right) d\Omega - \langle p, \nabla \cdot \langle\mathbf{\Phi}\rangle\rangle_S \;=\; \langle \mathbf{f}, \mathbf{\Phi}\rangle_\Omega + \langle \tau, \mathbf{\Phi}\rangle_{\Gamma_s}, \\ \forall \mathbf{\Phi} \in H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2 \quad \text{and} \quad \partial_z \mathbf{\Phi}|_{\Gamma_s} = 0, \quad \forall \pi \in H^1(S). \end{array}\right. \tag{14}$$

Notice that from $\mathbf{\Phi} \in H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2$ one has $\nabla \cdot \langle\mathbf{\Phi}\rangle \in H^1(S)$ and $\int_S \nabla \cdot \langle\mathbf{\Phi}\rangle = 0$. On the other hand, $p \in (H^1(S))'/\mathbb{R}$ means that $p \in (H^1(S))'$ is defined up to an additive constant.

Now, we are able to show the result about existence (and uniqueness) of very weak solution:

**Theorem 2.11** *Under the regularity hypothesis* $(H)$, *there exists a unique solution* $(\mathbf{u}, p)$ *of* (14) *in* $L^2(\Omega)^2 \times (H^1(S))'/\mathbb{R}$. *Moreover, it verifies:*

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{(H^1(S))'/\mathbb{R}} \le C \left\{ \|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\tau\|_{H^{-3/2}(\Gamma_s)} \right\}. \tag{15}$$

**Proof:** From Lemma 2.9, there exists a unique pair $(\mathbf{u}, \widetilde{p}) \in L^2(\Omega)^2 \times \mathcal{H}'$ such that:

$$\int_\Omega \mathbf{u} \cdot \mathbf{g}\, d\Omega + \langle \widetilde{p}, \varphi\rangle_{\mathcal{H}',\mathcal{H}} = l(\mathbf{g}, \varphi) \quad \forall \mathbf{g} \in L^2(\Omega)^2, \, \forall \varphi \in \mathcal{H}.$$

As by Proposition 2.10, $(H^1(S))'/\mathbb{R}$ is isomorphic to $\mathcal{H}'$, we can identify $\widetilde{p}$ with a distribution $p$ in $(H^1(S))'/\mathbb{R}$ such that $\langle \widetilde{p}, \varphi\rangle_{\mathcal{H}',\mathcal{H}} = \langle p, \varphi\rangle_{(H^1(S))',H^1(S)}$, $\forall \varphi \in \mathcal{H}$. Therefore, we conclude that $(\mathbf{u}, p)$ is a solution of (14), and we finish the proof of existence of solution. The uniqueness is deduced from the linearity of the problem.

In order to obtain the continuous dependence of the very weak solution from the data, we use the continuous dependence for the dual problem. Indeed, one has:

$$\begin{aligned}|\langle(\mathbf{u},p),(\mathbf{g},\varphi)\rangle_{L^2(\Omega)\times(H^1(S))'/\mathbb{R},L^2(\Omega)\times H^1(S)}| \;&=\; |l(\mathbf{g},\varphi)| = |\langle \mathbf{f},\mathbf{\Phi}\rangle_\Omega + \langle\tau,\mathbf{\Phi}\rangle_{\Gamma_s}| \\ &\le\; \|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'}\|\mathbf{\Phi}\|_{H^2(\Omega)} + \|\tau\|_{H^{-3/2}(\Gamma_s)}\|\mathbf{\Phi}|_{\Gamma_s}\|_{H^{3/2}(\Gamma_s)} \\ &\le\; \left(\|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\tau\|_{H^{-3/2}(\Gamma_s)}\right)\|\mathbf{\Phi}\|_{H^2(\Omega)} \\ &\le\; \left(\|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\tau\|_{H^{-3/2}(\Gamma_s)}\right)\left(\|\mathbf{g}\|_{L^2(\Omega)} + \|\varphi\|_{H^1(S)}\right).\end{aligned}$$

Therefore,

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{(H^1(S))'/\mathbb{R}} \le C\left(\|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\tau\|_{H^{-3/2}(\Gamma_s)}\right),$$

that concludes the proof of the Theorem. ■

# 3 Interpretation for the differential problem

## 3.1 Differential equation

¿From now on, we fix $\mathbf{f} \in L^2(\Omega)^2$ instead of $(H^2(\Omega)\cap H^1_{b,l})'$ (in order to have a space of distributions) and $(\mathbf{u},p)$ the very weak solution of (1). Taking $\pi = 0$ and $\mathbf{\Phi} \in \mathcal{D}(\Omega)$ in (14),

$$\int_\Omega \mathbf{u}\cdot\left(-\nu\Delta\mathbf{\Phi} - \nu_3\partial^2_{zz}\mathbf{\Phi}\right)d\Omega - \langle p, \nabla\cdot\langle\mathbf{\Phi}\rangle\rangle_S \;=\; \int_\Omega \mathbf{f}\cdot\mathbf{\Phi}d\Omega,$$

hence we can deduce $(1)_1$ in the distributional sense. Taking $\mathbf{\Phi} = \mathbf{0}$ and $\pi \in \mathcal{D}(S)$ in (14),

$$\int_\Omega \mathbf{u}\cdot\nabla\pi\, d\Omega \;=\; 0$$

hence we can deduce $\nabla\cdot\langle\mathbf{u}\rangle = 0$ in $\mathcal{D}'(S)$. Therefore, we have:

**Proposition 3.1** *Let $(\mathbf{u},p) \in L^2(\Omega)^2 \times (H^1(S))'/\mathbb{R}$ be the unique very weak solution of* (1)*. Then $\mathbf{u}$ and $p$ verify $(1)_{1-2}$ in the distributional sense in $\Omega$ and $S$, respectively.*

## 3.2 Sense for the boundary conditions.

¿From $\mathbf{u} \in L^2(\Omega)^2$, it is easy to deduce that $\langle\mathbf{u}\rangle \in L^2(S)^2$. On the other hand, since $\nabla\cdot\langle\mathbf{u}\rangle = 0$, in particular $\nabla\cdot\langle\mathbf{u}\rangle \in L^2(S)$. Then we can conclude that $\langle\mathbf{u}\rangle\cdot\mathbf{n}_{\partial S} \in H^{-1/2}(S)$. Moreover, taking $\mathbf{\Phi} = \mathbf{0}$ and $\pi \in H^1(S)$ in (14), we get that:

$$\int_\Omega \mathbf{u}\cdot\nabla\pi\, d\Omega =\; 0,$$

hence

$$0 \;=\; \int_S \langle\mathbf{u}\rangle\cdot\nabla\pi\, dS = \langle\langle\mathbf{u}\rangle\cdot\mathbf{n},\pi\rangle_{\partial S}, \quad \forall\pi\in H^1(S)$$

where $\langle\cdot,\cdot\rangle_{\partial S}$ denotes the duality $H^{-1/2}(\partial S), H^{1/2}(\partial S)$. We can then deduce that:

$$\langle\mathbf{u}\rangle\cdot\mathbf{n}_{\partial S} = 0 \quad \text{in } H^{-1/2}(\partial S). \tag{16}$$

### 3.2.1 Dirichlet boundary conditions

In order to give a sense at the boundary conditions $(1)_4$, we define the operator

$$\mathbf{D}_{(\mathbf{u},p)} \quad : \quad H^{1/2}(\Gamma_b \cup \Gamma_l) \longrightarrow \mathbb{R},$$
$$\psi \mapsto \mathbf{D}_{(\mathbf{u},p)}(\psi),$$

such that

$$\mathbf{D}_{(\mathbf{u},p)}(\psi) = \int_\Omega \mathbf{u}\cdot(-\nu\Delta\mathbf{\Phi} - \nu_3\partial^2_{zz}\mathbf{\Phi})d\Omega - \langle p, \nabla\cdot\langle\mathbf{\Phi}\rangle\rangle_S - \langle \mathbf{f}, \mathbf{\Phi}\rangle_\Omega,$$

where $\mathbf{\Phi} = \mathbf{\Phi}(\psi)$ is the unique weak solution, $\mathbf{\Phi} \in H^2(\Omega) \cap H^1_0(\Omega)$, for the problem:

$$\begin{cases} \Delta^2\mathbf{\Phi} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{\Phi} = \mathbf{0} & \text{on } \partial\Omega, \\ \dfrac{\partial\mathbf{\Phi}}{\partial(\mathbf{n}, n_3)} = \psi & \text{on } \Gamma_b \cup \Gamma_l, \quad \partial_z\mathbf{\Phi} = \mathbf{0} \text{ on } \Gamma_s. \end{cases}$$

It is easy to show that $\mathbf{D}_{(\mathbf{u},p)}$ is a linear continuous operator. Then, we can define the map:

$$(\mathbf{u}, p) \in L^2(\Omega)^2 \times (H^1(S))'/\mathbb{R} \longrightarrow \mathbf{D}_{(\mathbf{u},p)} \in H^{1/2}(\Gamma_b \cup \Gamma_l)'.$$

Replacing $\pi = 0$ and $\mathbf{\Phi} = \mathbf{\Phi}(\psi)$ in (14), we obtain that:

$$\mathbf{D}_{(\mathbf{u},p)}(\psi) = 0, \quad \forall\psi \in H^{1/2}(\Gamma_b \cup \Gamma_l)$$

Thus, it follows that $\mathbf{D}_{(\mathbf{u},p)} = 0$ as an element of $(H^{1/2}(\Gamma_b \cup \Gamma_l))'$. We denote this operator the **generalised trace over** $\Gamma_b \cup \Gamma_l$.

### 3.2.2 Neumann boundary condition

In this case, in order to give a sense at the boundary conditions $(1)_3$, we define the operator

$$\mathbf{N}_{(\mathbf{u},p)} \quad : \quad H^{3/2}_0(S)^2 \longrightarrow \mathbb{R},$$
$$\psi \mapsto \mathbf{N}_{(\mathbf{u},p)}(\psi),$$

such that:

$$\mathbf{N}_{(\mathbf{u},p)}(\psi) = \int_\Omega \mathbf{u}\cdot(-\nu\Delta\mathbf{\Phi} - \nu_3\partial^2_{zz}\mathbf{\Phi})d\Omega - \langle p, \nabla\cdot\langle\mathbf{\Phi}\rangle\rangle_S - \langle \mathbf{f}, \mathbf{\Phi}\rangle_\Omega,$$

where $\mathbf{\Phi} = \mathbf{\Phi}(\psi)$ is the unique weak solution,$\mathbf{\Phi} \in H^2(\Omega)$, for the problem:

$$\begin{cases} \Delta^2\mathbf{\Phi} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{\Phi} = \psi & \text{on } \Gamma_s, \quad \mathbf{\Phi} = \mathbf{0} \text{ on } \Gamma_b \cup \Gamma_l, \\ \dfrac{\partial\mathbf{\Phi}}{\partial(\mathbf{n}, n_3)} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Replacing $\pi = 0$ and this new $\mathbf{\Phi} = \mathbf{\Phi}(\psi)$ in (14), we obtain that:

$$\mathbf{N}_{(\mathbf{u},p)}(\psi) = \langle\tau, \psi\rangle_{\Gamma_s}, \quad \forall\psi \in H^{3/2}_0(S)$$

Thus, it follows that $\mathbf{N}_{(\mathbf{u},p)} = \tau$ as an element of $(H^{3/2}_0(S))' \equiv H^{-3/2}(S)$. We denote this operator the **generalised normal trace over** $\Gamma_s$.

# 4 Application to the non-stationary problem

According to the purpose stated in the Introduction of this work, we want to weaken the data hypothesis necessary for getting a strong solution for the linear non-stationary Primitive Equations, also called non-stationary hydrostatic Stokes problem:

$$\left\{\begin{array}{rcll} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} - \nu_3\partial^2_{zz}\mathbf{u} + \nabla p &=& \mathbf{F} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\langle\mathbf{u}\rangle &=& 0 & \text{in } (0,T)\times S, \\ \mathbf{u}|_{t=0} &=& \mathbf{u}_0 & \text{in } \Omega, \\ \nu_3\partial_z\mathbf{u} &=& \tau & \text{on } (0,T)\times\Gamma_s, \\ \mathbf{u} &=& \mathbf{0} & \text{on } (0,T)\times(\Gamma_b\cup\Gamma_l). \end{array}\right. \tag{17}$$

The following result is given in [6]:

**Theorem 4.1** *Let $S \subseteq \mathbb{R}^2$ be a domain $C^3$ and $h \in C^3(\overline{S})$ with $h \geq h_{min} > 0$ in $\overline{S}$. If $\mathbf{F} \in L^2((0,T)\times\Omega)^2$, $\mathbf{u}_0 \in V$, $\tau \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^2)$, for some $\varepsilon > 0$ with $\partial_t\tau \in L^2(0,T;H^{-1/2}(\Gamma_s)^2)$, then there exists a unique strong solution $\mathbf{u}$ of (17) in $(0,T)$. Moreover, there exists a constant $C > 0$ such that:*

$$\begin{array}{rl} \|\mathbf{u}\|^2_{L^\infty(V)} &+ \quad \|\mathbf{u}\|^2_{L^2(H^2(\Omega))} + \|\partial_t\mathbf{u}\|^2_{L^2(H)} \leq C\Big\{\|\mathbf{u}_0\|^2_V + \|\tau(0)\|^2_{H^{-1/2}(\Gamma_s)} \\ &+ \quad \|\mathbf{f}\|^2_{L^2(L^2(\Omega))} + \|\tau\|^2_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s))} + \|\partial_t\tau\|^2_{L^2(H^{-1/2}(\Gamma_s))}\Big\} \end{array} \tag{18}$$

The proof of this theorem is based on the following weak regularity result due to Lions-Temam-Wang [10], and the strong regularity result due to M. Ziane [12], respectively:

**Lemma 4.2** *Suppose $S \subseteq \mathbb{R}^2$ such that $\Omega \subseteq \mathbb{R}^3$ be Lipschitz-continuous. If $\mathbf{f} \in H^{-1}_{b,l}(\Omega)^2$ and $\tau \in H^{-1/2}(\Gamma_s)^2$, Then the problem (1) has a unique solution $\mathbf{u} \in H^1(\Omega)^2$. Moreover, there are continuous dependence of the solution from the data, i.e. there exists a constant $K_1 = K_1(\Omega,\nu,\nu_3) > 0$ such that:*

$$\|\mathbf{u}\|^2_V \leq K_1\left\{\|\tau\|^2_{H^{-1/2}(\Gamma_s)} + \|\mathbf{f}\|^2_{H^{-1}_{b,l}(\Omega)}\right\}. \tag{19}$$

**Lemma 4.3** *Let $S \subseteq \mathbb{R}^2$ be a $C^3$ domain and $h \in C^3(\overline{S})$ with $h \geq h_{min} \geq 0$ in $\overline{S}$. If $\mathbf{f} \in L^2(\Omega)^2$ and $\tau \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$, for some $\varepsilon > 0$, the unique solution $\mathbf{u}$ of the problem (1) belongs to $H^2(\Omega)^2 \cap V$. Moreover, there are continuous dependence of the solution from the data, i.e. there exists a constant $K_2 = K_2(\Omega,\nu,\nu_3) > 0$ such that:*

$$\|\mathbf{u}\|^2_{H^2(\Omega)} \leq K_2\left\{\|\tau\|^2_{H_0^{1/2+\varepsilon}(\Gamma_s)} + \|\mathbf{f}\|^2_{L^2(\Omega)}\right\}. \tag{20}$$

In this section, using Theorem 2.11 instead of Lemma 4.2, we will obtain the estimate (18) imposing less regularity over $\partial_t\tau$ (replacing $H^{-1/2}(\Gamma_s)$ by $H^{-3/2}(\Gamma_s)$). More precisely, the new result is:

**Theorem 4.4** *Let $S \subseteq \mathbb{R}^2$ be a $C^3$ domain and $h \in C^3(\overline{S})$ with $h \geq h_{min} > 0$ in $\overline{S}$. If $\mathbf{F} \in L^2((0,T)\times\Omega)^2$, $\mathbf{u}_0 \in V$, $\tau \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^2) \cap L^\infty(0,T;H^{-1/2}(\Gamma_s)^2)$, for some $\varepsilon > 0$ with $\partial_t\tau \in L^2(0,T;H^{-3/2}(\Gamma_s)^2)$ and $\tau(0) \in H^{-1/2}(\Gamma_s)^2$, then there exists a unique strong solution $\mathbf{u}$ of the problem (17) in $(0,T)$. Moreover, there exists a constant $C > 0$ such that:*

$$\begin{aligned}\|\mathbf{u}\|^2_{L^\infty(V)} &+ \|\mathbf{u}\|^2_{L^2(H^2(\Omega))} + \|\partial_t\mathbf{u}\|^2_{L^2(H)} \leq C\Big\{\|\mathbf{u}_0\|^2_V + \|\mathbf{f}\|^2_{L^2(L^2(\Omega))} \\ &+ \|\tau(0)\|^2_{H^{-1/2}(\Gamma_s)} + \|\tau\|^2_{L^\infty(H^{-1/2}(\Gamma_s))} + \|\tau\|^2_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s))} + \|\partial_t\tau\|^2_{L^2(H^{-3/2}(\Gamma_s))}\Big\}\end{aligned} \tag{21}$$

**Remark 4.1** *In case of $S$ smooth enough, the hypothesis $\tau \in L^\infty(0,T;H^{-1/2}(\Gamma_s)^d)$ and $\tau(0) \in H^{-1/2}(\Gamma_s)^d$ are not necessary, because from $\tau \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^d)$ and $\partial_t\tau \in L^2(0,T;H^{-3/2}(\Gamma_s)^d)$ we can deduce $\tau \in C([0,T];H^{-1/2}(\Gamma_s)^d)$ with continuous dependence (see* **Appendix B***).*

**Proof:** Uniqueness is obtained thanks to the linearity of the problem. To get the existence, we will separate the proof in the same steps of the proof made in [5, 6]:
**Step 1: Existence of weak solution.** It can be obtained as the limit for the Galerkin approximates solutions $\mathbf{u}_m \in C^1([0,T];V_m)$ (being $V_m$ a m-dimensional subspace of $V$) such that:

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \mathbf{u}_m\cdot\varphi\,d\Omega + \nu\int_\Omega \nabla\mathbf{u}_m : \nabla\varphi\,d\Omega + \nu_3\int_\Omega \partial_z\mathbf{u}_m\cdot\partial_z\varphi\,d\Omega \\ \qquad = \displaystyle\int_\Omega \mathbf{F}_m\cdot\varphi\,d\Omega + \int_{\Gamma_s}\tau_m\cdot\varphi|_{\Gamma_s}\,d\sigma \quad \forall\varphi\in V_m, \\ \mathbf{u}_m(0) \equiv \text{ projection of } \mathbf{u}_0 \text{ over } V_m, \end{cases}$$

where $\mathbf{F}_m \in C^0([0,T];H^{-1}_{b,l}(\Omega)^2)$ and $\tau_m \in C^0([0,T];H^{-1/2}(\Gamma_s)^2)$ are smooth approximates of $\mathbf{F}$ and $\tau$, respectively.

Taking $\varphi = \mathbf{u}_m$, we can deduce that the sequence $\mathbf{u}_m$ is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$. Reasoning in a standard way, we obtain weak regularity for the limit $\mathbf{u}$.

**Step 2: Lifting of the Neumann boundary conditions.** We define $B : \tau \in H^{-1/2}(\Gamma_s)^d \to \mathbf{u} = B\tau \in V$ as the operator that solves the stationary hydrostatic Stokes problem (1) with $\mathbf{f} = \mathbf{0}$.
We also define $\mathbf{e}(t) = B(\tau(t))$ *a.e.* $t \in (0,T)$. Using Lemma 4.2 and Lemma 4.3, taking into account that $\tau(t) \in H_0^{1/2+\varepsilon}(\Gamma_s)^d$ a.e. $t \in (0,T)$, we obtain that $\mathbf{e}(t) \in H^2(\Omega)^d \cap V$ a.e. $t \in (0,T)$, and

$$\|\mathbf{e}(t)\|^2_V \leq K_1\|\tau(t)\|^2_{H^{-1/2}(\Gamma_s)}, \quad \|\mathbf{e}(t)\|^2_{H^2(\Omega)} \leq K_2\|\tau(t)\|^2_{H_0^{1/2+\varepsilon}(\Gamma_s)}.$$

Therefore $\mathbf{e} \in L^2(0,T;H^2(\Omega)^d\cap V)\cap L^\infty(0,T;V)$ and

$$\|\mathbf{e}\|^2_{L^\infty(V)} \leq K_1\|\tau\|^2_{L^\infty(H^{-1/2}(\Gamma_s))} \tag{22}$$

$$\|\mathbf{e}\|^2_{L^2(H^2(\Omega))} \leq K_2\|\tau\|^2_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s))}. \tag{23}$$

On the other hand, using Theorem 2.11 we have that, as $\partial_t\tau(t) \in H^{-3/2}(\Gamma_s)$ a.e. $t \in (0,T)$, we can define $\widetilde{\mathbf{e}}(t) = B(\partial_t\tau(t))$ a.e. $t \in (0,T)$, with $\widetilde{\mathbf{e}}(t) \in L^2(\Omega)$ and $\|\widetilde{\mathbf{e}}(t)\|_{L^2(\Omega)} \leq C\|\partial_t\tau\|_{H^{-3/2}(\Gamma_s)}$.

Now, let us see that $\widetilde{\mathbf{e}}(t) = \partial_t \mathbf{e}(t)$; taking

$$\mathbf{u}_\delta = \frac{\mathbf{e}(t+\delta) - \mathbf{e}(t)}{\delta} - \widetilde{\mathbf{e}} = B\left(\frac{\tau(t+\delta)-\tau(t)}{\delta} - \partial_t \tau(t)\right),$$

as $\dfrac{\tau(t+\delta)-\tau(t)}{\delta} - \partial_t\tau(t) \in H^{-3/2}(\Gamma_s)$, from Theorem 2.11, we deduce that:

$$\|\mathbf{u}_\delta(t)\|_{L^2(\Omega)} \le C \left\| \frac{\tau(t+\delta)-\tau(t)}{\delta} - \partial_t\tau(t)\right\|_{H^{-3/2}(\Gamma_s)} \longrightarrow 0 \quad \text{as } \delta \to 0.$$

Therefore, we can conclude that $\widetilde{\mathbf{e}}(t) = \partial_t \mathbf{e}(t)$ in $L^2(\Omega)$ a.e. $t \in (0,T)$. In this way, we obtain the bound:

$$\|\partial_t \mathbf{e}(t)\|_{L^2(\Omega)} \le C\|\partial_t\tau(t)\|_{H^{-3/2}(\Gamma_s)}$$

and

$$\|\partial_t \mathbf{e}\|_{L^2(L^2(\Omega))} \le C_1 \|\partial_t \tau\|_{L^2(H^{-3/2}(\Gamma_s))}. \tag{24}$$

**Remark 4.2** *Observe that previous step is the fundamental step in the proof, because the fact of using estimate* (15) *(and Theorem* 2.11*) instead of estimate* (19) *(and Lemma* 4.2*) allows us to weaken the hypothesis on* $\partial_t\tau$.

**Step 3: Strong solution for the homogeneous problem.** The function $\mathbf{y} = \mathbf{u} - \mathbf{e}$ verifies the following system:

$$\begin{cases} \partial_t \mathbf{y} - \nu\Delta\mathbf{y} - \nu_3 \partial^2_{zz}\mathbf{y} + \nabla\pi_s = \mathbf{h} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\langle\mathbf{y}\rangle = 0 & \text{in } (0,T)\times S, \\ \mathbf{y}|_{t=0} = \mathbf{y}_0 & \text{in } S, \\ \nu_3\partial_z\mathbf{y} = 0 & \text{on } (0,T)\times\Gamma_s, \\ \mathbf{y} = 0 & \text{on } (0,T)\times(\Gamma_b\cup\Gamma_l), \end{cases}$$

where $\mathbf{h} = \mathbf{F} - \partial_t\mathbf{e} \in L^2(0,T;L^2(\Omega)^d)$ and $\mathbf{y}_0 = \mathbf{u}_0 - \mathbf{e}(0) \in V$. Again, arguing by a Galerkin procedure, we denote by $\mathbf{y}_m : [0,T] \to V_m$ the Galerkin approximate functions, where $V_m$ is the subspace of $V = \{w^1, w^2, ..., w^m\}$ spanned by the eigenfunctions of the hydrostatic Stokes operator. These approximates solve the ordinary differential problem:

$$(R)\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \mathbf{y}_m(t)\cdot\mathbf{v}_m d\Omega + \nu\int_\Omega \nabla\mathbf{y}_m : \nabla\mathbf{v}_m d\Omega + \nu_3\int_\Omega \partial_z\mathbf{y}_m\cdot\mathbf{v}_m d\Omega \\ = \displaystyle\int_\Omega \mathbf{h}_m\cdot\mathbf{v}_m d\Omega, \quad \forall \mathbf{v}_m \in V_m, \\ \mathbf{y}_m(0) = \mathbf{y}_{0m} = \sum_{j=1}^m \left(\displaystyle\int_\Omega (\nabla\mathbf{y}_0 : \nabla\mathbf{v}^j + \partial_z\mathbf{y}_0\cdot\partial_z\mathbf{v}^j\right)\mathbf{w}^j, \end{cases}$$

being $\mathbf{h}_m$ a smooth approximated function of $\mathbf{h}$. Let us now obtain strong estimates for $\mathbf{y}_m$. First, we construct $\mathbf{y}_{0m} \in V_m$ and such that $\mathbf{y}_{0m} \to \mathbf{y}_0$ in $V$. Taking $\mathbf{v}_m = A\mathbf{y}_m(t) \in V_m$ as test functions in $(R)$, we deduce the inequality: $\forall t \in [0,T]$,

$$\frac{d}{dt}\|\mathbf{y}_m(t)\|_V^2 + \|A\mathbf{y}_m(t)\|_{L^2(\Omega)}^2 \leq \|\mathbf{h}_m\|_{L^2(\Omega)}^2.$$

Integrating in time, we get:

$$\|\mathbf{y}_m(t)\|_V^2 + \int_0^T \|A\mathbf{y}_m(t)\|_{L^2(\Omega)}^2 dt \leq \|\mathbf{y}_{0m}\|_V^2 + \int_0^T \|\mathbf{h}_m(t)\|_{L^2(\Omega)}^2 dt.$$

Therefore, the sequence $(\mathbf{y}_m)_m$ is bounded in $L^2(0,T;D(A)) \cap L^\infty(0,T;V)$, so its limit belongs to the same space and

$$\|\mathbf{y}\|_{L^\infty(V)}^2 + \|\mathbf{y}\|_{L^2(D(A))}^2 \leq \|\mathbf{y}_0\|_V^2 + \|\mathbf{h}\|_{L^2(L^2(\Omega))}^2. \tag{25}$$

Now, taking $\partial_t \mathbf{y}_m \in V_m$ as test functions in $(R)$ and integrating in time, we have:

$$\|\partial_t \mathbf{y}_m\|_{L^2(H)}^2 \leq \|\mathbf{y}_{0m}\|_V^2 + \|\mathbf{h}_m\|_{L^2(L^2\Omega))}^2,$$

and the limit $\partial_t \mathbf{y} \in L^2(H)$ and verifies the same inequality:

$$\|\partial_t \mathbf{y}\|_{L^2(H)}^2 \leq \|\mathbf{y}_0\|_V^2 + \|\mathbf{h}\|_{L^2(L^2\Omega))}^2 \tag{26}$$

Adding (25) to (26), and using that $\mathbf{y}_0 = \mathbf{u}_0 - \mathbf{e}(0)$ and $\mathbf{h} = \mathbf{F} - \partial_t \mathbf{e}$, we conclude that:

$$\begin{aligned} \|\mathbf{y}\|_{L^\infty(V)}^2 &+ \|\mathbf{y}\|_{L^2(D(A))}^2 + \|\partial_t \mathbf{y}\|_{L^2(H)}^2 \leq 4\left\{\|\mathbf{u}_0\|_V^2 + \|\mathbf{e}(0)\|_V^2 \right. \\ &+ \left. \|\mathbf{F}\|_{L^2(L^2(\Omega))}^2 + \|\partial_t \mathbf{e}\|_{L^2(L^2(\Omega))}^2 \right\}. \end{aligned}$$

Finally, replacing estimates (19) for $\mathbf{e}(0)$ in $V$ and estimates (24) for $\partial_t \mathbf{e}$ in $L^2(0,T;L^2(\Omega))$, we obtain (18). ■

**Remark 4.3 Aplications to the non-stationary nonlinear Primitive Equations.** *The extension of the result from Theorem* 4.4 *to the nonlinear case is identical to that obtained in [5, 6], replacing the use made there of Theorem* 4.1 *by Theorem* 4.4*.*

# A The isomorphism

**Proposition A.1 (before cited as Proposition 2.10)** *The space $(H^1(S))'/\mathbb{R}$ is isomorphic to $\mathcal{H}'$.*

**Proof:** Recall that $\mathcal{H} = \{\varphi \,/\varphi \in H^1(S),\ \int_S \varphi\, d\mathbf{x} = 0\}$.

**a)** $H^1(S) = \mathcal{H} \oplus \mathcal{H}^\perp$**,** identifying $\mathcal{H}^\perp$ with the space spanned by the constant functions over $S$. In fact, $\omega = 1$ is the unique solution for the linear problem:

$$\begin{cases} -\Delta\omega + \omega = 1 & \text{in } S, \\ \dfrac{\partial\omega}{\partial\mathbf{n}} = 0 & \text{on } \partial S. \end{cases} \tag{27}$$

Using $(27)_1$, we deduce that:

$$((\omega,\varphi))_1 = \int_S \nabla\omega\cdot\nabla\varphi\,d\mathbf{x} + \int_S \omega\,\varphi\,d\mathbf{x} = \int_S \varphi\,d\mathbf{x} = 0, \quad \forall\varphi\in\mathcal{H},$$

where $((\cdot,\cdot))_1$ denotes the inner product in $H^1(S)$. Therefore, $\omega = 1$ is perpendicular (in $H^1(S)$) to $\mathcal{H}$.
On the other hand, every function $v \in H^1(S)$ can be written in a unique manner as:

$$v = \varphi + \alpha,\ \alpha\in\mathbb{R},$$

with $\alpha = \frac{1}{|S|}\int_S v\,d\mathbf{x}$ and $\varphi = v - \alpha \in \mathcal{H}$.

**b) The isomorphism:** We define the operator

$$\begin{array}{rcrcl} T & : & (H^1(S))'/\mathbb{R} & \longrightarrow & \mathcal{H}' \\ & & q & \longmapsto & Tq \end{array}$$

as

$$\langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}} = \langle q,\varphi\rangle_{(H^1(S))',H^1(S)}, \quad \forall\varphi\in\mathcal{H}.$$

Such operator is **well-defined** because if $c$ is a constant, $T(q+c) = Tq$. Indeed, $\langle T(q+c),\varphi\rangle = \langle q+c,\varphi\rangle = \langle q,\varphi\rangle$, because of $\int_S \varphi\,d\mathbf{x}$. It suffices to prove that $T$ is a continuous bijection.

***T* is one-to-one:** Suppose that $Tq = 0$, i.e. $\langle Tq,\varphi\rangle = 0$, $\forall\varphi\in\mathcal{H}$. Let $v\in H^1(S)$ and write, following the previous decomposition, $v = \varphi+\alpha$ with $\alpha = \frac{1}{|S|}\int_S v\,d\mathbf{x}$ and $\varphi\in\mathcal{H}$ . Then,

$$\begin{aligned} \langle q,v\rangle_{(H^1(S))',H^1(S)} &= \langle q,\varphi+\alpha\rangle_{(H^1(S))',H^1(S)} = \langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}} + \frac{1}{|S|}\langle q,1\rangle_{(H^1(S))',H^1(S))}\int_S v\,d\mathbf{x} \\ &= \frac{1}{|S|}\langle q,1\rangle_{(H^1(S))',H^1(S)}\int_S v\,d\mathbf{x}, \end{aligned}$$

therefore $\langle q - \frac{1}{|S|}\langle q,1\rangle_{(H^1(S))',H^1(S)}, v\rangle_{(H^1(S))',H^1(S)} = 0\ \forall v\in H^1(S)$, and thus $q - \frac{1}{|S|}\langle q,1\rangle_{(H^1(S))',H^1(S)}$ belongs to the zero equivalent class in $(H^1(S))'/\mathbb{R}$, and therefore $q = 0$ in $(H^1(S))'/\mathbb{R}$.

***T* is surjective:** If $l\in\mathcal{H}'$, we have to prove that there exists an element $q\in(H^1(S))'/\mathbb{R}$ such that $Tq = l$, i. e., $\langle q,\varphi\rangle_{(H^1(S))',H^1(S)} = \langle l,\varphi\rangle_{\mathcal{H}',\mathcal{H}}\ \forall\varphi\in\mathcal{H}$. Indeed, it suffices to define $\langle q,v\rangle_{(H^1(S))',H^1(S)} = \langle l,\varphi\rangle_{\mathcal{H}',\mathcal{H}}$ if $v = \varphi+\alpha$.

***T* is continuous:** Using the standard norm definitions,

$$\begin{aligned} \|q\|_{(H^1(S))'/\mathbb{R}} &= \inf_{c\in\mathbb{R}}\|q+c\|_{(H^1(S))'} = \inf_{c\in\mathbb{R}}\sup_{v\in H^1(S)}\frac{1}{\|v\|_{H^1(S)}}\langle q+c,v\rangle_{(H^1(S))',H^1(S)} \\ &\geq \inf_{c\in\mathbb{R}}\sup_{\varphi\in\mathcal{H}}\frac{1}{\|\varphi\|_{H^1(S)}}\langle q+c,\varphi\rangle_{(H^1(S))',H^1(S)} \\ &= \sup_{\varphi\in\mathcal{H}}\frac{1}{\|\varphi\|_{H^1(S)}}\langle q,\varphi\rangle_{(H^1(S))',H^1(S)} \\ &= \sup_{\varphi\in\mathcal{H}}\frac{1}{\|\varphi\|_{H^1(S)}}\langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}} = \|Tq\| \end{aligned}$$

■

# B Regularity in $S$

**Lemma B.1** *Suppose that* $w \in H^s(\Omega)$, $1 \le s \le 2$, *and* $h \in H^2(S)$. *Then,* $\langle w \rangle \in H^s(S)$.

**Proof:** If $\mathbf{w} \in L^2(\Omega)^2$, then integrating in $(-h(\mathbf{x}), 0)$ we obtain:

$$|\langle w \rangle|^2 = \left| \int_{-h(\mathbf{x})}^0 w(\mathbf{x}, z) dz \right|^2 \le \left( \int_{-h(\mathbf{x})}^0 |w(\mathbf{x}, z)|^2 dz \right) h(\mathbf{x}).$$

Integrating now in $S$, we obtain:

$$\begin{aligned} \int_S |\langle w \rangle|^2 d\mathbf{x} &\le \int_S h(\mathbf{x}) \left( \int_{-h(\mathbf{x})}^0 |w(\mathbf{x}, z)|^2 ds \right) d\mathbf{x} \\ &\le h_{\max} \int_\Omega |w(\mathbf{x}, z)|^2 d\Omega = h_{\max} \|w\|^2_{L^2(\Omega)}, \end{aligned}$$

that implies:

$$\|\langle w \rangle\|_{L^2(S)} \le \sqrt{h_{\max}}\, \|w\|_{L^2(\Omega)}. \tag{28}$$

Deriving $\langle w \rangle$, we get that $\nabla \langle w \rangle = \langle \nabla w \rangle + w|_{\Gamma_b} \nabla h$. Taking the $L^2$-norm over $S$ and using (28), we obtain:

$$\|\nabla \langle w \rangle\|_{L^2(S)} \le \|\langle \nabla w \rangle\|_{L^2(S)} + \|w|_{\Gamma_b} \nabla h\|_{L^2(S)} \le \sqrt{h_{\max}} \|w\|_{L^2(\Omega)} + \|w|_{\Gamma_b}\|^{1/2}_{L^4(S)} \|\nabla h\|^{1/2}_{L^4(S)}.$$

Let us focus our attention on the terms of $L^4(S)$-type:

$$\begin{aligned} \|w|_{\Gamma_b}\|_{L^4(S)} &= \left( \int_S |w(\mathbf{x}, -h(\mathbf{x}))|^4 \, d\mathbf{x} \right)^{1/4} = \left( \int_{\Gamma_b} |w(\mathbf{x}, -h(\mathbf{x}))|^4 \left(1 + |\nabla h(\mathbf{x})|^2\right)^{-1/2} d\sigma \right)^{1/4} \\ &\le \|w\|_{L^4(\Gamma_b)} \le C \, \|w\|_{H^1(\Omega)}, \end{aligned}$$

where we have used that $H^1(\Omega) \hookrightarrow L^4(\partial\Omega)$. On the other hand,

$$\|\nabla h\|_{L^4(S)} \le C \, \|h\|_{H^2(S)},$$

where now we use that $H^2(S) \hookrightarrow W^{1,4}(S)$. Therefore,

$$\|\nabla \langle w \rangle\|^2_{L^2(S)} \le h_{\max} \, \|\nabla w\|^2_{L^2(\Omega)} + C \, \|w\|_{H^1(\Omega)} \|h\|_{H^2(\Omega)}. \tag{29}$$

The estimates (28) and (29) let us deduce that if $w \in H^1(\Omega)$ and $h \in H^2(S)$, then $\langle w \rangle \in H^1(S)$.

Now, we study the case where $w \in H^2(\Omega)$ and $h \in H^2(S)$. ¿From the previous estimates, we only need to estimate second order derivatives for $w$. For simplicity, we reason for $\partial^2_{xx} \langle w \rangle$, and then we will extend the result for the other derivatives. We have:

$$\partial^2_{xx} \langle w \rangle = \langle \partial^2_{xx} w \rangle + 2(\partial_x w)|_{\Gamma_b} \partial_x h(\mathbf{x}) - (\partial_z w)|_{\Gamma_b} |\partial_x h(\mathbf{x})|^2 + w|_{\Gamma_b} \partial^2_{xx} h(\mathbf{x}).$$

Therefore, using the same arguments as before:

$$\begin{aligned} \|\partial^2_{xx} \langle w \rangle\|^2_{L^2(S)} &\le h_{\max} \, \|\partial^2_{xx} w\|^2_{L^2(\Omega)} \\ &+ C \left( \|\partial_x w\|_{H^1(\Omega)} \|h\|_{H^2(S)} + \|\partial_z w\|_{H^1(\Omega)} \|h\|^2_{H^2(S)} + \|w|_{\Gamma_b}\|_{L^\infty(S)} \|h\|_{H^2(S)} \right) \\ &\le h_{\max} \, \|\partial^2_{xx} w\|^2_{L^2(\Omega)} + C \|w\|_{H^2(\Omega)} \|h\|_{H^2(S)} \left(1 + \|h\|_{H^2(S)}\right), \end{aligned} \tag{30}$$

where we have used in the last term that if $w \in H^2(\Omega)$, then $w|_{\partial\Omega} \in H^{3/2}(\partial\Omega) \hookrightarrow L^\infty(\partial\Omega)$. The expression (30) together with (28) and (29), let us deduce that if $w \in H^2(\Omega)$ and $h \in H^2(S)$, then $\langle w\rangle \in H^2(S)$.

We have just proved that, supposing that $h \in H^2(S)$, if $w \in H^1(\Omega)$ then $\langle w\rangle \in H^1(S)$, and if $w \in H^2(\Omega)$ then $\langle w\rangle \in H^2(S)$. In the case when $s \in (1,2)$, interpolation results let us see $H^s(\Omega) = [H^1(\Omega), H^2(\Omega)]_\theta$ and $H^s(S) = [H^1(S), H^2(S)]_\theta$ with $\theta = s-1$. Then, under the hypothesis $h \in H^2(S)$, we can deduce that if $w \in H^s(\Omega)$, then $\langle w\rangle \in H^s(S)$. ∎

# C Interpolation results

**Lemma C.1** *Let $S$ be a bounded open set of $\mathbb{R}^2$, with smooth enough boundary $\partial S \in C^\infty$. If $\tau \in L^2(0,T;H_0^{1/2+\varepsilon}(S))$ for some $\varepsilon > 0$ and $\partial_t\tau \in L^2(0,T;H^{-3/2}(S))$, then $\tau \in C^0([0,T];H^{-1/2}(S))$.*

In order to proof this lemma, we will use some interpolation results appearing in Lions-Magenes [8]:

**Theorem C.2 ([8], page 79)**
*Suppose that $\Gamma_s \in C^\infty$. Let $s_1$, $s_2 \geq 0$ such that $s_i \neq \lambda + 1/2$ ($\lambda$ integer, $i = 1,2$). Let $\theta \in [0,1]$ such that:*

$$(1-\theta)s_1 - \theta s_2 \neq \mu + 1/2 \quad \text{and} \quad \neq -\mu - 1/2 \quad (\mu \text{ integer} \geq 0). \tag{31}$$

*Then,*

$$\left[H_0^{s_1}(\Gamma_s), H^{-s_2}(\Gamma_s)\right]_\theta = \begin{cases} H_0^{(1-\theta)s_1-\theta s_2}(\Gamma_s) & \text{if } (1-\theta)s_1 - \theta s_2 \geq 0, \\ H^{(1-\theta)s_1-\theta s_2}(\Gamma_s) & \text{if } (1-\theta)s_1 - \theta s_2 \leq 0. \end{cases} \tag{32}$$

**Proposition C.3 ([8], page 53)** *Let $X$ and $Y$ two separable Hilbert spaces such that $X \subset Y$, $X$ dense in $Y$ with continuous embedding. Then:*

$$\left[H^{t_1}(\Omega;X), H^{t_2}(\Omega;Y)\right]_\theta = H^{(1-\theta)t_1+\theta t_2}(\Omega;[X,Y]_\theta).$$

**Proof of Lemma** $C.1$**:** In our case, taking in Theorem B.2, $\mu = 0$, $s_1 = 1/2+\varepsilon$, $s_2 = 3/2+\alpha$ with $\alpha < \varepsilon$ one has

$$\left[H_0^{1/2+\varepsilon}(\Gamma_s), H^{-3/2-\alpha}(\Gamma_s)\right]_\theta = H^{-1/2+\beta}(\Gamma_s),$$

if one imposes that $(1-\theta)s_1 - \theta s_2 = -1/2+\beta < 0$ where $\beta$ is small enough, then:

$$\theta = \frac{1+\varepsilon-\beta}{2+\varepsilon+\alpha} > \frac{1}{2} \quad (\text{taking } \varepsilon > \alpha + 2\beta).$$

By hypothesis for $\tau$, $\tau \in H^1(0,T;H^{-3/2}(\Gamma_s)) \cap H^0(0,T;H_0^{1/2+\varepsilon}(\Gamma_s))$. We use then Proposition $C.3$ for $t_1 = 0$, $t_2 = 1$, $X = H_0^{1/2+\varepsilon}(\Gamma_s)$, $Y = H^{-3/2-\alpha}(\Gamma_s)$ and $\Omega = (0,T)$. Then, $\tau \in H^\theta(0,T;H^{-1/2+\beta}(\Gamma_s))$ (with $\theta > 1/2$), hence in particular one has $\tau \in C^0([0,T];H^{-1/2}(\Gamma_s))$. ∎

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