A SLLN for random closed sets

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Abstract

In this paper we establish a strong law of large numbers for independent and identically nonempty random closed sets in $\mathbb{R}^p$.

Key words: Strong law of large numbers, Random closed sets, Hausdorff distance, Integrably bounded.

1 Introduction

Strong laws of large numbers have been stated in the literature for measurable functions taking values on different spaces. In this paper we give a strong law of large numbers (SLLN) for random sets in $\mathbb{R}^p$.

The first SLLN for random sets was proved by Artstein and Vitale (3) for independent and identically distributed (i.i.d.) random compact subsets of $\mathbb{R}^p$. This result have been extended to i.i.d. random compact subsets of a separable Banach space by Puri and Ralescu (10), Giné, Hahn and Zinn (5) and Hiai (7). The SLLN in these papers is established with the Hausdorff distance and assuming that the random sets are integrably bounded.

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Motivated by an optimization problem arising in allocation processes under uncertainty, Artstein and Hart (2) established a SLLN for i.i.d. random closed subsets of $\mathbb{R}^p$. This result have been also extended to i.i.d. random closed subsets of a separable Banach space by Hiai (8) and Hess (6). The integrability condition on the random sets assumed in these papers is that the mean is not empty. This condition is weaker than being integrably bounded and so, the covergences in these papers (the Kuratowski convergence in Artstein and Hart (2), the Mosco convergence in Hiai (8) and the convergence in the Wijsman topology in Hess (6)) are less stronger than the convergence with the Hausdorff distance.

In this paper we consider i.i.d. random closed subsets of $\mathbb{R}^p$ and we show that if they are integrably bounded, then the SLLN is satisfied with the Hausdorff distance. Hence our result is a generalization of the one in Artstein and Vitale (3) for not necessarily bounded random closed sets, and it is also a strengthening of the result in Artstein and Hart (2) when one assume a stronger condition on the integrability of the random closed sets.

To achieve our aim we have organized the paper as follows. In Section 2 we give definitions and introduce some notation. In Section 3 we show some results that will be used for the proof of the main result, the SLLN for nonempty random closed sets, which is given in Section 4.

2 Notation

Let $\mathcal{C}(\mathbb{R}^p)$ (resp. $\mathcal{C}_c(\mathbb{R}^p)$) denote the collection of nonempty closed (resp. nonempty closed convex) subsets of the Euclidean space $\mathbb{R}^p$, and let $\|.\|$ denote the euclidean norm. For $X, Y \in \mathcal{C}(\mathbb{R}^p)$, the distance $d(y, X)$ between X and $y \in \mathbb{R}^p$, the Hausdorff distance $H(X, Y)$ between X and Y, and the norm $\|X\|_H$ of X are defined by

$$d(y, X) = \inf_{x \in X} \|x - y\|,$$
$$H(X, Y) = \max\left\{ \sup_{x \in X} d(x, Y),\ \sup_{y \in Y} d(y, X) \right\},$$
$$\|X\|_H = H(X, \{0\}) = \sup_{x \in X} \|x\|.$$

For $A, B \in \mathcal{C}(\mathbb{R}^p)$ and $\lambda \in \mathbb{R}^p$, the addition and scalar multiplications are defined as

$$A + B = \{a + b : \ a \in A,\, b \in B\}, \quad \lambda A = \{\lambda a : \ a \in A\}.$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A function $X : \Omega \to \mathcal{C}(\mathbb{R}^p)$ is said to be a random set if it satisfies the following condition

$$X^{-1}(O) = \{\omega \in \Omega : \ X(\omega) \cap O \neq \emptyset\} \in \mathcal{A},$$

for every open subset O of $\mathbb{R}^p$, that is equivalent to $d(x, X(\omega))$ is a measurable function of ω, for every $x \in \mathbb{R}^p$.

The expected value of a random set X is defined by $E(X) = \{E(f) : f \in S_X^1\}$, where $S_X^1 = \{f : \Omega \to \mathbb{R}^p$ is measurable, $f(\omega) \in X(w)$ *a.e.* and $E(\|f\|) < \infty\}$.

For each $A \in \mathcal{C}(\mathbb{R}^p)$ let $\overline{co}A$ denote the closed convex hull of A. If $X : \Omega \to \mathcal{C}(\mathbb{R}^p)$ is a random set, then $\overline{co}X : \Omega \to \mathcal{C}_c(\mathbb{R}^p)$ defined by $\overline{co}X(\omega) = \overline{co}\{X(\omega)\}$, is also a random set.

3 Preliminary results

To prove our SLLN we will proceed in a way that, in a certain sense, is similar to the classical theory for random variables. Specifically, given a sequence of nonempty random closed sets in $\mathbb{R}^p$, $X_1, X_2, ...$, we construct the sequence $Y_1, Y_2, ...$ by "truncating" the original random sets. Then we prove two facts: first, that $Y_1, Y_2, ...$ satisfies the SLLN, and second, that we can replace $X_1, X_2, ...$ by $Y_1, Y_2, ...$ without changing the asymptotics. In this section we give results that will be useful in proving this second part.

Let $X_1, X_2, ...$ be a sequence of nonempty random closed sets in $\mathbb{R}^p$, i.i.d. and such that $E\{\|X_1\|_H\} < \infty$. For each $n \in \mathbb{N}$, let

$$Y_n = \begin{cases} X_n & \text{if } X_n \subseteq K(0; n), \\ \{0\} & \text{otherwise,} \end{cases}$$

where $K(0; n) = \{x \in \mathbb{R}^p : \|x\| \leq n\}$. Clearly $Y_1, Y_2, ...$ is a sequence of nonempty independent compact random sets in $\mathbb{R}^p$ with $E\{\|Y_n\|_H\} \leq E\{\|X_1\|_H\} < \infty$, $\forall n \in \mathbb{N}$.

Lemma 1 *Let $X_1, X_2, ...$ be a sequence of nonempty random closed sets in $\mathbb{R}^p$, i.i.d. with $E\{\|X_1\|_H\} < \infty$. Then*

$$H\left(\frac{X_1 + X_2 + ... + X_n}{n}, \frac{Y_1 + Y_2 + ... + Y_n}{n}\right) \to 0, \quad a.s.$$

as $n \to \infty$.

Proof We have that

$$H\left(\frac{X_1+X_2+...+X_n}{n},\frac{Y_1+Y_2+...+Y_n}{n}\right)\leq\frac{1}{n}\sum_{i=1}^{n}W_i, \tag{1}$$

where $W_i = I(Y_i \neq X_i)\|X_i\|_H$ and

$$I(A)=\begin{cases}1 & \text{if } A \text{ holds,}\\ 0 & \text{otherwise.}\end{cases}$$

Since $E\left\{\|X_1\|_H\right\} < \infty$,

$$\sum_{i\geq 1}P(W_i>0)=\sum_{i\geq 1}P(\|X_i\|_H>i)<\infty,$$

and hence, by Lemma 1 in Rohatgi (11) (page 266), we have that

$$\frac{1}{n}\sum_{i=1}^{n}W_i\to 0,\quad a.s.$$

as $n\to\infty$. This joint with inequality (1) give the result. □

Lemma 2 *Let $X_1, X_2, ...$ be a sequence of nonempty random closed sets in $\mathbb{R}^p$, i.i.d. with $E\left\{\|X_1\|_H\right\} < \infty$. Then for any $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that*

a) $\forall\ x \in E(\overline{co}X_1)$, there exists $y = y(x) \in E(\overline{co}Y_n)$ verifying $\|x - y\| < \varepsilon$, $\forall n \geq M$.
b) $\forall\ y \in E(\overline{co}Y_n)$, there exists $x = x(y) \in E(\overline{co}X_1)$ verifying $\|x - y\| < \varepsilon$, $\forall n \geq M$.

Proof Let

$$Z_n=\begin{cases}1 & \text{if } X_n\subseteq K(0;n)\ \Leftrightarrow\ \overline{co}X_n\subseteq K(0;n),\\ 0 & \text{otherwise.}\end{cases}$$

Let $x \in E(\overline{co}X_n)$. Then there exists $f \in S^1_{\overline{co}X_n}$ such that $x = E(f)$. Let $y = E(g)$ with $g = Z_nf$. Clearly, $y \in E(\overline{co}Y_n)$ since $g \in S^1_{\overline{co}Y_n}$. We have that

$$E(f)=E(Z_nf)+E\{(1-Z_n)f\}=E(g)+E\{(1-Z_n)f\},$$

and hence,

$$\|x-y\|=\|E\{(1-Z_n)f\}\|\leq E\{\|X_n\|_H I(\|X_n\|_H>n)\}. \tag{2}$$

Since $E\{\|X_1\|_H\} < \infty$, this implies that $\forall \varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that $E\{\|X_n\|_H I(\|X_n\|_H > n)\} < \varepsilon$, $\forall n \geq M$. This joint with (2) completes the proof of part a).

Now, let $y \in E(\overline{co}Y_n)$. Then there exists $g \in S^1_{\overline{co}Y_n}$ such that $y = E(g)$. Let $x = E(f)$ with $f = Z_n g + (1 - Z_n)h$, for some $h \in S^1_{\overline{co}X_n}$. Clearly, $x \in E(\overline{co}X_n)$ since $f \in S^1_{\overline{co}X_n}$. We have that

$$\|x - y\| = \|E\{(1 - Z_n)h\}\| \leq E\{\|X_n\|_H I(\|X_n\|_H > n)\}. \tag{3}$$

Reasoning as before, the result in part b) follows from (3) and that $E\{\|X_1\|_H\} < \infty$. □

Lemma 3 *Let $X_1, X_2, ...$ be a sequence of nonempty random closed sets in $\mathbb{R}^p$, i.i.d. with $E\{\|X_1\|_H\} < \infty$. Then*

$$H\left(\frac{E(\overline{co}Y_1) + E(\overline{co}Y_2) + ... + E(\overline{co}Y_n)}{n}, E(\overline{co}X_1)\right) \to 0,$$

as $n \to \infty$.

Proof Let $x \in E(\overline{co}X_1)$ and $\varepsilon > 0$ be fixed. By Lemma 2 part a), there exists $\hat{y}_m \in E(\overline{co}Y_m)$ such that $\|x - \hat{y}_m\| < \varepsilon$, $\forall m \geq M$, for some $M \in \mathbb{N}$. Assume that $n \geq M$. Let $\hat{y} = (y_1 + ... + y_{M-1} + \hat{y}_M + ... + \hat{y}_n)/n$, for arbitrary $y_j \in E(\overline{co}Y_j)$, $i = 1, 2, ..., M - 1$, and let $\Psi_n = \{E(\overline{co}Y_1) + E(\overline{co}Y_2) + ... + E(\overline{co}Y_n)\}/n$. Then,

$$\begin{aligned} \inf_{y \in \Psi_n} \|x - y\| \leq \|x - \hat{y}\| &\leq \frac{1}{n} \sum_{j=1}^{M-1} \|x - y_j\| + \frac{n - M + 1}{n} \varepsilon \\ &\leq \frac{2(M-1)}{n} E\{\|X_1\|_H\} + \frac{n - M + 1}{n} \varepsilon \leq 2\varepsilon, \end{aligned} \tag{4}$$

for any $n \geq M_0 = \max\{M, M_1\}$, where $M_1 = 2(M - 1)E\{\|X_1\|_H\}/\varepsilon$. Now let $y \in \Psi_n$ be fixed. By using Lemma 2 part b) and proceeding analogously we have that

$$\inf_{x \in E(\overline{co}X_1)} \|x - y\| \leq 2\varepsilon, \tag{5}$$

for any $n \geq M_0$. Finally, the result follows from (4) and (5). □

4 A SLLN for closed random sets

Theorem 1 *Let $X_1, X_2, ...$ be a sequence of nonempty random closed sets in $\mathbb{R}^p$, i.i.d. with $E\{\|X_1\|_H\} < \infty$. Then*

$$H\left(\frac{X_1 + X_2 + ... + X_n}{n}, E\{\overline{co}X_1\}\right) \to 0, \quad a.s.$$

as $n \to \infty$.

Proof We have that

$$\begin{aligned} H\left(\frac{X_1 + ... + X_n}{n}, E\{\overline{co}X_1\}\right) \leq H\left(\frac{X_1 + ... + X_n}{n}, \frac{Y_1 + ... + Y_n}{n}\right) + \\ H\left(\frac{Y_1 + ... + Y_n}{n}, \frac{\overline{co}Y_1 + ... + \overline{co}Y_n}{n}\right) + \\ H\left(\frac{\overline{co}Y_1 + ... + \overline{co}Y_n}{n}, \frac{E\{\overline{co}Y_1\} + ... + E\{\overline{co}Y_n\}}{n}\right) + \\ H\left(\frac{E\{\overline{co}Y_1\} + ... + E\{\overline{co}Y_n\}}{n}, E\{\overline{co}X_1\}\right). \end{aligned} \tag{6}$$

To prove the result we will show that each term in the right side of (6) converges to 0 a.s. when $n \to \infty$.

First, by Lemma 1 we have that

$$H\left(\frac{X_1 + ... + X_n}{n}, \frac{Y_1 + ... + Y_n}{n}\right) \to 0, \quad a.s.$$

as $n \to \infty$.

Second, by the Shapley-Folkman Lemma (see (1)) we have that

$$H\left(\frac{Y_1 + ... + Y_n}{n}, \frac{\overline{co}Y_1 + ... + \overline{co}Y_n}{n}\right) \leq \frac{\sqrt{p}}{n} \max_{1 \leq i \leq n} \|Y_i\|_H. \tag{7}$$

Since $\|Y_i\|_H \leq \|X_i\|_H$, $\forall i$, and $E\{\|X_1\|_H\} < \infty$, the right side of (7) converges to 0 a.s. when $n \to \infty$ (see for example Lemma 1 in Babu (1986)).

Third, by Theorem 20 in Lyashenko (9), to show that

$$H\left(\frac{\overline{co}Y_1 + ... + \overline{co}Y_n}{n}, \frac{E\{\overline{co}Y_1\} + ... + E\{\overline{co}Y_n\}}{n}\right) \to 0, \quad a.s.$$

as $n \to \infty$ it suffices to see that

$$\sum_{i \geq 1} \frac{E\left[H^2\left(\overline{co}Y_i, E\{\overline{co}Y_i\}\right)\right]}{i^2} < \infty, \tag{8}$$

and that all $E\{\overline{co}Y_i\}$ are bounded.

All $E\{\overline{co}Y_i\}$ are bounded because

$$\|E\{\overline{co}Y_i\}\|_H \leq E\{\|Y_i\|_H\} \leq E\{\|X_1\|_H\} < \infty, \quad \forall i. \tag{9}$$

Since $H\left(\overline{co}Y_i, E\{\overline{co}Y_i\}\right) \leq \|\overline{co}Y_i\|_H + \|E\{\overline{co}Y_i\}\|_H$, we have that

$$\sum_{i \geq 1} \frac{E\left[H^2\left(\overline{co}Y_i, E\{\overline{co}Y_i\}\right)\right]}{i^2} \leq \sum_{i \geq 1} \frac{E\left\{\|\overline{co}Y_i\|_H^2\right\}}{i^2} +$$
$$2\sum_{i \geq 1} \frac{\|E\{\overline{co}Y_i\}\|_H E\left\{\|\overline{co}Y_i\|_H\right\}}{i^2} + \sum_{i \geq 1} \frac{\|E\{\overline{co}Y_i\}\|_H^2}{i^2}.$$

From (9),

$$\sum_{i \geq 1} \frac{\|E\{\overline{co}Y_i\}\|_H E\left\{\|\overline{co}Y_i\|_H\right\}}{i^2} < \infty \quad \text{and} \quad \sum_{i \geq 1} \frac{\|E\{\overline{co}Y_i\}\|_H^2}{i^2} < \infty.$$

Since $\|\overline{co}Y_i\|_H \leq i$, $\forall i$ and $\sum_{i \geq k} 1/i^2 \leq 2/k$, we have that

$$\sum_{i \geq 1} \frac{E\left\{\|\overline{co}Y_i\|_H^2\right\}}{i^2} \leq \sum_{i \geq 1} \sum_{k=0}^{i-1} \frac{(k+1)^2}{i^2} P\left(k \leq \|X_1\|_H < k+1\right) =$$
$$\sum_{k \geq 1} k^2 P\left(k-1 \leq \|X_1\|_H < k\right) \sum_{i \geq k} \frac{1}{i^2} \leq 2 \sum_{k \geq 1} k P\left(k-1 \leq \|X_1\|_H < k\right)$$
$$\leq 2E\left\{\|X_1\|_H\right\} + 1 < \infty.$$

and therefore (8) holds.

Finally, by Lemma 3

$$H\left(\frac{E(\overline{co}Y_1) + E(\overline{co}Y_2) + \ldots + E(\overline{co}Y_n)}{n}, E(\overline{co}X_1)\right) \to 0,$$

as $n \to \infty$. This completes the proof. □

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