A Tridimensional Phase-Field Model with Convection for Change Phase of an Alloy

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Abstract

We consider a tridimensional phase-field model for a solidification/melting non-stationary process, which incorporates the physics of binary alloys, thermal properties and fluid motion of non-solidified material. The model is a free-boundary value problem consisting of a non-linear parabolic system including a phase-field equation, a heat equation, a concentration equation and a variant of the Navier-Stokes equations modified by a penalization term of Carman-Kozeny type to model the flow in mushy regions and a Boussinesq type term to take into account the effects of the differences in temperature and concentration in the flow. A proof of existence of generalized solutions for the system is given. For this, the problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder's fixed point theorem. A solution of the original problem is then found by using compactness arguments.

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1 Introduction

This paper is concerned with a non-isothermal phase-field model that accounts for both solidification/melting of a binary alloy and fluid motion.

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The present approach is based on ideas of Blanc et al.[1] and Voller et al. [15] to model the possibility of flow and those of Caginalp et al. [2] for the phase-field and the thermal properties of the alloy, and simpler versions were also considered in [11] and [12]. It is described as the following coupled system,

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \text{ in } Q, \quad (1)$$

$$C_{\rm v}\theta_t + C_{\rm v} v\cdot\nabla\theta = \nabla\cdot K_1(\phi)\nabla\theta + \frac{l}{2}f_s(\phi)_t \quad \text{in } Q, \quad (2)$$

$$c_t + v\cdot\nabla c = K_2\left(\Delta c + M\nabla\cdot c(1-c)\nabla\phi\right) \text{ in } Q, \quad (3)$$

$$v_t + \nu_o Av - \nu\Delta v + v\cdot\nabla v + \nabla p + k(f_s(\phi))v = \mathcal{F}(c,\theta) \text{ in } Q_{ml}, \quad (4)$$

$$\text{div } v = 0 \text{ in } Q_{ml}, \quad (5)$$

$$v = 0 \text{ in } Q_s, \quad (6)$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega\times(0,T), \quad v = 0 \text{ on } \partial Q_{ml}, \quad (7)$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \text{ in } \Omega, \quad v(0) = v_0 \quad \text{in } \Omega_{ml}(0), \quad (8)$$

where $Q = \Omega\times(0,T)$, $0 < T < +\infty$ and Ω is an open bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Here, ϕ is the phase-field which is the state variable characterizing the different phases; θ denotes the temperature, and $c \in [0,1]$ denotes the concentration, which is the fraction of one of the two materials in the mixture; v is the velocity field; p is the associated hydrostatic pressure; ν and ν_0 are positive constants corresponding to viscosities associated to the fluid material; $f_s \in [0,1]$ is the solid fraction.

The operator A is defined by

$$Av = -\text{div}\left(|\nabla v|^{p-2}\nabla v\right), \qquad p \geq 3,$$

The penalization term $k(f_s)$ is the Carman-Kozeny type term and accounts for mushy effects in the flow; its usual expression is $k(f_s) = C_0{f_s}^2/(1-f_s)^3$, but more general expressions will be allowed in this paper. $\mathcal{F}(c,\theta)$ is the buoyancy force, which by using Boussinesq approximation is given by $\mathcal{F}(c,\theta) = \rho\mathbf{g}\left(c_1(\theta-\theta_r) + c_2(c-c_r)\right) + F$, where ρ is the mean value of the density; $\mathbf{g}$ is the acceleration of gravity; c_1 and c_2 are two real constants; θ_r, c_r are respectively the reference temperature and concentration, which for simplicity of exposition are assumed to be zero; F is an external force. The following physical parameter are assumed to be constant: $\alpha > 0$ the relaxation scaling; $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width; σ is the surface tension, and $[s]$ is the entropy density difference between phases. $C_{\rm v} > 0$ is the specific heat; $l > 0$ is associated to the latent heat; θ_A and θ_B are the respective melting temperatures of two materials composing the alloy; $K_2 > 0$ is the solute diffusivity, and M a constant related to the slopes of solidus and liquidus lines. Finally, $K_1 > 0$ denotes the thermal conductivity, which we will assume to depend on the phase-field.

We observe that equation (4) is associated to a modified form of the classical form of the Navier-Stokes equations as proposed by Ladyzenskaja in [6], in which the effective fluid viscosity depends on the gradient of the velocity.

The domain Q is composed of three regions: Q_s, Q_m and Q_l. The first region is fully solid; the second is mushy, and the third is fully liquid. They are defined by

$$\begin{array}{rcl} Q_s &=& \{\ (x,t) \in Q \quad / \quad f_s(\phi(x,t)) = 1\ \}, \\ Q_m &=& \{\ (x,t) \in Q \quad / \quad 0 < f_s(\phi(x,t)) < 1\ \}, \\ Q_l &=& \{\ (x,t) \in Q \quad / \quad f_s(\phi(x,t)) = 0\ \}, \end{array} \tag{9}$$

and Q_{ml} will refer to the not fully solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{\ (x,t) \in Q \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{10}$$

At each time $t \in [0,T]$, $\Omega_{ml}(t)$ is defined by

$$\Omega_{ml}(t) = \{\ x \in \Omega \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{11}$$

In view of these regions are a priori unknown, the model is a free boundary problem.

Throughout this paper we assume the conditions,

(H1) f_s is a Lipschitz continuous function defined on $\mathbb{R}$ and satisfying $0 \le f_s(r) \le 1$ for all $r \in \mathbb{R}$; moreover f_s' is measurable,

(H2) k is a non decreasing function of class $C^1[0,1)$, satisfying $k(0) = 0$, $\lim_{x \to 1^-} k(x) = +\infty$,

(H3) K_1 is a Lipschitz continuous function defined on $\mathbb{R}$; there exist $0 < a \le b$ such that $0 < a \le K_1(r) \le b$ for all $r \in \mathbb{R}$,

(H4) F is a given function in $L^2(Q)$.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq \mathbb{R}^3$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0,T)$ Then,

$$\begin{aligned}
\mathcal{V}(G) &= \left\{ w \in \left(C_0^\infty(G)\right)^3,\ \text{div } w = 0 \right\}, \\
H(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^3, \\
V^p(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(W_0^{1,p}(G)\right)^3, \\
V(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^3, \\
H^{\tau,\tau/2}(\overline{Q}_G) &= \text{Hölder continuous functions of exponent } \tau \text{ in } x \\
& \quad\ \text{and exponent } \tau/2 \text{ in } t, \\
W_q^{2,1}(Q_G) &= \left\{ w \in L^q(Q_G) / \, D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \right\}.
\end{aligned}$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$, $V^p = V^p(\Omega)$. Properties of these functional spaces can be found for instance in [7, 9, 14]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^3$.

Our purpose in this work is to show that problem (1)-(8) is solvable in a generalized sense to be made precise below.

The main result of this paper is the following.

Theorem 1 *Let be $T > 0$, $p \geq 3$, $5/2 < q \leq 10/3$, $\Omega \subseteq \mathbb{R}^3$ an open bounded domain de class C^3. Suppose that $v_0 \in H(\Omega_{ml}(0))$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, $\theta_0 \in L^2(\Omega)$ and $c_0 \in L^2(\Omega)$, $0 \leq c_0 \leq 1$ a.e. in $\bar{\Omega}$, satisfying the compatibility conditions $\dfrac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$. Under the assumptions* **(H1)-(H4)**, *there exist functions $(\phi, \theta, c, v, \chi)$ satisfying*

(i) $\phi \in W_q^{2,1}(Q)$, $\phi(0) = \phi_0$,

(ii) $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta(0) = \theta_0$,

(iii) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q,

(iv) $v \in L^p(0,T;V^p) \cap L^\infty(0,T;H)$, $v = 0$ *a.e. in* $\overset{\circ}{Q}_s$, $v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where Q_s is defined by (9) and $\Omega_{ml}(0)$ by (11),*

(v) $\chi \in L^{p'}(0,T;(V^p)')$

and such that for any $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\zeta_t \in L^2(0,T;L^2(\Omega))$ *and* $\zeta(T)=0$ *in* Ω*, we have*

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3) + \beta\left(\theta + (\theta_B-\theta_A)c - \theta_B\right) \quad a.e.\ in\ Q, \tag{12}$$

$$\frac{\partial\phi}{\partial n} = 0 \qquad a.e.\ on\ \partial\Omega\times(0,T), \tag{13}$$

$$\begin{aligned} -C_{\rm v}\int_0^T\int_\Omega \theta\zeta_t dxdt - C_{\rm v}\int_0^T\int_\Omega v\theta\cdot\nabla\zeta dxdt + \int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\zeta\, dxdt \\ = \frac{l}{2}\int_0^T\int_\Omega f_s(\phi)_t\zeta\, dxdt + C_{\rm v}\int_\Omega \theta_0\zeta(0)dx, \end{aligned} \tag{14}$$

$$\begin{aligned} -\int_0^T\int_\Omega c\zeta_t dxdt - \int_0^T\int_\Omega vc\cdot\nabla\zeta dxdt + K_2\int_0^T\int_\Omega \nabla c\cdot\nabla\zeta\, dxdt \\ + K_2 M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta\, dxdt = \int_\Omega c_0\zeta(0)dx. \end{aligned} \tag{15}$$

Also, for any $t\in(0,T)$ *and* $\eta\in L^p(0,T;V^p)$ *with compact support contained in* $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ *and such that* $\eta_t\in L^{p'}(0,T;(V^p)')$*, where* Q_{ml} *is defined by (10) and* $\Omega_{ml}(t)$ *by (11), there hold*

$$\begin{aligned} (v(t),\eta(t)) - \int_0^t (v,\eta_t)ds + \nu\int_0^t(\nabla v,\nabla\eta)ds + \nu_o\int_0^t(\chi,\eta)ds \\ + \int_0^t(v\cdot\nabla v,\eta)ds + \int_0^t(k(f_s(\phi))v,\eta)ds \\ = \int_0^t(\mathcal{F}(c,\theta),\eta)ds + (v_0,\eta(0)). \end{aligned} \tag{16}$$

Moreover, (ϕ,θ,c,v,χ) *is a generalized solution of (1)-(8), in the sense that under the following additional regularity and integrability assumptions:*

- *for a.e.* $t\in(0,T)$*, the boundary* $\partial\Omega_{ml}(t)$ *of* Ω_{ml} *in* Ω *has zero Lebesgue measure*
- *suppose that* $k(f_s(\phi))\in L^s(0,T;L^{1+\delta}(\Omega_{ml}(t)))$*, with* $s=p/(p-2)$*, any* $\delta>0$ *when* $p=3$ *and* $\delta=0$*, when* $p>3$*,*

then $\chi = Av$ *in the sense of distribution in* Q_{ml}*.*

Remark: Interpreting the modified Navier-Stokes equations requires some topological information about the set occupied by the fluid. In fact, one should know that such a set is open to interpret the modified Navier-Stokes equations at least in the sense of distributions. This information is in particular implied by the continuity of phase-field, which in turn depends on the degree of smoothness of the other variables. In the two dimensional case, that is $\Omega \subseteq \mathbb{R}^2$, and when $\nu_o = 0$, an existence theorem for system (1)-(8) was obtained in [12]. The main feature of [12] is that the smoothness of v, θ and c suffice to yield the continuity of ϕ. In the three dimensional case, this does not appear possible. To stress this point, consider weak solutions of the classical Navier-Stokes equations with external force in $L^2(Q)$. It is well known that such solution satisfies $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ (see e.g. [14]). If $\Omega \subseteq \mathbb{R}^2$, this regularity implies that $|v| \in L^4(Q)$. This fact together with (12)-(13) suffices to prove that ϕ is continuous, and therefore the set Q_{ml} is open. In the three dimensional case, we just have that $v \in L^{10/3}(Q)$. This modest degree of integrability of velocity prevents us from proving that ϕ is continuous. When $\nu_o > 0$ and p is large enough, as it is the case of the present paper, it is possible to get more regularity of v and then the required continuity of ϕ. In fact, if $v \in L^p(0,T;V^p) \cap L^\infty(0,T;H)$, by interpolation ([8] p. 207), we conclude that $v \in L^{p5/3}(Q)$. Taking $p \geq 3$ is then enough to yield the continuity of ϕ (see Thm 2) together the additional restriction $q > 5/2$, because in this case $W^{2,1}_q(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$, with $\tau = 2 - 5/q$ ([7] p. 80). Therefore the set Q_{ml} is open, giving a meaningful interpretation to the velocity equation. The restriction $q \leq 10/3$ is consequence of the obtained regularity of temperature. This will be clear in the next section.

The previous existence result will be obtained by using a regularization technique similar to the one already used in [1] and [11, 12]. The idea is to use a auxiliary parameter to transform the original free-boundary value problem in a penalized but more standard problem. This will be called the regularized problem and will be studied by using fixed point arguments. Then, by using compactness arguments as the auxiliary parameter goes to zero, we obtain a generalized solution of the original problem.

The outline of this paper is as follows. In Section 2, we study an auxiliary problem. Then, in Section 3, we study a regularized problem. Section 4 is devoted to the proof of the main existence theorem.

2 An auxiliary problem

We consider the initial boundary value problem,

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3)+g \quad \text{in } Q, \tag{17}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{18}$$

$$\phi(0) = \phi_0 \quad \text{in } \Omega, \tag{19}$$

and prove the following result using a technique similar to the one already used in [5] to treat a phase-field equation without convective term or in [12] to treat the two dimensional problem.

Theorem 2 *Let be $T>0$, $q\geq 2$, $p\geq 3$. Suppose that $g\in L^q(Q)$, $v\in L^p(0,T;V^p)\cap L^\infty(0,T;H)$ and $\phi_0\in W^{2-2/q,q}(\Omega)$ satisfying the compatibility condition $\frac{\partial\phi_0}{\partial n}=0$ on $\partial\Omega$. Then*

i) *If $2\leq q<5$, there exists a unique $\phi\in W_q^{2,1}(Q)$ solution of problem (17)-(19) , which satisfies the estimate*

$$\|\phi\|_{W_q^{2,1}(Q)} \leq C\left(\|\phi_0\|_{W^{2-2/q,q}(\Omega)} + \|g\|_{L^q(Q)} + \|\phi_0\|^3_{W^{2-2/q,q}(\Omega)} + \|g\|^3_{L^q(Q)}\right) \tag{20}$$

where C depends on $\|v\|_{L^5(Q)}$, on Ω and T,

ii) *If $q\geq 5$ and $p>3$, there exists a unique $\phi\in W_r^{2,1}(Q)$, $r=\min\{q,p5/3\}$ solution of problem (17)-(19), which satisfies the estimate (20) where C depends on $\|v\|_{L^{p5/3}(Q)}$, on Ω and T.*

Proof: In order to apply Leray-Schauder fixed point theorem ([3] p. 189) we consider the operator T_λ, $0\leq\lambda\leq 1$, on the Banach space $B=L^6(Q)$, which maps $\hat\phi\in B$ into ϕ by solving the problem

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{\lambda}{2}(\hat\phi-\hat\phi^3)+\lambda g \quad \text{in } Q, \tag{21}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{22}$$

$$\phi(0) = \phi_0 \quad \text{in } \Omega. \tag{23}$$

We define $G_\lambda = \frac{\lambda}{2}(\hat{\phi} - \hat{\phi}^3) + \lambda g$ and we observe that $G_\lambda \in L^2(Q)$. Since $v \in L^5(Q)$, we infer from L^p-theory of parabolic equations ([7] Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution ϕ of problem (21)-(23) with $\phi \in W_2^{2,1}(Q)$. Due to the embedding of $W_2^{2,1}(Q)$ into $L^{10}(Q)$ ([9] p.15), the operator T_λ is well defined from B into B.

To prove continuity of T_λ, let $\hat{\phi}_n \in B$ strongly converging to $\hat{\phi} \in B$; for each n, let $\phi_n = T_\lambda(\hat{\phi}_n)$. We have that ϕ_n satisfies the following estimate ([7] p. 341)

$$\|\phi_n\|_{W_2^{2,1}(Q)} \le C\left(\|\hat{\phi}_n\|_{L^2(Q)} + \|\hat{\phi}_n\|^3_{L^6(Q)} + \|g\|_{L^2(Q)} + \|\phi_0\|_{H^1(\Omega)}\right)$$

for some constant C independent of n. Since $W_2^{2,1}(Q)$ is compactly embedded in $L^2(0,T;H^1(\Omega))$ ([13] Cor.4) and in $L^9(Q)$, it follows that there exist a subsequence of ϕ_n (which we still denote by ϕ_n) strongly converging to $\phi = T_\lambda(\hat{\phi})$ in B. Hence T_λ is continuous for all $0 \le \lambda \le 1$. At the same time, T_λ is bounded in $W_2^{2,1}(Q)$, and the embedding of this space in B is compact. Thus, we conclude that T_λ is a compact operator for each $\lambda \in [0,1]$.

To prove that for $\hat{\phi}$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $\phi_i\,(i = 1,2)$ be the corresponding solutions of (21)-(23). For $\phi = \phi_1 - \phi_2$ the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q)} \le C|\lambda_1 - \lambda_2|\left(\|\hat{\phi}\|_{L^2(Q)} + \|\hat{\phi}\|^3_{L^6(Q)} + \|g\|_{L^2(Q)}\right)$$

where C is independent of λ_i. Therefore, T_λ is uniformly continuous in λ.

Now we have to estimate the set of all fixed points of T_λ, let $\phi \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{\lambda}{2}(\phi - \phi^3) + \lambda g \quad \text{in } Q, \tag{24}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{25}$$

$$\phi(0) = \phi_0 \quad \text{in } \Omega. \tag{26}$$

We multiply (24) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over $\Omega\times(0,t)$. After integration by parts and the use the Hölder's, Young's and Gagliardo-

Nirenberg's inequalities, we obtain in the usual manner the following estimate

$$\begin{aligned}\int_\Omega \left(\phi^2+|\nabla\phi|^2\right)dx+\|\phi\|^2_{W_2^{2,1}(Q)} &\le C\left(\|g\|^2_{L^2(Q)}+\|\phi_0\|^2_{H^1(\Omega)}\right)\\ &+C\int_0^t\left(1+\|v\|^5_{L^5(\Omega)}\right)\left(\|\phi\|^2_{L^2(\Omega)}+\|\nabla\phi\|^2_{L^2(\Omega)}\right)dt\end{aligned} \tag{27}$$

where C is independent of λ. By applying Gronwall's Lemma we get

$$\|\phi\|_{L^6(Q)}\le C\|\phi\|_{W_2^{2,1}(Q)}\le C'$$

where C and C' are constants independent of λ. Therefore, all fixed points of T_λ in B are bounded independently of $\lambda\in[0,1]$.

Finally, for $\lambda=0$, it is clear that problem (21)-(23) has a unique solution. Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $\phi\in B\cap W_2^{2,1}(Q)$ of the operator T_1, i.e., $\phi=T_1(\phi)$. This corresponds to a solution of problem (17)-(19). Now we have to examine the regularity of ϕ. To prove **i)** we discuss the cases $2\le q\le 3$ and $3<q<5$ separately.

If $2\le q\le 3$, since $W_2^{2,1}(Q)$ is embedded into $L^9(Q)$, we have that $G=\frac{1}{2}(\phi-\phi^3)+g\in L^q(Q)$ and this implies $\phi\in W_q^{2,1}(Q)$. If $3<q<5$, we have that $G\in L^3(Q)$ and as a consequence $\phi\in W_3^{2,1}(Q)$. According to embedding ([9] p.15) we can conclude that $\phi\in L^\infty(Q)$ and consequently $\phi\in W_q^{2,1}(Q)$. To prove estimate (20) we restrict to the case $2\le q\le 3$. The proof for $3<q<5$ is similar. Observe that from L^p-theory of parabolic equations we have

$$\begin{aligned}\|\phi\|_{W_q^{2,1}(Q)} &\le C\left(\|G\|_{L^q(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\le C\left(\|g\|_{L^q(Q)}+\|\phi\|_{L^q(Q)}+\|\phi\|^3_{L^{3q}(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\le C\left(\|g\|_{L^q(Q)}+\|\phi\|_{W_2^{2,1}(Q)}+\|\phi\|^3_{W_2^{2,1}(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right).\end{aligned}$$

Using estimate (27) we deduce (20).

If $q\ge 5$ and $p>3$, we have that $G\in L^q(Q)$ and since $v\in L^{p5/3}(Q)$, from L^p-theory of parabolic equations we can conclude that $\phi\in W_r^{2,1}(Q)$ where $r=\min\{q,p5/3\}$. The estimative (20) is proved by analogous reasoning.

It remains to show uniqueness of the solution. Let us assume that ϕ_1 and ϕ_2 are two solutions of problem (17)-(19). Then the difference $\phi=\phi_1-\phi_2$

satisfies the following initial boundary value problem

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{1}{2}\phi\left(1-(\phi_1^2+\phi_1\phi_2+\phi_2^2)\right) \quad \text{in } Q, \quad (28)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), \quad (29)\\ \phi(0) &= 0 \quad \text{in } \Omega, \quad (30) \end{aligned}$$

We remark that $d := \phi_1^2+\phi_1\phi_2+\phi_2^2 \geq 0$. Multiplying (28) by ϕ and using the usual method of Gronwall's Lemma give us $\phi \equiv 0$. Therefore, the solution of problem (17)-(19) is unique and the proof of Theorem 2 is then complete. □

3 A regularized problem

In this section we turn to the full problem and introduce a regularized problem to lead with the modified Navier-Stokes equations in the whole domain instead of unknown regions, as well as with suitable regularity to the coefficients. We prove an existence result using Leray-Schauder Fixed Point Theorem ([3] p. 189).

Before doing so, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(\mathbb{R}^4)$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(\mathbb{R}^4)} \leq C\,\|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [10] p.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(\mathbb{R}^3)$ be a family of symmetric positive mollifier functions converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(\mathbb{R}^3)$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $\mathbb{R}^4\backslash Q$. Then, as in [10] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $\mathbb{R}^4$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(\mathbb{R}^4)$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in \mathbb{R}^3 : d(x, \Omega) < \delta\}. \tag{31}$$

We also define the associated space-time cylinder

$$Q^\delta = \Omega^\delta \times (0, T). \tag{32}$$

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0, T; H) \cap L^2(0, T; V)$, we conclude that $\rho_\delta(v) \in L^\infty(0, T; H(\Omega^\delta)) \cap L^2(0, T; V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \leq \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \leq \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. For $\delta \in (0, \delta(\Omega))$, we consider the system

$$\begin{aligned}(v_t^\delta, u) + \nu(\nabla v^\delta, \nabla u) + \nu_o(Av^\delta, u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, u) \\ = (\mathcal{F}(c^\delta, \theta^\delta), u) \quad \forall\, u \in V^p, \text{ a.e. } t \in (0, T),\end{aligned} \tag{33}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t^\delta + \alpha\epsilon^2\rho_\delta(v^\delta) \cdot \nabla\phi^\delta - \epsilon^2\Delta\phi^\delta \\ = \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) + \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{34}$$

$$C_v\theta_t^\delta + C_v\rho_\delta(v^\delta)\cdot\nabla\theta^\delta = \nabla\cdot\big(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\big) + \frac{l}{2}f_s^\delta(\phi^\delta)_t \text{ in } Q^\delta, \tag{35}$$

$$c_t^\delta - K_2\Delta c^\delta + \rho_\delta(v^\delta)\cdot\nabla c^\delta = K_2 M\nabla\cdot\big(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\big) \text{ in } Q^\delta, \tag{36}$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \quad \frac{\partial\theta^\delta}{\partial n} = 0, \quad \frac{\partial c^\delta}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta\times(0,T), \tag{37}$$

$$v^\delta(0) = v_0^\delta \text{ in } \Omega, \quad \phi^\delta(0) = \phi_0^\delta, \quad \theta^\delta(0) = \theta_0^\delta, \quad c^\delta(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{38}$$

We then have the following existence result.

Proposition 1 *Let be $T > 0$, $p \geq 3$. For each $\delta \in (0, \delta(\Omega))$, let $v_0^\delta \in H$, $\phi_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $\theta_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $1/2 < \gamma \leq 1$ and $c_0^\delta \in C^1(\bar{\Omega}^\delta), 0 < c_0^\delta < 1$ in $\bar{\Omega}^\delta$, satisfying the compatibility conditions $\dfrac{\partial\phi_0^\delta}{\partial n} = \dfrac{\partial\theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$. Assume that* **(H1)-(H4)** *hold. Then there exist functions $(v^\delta, \phi^\delta, \theta^\delta, c^\delta)$ which satisfy (33)-(38) and*

i) $v^\delta \in L^p(0,T;V^p)\cap L^\infty(0,T;H)$, $\quad v_t^\delta \in L^{p'}(0,T;(V^p)')$,

ii) $\phi^\delta \in L^2(0,T;H^2(\Omega^\delta))$, $\quad \phi_t^\delta \in L^2(Q^\delta)$,

iii) $\theta^\delta \in L^2(0,T;H^2(\Omega^\delta))$, $\quad \theta_t^\delta \in L^2(Q^\delta)$,

iv) $c^\delta \in C^{2,1}(Q^\delta)$, $\quad 0 < c^\delta < 1$.

Remark: It is possible to obtain more regularity for ϕ^δ when the initial data are more regular. This will be done in the last section.

Proof: For simplicity we shall omit the superscript δ at v^δ, ϕ^δ, θ^δ, c^δ. First of all, we consider the following family of operators, indexed by the parameter $0 \leq \lambda \leq 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^p(0,T;H)\times L^2(Q^\delta)\times L^2(Q^\delta)\times L^2(Q^\delta),$$

and defined as follows: given $(\hat{v},\hat{\phi},\hat{\theta},\hat{c}) \in B$, let $\mathcal{T}_\lambda(\hat{v},\hat{\phi},\hat{\theta},\hat{c}) = (v,\phi,\theta,c)$, where (v,ϕ,θ,c) is obtained by solving the problem

$$\begin{aligned}(v_t,u) + \nu(\nabla v,\nabla u) + \nu_0(Av,u) + (v\cdot\nabla v,u)&\\ = \lambda(\mathcal{F}(\hat{c},\hat{\theta}),u) - \lambda(k(f_s^\delta(\hat{\phi})-\delta)\hat{v},u)\ \forall u\in V^p,\ t\in(0,T),&\end{aligned} \tag{39}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi-\phi^3) \\ = \lambda\beta\left(\hat{\theta} + (\theta_B-\theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{40}$$

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{41}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \text{ in } Q^\delta, \tag{42}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta\times(0,T), \tag{43}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{44}$$

Clearly (v,ϕ,θ,c) is a solution of (33)-(38) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem ([3] p. 189).

To verify that $\mathcal{T}_\lambda$ is well defined, observe that equation (39) is a variant of Navier-Stokes equation and since $k(f_s^\delta(\hat{\phi})-\delta)\hat{v} \in L^2(Q)$, there exist a unique solution $v \in L^p(0,T;V^p)\cap L^\infty(0,T;H)\cap L^{p5/3}(Q)$ ([8] p. 207).

Since $\hat{\theta}$, $\hat{c} \in L^2(Q^\delta)$ and $\rho_\delta(v) \in L^5(Q^\delta)$, we infer from Theorem 2 that there is a unique solution ϕ of equation (40) with $\phi \in W_2^{2,1}(Q^\delta)$.

Since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi) \in C^\infty(Q^\delta)$, we have that $K_1(\rho_\delta(\phi)) \in W_r^{1,1}(Q^\delta)$, $1 \le r \le \infty$, and since $\rho_\delta(v) \in L^5(Q^\delta)$ and $f_s^\delta(\phi)_t = f_s^{\delta\prime}(\phi)\phi_t \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([7] Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ of equation (41) with $\theta \in W_2^{2,1}(Q^\delta)$.

We observe that equation (42) is a semilinear parabolic equation with smooth coefficients and growth conditions on the non-linear forcing terms to apply semigroup results of Henry [4] p.75. Thus, there is a unique global classical solution c.

In addition, note that equation (42) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principles together with conditions $0 < c_0^\delta < 1$ and $\dfrac{\partial c^\delta}{\partial n} = 0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\,(x,t) \in Q^\delta. \tag{45}$$

Therefore, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$ let $(\hat{v}^k, \hat{\phi}^k, \hat{\theta}^k, \hat{c}^k)$, $k \in \mathbb{N}$ be a sequence in B such that converges strongly in B to $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c})$ and let $(v^k, \phi^k, \theta^k, c^k)$ be the solution of the problem:

$$\begin{aligned}(v_t^k, u) + \nu(\nabla v^k, \nabla u) + \nu_o(Av^k, u) + (v^k \cdot \nabla v^k, u) \\ = \lambda(\mathcal{F}(\hat{c}^k, \hat{\theta}^k), u) - \lambda(k(f_s^\delta(\hat{\phi}^k) - \delta)\hat{v}^k, u) \; \forall\, u \in V^p, \; t \in (0,T),\end{aligned} \tag{46}$$

$$\begin{aligned}\alpha\epsilon^2 \phi_t^k + \alpha\epsilon^2 \rho_\delta(v^k) \cdot \nabla\phi^k - \epsilon^2 \Delta\phi^k \\ = \frac{1}{2}(\phi^k - (\phi^k)^3) + \lambda\beta\left(\hat{\theta}^k + (\theta_B - \theta_A)\hat{c}^k - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{47}$$

$$C_{\mathrm{v}}\theta_t^k + C_{\mathrm{v}}\rho_\delta(v^k) \cdot \nabla\theta^k = \nabla \cdot \left(K_1(\rho_\delta(\phi^k))\nabla\theta^k\right) + \frac{l}{2} f_s^\delta(\phi^k)_t \text{ in } Q^\delta, \tag{48}$$

$$c_t^k - K_2\Delta c^k + \rho_\delta(v^k) \cdot \nabla c^k = K_2 M \nabla \cdot \left(c^k(1 - c^k)\nabla\rho_\delta(\phi^k)\right) \text{ in } Q^\delta, \tag{49}$$

$$\frac{\partial\phi^k}{\partial n} = 0, \quad \frac{\partial\theta^k}{\partial n} = 0, \quad \frac{\partial c^k}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \tag{50}$$

$$v^k(0) = v_0^\delta \text{ in } \Omega, \quad \phi^k(0) = \phi_0^\delta, \quad \theta^k(0) = \theta_0^\delta, \quad c^k(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{51}$$

We show that the sequence $(v^k, \phi^k, \theta^k, c^k)$ converges strongly in B to (v, ϕ, θ, c) $= \mathcal{T}_\lambda(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c})$. For that purpose, we will obtain estimates to $(v^k, \phi^k, \theta^k, c^k)$ independent of k. We denote by C_i any positive constant independent of k.

We take $u = v^k$ in equation (46). Using Hölder's and Young's inequalities we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |v^k|^2 dx \quad &+ \nu_o \int_\Omega |\nabla v^k|^p dx + \nu \int_\Omega |\nabla v^k|^2 dx \\ &\leq C_1 \int_\Omega \left(|F|^2 + |\hat{v}^k|^2 + |\hat{\theta}^k|^2 + |\hat{c}^k|^2 + |v^k|^2\right) dx.\end{aligned}$$

Then, by the usual method of Gronwall's inequality, we get

$$\|v^k\|_{L^\infty(0,T;H) \cap L^p(0,T;V^p)} \leq C_1. \tag{52}$$

Observe that operator A satisfies $\|Av\| \leq C\|v\|_{V^p}^{p-1}$. Now, from the equation (46) we infer that

$$\begin{aligned}\|v_t^k\|_{(V^p)'} \leq \quad & C_1 \Big(\|v^k\|_{V^p}^{p-1} + \|v^k\|_V + \|v^k\|_{L^{2p'}(\Omega)}^2 + \|F\|_{L^2(\Omega)} \\ & + \|\hat{v}^k\|_{L^2(\Omega)} + \|\hat{\theta}^k\|_{L^2(\Omega^\delta)} + \|\hat{c}^k\|_{L^2(\Omega^\delta)}\Big),\end{aligned}$$

then, using (52) and since $2p' \leq p5/3$, we obtain

$$\|v_t^k\|_{L^{p'}(0,T;(V^p)')} \leq C_1. \tag{53}$$

From estimate (20) we have that

$$\begin{aligned}\|\phi\|_{W_2^{2,1}(Q^\delta)} \leq \ & C\Big(\|\phi_0\|_{H^1(\Omega^\delta)} + \|\hat{\theta}^k\|_{L^2(Q^\delta)} + \|\hat{c}^k\|_{L^2(Q^\delta)} + \|\phi_0\|^3_{H^1(\Omega^\delta)} \\ & + \|\hat{\theta}^k\|^3_{L^2(Q^\delta)} + \|\hat{c}^k\|^3_{L^2(Q^\delta)} + 1\Big)\end{aligned}$$

where C depends on $\|\rho_\delta(v^k)\|_{L^5(Q^\delta)}$. Therefore, using (52) we conclude that

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \leq C_1. \tag{54}$$

Now, multiplying (48) by θ^k one obtains

$$\int_{\Omega^\delta} |\theta^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla\theta^k|^2 dxdt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\phi_t^k|^2 + |\theta^k|^2\right) dxdt \tag{55}$$

and we infer from (54) and Gronwall's Lemma that

$$\|\theta^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1, \tag{56}$$

hence, it follows from (55) that

$$\|\theta^k\|_{L^2(0,T;H^1(\Omega^\delta))} \leq C_2. \tag{57}$$

We take scalar product of (48) with $\eta \in H^1(\Omega^\delta)$, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_t^k\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla\theta^k\|_{L^2(\Omega^\delta)} + \|\rho_\delta\|_{L^\infty(Q^\delta)}\|v^k\|_{L^2(\Omega)}\|\theta^k\|_{L^2(\Omega^\delta)} + \|\phi_t^k\|_{L^2(\Omega^\delta)}\right)$$

and we infer from (52),(54) and (57) that

$$\|\theta_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{58}$$

Next, multiplying (49) by c^k we conclude by analogous reasoning and using (45) that

$$\int_{\Omega^\delta} |c^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c^k|^2 dxdt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla\phi^k|^2 dxdt,$$

hence, from (54) we have,

$$\|c^k\|_{L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{59}$$

In order to get an estimate for (c_t^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we return to the equation (49) and use similar techniques, then

$$\|c_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{60}$$

We now infer from (52)-(60) that the sequence (v^k) is uniformly bounded with respect to k in

$$W_1 = \left\{ w \in L^p(0,T;V^p),\ w_t \in L^{p'}(0,T;(V^p)') \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H),\ w_t \in L^{p'}(0,T;(V^p)') \right\},$$

the sequence (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$ and the sequences (θ^k) and (c^k) are bounded in

$$W_3 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Since W_1 is compactly embedded in $L^p(0,T;H)$, W_2 in $C([0,T];(V^p)')$, $W_2^{2,1}(Q^\delta)$ in $L^2(0,T;H^1(\Omega^\delta))$, W_3 in $L^2(Q^\delta)$ and W_4 in $C([0,T];H^1(\Omega^\delta)')$ ([13] Cor.4), it follows that there exist

$$\begin{array}{lcl}
v & \in & L^p(0,T;V^p)\cap L^\infty(0,T;H) \text{ with } v_t \in L^{p'}(0,T;(V^p)'),\\
\chi & \in & L^{p'}(0,T;(V^p)'),\\
\phi & \in & L^2(0,T;H^2(\Omega^\delta)) \text{ with } \phi_t \in L^2(Q^\delta),\\
\theta & \in & L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } \theta_t \in L^2(0,T;H^1(\Omega^\delta)'),\\
c & \in & L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } c_t \in L^2(0,T;H^1(\Omega^\delta)'),
\end{array}$$

and a subsequence of $(v^k, \phi^k, \theta^k, c^k)$ (which we still denote by $(v^k, \phi^k, \theta^k, c^k)$), such that, as $k \to +\infty$,

$$\begin{array}{rcclr}
v^k & \to & v & \text{in} & L^p(0,T;H) \cap C([0,T];(V^p)') \ \text{strongly}, \\
v^k & \rightharpoonup & v & \text{in} & L^p(0,T;V^p) \ \text{weakly}, \\
Av^k & \rightharpoonup & \chi & \text{in} & L^{p'}(0,T;(V^p)') \ \text{weakly}, \\
\phi^k & \to & \phi & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)) \ \text{strongly}, \\
\phi^k & \rightharpoonup & \phi & \text{in} & L^2(0,T;H^2(\Omega^\delta)) \ \text{weakly}, \\
\theta^k & \to & \theta & \text{in} & L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \ \text{strongly}, \\
\theta^k & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \ \text{weakly}, \\
c^k & \to & c & \text{in} & L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \ \text{strongly}, \\
c^k & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \ \text{weakly}.
\end{array} \tag{61}$$

It now remains to pass to the limit as k tends to $+\infty$ in (46)-(51).

We observe that $k(f_s^\delta(\cdot) - \delta)$ is bounded Lipschitz continuous function from $\mathbb{R}$ in $\mathbb{R}$ then $k(f_s^\delta(\hat{\phi}^k) - \delta)$ converges to $k(f_s^\delta(\hat{\phi}) - \delta)$ in $L^p(Q)$, for any $p \in [1, \infty)$. We then pass to the limit in the usual form as k tends to $+\infty$ in (46) and get

$$\frac{d}{dt}(v,u) + \nu_o(\chi,u) + \nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(\hat{c}, \hat{\theta}), u) - \lambda(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u)$$

for all $u \in V^p$, $t \in (0,T)$. Using that the operator A is monotone we can conclude that $\chi = Av$.

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^9(Q^\delta)$ is compact ([9] p.15), and (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi^k)^3$ converges to ϕ^3 in $L^2(Q^\delta)$. Also, since v^k converges to v in $L^p(0,T;H)$ we have that $\rho_\delta(v^k)$ converges to $\rho_\delta(v)$ in $L^p(0,T;H(\Omega^\delta))$. We then pass to the limit as k tends to $+\infty$ in (47) and get

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \ \text{in } Q^\delta.$$

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions and ϕ^k converges to ϕ in $L^9(Q^\delta)$ we have that $K_1(\rho_\delta(\phi^k))$ converges to $K_1(\rho_\delta(\phi))$ and $f_s^{\delta\prime}(\phi^k)$ converges to $f_s^{\delta\prime}(\phi)$ in $L^p(Q^\delta)$ for any $p \in [1, \infty)$. These facts and (61) yield the weak convergence of $K_1(\rho_\delta(\phi^k))\nabla\theta^k$ to $K_1(\rho_\delta(\phi))\nabla\theta$ and $f_s^{\delta\prime}(\phi^k)\phi_t^k$ to $f_s^{\delta\prime}(\phi)\phi_t$ in $L^{3/2}(Q^\delta)$. Now, multiplying (48) by $\eta \in \mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta \times (0,T)$ and by parts, we obtain

$$\int_0^T \int_{\Omega^\delta} C_{\rm v} \left(\theta_t^k + \rho_\delta(v^k)\cdot\nabla\theta^k\right)\eta \; + K_1(\rho_\delta(\phi^k))\nabla\theta^k\cdot\nabla\eta\, dxdt$$
$$= \int_0^T \int_{\Omega^\delta} \frac{l}{2} f_s^{\delta\prime}(\phi^k)\phi_t^k\eta\, dxdt,$$

then we may pass to the limit and find that,

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2} f_s^{\delta\prime}(\phi)\phi_t \quad \text{in } \mathcal{D}'(Q^\delta), \quad (62)$$

and using L^p-theory of parabolic equations we have that (62) holds almost everywhere in Q^δ.

It remains to pass to the limit in (49). We infer from (61) that $\nabla\rho_\delta(\phi^k)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q^\delta)$ and since $\|c^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c^k(1-c^k)$ converges to $c(1-c)$ in $L^p(Q^\delta)$ for any $p\in[1,\infty)$. Similarly, we may pass to the limit in (49) to obtain

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \qquad \text{in } Q^\delta.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0\le\lambda\le 1$. At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1\times W_2^{2,1}(Q^\delta)\times W_3\times W_3$ and the embedding of this space in B is compact; then we conclude that $\mathcal{T}_\lambda$ is a compact operator.

To prove that for $(\hat v,\hat\phi,\hat\theta,\hat c)$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous in λ, let $0\le\lambda_1,\lambda_2\le 1$ and $(v_i,\phi_i,\theta_i,c_i)$ $(i=1,2)$ be the corresponding solutions of (39)-(44). We observe that $v=v_1-v_2$, $\phi=\phi_1-\phi_2$, $\theta=\theta_1-\theta_2$ and $c=c_1-c_2$ satisfy the following problem:

$$\begin{gathered}(v_t,u)+\nu(\nabla v,\nabla u)+\nu_o(Av_1-Av_2,u)+(v_1\cdot\nabla v,u)-(v\cdot\nabla v_2,u)\\ =(\lambda_1-\lambda_2)(\mathcal{F}(\hat c,\hat\theta),u)+(\lambda_2-\lambda_1)(k(f_s^\delta(\hat\phi)-\delta)\hat v,u),\\ \text{for all } u\in V^p,\ t\in(0,T),\end{gathered} \quad (63)$$

$$\begin{gathered}\alpha\epsilon^2\phi_t-\epsilon^2\Delta\phi+\alpha\epsilon^2\rho_\delta(v_1)\cdot\nabla\phi-\frac12\phi\left(1-(\phi_1^2+\phi_1\phi_2+\phi_2^2)\right)\\ =\alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi_2+(\lambda_1-\lambda_2)\beta\left(\hat\theta+(\theta_B-\theta_A)\hat c-\theta_B\right) \text{ in } Q^\delta,\end{gathered} \quad (64)$$

$$\begin{gathered}C_{\rm v}\theta_t-\nabla\cdot K_1(\rho_\delta(\phi_1))\nabla\theta-\nabla\cdot\left[K_1(\rho_\delta(\phi_1))-K_1(\rho_\delta(\phi_2))\right]\nabla\theta_2\\ +C_{\rm v}\rho_\delta(v_1)\cdot\nabla\theta=C_{\rm v}\rho_\delta(v)\cdot\nabla\theta_2\\ +\frac{l}{2}f_s^{\delta\prime}(\phi_1)\phi_t+\frac{l}{2}\left[f_s^{\delta\prime}(\phi_1)-f_s^{\delta\prime}(\phi_2)\right]\phi_{2t} \text{ in } Q^\delta,\end{gathered} \quad (65)$$

$$c_t - K_2\Delta c + \rho_\delta(v_1)\cdot\nabla c = K_2 M\nabla\cdot\left(c_1(1-c_1)\left[\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)\right]\right)$$
$$+\rho_\delta(v)\cdot\nabla c_2 + K_2 M\nabla\cdot(c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2)) \text{ in } Q^\delta, \tag{66}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta\times(0,T), \tag{67}$$

$$v(0) = 0 \text{ in } \Omega, \quad \phi(0) = 0, \quad \theta(0) = 0, \quad c(0) = 0 \text{ in } \Omega^\delta. \tag{68}$$

Taking $u = v$ in equation (63), using Hölder's, Young's and interpolation inequalities and the monotonicity of operator A we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2dx \;&+\; \int_\Omega \nu|\nabla v|^2dx \le \int_\Omega |v||\nabla v_2||v|dx\\
&+\; |\lambda_1-\lambda_2|\int_\Omega\left(|\mathcal{F}(\hat{c},\hat{\theta})||v| + k(f_s^\delta(\hat{\phi})-\delta)|\hat{v}||v|\right)dx\\
&\le\; C_1\|v_2\|^s_{L^r(\Omega)}\|v\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|v\|^2_V\\
&+\; C_2|\lambda_1-\lambda_2|^2\int_\Omega |F|^2 + |\hat{\theta}|^2 + |\hat{c}|^2 + |\hat{v}|^2dx + C_3\int_\Omega |v|^2dx.\end{aligned}$$

where $2/s+3/r = 1$ and $r > 3$. Observe that due to assumption $p \ge 3$ we have that $v \in L^s(0,T;L^r(\Omega))$. Then, integration with respect t and Gronwall's Lemma give us

$$\|v\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C_1\,|\lambda_1-\lambda_2|^2. \tag{69}$$

Applying L^p-theory of parabolic equations ([7] p. 341) to equation (64), the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C_1\left(\|\rho_\delta(v)\cdot\nabla\phi_2\|_{L^2(Q^\delta)} + |\lambda_1-\lambda_2|\left(\|\hat{\theta}\|_{L^2(Q^\delta)} + \|\hat{c}\|_{L^2(Q^\delta)} + 1\right)\right)$$

where C_1 depends on $\|\rho_\delta(v_1)\|_{L^5(Q^\delta)}$ and $\|\phi_1^2+\phi_1\phi_2+\phi_2^2\|_{L^{5/2}(Q^\delta)}$, which are independent of λ_i. Therefore, using (69) we arrive at

$$\|\phi\|^2_{W_2^{2,1}(Q^\delta)} \le C_1\,|\lambda_1-\lambda_2|^2. \tag{70}$$

Multiplying (65) by θ, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega^\delta}|\theta|^2dx + a\int_{\Omega^\delta}|\nabla\theta|^2dx\\
&\le C_1\int_{\Omega^\delta}|\rho_\delta(\phi)||\nabla\theta_2||\nabla\theta| + |\rho_\delta(v)||\nabla\theta_2||\theta| + |\phi_t||\theta| + |\phi||\phi_{2t}|\,|\theta|dx\\
&\le C_1\|\phi\|^2_{L^\infty(0,T;L^2(\Omega^\delta))}\|\nabla\theta_2\|_{L^2(\Omega^\delta)} + C_2\|v\|^2_{L^\infty(0,T;H)}\|\nabla\theta_2\|^2_{L^2(\Omega^\delta)}\\
&+C_3\int_{\Omega^\delta}|\phi_t|^2 + |\theta|^2dx + C_4\|\phi\|^2_{L^\infty(0,T;H^1(\Omega^\delta))}\|\phi_{2t}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta}|\nabla\theta|^2dx.\end{aligned}$$

Integration with respect to t and the use of Gronwall's Lemma and (69)-(70) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{71}$$

We multiply (66) by c, integrate over $\Omega^\delta \times (0,t)$ and by parts, and we use Hölder's and Young's inequalities and (45) to obtain

$$\begin{aligned}\int_{\Omega^\delta} |c|^2 dx \;&+\; \int_0^t \int_{\Omega^\delta} |\nabla c|^2 dxdt \\ &\leq\; C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)|^2 + |\rho_\delta(v)|^2 + |c|^2\right) dxdt \\ &\leq\; C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla\phi|^2 + |c|^2\right) dxdt + C_1 \int_0^t \int_{\Omega} |v|^2 dxdt.\end{aligned}$$

Applying Gronwall's Lemma and using (69)-(70) we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{72}$$

Therefore, it follows from (69)-(72) that $\mathcal{T}_\lambda$ is uniformly continuous in λ.

To estimate the set of all fixed points of $\mathcal{T}_\lambda$ let $(v, \phi, \theta, c) \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\begin{aligned}&(v_t, u) + \nu(\nabla v, \nabla u) + \nu_o(Av, u) + (v \cdot \nabla v, u) \\ &\quad = \lambda(\mathcal{F}(c, \theta), u) - \lambda(k(f_s^\delta(\phi) - \delta)v, u)\ \forall u \in V,\ t \in (0,T),\end{aligned} \tag{73}$$

$$\begin{aligned}&\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ &\qquad = \lambda\beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{74}$$

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2} f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{75}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M \nabla\cdot\left(c(1-c)\nabla\left(\rho_\delta(\phi)\right)\right) \text{ in } Q^\delta, \tag{76}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \tag{77}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{78}$$

We take $u = v$ in equation (73). Then

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx \quad &+ \int_\Omega \left(\nu_o|\nabla v|^p + \nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right) dx \\
&\leq C_1 \int_\Omega |F|^2 + |\theta|^2 + |c|^2 + |v|^2 dx \\
&\leq C_1 \int_\Omega |F|^2 + |v|^2 dx + C_1 \int_{\Omega^\delta} |\theta|^2 + |c|^2 dx.
\end{aligned} \tag{79}
$$

Multiplying equation (74) by ϕ, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities we obtain,

$$
\begin{aligned}
\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_{\Omega^\delta} |\phi|^2 dx \quad &+ \quad \int_{\Omega^\delta} \left(\epsilon^2|\nabla\phi|^2 + \frac{1}{2}\phi^4\right) dx \\
&\leq \quad C_1 + C_1 \int_{\Omega^\delta} \left(|\theta|^2 + |c|^2 + |\phi|^2\right) dx.
\end{aligned} \tag{80}
$$

By multiplying (75) by $e = C_{\mathrm{v}}\theta - \frac{l}{2}f_s^\delta(\phi)$ and (76) by c, arguments similar to the previous ones lead to the following estimates

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta} |e|^2 dx + \frac{C_{\mathrm{v}}a}{2}\int_{\Omega^\delta} |\nabla\theta|^2 dx \leq C_2 \int_{\Omega^\delta} |\nabla\phi|^2 dx + C_1 \int_\Omega |v|^2 dx, \tag{81}
$$

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta} |c|^2 dx + \frac{K_2}{2}\int_{\Omega^\delta} |\nabla c|^2 dx \leq C_2 \int_{\Omega^\delta} |\nabla\phi|^2 dx, \tag{82}
$$

where (45) was used to obtain the last inequality.

Now, multiplying (80) by A and adding the result to (79),(81)-(82), gives us

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2 + \frac{d}{dt}\int_{\Omega^\delta} \left(\frac{A\alpha\epsilon^2}{4}|\phi|^2 + \frac{1}{2}|e|^2 + \frac{1}{2}|c|^2\right) dx \\
&\quad + \int_\Omega \left(\nu_o|\nabla v|^p + \nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right) dx \\
&\quad + \int_{\Omega^\delta} \left((A\epsilon^2 - 2C_2)|\nabla\phi|^2 + \frac{A}{2}\phi^4 + \frac{C_{\mathrm{v}}a}{2}|\nabla\theta|^2 + \frac{K_2}{2}|\nabla c|^2\right) dx \\
&\qquad \leq C_1 + C_1 \int_\Omega |v|^2 dx + C_1 \int_{\Omega^\delta} \left(|\phi|^2 + |\theta|^2 + |c|^2\right) dx
\end{aligned} \tag{83}
$$

where C_1 is independent of λ and δ, being $A \in \mathbb{R}$ an arbitrary parameter. Taking A large enough and using Gronwall's Lemma to obtain

$$\|v\|_{L^\infty(0,T;H)} + \|\phi\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|e\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1,$$

where C_1 is independent of λ. Since $\theta = \dfrac{1}{C_{\rm v}}\left(e + \dfrac{l}{2} f_s^\delta(\phi)\right)$ and $f_s^\delta(\phi)$ is bounded in $L^\infty(Q^\delta)$, we also have that $\|\theta\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1$. Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0,1]$.

Finally, for $\lambda = 0$, we can reason as in the proof that $\mathcal{T}_\lambda$ is well defined to conclude that the problem (39)-(44) has a unique solution. Therefore, we can apply Leray-Schauder's Theorem and so there is at least one fixed point $(v,\phi,\theta,c) \in B \cap \{L^p(0,T;V^p) \cap L^\infty(0,T;H)\} \times W_2^{2,1}(Q^\delta) \times W_2^{2,1}(Q^\delta) \times C^{2,1}(Q^\delta)$ of the operator $\mathcal{T}_1$, i.e. $(v,\phi,\theta,c) = \mathcal{T}_1(v,\phi,\theta,c)$. These functions are a solution of problem (33)-(38) and the proof of Proposition 1 is complete. □

4 The proof of Theorem 1

To prove Theorem 1, we start by taking the initial condition in the previous regularized problem as follows. For $0 < \delta \le \delta(\Omega)$ as in the statement of Theorem 1, we choose $\phi_0^\delta \in W^{2-2/q,q}(\Omega^\delta) \cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta \in H$, $\theta_0^\delta \in H^{1+\gamma}(\Omega)$, $1/2 < \gamma \le 1$, $c_0^\delta \in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial \phi_0^\delta}{\partial n} = \dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 < c_0^\delta < 1$ in $\overline{\Omega^\delta}$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0))$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (33)-(38).

In the following, we will derive bounds, independent of δ, for such solutions and then use compactness arguments to pass to the limit as δ approach 0 to establish the desired existence result. Such estimates will stated in following in a sequence of lemmas; however, most of them are ease consequence of the estimates obtained in the last section (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 1 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\|v^\delta\|_{L^\infty(0,T;H) \cap L^p(0,T;V^p)} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \le C_1, \qquad (84)$$

$$\|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \quad (85)$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \quad (86)$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1. \quad (87)$$

Proof: The result follows from inequality (83). □

Lemma 2 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\|\phi^\delta\|_{W_q^{2,1}(Q)} \le C_1, \quad \text{for any } 2 \le q \le 10/3, \quad (88)$$

$$\|\theta_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} \le C_1, \quad (89)$$

$$\|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} \le C_1. \quad (90)$$

Proof: Note that (88) follows from estimate (20) of Theorem 2 and Lemma 1.

Next, we take the scalar product of (35) with $\eta \in H_o^1(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_{\rm v}\|\theta_t^\delta\|_{H_o^1(\Omega)'} \le C_1 \left(\|\nabla\theta^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^{10/3}(\Omega)}\|v^\delta\|_{L^5(\Omega)} + \|\phi_t^\delta\|_{L^2(\Omega)}\right).$$

Then, (89) follows from Lemma 1 and (88).

Using that $0 < c^\delta < 1$ in Q, we infer from (36) that,

$$\|c_t^\delta\|_{H_o^1(\Omega)'} \le C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla\phi^\delta\|_{L^2(\Omega)}\right).$$

Then, (90) follows from Lemma 1. □

Lemma 3 *There exist a constant C_1 and $\delta_0 \in (0, \delta(\Omega))$ such that, for any $\delta < \delta_0$,*

$$\|v_t^\delta\|_{L^{p'}(t_1,t_2;V^p(U)')} \le C_1 \quad (91)$$

where $0 \le t_1 < t_2 \le T$, $U \subseteq \Omega_{ml}(t_1)$ and such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \le t_1 < t_2 \le T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (33) that for a.e. $t \in (t_1, t_2)$,

$$(v_t^\delta, u) = -\nu_o \int_U Av^\delta\, u dx - \nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx$$
$$- \int_U k(f_s^\delta(\phi^\delta) - \delta) v^\delta u dx + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx, \quad u \in V^p(U).$$

In order to estimate $\|v_t^\delta\|_{V^p(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W_q^{2,1}(Q)$, for $2 \leq q < 5$, in particular, for $q > 5/2$ we have that $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 5/q$ ([7] p.80). Consequently, because of Arzela-Ascoli's theorem, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q \,/\, 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega \,/\, 0 \leq f_s(\phi(x,t)) < 1\}$. Note that for a certain $\overline{\gamma} \in (0,1)$ and for $(x,t) \in [t_1,t_2] \times \bar{U}$,

$$f_s(\phi(x,t)) < 1 - \overline{\gamma}.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0,\delta_0)$ and for all $(x,t) \in [t_1,t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \overline{\gamma}/2.$$

By assumption **(H2)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \overline{\gamma}/2) \qquad \text{for } (x,t) \in [t_1,t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned} \|v_t^\delta\|_{V^p(U)'} \;\leq\; & C_1\big(\|v^\delta\|_{V^p}^{p-1} + \|v^\delta\|_V + \|v^\delta\|_{L^s(\Omega)}^2 + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} \\ & + \|\theta^\delta\|_{L^2(\Omega)} + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)}\|v^\delta\|_{L^2(\Omega)}\big), \end{aligned}$$

where $2/s + 1/p = 1$. Hence, (91) follows from Lemma 1. □

From (84), the sequence (v^δ) is also bounded in $L^p(t_1,t_2;W^{1,p}(U))$; then, by compact embedding ([13] Cor. 4), there exist v and a subsequence of (v^δ) (which we still denote v^δ), such that

$$v^\delta \to v \quad \text{strongly in } L^p((t_1,t_2) \times U).$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i,t_{i+1}) \times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{strongly in } L^p_{loc}(Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)). \tag{92}$$

Moreover, from (84) we have that $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{aligned} v^\delta \;&\rightharpoonup\; v \quad \text{weakly in} \quad L^2(0,T;V), \\ v^\delta \;&\overset{*}{\rightharpoonup}\; v \quad \text{weakly * in} \quad L^\infty(0,T;H). \end{aligned} \tag{93}$$

Since Av^δ is bounded in $L^{p'}(0,T;(V^p)')$ there exists $\chi \in L^{p'}(0,T;(V^p)')$ such that

$$Av^\delta \rightharpoonup \chi \in L^{p'}(0,T;(V^p)') \text{ weakly.} \tag{94}$$

We now infer from Lemma 1 and Lemma 2, using compact embedding ([13] Cor.4), that there exist

$$\begin{array}{rcl} \phi & \in & W_q^{2,1}(Q) \text{ for } 2 \leq q \leq 10/3, \\ \theta & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \\ c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \end{array}$$

and a subsequence of $(\phi^\delta, \theta^\delta, c^\delta)$ (which we still denote by $(\phi^\delta, \theta^\delta, c^\delta)$) such that, as $\delta \to 0$,

$$\begin{array}{rcll} \phi^\delta & \to & \phi & \text{uniformly in } Q, \\ \phi^\delta & \to & \phi & \text{strongly in } L^q(0,T;W^{1,q}(\Omega)), \\ \phi_t^\delta & \rightharpoonup & \phi_t & \text{weakly in } L^q(Q), \\ \theta^\delta & \to & \theta & \text{strongly in } L^2(Q) \cap C([0,T];H_o^1(\Omega)'), \\ \theta^\delta & \rightharpoonup & \theta & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ c^\delta & \to & c & \text{strongly in } L^2(Q) \cap C([0,T];H_o^1(\Omega)'), \\ c^\delta & \rightharpoonup & c & \text{weakly in } L^2(0,T;H^1(\Omega)). \end{array} \tag{95}$$

It now remains pass to the limit as δ decreases to zero in (33)-(38). We start with the velocity equation.

We take $u = \eta(t)$ in (33) where $\eta \in L^p(0,T;V^p)$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and $\eta_t \in L^{p'}(0,T;V^p(\Omega_{ml}(t))')$; after integration over $(0,t)$, we find

$$\begin{aligned} \int_0^t \big((v_t^\delta, \eta) + \nu(\nabla v^\delta, \nabla\eta) + \nu_o(Av^\delta, \eta) + (v^\delta \cdot \nabla v^\delta, \eta) \\ + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, \eta)\big)\, ds = \int_0^t (\mathcal{F}(c^\delta, \theta^\delta), \eta) ds. \end{aligned} \tag{96}$$

Moreover, we observe that

$$\int_0^t (v_t^\delta, \eta) ds = -\int_0^t (v^\delta, \eta_t) ds + (v^\delta(t), \eta(t)) - (v_0^\delta, \eta(0)).$$

Also, because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H2)**, it follows that $k(f_s^\delta(\phi^\delta) - \delta)$ converges to $k(f_s(\phi))$

uniformly on compact subsets of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. These facts, together with (92)-(95), ensure that we can pass to the limit in (96) and get

$$(v(t), \eta(t)) - \int_0^t (v, \eta_t) ds + \nu \int_0^t (\nabla v, \nabla \eta) ds + \nu_o \int_0^t (\chi, \eta) ds \tag{97}$$
$$+ \int_0^t (v \cdot \nabla v, \eta) ds + \int_0^t (k(f_s(\phi)) v, \eta) ds = \int_0^t (\mathcal{F}(c, \theta), \eta) ds + (v_0, \eta(0)),$$

Since $v^\delta(0) \to v(0)$ in $(V^p(U))'$, for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$, by using (97) it is easy to see that $v(0) = v_0$ in $\Omega_{ml}(0)$.

Now, we check that $v = 0$ a.e. in $\overset{o}{Q}_s$. For this, take a compact set $K \subseteq \overset{o}{Q}_s$. Then there is an $\delta_K \in (0, \delta(\Omega))$ such that

$$f_s^\delta(\phi^\delta(x,t)) = 1 \quad \text{ in } K \text{ for } \delta < \delta_K.$$

Hence, $k(f_s^\delta(\phi^\delta(x,t) - \delta) = k(1-\delta)$ in K for $\delta < \delta_K$. From (84) we infer that

$$k(1-\delta) \| v^\delta \|^2_{L^2(K)} \leq C_1 \quad \text{for } \delta < \delta_K,$$

where C_1 is independent of δ. Thus, as δ tends to 0, by assumption **(H2)**, $k(1-\delta)$ blows up and, consequently, $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K, and since K is an arbitrary compact subset, we conclude that

$$v = 0 \quad \text{a.e. in } \overset{o}{Q}_s$$

Now, we proceed with the other equations.

It follows from (93)-(95) that we may pass to the limit in (34), and find that (12) holds almost everywhere.

In order to pass to the limit in (35), we note that given $\zeta \in L^2(0,T; H^1(\Omega))$ with $\zeta_t \in L^2(0,T; L^2(\Omega))$ satisfying $\zeta(T) = 0$, we can consider an extension of ζ such that $\zeta^\delta \in L^2(0,T; H^1(\Omega^\delta))$ with $\zeta^\delta_t \in L^2(0,T; L^2(\Omega^\delta))$ satisfying $\zeta^\delta(T) = 0$. Now, we take the scalar product of (35) with ζ^δ,

$$-C_{\rm v} \int_{\Omega^\delta} \theta_0^\delta \zeta^\delta(0) dx - C_{\rm v} \int_0^T \int_{\Omega^\delta} \theta^\delta \zeta_t^\delta dxdt \; - \; C_{\rm v} \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta) \theta^\delta \cdot \nabla \zeta^\delta dxdt$$
$$+ \int_0^T \int_{\Omega^\delta} K_1(\rho_\delta(\phi^\delta)) \nabla \theta^\delta \cdot \nabla \zeta^\delta dxdt \; = \; \frac{l}{2} \int_0^T \int_{\Omega^\delta} f_s^{\delta\prime}(\phi^\delta) \phi_t^\delta \zeta^\delta dxdt \tag{98}$$

Observe that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0,T; H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0,T]; H^1_o(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$.

Observe that ${f_s^\delta}' \to f_s'$ in $L^q(\mathbb{R})$ for $2 \le q < \infty$, then from (95) we infer that ${f_s^\delta}'(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 1 the integrals over $\Omega^\delta \backslash \Omega$ are bounded independent of δ and since $|\Omega^\delta \backslash \Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (98) and obtain

$$-C_{\mathrm{v}} \int_0^T \int_\Omega \theta \zeta_t dxdt - C_{\mathrm{v}} \int_0^T \int_\Omega v\theta \cdot \nabla\zeta dxdt + \int_0^T \int_\Omega K_1(\phi)\nabla\theta \cdot \nabla\zeta \, dxdt \\ = \frac{l}{2} \int_0^T \int_\Omega f_s'(\phi)\phi_t \zeta \, dxdt + C_{\mathrm{v}} \int_\Omega \theta_0 \zeta(0) dx$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$.

It remains to pass to the limit in (36). We proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$-\int_0^T \int_{\Omega^\delta} c^\delta \zeta_t^\delta dxdt - \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta)c^\delta \cdot \nabla\zeta^\delta dxdt + K_2 \int_0^T \int_{\Omega^\delta} \nabla c^\delta \cdot \nabla\zeta^\delta dxdt \\ +K_2 M \int_0^T \int_{\Omega^\delta} c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta) \cdot \nabla\zeta^\delta dxdt = \int_{\Omega^\delta} c_0^\delta \zeta^\delta(0) dx,$$

then from (93),(95) and using that the sequence (c^δ) is bounded in $L^\infty(Q)$ we may pass to the limit as $\delta \to 0$ and obtain

$$-\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega vc \cdot \nabla\zeta dxdt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla\zeta \, dxdt \\ +K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\zeta \, dxdt = \int_\Omega c_0 \zeta(0) dx$$

holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$. Observe that since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \le c \le 1$ a.e. in Q.

Now, it follows from (95) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$, and the first part of the proof of Theorem 1 is complete.

Under the additional regularity and integrability hypotheses stated in the second part of the statement of Theorem 1, in the following we will show that $\chi = Av$. We will use the monotonicity and the hemicontinuity of operator A ([9] Chp. 2) by adapting an argument that is usual in the theory of monotone operators. For this, we take any $\psi \in L^p(0,T;V^p)$ such that $\operatorname{supp}\psi$ is contained in the closure of $\Omega_{ml}(0) \cup Q_m \cup \Omega_{ml}(T)$ and define

$$\begin{aligned} X_\delta^t &= \nu_o \int_0^t \left(Av^\delta - A\psi, v^\delta - \psi\right) ds + \frac{1}{2}\|v^\delta(t)\|^2_{L^2(\Omega)} \\ &\quad +\nu \int_0^t \|\nabla v^\delta\|^2_{L^2(\Omega)} ds + \int_0^t \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dx ds. \end{aligned} \tag{99}$$

Since A is monotone and $\Omega_{ml}(t) \subseteq \Omega$,

$$\begin{aligned} X_\delta^t \geq \ & \frac{1}{2}\|v^\delta(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v^\delta\|^2_{L^2(\Omega_{ml}(s))} ds \\ & + \int_0^t \|k^{1/2}(f_s^\delta(\phi^\delta) - \delta)|v^\delta|\|^2_{L^2(\Omega_{ml}(s))} ds. \end{aligned} \tag{100}$$

Observe that $v^\delta(t) \rightharpoonup v(t)$ weakly in H, $v^\delta \rightharpoonup v$ weakly in $L^p(Q)$; thus, thanks to (92), $v^\delta \to v$ a.e. in Q_{ml}. Note also that $k^{1/2}(f_s^\delta(\phi^\delta) - \delta) \to k^{1/2}(f_s(\phi))$ a.e. in Q_{ml}; hence

$$k^{1/2}(f_s^\delta(\phi^\delta) - \delta)v^\delta \to k^{1/2}(f_s(\phi))v \quad \text{a.e. in } Q_{ml}$$

From (84) we have that $\displaystyle\int_0^t \|k^{1/2}(f_s^\delta(\phi^\delta) - \delta)|v^\delta|\|^2_{L^2(\Omega_{ml}(s))} ds$ is bounded. Therefore ([8] Lemma 1.3),

$$k^{1/2}(f_s^\delta(\phi^\delta) - \delta)v^\delta \rightharpoonup k^{1/2}(f_s(\phi))v \quad \text{weakly in } L^2(Q_{ml}).$$

Thus, we conclude from (100) that

$$\begin{aligned} \liminf_{\delta\to 0} X_\delta^t \geq \ & \frac{1}{2}\|v(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v\|^2_{L^2(\Omega_{ml}(s))} ds \\ & + \int_0^t \|k^{1/2}(f_s(\phi))|v|\|^2_{L^2(\Omega_{ml}(s))} ds. \end{aligned} \tag{101}$$

On the other hand, by using (33) with $u = v^\delta$, after integrating in $[0,t]$, we obtain an expression for $\nu_0 \displaystyle\int_0^t (Av^\delta, v^\delta) ds$ that substituted into (99), gives

$$X_\delta^t = \frac{1}{2}\|v_0^\delta\|^2_{L^2(\Omega)} + \int_0^t (\mathcal{F}(c^\delta, \theta^\delta), v^\delta) ds - \nu_o \int_0^t (Av^\delta, \psi) ds - \nu_o \int_0^t (A\psi, v^\delta - \psi) ds.$$

By letting $\delta \to 0$ in this last expression, we conclude that

$$X_\delta^t \to X^t = \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \int_0^t (\mathcal{F}(c,\theta), v)ds - \nu_o \int_0^t (\chi, \psi)ds - \nu_o \int_0^t (A\psi, v-\psi)ds$$

Now, from the fact that $v = 0$ a.e. in $\overset{o}{Q}_s$ and our additional hypothesis that the measure of $\partial\Omega_{ml}$ is zero for a.e. $t \in (0,t)$, we can write X^t as

$$\begin{aligned} X^t = \; & \frac{1}{2}\|v_0\|^2_{L^2(\Omega_{ml}(0))} + \int_0^t (\mathcal{F}(c,\theta), v)_{\Omega_{ml}(s)}ds - \nu_o \int_0^t (\chi, \psi)ds \\ & -\nu_o \int_0^t (A\psi, v-\psi)ds. \end{aligned}$$

This and (101) imply that

$$\begin{aligned} & \frac{1}{2}\|v_0\|^2_{L^2(\Omega_{ml}(0))} + \int_0^t (\mathcal{F}(c,\theta), v)_{\Omega_{ml}(s)}ds - \nu_o \int_0^t (\chi, \psi)ds - \nu_o \int_0^t (A\psi, v-\psi)ds \\ & \geq \frac{1}{2}\|v(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v\|^2_{L^2(\Omega_{ml}(s))}ds + \int_0^t \|k^{1/2}(f_s(\phi))|v|\|^2_{L^2(\Omega_{ml}(s))}ds. \end{aligned}$$

Now, we recall that (97) holds for a.e. $t \in (0,T)$ and any $\eta \in L^p(0,T;V^p)$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and such that $\eta_t \in L^{p'}(0,T;(V^p)')$. Thus, our previous estimates and our additional hypothesis on the integrability of $k(f_s(\phi))$ allow us to use density arguments to conclude that (97) holds for any $\eta \in L^p(0,T;V^p)$ with support contained in the closure of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and such that $\eta_t \in L^{p'}(0,T;(V^p)')$. In particular, v has this properties, and we can take $\eta = v$ in (97) and integrate in time on the interval $[0,t]$ to find an energy identity that used with the last inequality furnishes

$$\nu_o \int_0^t (\chi - A\psi, v - \psi)ds \geq 0 \quad \text{a.e. } t.$$

Therefore, by standard arguments using the hemicontinuity of operator A ([8] Chp.2), we can conclude that $\chi = Av$, and the proof of Theorem 1 is then complete. □

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