A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy

Gabriela Planas *
gplanas@ime.unicamp.br

José Luiz Boldrini †
boldrini@ime.unicamp.br

IMECC - UNICAMP
Caixa Postal 6065, CEP 13081-970, Campinas-SP, Brazil

Abstract

The article analyzes a two-dimensional phase-field model for a non-stationary process of solidification of a binary alloy with thermal properties. The model allows the occurrence of fluid flow in non-solid regions, which are a priori unknown, and is thus associated to a free boundary value problem for a highly non-linear system of partial differential equations. These equations are the phase-field equation, the heat equation, the concentration equation and a modified Navier-Stokes equations obtained by the addition of a penalization term of Carman-Kozeny type, which accounts for the mushy effects, and also of a Boussinesq term to take in care of the effects of variations of temperature and concentration in the flow. A proof of existence of weak solutions for such system is given. The problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder fixed point theorem. A solution is then found by using compactness argument.

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1 Introduction

Through the introduction of an extra variable to distinguish among physical phases, the phase-field methodology provides a continuum description of phase change processes. This method has proved itself to be a powerful tool for the study of situations with complex growth structures like dendrites, and recently phase-field models for solidification have been extended to include melt convection, bringing interesting new mathematical aspects to the methodology.

In an attempt to understand such mathematical aspects, we consider here a two-dimensional phase-field model for a non-stationary process of solidification with convection of a binary alloy with thermal properties. Our objective is to prove the existence of solutions of a mathematical model that combines ideas of Voller et al. [12, 13] and of Blanc et al. [1] for taking in consideration the possibility of flow, with those of Caginalp et al. [2] for the phase-field and the thermal properties of the alloy. The resulting system will be described in detail in the next section. Here, we just observe that, besides having a phase-field equation, a heat equation and a concentration equation, it also includes the Navier-Stokes equations modified by the addition of a Carman-Kozeny type term to take care of the flow in mushy regions and also by the addition a Boussinesq type term to take in consideration buoyancy forces due to thermal and concentration differences. Since these equations for the flow only hold in an a priori unknown non-solid region, the model corresponds to a free-boundary value problem. Moreover, since the Carman-Kozeny term is dependent on the local solid fraction, this is assumed to be functionally related to the phase-field.

The phase-field model with convection considered here includes advection terms in each of its equations. In a recent paper, [9], a simplified version of this model, which did not include the advection term in the phase-field equation, was analyzed. We should say that the inclusion of this term brings several new technical difficulties to an already hard problem. To overcome these difficulties is the purpose of this paper; for this, we had to adapt to our case the results presented in Hoffman and Jiang [5] concerning the phase-field equation. We also had to restrict the analysis to the two-dimensional situation. We will comment more about this point in the next section; here we just remark that this restriction in the dimension of the space is clearly of technical nature. We hope to remove it in the future.

Existence of solutions will be obtained by using a regularization technique similar to the one already used in [1] and [9]: with the help of an auxiliary parameter, we will transform the original free-boundary value problem into a more standard penalized one. This regularized problem will then be studied by using fixed point arguments, and then we pass to the limit to obtain a solution of the original problem.

The outline of this paper is as follows. In Section 2 we detail the model we consider; we also fix the notation and state our main result. In Section 3 we study an auxiliary phase-field problem. The description and the analysis of the regularized problem is done in Section 4. Section 5 is devoted to proof the main existence theorem.

2 The model and the main result

Consider $0 < T < +\infty$, a bounded open domain $\Omega \subset I\!R^2$ with smooth boundary $\partial\Omega$, and denote $Q = \Omega \times (0,T)$. Then, consider the following problem:

$$v_t - \nu\Delta v + \nabla p + v\cdot\nabla v + k(f_s(\phi))v = \mathcal{F}(c,\theta) \text{ in } Q_{ml}, \tag{1}$$

$$\text{div } v = 0 \quad \text{in } Q_{ml}, \tag{2}$$

$$v = 0 \quad \text{in } Q_s, \tag{3}$$

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \text{ in } Q, \tag{4}$$

$$C_{\rm v}\theta_t + C_{\rm v}v\cdot\nabla\theta = \nabla\cdot K_1(\phi)\nabla\theta + \frac{l}{2}f_s(\phi)_t \quad \text{in } Q, \tag{5}$$

$$c_t + v\cdot\nabla c = K_2\left(\Delta c + M\nabla\cdot c(1-c)\nabla\phi\right) \text{ in } Q, \tag{6}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega\times(0,T), \quad v = 0 \text{ on } \partial Q_{ml}, \tag{7}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \text{ in } \Omega, \quad v(0) = v_0 \text{ in } \Omega_{ml}(0), \tag{8}$$

In the previous equations, the order parameter (phase-field) ϕ is the state variable characterizing the different phases; v is the velocity field, and p is the associated hydrostatic pressure; $f_s \in [0,1]$ is the solid fraction; θ is the temperature; $c \in [0,1]$ is the concentration (the fraction of one of the two materials in the mixture.) The Carman-Kozeny type term $k(f_s)$ accounts for the mushy effect on the flow, and its usual form is $k(f_s) = C_0 {f_s}^2/(1-f_s)^3$. We do not restrict to this form and allow more general expressions. $\mathcal{F}(c,\theta)$ denotes the buoyancy forces, which by using Boussinesq approximation, we assume to be of form $\mathcal{F}(c,\theta) = \rho\mathbf{g}\,(c_1(\theta-\theta_r) + c_2(c-c_r)) + F$. Here, $\rho > 0$ is the mean value of the density (constant); $\mathbf{g}$ is the acceleration of gravity; c_1 and c_2 are two real constants; θ_r and c_r are respectively the reference temperature and concentration, which for simplicity of exposition will be assumed to be zero, and F is a given external force field. Also, $\alpha > 0$ is the relaxation scaling; $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width; σ is the surface tension, and $[s]$ is the entropy density difference between phases; $\nu > 0$ is the viscosity; $C_{\mathrm{v}} > 0$ is the specific heat; $l > 0$ the latent heat (constant); θ_A, θ_B are the melting temperatures of two materials composing the alloy; $K_2 > 0$ is the solute diffusivity, and M is a constant related to the slopes of solidus and liquidus lines. Finally, $K_1 > 0$ denotes the thermal conductivity which is assumed to depend on the phase-field.

The domain Q is composed of three regions, Q_s, Q_m and Q_l. The first region is fully solid, the second is mushy and the third is fully liquid . They are defined by

$$\begin{array}{rcl} Q_s &=& \{\ (x,t)\in Q \quad / \quad f_s(\phi(x,t)) = 1\ \}, \\ Q_m &=& \{\ (x,t)\in Q \quad / \quad 0 < f_s(\phi(x,t)) < 1\ \}, \\ Q_l &=& \{\ (x,t)\in Q \quad / \quad f_s(\phi(x,t)) = 0\ \}, \end{array} \tag{9}$$

and Q_{ml} will refer to the not-solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{\ (x,t)\in Q \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{10}$$

At each time $t \in [0,T]$, $\Omega_{ml}(t)$ is defined by

$$\Omega_{ml}(t) = \{\ x\in\Omega \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}\,. \tag{11}$$

In view of these regions are a priori unknown, the model is a free boundary problem.

Throughout this paper we assume the conditions,

(H1) k is a non decreasing function of class $C^1[0,1)$ satisfying $k(0) = 0$ and $\lim_{x\to 1^-} k(x) = +\infty$,

(H2) f_s is a Lipschitz continuous function defined on $I\!R$ and satisfying $0 \le f_s(r) \le 1$ for $r \in I\!R$; f_s' is measurable,

(H3) K_1 is a Lipschitz continuous function defined on $I\!R$ such that there exist $a > 0$ and $b > 0$ for which

$$0 < a \leq K_1(r) \leq b \qquad \text{for all } r \in I\!R,$$

(H4) F is a given function in $L^2(Q)$.

Our purpose in this work is to show that problem (1)-(8) admits at least one solution in a sense to be made precise below.

Before that, we comment on the restriction on the spatial dimension. Since the modified Navier-Stokes equations only hold in the non-solid region Q_{ml}, this set must be open for these equations to be understood at least in the sense of distributions. This information is in particular implied by the continuity of phase-field ϕ which in turn depends on the smoothness of v. It turns out that only for the bidimensional case we are able to show enough regularity of v to yield the continuity of ϕ. As we wrote in the Introduction, such limitations are quite clearly of technical nature, and it is our hope to remove them in the future.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq I\!R^2$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0, T)$ Then,

$$\begin{array}{rcl} \mathcal{V}(G) & = & \left\{ w \in \left(C_0^\infty(G)\right)^2, \ \text{div } w = 0 \right\}, \\ H(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^2, \\ V(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^2, \\ H^{\tau,\tau/2}(\overline{Q}_G) & = & \text{Hölder continuous functions of exponent } \tau \text{ in } x \\ & & \text{and exponent } \tau/2 \text{ in } t, \\ W_q^{2,1}(Q_G) & = & \{ w \in L^q(Q_G) / \, D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \}. \end{array}$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$. Properties of these functional spaces can be found for instance in [6, 11]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^2$.

The main result of this paper is the following.

Theorem 1 *Let be $T > 0$, $\Omega \subseteq I\!R^2$ a bounded open domain of class C^3. Suppose that $v_0 \in H(\Omega_{ml}(0))$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $2 < q < 4$, $1/2 < \gamma \leq 1$, satisfying the compatibility condition $\dfrac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$, $\theta_0 \in L^2(\Omega)$*

and $c_0 \in L^2(\Omega)$ *satisfying* $0 \leq c_0 \leq 1$ *a.e. in* $\bar{\Omega}$. *Under the assumptions* **(H1)-(H4)**, *there exist functions* (v, ϕ, θ, c) *such that*

i) $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\quad v = 0$ *a.e. in* $\mathring{Q}_s$, $\quad v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where* Q_s *is defined by (9) and* $\Omega_{ml}(0)$ *by (11),*

ii) $\phi \in W_q^{2,1}(Q)$, $\quad \phi(0) = \phi_0$,

iii) $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta(0) = \theta_0$,

iv) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q,

and such that

$$\begin{aligned}(v(t), \eta(t))_{\Omega_{ml}(t)} &- \int_0^t (v, \eta_t)_{\Omega_{ml}(s)} ds + \nu \int_0^t (\nabla v, \nabla \eta)_{\Omega_{ml}(s)} ds \\ &+ \int_0^t (v \cdot \nabla v, \eta)_{\Omega_{ml}(s)} ds + \int_0^t (k(f_s(\phi)) v, \eta)_{\Omega_{ml}(s)} ds \quad (12) \\ &= \int_0^t (\mathcal{F}(c, \theta), \eta)_{\Omega_{ml}(s)} ds + (v_0, \eta(0))_{\Omega_{ml}(0)},\end{aligned}$$

$t \in (0,T)$, *for any* $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ *with compact support contained in* $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ *and* $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$ *where* Q_{ml} *is defined by (10) and* $\Omega_{ml}(t)$ *by (11),*

$$\alpha\epsilon^2 \phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2 \Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \quad a.e. \text{ in } Q, \tag{13}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad a.e. \text{ on } \partial\Omega \times (0,T), \tag{14}$$

$$\begin{aligned}-C_{\mathrm{v}} \int_0^T \int_\Omega \theta \zeta_t dx dt &- C_{\mathrm{v}} \int_0^T \int_\Omega v\, \theta \cdot \nabla\zeta \, dx dt + \int_0^T \int_\Omega K_1(\phi) \nabla\theta \cdot \nabla\zeta \, dx dt \\ &= \frac{l}{2} \int_0^T \int_\Omega f_s(\phi)_t \zeta \, dx dt + C_{\mathrm{v}} \int_\Omega \theta_0 \zeta(0) dx \end{aligned} \tag{15}$$

$$\begin{aligned}-\int_0^T \int_\Omega c\zeta_t dx dt - \int_0^T \int_\Omega v\, c \cdot \nabla\zeta \, dx dt &+ K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla \zeta \, dx dt \\ +K_2 M \int_0^T \int_\Omega c(1-c) \nabla\phi \cdot \nabla\zeta \, dx dt &= \int_\Omega c_0 \zeta(0) dx,\end{aligned} \tag{16}$$

for any $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\zeta_t \in L^2(0,T;L^2(\Omega))$ *and* $\zeta(T) = 0$ *in* Ω.

Remark. The restriction $q > 2$ ensure the continuity of phase-field because $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 4/q$ if $q > 2$ ([6] p. 80). Therefore the set Q_{ml} is open giving a meaningful interpretation to equation of velocity field. The restriction $q < 4$ comes from the regularity of velocity field. It will be clear in the next section.

3 An auxiliary problem

We consider the initial boundary value problem,

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{1}{2}(\phi-\phi^3) + g \quad \text{in } Q, && (17)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), && (18)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega, && (19) \end{aligned}$$

and prove the following result using a technique similar to the one already used in [5] to treat a phase-field equation without convective term.

Theorem 2 *Suppose that $g \in L^q(Q)$ with $2 \leq q < 4$, $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $\phi_0 \in W^{2-2/q,q}(\Omega)$ satisfying the compatibility conditions $\dfrac{\partial\phi_0}{\partial n} = 0$ on $\partial\Omega$. Then there exist a unique $\phi \in W_q^{2,1}(Q)$ solution of problem (17)-(19) for any $T > 0$, which satisfies the estimate*

$$\|\phi\|_{W_q^{2,1}(Q)} \leq C\left(\|\phi_0\|_{W^{2-2/q,q}(\Omega)} + \|g\|_{L^q(Q)} + \|\phi_0\|^3_{W^{2-2/q,q}(\Omega)} + \|g\|^3_{L^q(Q)}\right) \qquad (20)$$

where C depends on $\|v\|_{L^4(Q)}$, on Ω and T.

Proof: In order to apply Leray-Schauder fixed point theorem ([3] p. 189) we consider the operator T_λ, $0 \leq \lambda \leq 1$, on the Banach space $B = L^6(Q)$, which maps $\hat{\phi} \in B$ into ϕ by solving the problem

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{\lambda}{2}(\hat{\phi}-\hat{\phi}^3) + \lambda g \quad \text{in } Q, && (21)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), && (22)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega. && (23) \end{aligned}$$

We define $G_\lambda = \dfrac{\lambda}{2}(\hat{\phi} - \hat{\phi}^3) + \lambda g$ and we observe that $G_\lambda \in L^2(Q)$. Since $v \in L^4(Q)$, we infer from L^p-theory of parabolic equations ([6], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution ϕ of problem (21)-(23) with $\phi \in W_2^{2,1}(Q)$. Due to the embedding of $W_2^{2,1}(Q)$ into $L^p(Q)$, for any $p \in [1, \infty)$ ([7] p.15), the operator T_λ is well defined from B into B.

To prove continuity of T_λ, let $\hat{\phi}_n \in B$ strongly converging to $\hat{\phi} \in B$; for each n, let $\phi_n = T_\lambda(\hat{\phi}_n)$. We have that ϕ_n satisfies the following estimate ([6] p. 341)

$$\|\phi_n\|_{W_2^{2,1}(Q)} \le C\left(\|\hat{\phi}_n\|_{L^2(Q)} + \|\hat{\phi}_n\|^3_{L^6(Q)} + \|g\|_{L^2(Q)} + \|\phi_0\|_{H^1(\Omega)}\right)$$

for some constant C independent of n. Since $W_2^{2,1}(Q)$ is compactly embedded in $L^2(0,T;W^{1,p}(\Omega))$ ([10] Cor.4) and in $L^p(Q)$, $p \in [1,\infty)$, it follows that there exist a subsequence of ϕ_n (which we still denote by ϕ_n) strongly converging to $\phi = T_\lambda(\hat{\phi})$ in B. Therefore T_λ is continuous for all $0 \le \lambda \le 1$. At the same time, T_λ is bounded in $W_2^{2,1}(Q)$, and the embedding of this space in B is compact. Thus, we conclude that T_λ is a compact operator for each $\lambda \in [0,1]$.

To prove that for $\hat{\phi}$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $\phi_i\,(i = 1,2)$ be the corresponding solutions of (21)-(23). For $\phi = \phi_1 - \phi_2$ the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q)} \le C|\lambda_1 - \lambda_2|\left(\|\hat{\phi}\|_{L^2(Q)} + \|\hat{\phi}\|^3_{L^6(Q)} + \|g\|_{L^2(Q)}\right)$$

where C is independent of λ_i. Therefore, T_λ is uniformly continuous in λ.

Now we have to estimate the set of all fixed points of T_λ, let $\phi \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2\Delta\phi = \frac{\lambda}{2}(\phi - \phi^3) + \lambda g \quad \text{in } Q, \tag{24}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{25}$$

$$\phi(0) = \phi_0 \quad \text{in } \Omega. \tag{26}$$

We multiply (24) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over $\Omega\times(0,t)$. After integration by parts and the use the Hölder's, Young's and interpolation inequalities, we obtain in the usual manner the following estimate

$$\begin{aligned}\int_\Omega \left(\phi^2+|\nabla\phi|^2\right)dx+\|\phi\|^2_{W_2^{2,1}(Q)} &\le C\left(\|g\|^2_{L^2(Q)}+\|\phi_0\|^2_{H^1(\Omega)}\right)\\ &+C\int_0^t\left(1+\|v\|^4_{L^4(\Omega)}\right)\left(\|\phi\|^2_{L^2(\Omega)}+\|\nabla\phi\|^2_{L^2(\Omega)}\right)dt\end{aligned} \tag{27}$$

where C is independent of λ. By applying Gronwall's Lemma we get

$$\|\phi\|_{L^6(Q)}\le C\|\phi\|_{W_2^{2,1}(Q)}\le C'$$

where C and C' are constants independent of λ. Therefore, all fixed points of T_λ in B are bounded independently of $\lambda\in[0,1]$.

Finally, for $\lambda=0$, it is clear that problem (21)-(23) has a unique solution. Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $\phi\in B\cap W_2^{2,1}(Q)$ of the operator T_1, i.e., $\phi=T_1(\phi)$. This corresponds to a solution of problem (17)-(19). Observe that $W_2^{2,1}(Q)$ is embedded into $L^p(Q)$ for any $p\in[1,\infty)$, this implies that $G=\dfrac{1}{2}(\phi-\phi^3)+g\in L^q(Q)$ and further $\phi\in W_q^{2,1}(Q)$.

To prove estimate (20), observe that from L^p-theory of parabolic equations we have

$$\begin{aligned}\|\phi\|_{W_q^{2,1}(Q)} &\le C\left(\|G\|_{L^q(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\le C\left(\|g\|_{L^q(Q)}+\|\phi\|_{L^q(Q)}+\|\phi\|^3_{L^{3q}(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\le C\left(\|g\|_{L^q(Q)}+\|\phi\|_{W_2^{2,1}(Q)}+\|\phi\|^3_{W_2^{2,1}(Q)}+\|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right).\end{aligned}$$

Using estimate (27) we deduce (20).

It remains to show uniqueness of the solution. Let us assume that ϕ_1 and ϕ_2 are two solutions of problem (17)-(19). Then the difference $\phi=\phi_1-\phi_2$ satisfies the following initial boundary value problem

$$\alpha\epsilon^2\phi_t+\alpha\epsilon^2 v\cdot\nabla\phi-\epsilon^2\Delta\phi = \frac{1}{2}\phi\left(1-(\phi_1^2+\phi_1\phi_2+\phi_2^2)\right) \quad \text{in } Q, \tag{28}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{29}$$

$$\phi(0) = 0 \quad \text{in } \Omega, \tag{30}$$

We remark that $d:=\phi_1^2+\phi_1\phi_2+\phi_2^2\ge 0$. Multiplying (28) by ϕ and using the usual method of Gronwall's Lemma give us $\phi\equiv 0$. Therefore, the solution of problem (17)-(19) is unique and the proof of Theorem 2 is then complete. ∎

4 A regularized problem

In this section we introduce a regularized version of the original problem. As in [1] and [9], the idea is to modify the problem in such way that the Navier-Stokes equations will hold in the whole domain Ω instead of only in a apriori unknown region. For technical reason, we also introduce a suitable regularization of the coefficients of the equations. For this regularized problem, we prove an existence result by using Leray-Schauder Fixed Point Theorem ([3] p. 189).

For this, we need to recall certain results. We start by recalling that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^3)$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^3)} \leq C\,\|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [8] p.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^3)$ be a family of symmetric positive mollifier functions converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^3)$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $I\!R^3 \backslash Q$. Then, as in [8] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $I\!R^3$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(I\!R^3)$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in I\!R^2 : d(x,\Omega) < \delta\}. \tag{31}$$

We also define the associated space-time cylinder $Q^\delta = \Omega^\delta \times (0,T)$.

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$, we conclude that $\rho_\delta(v) \in L^\infty(0,T;H) \cap L^2(0,T;V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \le \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \le \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. For $\delta \in (0, \delta(\Omega)]$, we consider the system

$$\begin{aligned}\frac{d}{dt}(v^\delta, u) \quad &+ \quad \nu(\nabla v^\delta, \nabla u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, u) \\ &= \quad (\mathcal{F}(c^\delta, \theta^\delta), u) \text{ for all } u \in V,\ t \in (0,T), \end{aligned} \tag{32}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t^\delta + \alpha\epsilon^2\rho_\delta(v^\delta)\cdot\nabla\phi^\delta \quad &- \quad \epsilon^2\Delta\phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) \\ &= \quad \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{33}$$

$$C_{\rm v}\theta_t^\delta + C_{\rm v}\rho_\delta(v^\delta)\cdot\nabla\theta^\delta = \nabla\cdot\left(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\right) + \frac{l}{2}f_s^\delta(\phi^\delta)_t \quad \text{ in } Q^\delta, \tag{34}$$

$$c_t^\delta - K_2\Delta c^\delta + \rho_\delta(v^\delta)\cdot\nabla c^\delta = K_2 M \nabla\cdot\left(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\right) \quad \text{ in } Q^\delta, \tag{35}$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \quad \frac{\partial\theta^\delta}{\partial n} = 0, \quad \frac{\partial c^\delta}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \tag{36}$$

$$v^\delta(0) = v_0^\delta \text{ in } \Omega, \quad \phi^\delta(0) = \phi_0^\delta, \quad \theta^\delta(0) = \theta_0^\delta, \quad c^\delta(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{37}$$

We then have the following existence result.

Proposition 1 *For each $\delta \in (0, \delta(\Omega)]$, let $v_0^\delta \in H$, $\phi_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $\theta_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $1/2 < \gamma \le 1$, and $c_0^\delta \in C^1(\bar\Omega^\delta)$, $0 < c_0^\delta < 1$ in $\bar\Omega^\delta$ satisfying the* *compatibility conditions* $\frac{\partial \phi_0^\delta}{\partial n} = \frac{\partial \theta_0^\delta}{\partial n} = \frac{\partial c_0^\delta}{\partial n} = 0$ *on* $\partial\Omega^\delta$. *Assume that* **(H1)-(H4)** *hold. Then there exist functions* $(v^\delta, \phi^\delta, \theta^\delta, c^\delta)$ *which satisfy (32)-(37) for any* $T > 0$ *and*

i) $v^\delta \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad v_t^\delta \in L^2(0,T;V')$,

ii) $\phi^\delta \in L^2(0,T;H^2(\Omega^\delta)), \quad \phi_t^\delta \in L^2(Q^\delta)$,

iii) $\theta^\delta \in L^2(0,T;H^2(\Omega^\delta)), \quad \theta_t^\delta \in L^2(Q^\delta)$,

iv) $c^\delta \in C^{2,1}(Q^\delta), \quad 0 < c^\delta < 1$.

Proof: For simplicity we shall omit the superscript δ at v^δ, ϕ^δ, θ^δ, c^δ. First of all, we consider the following family of operators, indexed by the parameter $0 \le \lambda \le 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(0,T;H) \times L^2(Q^\delta) \times L^2(Q^\delta) \times L^2(Q^\delta),$$

and defined as follows: given $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) \in B$, let $\mathcal{T}_\lambda(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) = (v, \phi, \theta, c)$, where (v, ϕ, θ, c) is obtained by solving the problem

$$\begin{aligned} \frac{d}{dt}(v,u) \;&+\; \nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(\hat{c},\hat{\theta}), u) \\ &-\; \lambda(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u) \text{ for all } u \in V,\ t \in (0,T), \end{aligned} \tag{38}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi \;&-\; \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ &= \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{39}$$

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{40}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M \nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \text{ in } Q^\delta, \tag{41}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{42}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{43}$$

We observe that clearly (v,ϕ,θ,c) is a solution of (32)-(37) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem ([3] p.189).

To verify that $\mathcal{T}_\lambda$ is well defined, observe that equation (38) is the classical Navier-Stokes equation and since $k(f_s^\delta(\hat\phi)-\delta)\hat v \in L^2(Q)$, there exist a unique solution $v \in L^2(0,T;V)\cap L^\infty(0,T;H)$ ([11] p.198).

Since $\hat\theta$, $\hat c \in L^2(Q^\delta)$ and $\rho_\delta(v) \in L^4(Q^\delta)$ we infer from Theorem 2 that there is a unique solution ϕ of equation (39) with $\phi \in W_2^{2,1}(Q^\delta)$.

Since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi) \in C^\infty(Q^\delta)$, we have that $K_1(\rho_\delta(\phi)) \in W_r^{1,1}(Q^\delta)$, $1 \le r \le \infty$, and since $\rho_\delta(v) \in L^4(Q^\delta)$ and $f_s^\delta(\phi)_t = f_s^{\delta'}(\phi)\phi_t \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([6], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ of equation (40) with $\theta \in W_2^{2,1}(Q^\delta)$.

We observe that equation (41) is a semilinear parabolic equation with smooth coefficients and growth conditions on the non-linear forcing terms to apply semigroup results of Henry [4], p.75. Thus, there is a unique global classical solution c. In addition, note that equation (41) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principles together with conditions $0 < c_0^\delta < 1$ and $\dfrac{\partial c^\delta}{\partial n} = 0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\,(x,t) \in Q^\delta. \tag{44}$$

Therefore, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$ let $(\hat v^k,\hat\phi^k,\hat\theta^k,\hat c^k)$, $k \in I\!N$ be a sequence in B such that converges strongly in B to $(\hat v,\hat\phi,\hat\theta,\hat c)$ and let $(v^k,\phi^k,\theta^k,c^k)$ the solution of the problem:

$$\begin{aligned}\frac{d}{dt}(v^k,u) \quad &+ \quad \nu(\nabla v^k,\nabla u) + (v^k\cdot\nabla v^k,u) = \lambda(\mathcal{F}(\hat c^k,\hat\theta^k),u)\\ &- \quad \lambda(k(f_s^\delta(\hat\phi^k)-\delta)\hat v^k,u) \text{ for all } u \in V,\ t \in (0,T),\end{aligned} \tag{45}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t^k + \alpha\epsilon^2\rho_\delta(v^k)\cdot\nabla\phi^k \quad &- \quad \epsilon^2\Delta\phi^k - \frac{1}{2}(\phi^k-(\phi^k)^3)\\ &= \quad \lambda\beta\left(\hat\theta^k + (\theta_B-\theta_A)\hat c^k - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{46}$$

$$C_{\rm v}\theta_t^k + C_{\rm v}\rho_\delta(v^k)\cdot\nabla\theta^k = \nabla\cdot\left(K_1(\rho_\delta(\phi^k))\nabla\theta^k\right) + \frac{l}{2}f_s^\delta(\phi^k)_t \text{ in } Q^\delta, \tag{47}$$

$$c_t^k - K_2\Delta c^k + \rho_\delta(v^k)\cdot\nabla c^k = K_2 M\nabla\cdot\Big(c^k(1-c^k)\nabla\rho_\delta(\phi^k)\Big) \text{ in } Q^\delta, \tag{48}$$

$$\frac{\partial\phi^k}{\partial n} = 0, \quad \frac{\partial\theta^k}{\partial n} = 0, \quad \frac{\partial c^k}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta\times(0,T), \tag{49}$$

$$v^k(0) = v_0^\delta \text{ in } \Omega, \quad \phi^k(0) = \phi_0^\delta, \quad \theta^k(0) = \theta_0^\delta, \quad c^k(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{50}$$

We show that the sequence $(v^k,\phi^k,\theta^k,c^k)$ converges strongly in B to $(v,\phi,\theta,c) = \mathcal{T}_\lambda(\hat v,\hat\phi,\hat\theta,\hat c)$. For that purpose, we will obtain estimates to $(v^k,\phi^k,\theta^k,c^k)$ independent of k. We denote by C_i any positive constant independent of k.

We take $u = v^k$ in equation (45). Using Hölder's and Young's inequalities we obtain

$$\frac{d}{dt}\int_\Omega |v^k|^2dx + \nu\int_\Omega |\nabla v^k|^2dx \le C_1\int_\Omega \Big(|F|^2 + |\hat v^k|^2 + |\hat\theta^k|^2 + |\hat c^k|^2 + |v^k|^2\Big)\,dx.$$

Then, by the usual method of Gronwall's inequality, we get

$$\|v^k\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C_1. \tag{51}$$

From the equation (45) we infer that

$$\|v_t^k\|_{V'} \le C_1\Big(\|v^k\|_V + \|v^k\|^2_{L^4(\Omega)} + \|F\|_{L^2(\Omega)} + \|\hat v^k\|_{L^2(\Omega)} + \|\hat\theta^k\|_{L^2(\Omega^\delta)} + \|\hat c^k\|_{L^2(\Omega^\delta)}\Big),$$

then, using (51) we obtain

$$\|v_t^k\|_{L^2(0,T;V')} \le C_1. \tag{52}$$

From estimate (20) we have that

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C\Big(\|\phi_0\|_{H^1(\Omega^\delta)} + \|\hat\theta^k\|_{L^2(Q^\delta)} + \|\hat c^k\|_{L^2(Q^\delta)} + \|\phi_0\|^3_{H^1(\Omega^\delta)} + \|\hat\theta^k\|^3_{L^2(Q^\delta)} + \|\hat c^k\|^3_{L^2(Q^\delta)} + 1\Big)$$

where C depends on $\|\rho_\delta(v^k)\|_{L^4(Q^\delta)}$. Therefore, using (51) we conclude that

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C_1. \tag{53}$$

Now, multiplying (47) by θ^k one obtains

$$\int_{\Omega^\delta} |\theta^k|^2dx + \int_0^t\int_{\Omega^\delta} |\nabla\theta^k|^2dxdt \le C_1 + C_2\int_0^t\int_{\Omega^\delta}\Big(|\phi_t^k|^2 + |\theta^k|^2\Big)\,dxdt \tag{54}$$

and we infer from (53) and Gronwall's Lemma that

$$\|\theta^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1, \tag{55}$$

hence, it follows from (54) that

$$\|\theta^k\|_{L^2(0,T;H^1(\Omega^\delta))} \leq C_2. \tag{56}$$

We take scalar product of (47) with $\eta \in H^1(\Omega^\delta)$, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_t^k\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla\theta^k\|_{L^2(\Omega^\delta)} + +\|v^k\|_{L^4(\Omega)}\|\theta^k\|_{L^4(\Omega^\delta)} + \|\phi_t^k\|_{L^2(\Omega^\delta)} \right)$$

and we infer from (51),(53) and (56) that

$$\|\theta_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{57}$$

Next, multiplying (48) by c^k we conclude by analogous reasoning and using (44) that

$$\int_{\Omega^\delta} |c^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c^k|^2 dxdt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla\phi^k|^2 dxdt,$$

hence, from (53) we have,

$$\|c^k\|_{L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{58}$$

In order to get an estimate for (c_t^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we return to the equation (48) and use similar techniques, then

$$\|c_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{59}$$

We now infer from (51)-(59) that the sequence (v^k) is bounded (uniformly with respect to k) in

$$W_1 = \left\{ w \in L^2(0,T;V),\, w_t \in L^2(0,T;V') \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H),\, w_t \in L^2(0,T;V') \right\},$$

the sequence (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$ and the sequences (θ^k) and (c^k) are bounded in

$$W_3 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\, w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Since W_1 is compactly embedded in $L^2(Q)$, W_2 in $C([0,T];V')$, $W_2^{2,1}(Q^\delta)$ in $L^2(0,T;W^{1,p}(\Omega^\delta))$, $p \in [1,\infty)$, W_3 in $L^2(Q^\delta)$ and W_4 in $C([0,T];H^1(\Omega^\delta)')$ ([10] Cor.4), it follows that there exist

$$\begin{array}{rcl} v & \in & L^2(0,T;V) \cap L^\infty(0,T;H) \ \text{ with } v_t \in L^2(0,T;V'), \\ \phi & \in & L^2(0,T;H^2(\Omega^\delta)) \ \text{ with } \phi_t \in L^2(Q^\delta), \\ \theta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \ \text{ with } \theta_t \in L^2(0,T;H^1(\Omega^\delta)'), \\ c & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \ \text{ with } c_t \in L^2(0,T;H^1(\Omega^\delta)'), \end{array}$$

and a subsequence of $(v^k,\phi^k,\theta^k,c^k)$ (which we still denote by $(v^k,\phi^k,\theta^k,c^k)$), such that, as $k \to +\infty$,

$$\begin{array}{lllll} v^k & \to & v & \text{in} & L^2(Q) \cap C([0,T];V') \ \text{ strongly}, \\ v^k & \rightharpoonup & v & \text{in} & L^2(0,T;V) \ \text{ weakly}, \\ \phi^k & \to & \phi & \text{in} & L^2(0,T;W^{1,p}(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)),\ p \in [1,\infty) \ \text{strongly}, \\ \phi^k & \rightharpoonup & \phi & \text{in} & L^2(0,T;H^2(\Omega^\delta)) \ \text{ weakly}, \\ \theta^k & \to & \theta & \text{in} & L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \ \text{ strongly}, \\ \theta^k & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \ \text{ weakly}, \\ c^k & \to & c & \text{in} & L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \ \text{ strongly}, \\ c^k & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \ \text{ weakly}. \end{array} \tag{60}$$

It now remains to pass to the limit as k tends to $+\infty$ in (45)-(50).

We observe that $k(f_s^\delta(\cdot)-\delta)$ is bounded Lipschitz continuous function from $I\!R$ in $I\!R$ then $k(f_s^\delta(\hat{\phi}^k)-\delta)$ converges to $k(f_s^\delta(\hat{\phi})-\delta)$ in $L^p(Q)$, for any $p \in [1,\infty)$. We then pass to the limit in standard ways as k tends to $+\infty$ in (45) and get

$$\begin{aligned} \frac{d}{dt}(v,u) \quad & +\nu(\nabla v,\nabla u) + (v\cdot\nabla v,u) = \lambda(\mathcal{F}(\hat{c},\hat{\theta}),u) \\ & -\lambda(k(f_s^\delta(\hat{\phi})-\delta)\hat{v},u) \text{ for all } u \in V,\ t \in (0,T). \end{aligned}$$

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^p(Q^\delta)$ for any $p \in [1,\infty)$ is compact ([7] p.15), and (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi^k)^3$ converges to ϕ^3 in $L^{p/3}(Q^\delta)$. Also, since v^k converges to v in $L^2(Q)$ we have that $\rho_\delta(v^k)$

converges to $\rho_\delta(v)$ in $L^2(Q^\delta)$. We then pass to the limit as k tends to $+\infty$ in (46) and get

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta.$$

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions and ϕ^k converges to ϕ in $L^p(Q^\delta)$, $p \in [1,\infty)$ we have that $K_1(\rho_\delta(\phi^k))$ converges to $K_1(\rho_\delta(\phi))$ and $f_s^{\delta\prime}(\phi^k)$ converges to $f_s^{\delta\prime}(\phi)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. These facts and (60) yield the weak convergence of $K_1(\rho_\delta(\phi^k))\nabla\theta^k$ to $K_1(\rho_\delta(\phi))\nabla\theta$ and $f_s^{\delta\prime}(\phi^k)\phi_t^k$ to $f_s^{\delta\prime}(\phi)\phi_t$ in $L^{3/2}(Q^\delta)$. Now, multiplying (47) by $\eta \in \mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta \times (0,T)$ and by parts, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega^\delta} C_{\rm v}\left(\theta_t^k + \rho_\delta(v^k)\cdot\nabla\theta^k\right)\eta \; + K_1(\rho_\delta(\phi^k))\nabla\theta^k\cdot\nabla\eta \, dxdt \\ = \int_0^T \int_{\Omega^\delta} \frac{l}{2} f_s^{\delta\prime}(\phi^k)\phi_t^k\eta \, dxdt,\end{aligned}$$

then we may pass to the limit and find that,

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^{\delta\prime}(\phi)\phi_t \quad \text{in } \mathcal{D}'(Q^\delta), \quad (61)$$

and using L^p-theory of parabolic equations we conclude that (61) holds almost everywhere in Q^δ.

It remains to pass to the limit in (48). We infer from (60) that $\nabla\rho_\delta(\phi^k)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q^\delta)$ and since $\|c^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c^k(1-c^k)$ converges to $c(1-c)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. Thus, we may pass to the limit in (48) to obtain

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \qquad \text{in } Q^\delta.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0 \le \lambda \le 1$.

At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1 \times W_2^{2,1}(Q^\delta) \times W_3 \times W_3$ but, the embedding of this space in B is compact, then we conclude that $\mathcal{T}_\lambda$ is a compact operator.

To prove that for $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c})$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous in λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $(v_i, \phi_i, \theta_i, c_i)\,(i = 1, 2)$ the corresponding solutions of (38)-(43). We observe that $v = v_1 - v_2$, $\phi = \phi_1 - \phi_2$, $\theta = \theta_1 - \theta_2$

and $c = c_1 - c_2$, satisfy the following problem:

$$\begin{aligned}\frac{d}{dt}(v,u) &+ \nu(\nabla v, \nabla u) + (v_1 \cdot \nabla v, u) - (v \cdot \nabla v_2, u) \\ &= (\lambda_1 - \lambda_2)(\mathcal{F}(\hat{c}, \hat{\theta}), u) + (\lambda_2 - \lambda_1)(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u), \\ &\quad \text{for all } u \in V,\ t \in (0,T),\end{aligned} \tag{62}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t &- \epsilon^2\Delta\phi + \alpha\epsilon^2\rho_\delta(v_1)\cdot\nabla\phi - \frac{1}{2}\phi\left(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right) \\ &= \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi_2 + (\lambda_1 - \lambda_2)\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{63}$$

$$\begin{aligned}C_{\mathrm{v}}\theta_t &- \nabla\cdot(K_1(\rho_\delta(\phi_1))\nabla\theta) - \nabla\cdot\left[K_1(\rho_\delta(\phi_1)) - K_1(\rho_\delta(\phi_2))\right]\nabla\theta_2 \\ &+ C_{\mathrm{v}}\rho_\delta(v_1)\cdot\nabla\theta = C_{\mathrm{v}}\rho_\delta(v)\cdot\nabla\theta_2 \\ &+ \frac{l}{2}f_s^{\delta\prime}(\phi_1)\phi_t + \frac{l}{2}\left[f_s^{\delta\prime}(\phi_1) - f_s^{\delta\prime}(\phi_2)\right]\phi_{2t} \text{ in } Q^\delta,\end{aligned} \tag{64}$$

$$\begin{aligned}c_t &- K_2\Delta c + \rho_\delta(v_1)\cdot\nabla c = K_2 M\nabla\cdot(c_1(1-c_1)\left[\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)\right]) \\ &+ \rho_\delta(v)\cdot\nabla c_2 + K_2 M\nabla\cdot(c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2)) \quad \text{in } Q^\delta,\end{aligned} \tag{65}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{66}$$

$$v(0) = 0 \text{ in } \Omega, \quad \phi(0) = 0, \quad \theta(0) = 0, \quad c(0) = 0 \text{ in } \Omega^\delta. \tag{67}$$

Taking $u = v$ in equation (62), using Hölder's, Young's and interpolation inequalities we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx &+ \int_\Omega \nu|\nabla v|^2 dx \\ &\leq \int_\Omega |v||\nabla v_2||v| dx \\ &+ |\lambda_1 - \lambda_2|\int_\Omega \left(|\mathcal{F}(\hat{c},\hat{\theta})||v| + k(f_s^\delta(\hat{\phi}) - \delta)|\hat{v}||v|\right) dx \\ &\leq C_1\|v_2\|_V^2\|v\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|v\|_V^2 + C_3\int_\Omega |v|^2 dx \\ &+ C_2|\lambda_1 - \lambda_2|^2\left(\int_\Omega |F|^2 + |\hat{v}|^2 dx + \int_{\Omega^\delta} |\hat{\theta}|^2 + |\hat{c}|^2 dx\right).\end{aligned}$$

Then, integration with respect t and Gronwall's Lemma give us

$$\|v\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}^2 \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{68}$$

Applying L^p-theory of parabolic equations ([6] p. 341) to equation (63), the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \leq C_1 \left(\|\rho_\delta(v) \cdot \nabla \phi_2\|_{L^2(Q^\delta)} + |\lambda_1 - \lambda_2| \left(\|\hat{\theta}\|_{L^2(Q^\delta)} + \|\hat{c}\|_{L^2(Q^\delta)} + 1 \right) \right)$$

where C_1 depends on $\|\rho_\delta(v_1)\|_{L^4(Q^\delta)}$ and $\|\phi_1^2 + \phi_1\phi_2 + \phi_2^2\|_{L^{2+\eta}(Q^\delta)}$, $\eta > 0$, which are independent of λ_i. Therefore, using (68) we arrive at

$$\|\phi\|^2_{W_2^{2,1}(Q^\delta)} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{69}$$

Multiplying (64) by θ, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega^\delta} |\theta|^2 dx \quad &+ \quad a\int_{\Omega^\delta} |\nabla\theta|^2 dx \\
&\leq \quad C_1 \int_{\Omega^\delta} |\rho_\delta(\phi)||\nabla\theta_2||\nabla\theta| + |\rho_\delta(v)||\nabla\theta_2||\theta| dx \\
&+ \quad C_2 \int_{\Omega^\delta} |\phi_t||\theta| + |\phi||\phi_{2t}|\,|\theta| dx \\
&\leq \quad C_1 \|\phi\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \|\nabla\theta_2\|_{L^2(\Omega^\delta)} \\
&+ \quad C_2 \|v\|^2_{L^\infty(0,T;H)} \|\nabla\theta_2\|^2_{L^2(\Omega^\delta)} + C_3 \int_{\Omega^\delta} \left(|\phi_t|^2 + |\theta|^2 \right) dx \\
&+ \quad C_4 \|\phi\|^2_{L^\infty(0,T;H^1(\Omega^\delta))} \|\phi_{2t}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta} |\nabla\theta|^2 dx.
\end{aligned}$$

Integration with respect to t and the use of Gronwall's Lemma and (68)-(69) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{70}$$

We multiply (65) by c, integrate over $\Omega^\delta \times (0,t)$ and by parts, and we use Hölder's and Young's inequalities and (44) to obtain

$$\begin{aligned}
\int_{\Omega^\delta} |c|^2 dx \quad &+ \quad \int_0^t \int_{\Omega^\delta} |\nabla c|^2 dx dt \\
&\leq \quad C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)|^2 + |\rho_\delta(v)|^2 + |c|^2 \right) dx dt \\
&\leq \quad C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla\phi|^2 + |c|^2 \right) dx dt + C_1 \int_0^t \int_{\Omega} |v|^2 dx dt.
\end{aligned}$$

Applying Gronwall's Lemma and using (68)-(69) we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{71}$$

Therefore, it follows from (68)-(71) that $\mathcal{T}_\lambda$ is uniformly continuous in λ.

To estimate the set of all fixed points of $\mathcal{T}_\lambda$ let $(v, \phi, \theta, c) \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\begin{aligned} \frac{d}{dt}(v,u) &+ \nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(c,\theta), u) \\ &- \lambda(k(f_s^\delta(\phi) - \delta)v, u) \text{ for all } u \in V,\ t \in (0,T), \end{aligned} \tag{72}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi &- \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ &= \lambda\beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{73}$$

$$C_\mathrm{v}\theta_t + C_\mathrm{v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{74}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot\left(c(1-c)\nabla\left(\rho_\delta(\phi)\right)\right) \text{ in } Q^\delta, \tag{75}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{ on } \partial\Omega^\delta \times (0,T), \tag{76}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{77}$$

We take $u = v$ in equation (72). Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx &+ \int_\Omega \left(\nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi) - \delta)|v|^2\right) dx \\ &\le C_1 \int_\Omega |F|^2 + |\theta|^2 + |c|^2 + |v|^2 dx \\ &\le C_1 \int_\Omega |F|^2 + |v|^2 dx + C_1\int_{\Omega^\delta} |\theta|^2 + |c|^2 dx. \end{aligned} \tag{78}$$

Multiplying equation (73) by ϕ, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities we obtain,

$$\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_{\Omega^\delta} |\phi|^2 dx + \int_{\Omega^\delta}\left(\epsilon^2|\nabla\phi|^2 + \frac{1}{2}\phi^4\right) dx \le C_1 + C_1\int_{\Omega^\delta}\left(|\theta|^2 + |c|^2 + |\phi|^2\right) dx. \tag{79}$$

By multiplying (74) by $e = C_\mathrm{v}\theta - \frac{l}{2}f_s^\delta(\phi)$ and (75) by c, arguments similar to the previous ones lead to the following estimates

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta} |e|^2 dx + \frac{C_\mathrm{v}a}{2}\int_{\Omega^\delta} |\nabla\theta|^2 dx \le C_2\int_{\Omega^\delta} |\nabla\phi|^2 dx + C_1\int_\Omega |v|^2 dx, \tag{80}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta} |c|^2 dx + \frac{K_2}{2}\int_{\Omega^\delta} |\nabla c|^2 dx \le C_2\int_{\Omega^\delta} |\nabla\phi|^2 dx, \tag{81}$$

where (44) was used to obtain the last inequality.

Now, multiplying (79) by A and adding the result to (78),(80)-(81), gives us

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2 dx + \frac{d}{dt}\int_{\Omega^\delta}\left(\frac{A\alpha\epsilon^2}{4}|\phi|^2 + \frac{1}{2}|e|^2 + \frac{1}{2}|c|^2\right)dx \\
&\quad + \int_\Omega \left(\nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right) \\
&\quad + \int_{\Omega^\delta}\left((A\epsilon^2 - 2C_2)|\nabla\phi|^2 + \frac{A}{2}\phi^4 + \frac{C_{\rm v}a}{2}|\nabla\theta|^2 + \frac{K_2}{2}|\nabla c|^2\right)dx \\
&\quad\le C_1 + C_1\int_\Omega |v|^2 dx + C_1\int_{\Omega^\delta}\left(|\phi|^2 + |\theta|^2 + |c|^2\right)dx \qquad (82)
\end{aligned}$$

where C_1 is independent of λ and δ, being $A \in I\!R$ an arbitrary parameter. Taking A large enough and using Gronwall's Lemma we obtain

$$\|v\|_{L^\infty(0,T;H)} + \|\phi\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|e\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1,$$

where C_1 is independent of λ. Since $\theta = \dfrac{1}{C_{\rm v}}\left(e + \dfrac{l}{2}f_s^\delta(\phi)\right)$ and $f_s^\delta(\phi)$ is bounded in $L^\infty(Q^\delta)$, we also have that $\|\theta\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1$. Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0,1]$.

Finally, for $\lambda = 0$, we can reason as in the proof that $\mathcal{T}_\lambda$ is well defined to conclude that the problem (38)-(43) has a unique solution. Therefore, we can apply Leray-Schauder's Theorem and so there is at least one fixed point $(v,\phi,\theta,c) \in B \cap \{L^2(0,T;V)\cap L^\infty(0,T;H)\} \times W_2^{2,1}(Q^\delta) \times W_2^{2,1}(Q^\delta) \times C^{2,1}(Q^\delta)$ of the operator $\mathcal{T}_1$, i.e. $(v,\phi,\theta,c) = \mathcal{T}_1(v,\phi,\theta,c)$. These functions are a solution of problem (32)-(37) and the proof of Proposition 1 is complete. ∎

5 Proof of Theorem 1

To prove Theorem 1, let $0 < \delta \le \delta(\Omega)$ be as in the statement of Theorem 1 and take $\phi_0^\delta \in W^{2-2/q,q}(\Omega^\delta) \cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta \in H$, $\theta_0^\delta \in H^{1+\gamma}(\Omega)$, $1/2 < \gamma \le 1$, $c_0^\delta \in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial\phi_0^\delta}{\partial n} = \dfrac{\partial\theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$, $\|\theta_0^\delta\|_{L^2(Q^\delta)} \le C$, $0 < c_0^\delta < 1$ in $\overline{\Omega^\delta}$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0)$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (32)-(37). We will derive bounds, independent of δ, for this solution and then use compactness arguments and passage to the limit procedure for δ tends to 0 to establish the desired existence result. They are stated in following in a sequence of lemmas; however, most of them are ease consequence of the previous estimates (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 1 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega)]$*

$$\|v^\delta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \le C_1, \tag{83}$$

$$\|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \tag{84}$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \tag{85}$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1. \tag{86}$$

Proof: Observe that it follows from inequality (82). ∎

Lemma 2 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega)]$*

$$\begin{aligned} \|\phi^\delta\|_{W_q^{2,1}(Q)} &\le C_1, \quad \textit{for any } 2 \le q < 4, & (87)\\ \|\theta_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\le C_1, & (88)\\ \|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\le C_1, & (89) \end{aligned}$$

Proof: Note that (87) follows from estimate (20) of Theorem 2 and Lemma 1.

Next, we take the scalar product of (34) with $\eta \in H_o^1(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_{\rm v}\|\theta_t^\delta\|_{H_o^1(\Omega)'} \le C_1 \left(\|\nabla\theta^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^4(\Omega)}\|v^\delta\|_{L^4(\Omega)} + \|\phi_t^\delta\|_{L^2(\Omega)}\right).$$

Then, (88) follows from Lemma 1 and (87).

Using that $0 < c^\delta < 1$ in Q, we infer from (35) that,

$$\|c_t^\delta\|_{H_o^1(\Omega)'} \le C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla\phi^\delta\|_{L^2(\Omega)}\right).$$

Then, (89) follows from Lemma 1. ∎

Lemma 3 *There exist a constant* C_1 *and* $\delta_0 \in (0, \delta(\Omega)]$ *such that, for any* $\delta < \delta_0$,

$$\|v_t^\delta\|_{L^2(t_1,t_2;V(U)')} \leq C_1 \tag{90}$$

where $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ *and such that* $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (32) that for a.e. $t \in (t_1, t_2)$,

$$(v_t^\delta, u) = -\nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx - \int_U k(f_s^\delta(\phi^\delta) - \delta) v^\delta u dx + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx \quad \text{for } u \in V(U).$$

In order to estimate $\|v_t^\delta\|_{V(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W_q^{2,1}(Q)$, for $2 \leq q < 4$, in particular, for $q > 2$ we have that $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 4/q$ ([6] p.80). Consequently, because of Arzela-Ascoli's theorem, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q \,/ 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega \,/ 0 \leq f_s(\phi(x,t)) < 1\}$. Note that for a certain $\gamma \in (0,1)$ and for $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s(\phi(x,t)) < 1 - \gamma.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0, \delta_0)$ and for all $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \gamma/2.$$

By assumption **(H1)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \gamma/2) \qquad \text{for } (x,t) \in [t_1, t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned} \|v_t^\delta\|_{V(U)'} \leq \; & C_1 \Big(\|v^\delta\|_V + \|v^\delta\|_{L^4(\Omega)}^2 + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^2(\Omega)} \\ & + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)} \|v^\delta\|_{L^2(\Omega)} \Big). \end{aligned}$$

Hence, (90) follows from Lemma 1. ∎

From (83) we conclude that the sequence (v^δ) is bounded in $L^2(t_1,t_2;H^1(U))$. Then, by the compact embedding ([10] Cor. 4), there exist v and a subsequence of (v^δ) (which we still denote by v^δ), such that

$$v^\delta \to v \quad \text{in } L^2((t_1,t_2)\times U) \text{ strongly.}$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i,t_{i+1})\times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{in } L^2_{loc}(Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)) \text{ strongly.} \tag{91}$$

Moreover, from (83) we have that $v\in L^2(0,T;V)\cap L^\infty(0,T;H)$ and

$$\begin{array}{lcccl} v^\delta & \rightharpoonup & v & \text{in} & L^2(0,T;V) \text{ weakly,} \\ v^\delta & \overset{*}{\rightharpoonup} & v & \text{in} & L^\infty(0,T;H) \text{ weakly star.} \end{array} \tag{92}$$

We now infer from Lemma 1 and Lemma 2 using the compact embedding ([10] Cor.4) that there exist

$$\begin{array}{rcl} \phi & \in & W^{2,1}_q(Q) \text{ for } 2\le q<4, \\ \theta & \in & L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)), \\ c & \in & L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)), \end{array}$$

and a subsequence of $(\phi^\delta,\theta^\delta,c^\delta)$ (which we still denote by $(\phi^\delta,\theta^\delta,c^\delta)$) such that, as $\delta\to 0$,

$$\begin{array}{lcccl} \phi^\delta & \to & \phi & & \text{uniformly in } Q, \\ \phi^\delta & \to & \phi & \text{in} & L^q(0,T;W^{1,q}(\Omega)) \text{ strongly,} \\ \phi^\delta_t & \rightharpoonup & \phi_t & \text{in} & L^q(Q) \text{ weakly,} \\ \theta^\delta & \to & \theta & \text{in} & L^2(Q)\cap C([0,T];H^1_o(\Omega)') \text{ strongly,} \\ \theta^\delta & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega)) \text{ weakly,} \\ c^\delta & \to & c & \text{in} & L^2(Q)\cap C([0,T];H^1_o(\Omega)') \text{ strongly,} \\ c^\delta & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega)) \text{ weakly.} \end{array} \tag{93}$$

It now remains pass to the limit as δ decreases to zero in (32)-(37).

Now, we take $u=\eta(t)$ in (32) where $\eta\in L^2(0,T;V(\Omega_{ml}(t)))$ with compact support contained in $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ and $\eta_t\in L^2(0,T;V(\Omega_{ml}(t))')$; after integration over $(0,t)$, we find

$$\begin{aligned} \int_0^t \Big((v^\delta_t,\eta)+(\nabla v^\delta,\nabla\eta)+(v^\delta\cdot\nabla v^\delta,\eta)+(k(f^\delta_s(\phi^\delta)-\delta)v^\delta,\eta)\Big)\,ds \\ = \int_0^t (\mathcal{F}(c^\delta,\theta^\delta),\eta)ds. \end{aligned} \tag{94}$$

Since supp $\eta \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ we have that supp $\eta(t) \subseteq \Omega_{ml}(t)$ a.e. $t \in [0,T]$. Moreover, we observe that

$$\int_0^t (v_t^\delta, \eta) ds = -\int_0^t (v^\delta, \eta_t)_{\Omega_{ml}(s)} ds + (v^\delta(t), \eta(t))_{\Omega_{ml}(t)} - (v_0^\delta, \eta(0))_{\Omega_{ml}(0)}.$$

Because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H1)**, it follows that $k(f_s^\delta(\phi^\delta) - \delta)$ converges to $k(f_s(\phi))$ uniformly on compact subsets of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. These facts together with (91)-(93) ensure that we may pass to the limit in (94) and get (12).

To check that $v = 0$ a.e. in $\overset{o}{Q}_s$, take a compact set $K \subseteq \overset{o}{Q}_s$. Then there is an $\delta_K \in (0,1)$ such that

$$f_s^\delta(\phi^\delta(x,t)) = 1 \quad \text{ in } K \text{ for } \delta < \delta_K,$$

hence, $k(f_s^\delta(\phi^\delta(x,t) - \delta) = k(1-\delta)$ in K for $\delta < \delta_K$. From (83) we infer that

$$k(1-\delta)\|v^\delta\|^2_{L^2(K)} \leq C_1 \quad \text{ for } \delta < \delta_K$$

where C_1 is independent of δ. As δ tends to 0, by assumption **(H1)**, $k(1-\delta)$ blows up and consequently $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K. Since K is an arbitrary subset, we conclude that $v = 0$ a.e. in $\overset{o}{Q}_s$.

It follows from (92)-(93) that we may pass to the limit in (33), and find that (13) holds almost everywhere.

In order to pass to the limit in (34), we note that given $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ satisfying $\zeta(T) = 0$, we can consider an extension of ζ such that $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ satisfying $\zeta^\delta(T) = 0$. Now, we take the scalar product of (34) with ζ^δ,

$$\begin{aligned} -C_{\text{v}} \int_{\Omega^\delta} \theta_0^\delta \zeta^\delta(0) dx - C_{\text{v}} \int_0^T \int_{\Omega^\delta} \theta^\delta \zeta_t^\delta dx dt \quad &- \quad C_{\text{v}} \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta)\theta^\delta \cdot \nabla \zeta^\delta dx dt \\ + \int_0^T \int_{\Omega^\delta} K_1(\rho_\delta(\phi^\delta)) \nabla \theta^\delta \cdot \nabla \zeta^\delta dx dt \quad &= \quad \frac{l}{2} \int_0^T \int_{\Omega^\delta} f_s^{\delta\prime}(\phi^\delta) \phi_t^\delta \zeta^\delta dx dt. \quad (95) \end{aligned}$$

Observe that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0,T;H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0,T];H_o^1(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$. Observe also that $f_s^{\delta\prime} \to f_s'$ in $L^q(I\!R)$ for $2 \leq q < \infty$, then from (93) we infer that $f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 1 the integrals over $\Omega^\delta \backslash \Omega$ are bounded independent of δ and since

$|\Omega^\delta\backslash\Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (95) and obtain

$$-C_{\rm v}\int_0^T\int_\Omega \theta\zeta_t dxdt - C_{\rm v}\int_0^T\int_\Omega v\,\theta\cdot\nabla\zeta\,dxdt + \int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\zeta\,dxdt \\ = \frac{l}{2}\int_0^T\int_\Omega f_s(\phi)_t\zeta\,dxdt + C_{\rm v}\int_\Omega \theta_0\zeta(0)dx$$

for $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$.

It remains to pass to the limit in (35). We proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$-\int_0^T\int_{\Omega^\delta} c^\delta\zeta_t^\delta dxdt - \int_0^T\int_{\Omega^\delta} \rho_\delta(v^\delta)c^\delta\cdot\nabla\zeta^\delta dxdt + K_2\int_0^T\int_{\Omega^\delta}\nabla c^\delta\cdot\nabla\zeta^\delta dxdt \\ + K_2 M\int_0^T\int_{\Omega^\delta} c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\cdot\nabla\zeta^\delta dxdt = \int_{\Omega^\delta} c_0^\delta\zeta^\delta(0)dx,$$

then from (92),(93) and using that the sequence (c^δ) is bounded in $L^\infty(Q)$ we may pass to the limit as $\delta \to 0$ and obtain

$$-\int_0^T\int_\Omega c\zeta_t dxdt - \int_0^T\int_\Omega v\,c\cdot\nabla\zeta\,dxdt + K_2\int_0^T\int_\Omega \nabla c\cdot\nabla\zeta\,dxdt \\ + K_2 M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta\,dxdt = \int_\Omega c_0\zeta(0)dx$$

holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$. Observe that since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \le c \le 1$ a.e. in Q.

Finally, it follows from (93) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$. Furthermore, $v(0) = v_0$ in $\Omega_{ml}(0)$ because $v^\delta(0) \to v(0)$ in $V'(U)$ for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$. The proof of Theorem 1 is then complete. ∎

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