Characterization of Solutions in Variational Problems. Duality

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Abstract: In this paper we introduce a new class of pseudoinvex functions for variational problems. Using this mew concept, we obtain a sufficient and necessary condition for a critical point of the variational problem to be an optimal solution. Weak, strong and converse duality are established.

Keywords: variational problem; pseudoinvexity; critical point; duality.

1 Introduction and Preliminaries

The Variational Problem found renewed attention in the 1960's. These early studies of the Variational Problem were set in the context of calculus of variations/optimal control theory and in connection with the solution of boundary value problems posed in the form of partial differential equations.

It is known that an optimal solution of a constrained variational problem verifies a critical point condition ([15], [2], [10]). But some conditions are needed in order that a critical point is an optimal solution of a constrained variational problem. In this sense, Chandra, Craven and Husain [2] proved under convexity that critical point is an optimal solution; analogous results were given by Mond and Hanson [10]; and Nahak and Nanda [13], under pseudoinvexity.

In section 2 we will define the new concepts of L-(KT/FJ)-pseudoinvex functions, enable us to establish a necessary and sufficient condition in order that a critical point of the Variational Problem to be an optimal solution,

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i. e., it is obtained a characterization which has not been obtained to date. In section 3 weak, strong and converse duality are established.

Let's introduce the variational problem and definitions.

Let $I = [a, b]$ be a real interval, and let $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ continuously differentiable functions with respect to each of their arguments. For notational convenience $f(t, x(t), \dot{x}(t))$ will be written $f(t, x, \dot{x})$, where $x : I \to R^n$, with derivative $\dot{x}$, denote the partial derivative of f with respect to t, x, and $\dot{x}$, by f_t, f_x, $f_{\dot{x}}$, respectively, such that

$$f_x = \left(\frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_{2},} \ldots, \frac{\partial f}{\partial x_n}\right), \qquad f_{\dot{x}} = \left(\frac{\partial f}{\partial \dot{x}_1}, \quad \frac{\partial f}{\partial \dot{x}_2}, \ldots, \frac{\partial f}{\partial \dot{x}_n}\right)$$

Similary, we write the partial derivatives of g_t, g_x, $g_{\dot{x}}$, using matrices with m rows instead of one.

Let $X = C(I, R^n)$ denote the space of piecewise smooth functions $x : I \to R^n$ with the norm

$$\|x\| = \|x\|_\infty + \|Dx\|_\infty$$

where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is a given boundary value. Therefor, $D = d/dt$ except at discontinuities.

Then, we can consider the scalar Constrained Variational Problem (CVP), in which the objective function is $F(x) = \displaystyle\int_a^b f(t, x, \dot{x})dt$,

$$(CVP) \qquad \text{Minimize} \int_a^b f(t, x, \dot{x})dt$$
$$\text{subject to:}$$
$$x(a) = \alpha, \quad x(b) = \beta$$
$$g(t, x(t), \dot{x}(t)) \leq 0, \quad t \in I$$

Let K be the set of feasible solutions of (CVP), i. e.,

$$K = \{x \in X = C(I, R^n) : x(a) = \alpha,\ x(b) = \beta,\ g(t, x(t), \dot{x}(t)) \leq 0, \quad t \in I\}$$

Under only boundary constrains, the Variational Problem can be written as follows.

$$(VP) \qquad \text{Minimize} \int_a^b f(t, x, \dot{x})dt$$
subject to:
$$x(a) = \alpha, \quad x(b) = \beta$$

DEFINITION 1 *$\bar{x} \in K$ is said to be an optimal solution or global minimum of (CVP) if*

$$F(\bar{x}) \leq F(x),$$

for all $x \in K$.

This definition is equivalent to write $F(x) < F(\bar{x})$ has no solution $x \in K$.

DEFINITION 2 *$x \in K$ is said to be a Fritz-John critical point if there exists $\tau \in R$, $y : I \to R^m$ a piecewise smooth function, such that*

$$\tau f_x(t, x, \dot{x}) + y(t)^T g_x(t, x, \dot{x}) = \frac{d}{dt}\left\{\tau f_{\dot{x}}(t, x, \dot{x}) + y(t)^T g_{\dot{x}}(t, x, \dot{x})\right\} \tag{1}$$

$$y(t)^T g(t, x, \dot{x}) = 0 \tag{2}$$

$$(\tau, y(t)) \geq 0, \quad (\tau, y(t)) \neq 0, \tag{3}$$

$\forall t \in I$, except at discontinuities.

If $\tau > 0$, a Fritz-John critical point is said to be a Kuhn-Tucker critical point , and we can write $\tau = 1$ (then x can be called *normal* [1]), as follows.

DEFINITION 3 *$x \in K$ is said to be a Kuhn-Tucker critical point if there exists $y : I \to R^m$ a piecewise smooth function, such that*

$$f_x(t, x, \dot{x}) + y(t)^T g_x(t, x, \dot{x}) = \frac{d}{dt}\left\{f_{\dot{x}}(t, x, \dot{x}) + y(t)^T g_{\dot{x}}(t, x, \dot{x})\right\} \tag{4}$$

$$y(t)^T g(t, x, \dot{x}) = 0 \tag{5}$$

$$y(t) \geq 0, \quad t \in I \tag{6}$$

In the Constrain Variational Problem (CVP), a critical point condition on $\bar{x}$ is a necessary condition in order that $\bar{x}$ is an optimal solution [15]. In this way, Chandra, Craven and Husain [2] introduced the next result, based on results from Craven [4], Craven and Mond [6], Mond [9], and Valentine [15]:

THEOREM 1 *If $\bar{x}$ is an optimal solution for (CVP) then $\bar{x}$ is a Fritz-John critical point.*

And therefore, it is obtained the next consequence.

THEOREM 2 *If $\bar{x}$ is an optimal normal solution for (CVP) then $\bar{x}$ is a Kuhn-Tucker type critical point.*

From theorem 2, if $\bar{x}$ is an optimal normal solution of (VP), then

$$f_x(t,x,\dot{x}) = \frac{d}{dt}\left\{f_{\dot{x}}(t,x,\dot{x})\right\} \tag{7}$$

2 Necessary and Sufficient Condition: L-(KT/FJ)-pseudoinvexity

In classical scalar optimization theory, the Kuhn-Tucker conditions are sufficient for optimality if the functions involved are convex. In the last few years, many effort has been made to weaken the convexity hypotheses and thus to explore the extent of optimality conditions applicability.

As it is known, invexity has been introduced in optimization theory by Hanson [7] as a substitute for convexity in constrained optimization. Craven and Glover [5] showed that any differentiable scalar function is invex if and only if every stationary point is a global minimizer. For constrained problems, the invexity defined by Hanson is a sufficient condition but not a necessary condition for every Kuhn-Tucker stationary point to be a global minimizer. Martin [8] defined a weaker invexity notion, called Kuhn-Tucker invexity of KT-invexity, which is both necessary and sufficient to establish the Kuhn-Tucker optimality conditions in scalar programming problems. Invexity was extended to variational problems by Mond, Chandra and Husain [9]:

DEFINITION 4 *The function $f(t, x, \dot{x})$ is said to be invex at $\bar{x} \in X$ with respect to η if there exists a vector function $\eta(t, \bar{x}, x)$, with $\eta(t, x, x) = 0$ if for all $x \in X$*

$$f(t, x, \dot{x}) - f(t, \bar{x}, \dot{\bar{x}})$$
$$\geq f_x(t, \bar{x}, \dot{\bar{x}})\eta(t, \bar{x}, x) + f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\frac{d}{dt}\eta(t, \bar{x}, x)$$

Here $\frac{d\eta}{dt}$ is the vector whose i-th component is $\frac{d}{dt}\eta^i(t, \bar{x}, x)$. If f does not depend explicitly on t, the previous definition essentially reduce to be the definition of invexity given by Hanson [7]. Nahak and Nanda [13] used pseudoinvexity instead of invexity. Later, Nahak and Nanda [14] extend pseudoinvexity to a functional.

Let the functional $F(x) = \displaystyle\int_a^b f(t, x, \dot{x})dt$.

Definition of invexity was extended from a function f to a functional F ([12], [11]).

DEFINITION 5 *F is invex at $\bar{x} \in K$, with respect to η if there exists a vector function $\eta(t, \bar{x}, x)$, with $\eta(t, x, x) = 0$ if for all $x \in X$*

$$F(x) - F(\bar{x}) \geq \int_a^b \left\{ f_x(t, \bar{x}, \dot{\bar{x}})\eta(t, \bar{x}, x) + f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\frac{d}{dt}\eta(t, \bar{x}, x) \right\} dt$$

We have the following characterization result for the class of invex functions, given in [12]:

THEOREM 3 *F is invex on the feasible set of (VP) if and only if all critical points are optimal solutions for (VP).*

Looking for optimality sufficient conditions for (CVP), Mond and Husain [11] resorted to generalize convexity:

DEFINITION 6 *F is pseudoinvex at $\bar{x} \in X$, with respect to η if there exists a vector function $\eta(t, \bar{x}, x)$, with $\eta(t, x, x) = 0$ if for all $x \in X$*

$$F(x) - F(\bar{x}) < 0 \Rightarrow \int_a^b \left\{ f_x(t, \bar{x}, \dot{\bar{x}})\eta(t, \bar{x}, x) + f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\frac{d}{dt}\eta(t, \bar{x}, x) \right\} dt < 0$$

Under these generalized invexity conditions, Mond and Husain [11] got sufficient Kuhn-Tucker conditions.

Let's consider the problem (CVP) and f, g, $F(x) = \int_a^b f(t, x, \dot{x})dt$ and $G(x) = \int_a^b g(t, x, \dot{x})dt$.

DEFINITION 7 *The pair (F, G) is said to be L-KT-pseudoinvex at $\bar{x} \in X$, if for all $x \in X$, $\bar{y} : I \to R^m$ piecewise smooth function, which verify (5) and (6), there exists a differentiable vector function $\eta(t, x, \bar{x}, \bar{y})$, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$, such that*

$$F(x) - F(\bar{x}) < 0 \Rightarrow$$

$$\int_a^b \Big\{(f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y})$$

$$+ (f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, \bar{x}, \dot{\bar{x}}, \bar{y})\Big\} dt < 0$$

or equivalently,

$$\int_a^b \{ (f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y})$$

$$+(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y}) \} dt \geq 0$$

$$\Rightarrow F(x) - F(\bar{x}) \geq 0$$

Note that given $\bar{x}, x \in X$, there exists $\bar{y}$ verifying (5) and (6). For example, consider $\bar{y}(t) = 0$, $t \in I$.

DEFINITION 8 *The pair (F, G) is said to be L-FJ-pseudoinvex at $\bar{x} \in X$, if for all $x \in X$, $\bar{\tau} \in R$, $\bar{y} : I \to R^m$ piecewise smooth function, which verify (2) and (3), there exists a differentiable vector function $\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$, with $\eta(a, x, \bar{x}, \bar{\tau}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$, such that*

$$F(x) - F(\bar{x}) < 0 \Rightarrow$$

$$\int_a^b \Big\{(\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$$

$$+ (\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, \bar{x}, \dot{\bar{x}}, \bar{\tau}, \bar{y})\Big\} dt < 0$$

or equivalently,

$$\int_a^b \{ \ (\bar{\tau} f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}}))\eta(t,x,\bar{x},\bar{\tau},\bar{y})$$

$$+(\bar{\tau} f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}))\frac{d}{dt}\eta(t,x,\bar{x},\bar{\tau},\bar{y}) \ \} \, dt \geq 0$$

$$\Rightarrow F(x) - F(\bar{x}) \geq 0$$

Note that given $\bar{x}, x \in X$, there exists $\bar{\tau}, \bar{y}$ verifying (2) and (3). For example, consider $\bar{\tau} = 1$, $\bar{y}(t) = 0$, $t \in I$.

If the previous definition are reduced on the set of feasible solutions of (CVP) K, we establish the following concepts:

DEFINITION 9 *The Constrained Variational Problem (CVP) is said to be L-KT-pseudoinvex if it is verified definition 9 for $x, \bar{x} \in K$.*

DEFINITION 10 *The Constrained Variational Problem (CVP) is said to be L-FJ-pseudoinvex if it is verified definition 10 for $x, \bar{x} \in K$.*

Some of the relationships between these definitions are as follows.

PROPOSITION 1 *It is verified:*
a) If (F,G) is L-FJ-pseudoinvex with respect to η_1, then (F,G) is L-KT-pseudoinvex with respect to η_2, with $\eta_2(t,x,\bar{x},\bar{y}) = \eta_1(t,x,\bar{x},1,\bar{y})$.
b) If (CVP) is L-FJ-pseudoinvex with respect to η_1, then (CVP) is L-KT-pseudoinvex with respect to η_2, with $\eta_2(t,x,\bar{x},\bar{y}) = \eta_1(t,x,\bar{x},1,\bar{y})$.

Proof. Consider $\bar{\tau} = 1$ in definitions of (F,G) L-FJ-pseudoinvex and (CVP) L-FJ-pseudoinvex. □

Let's see some relationships between L-KT-pseudoinvexity and other classes of functions.

PROPOSITION 2 *If (F,G) is L-KT-pseudoinvex at $\bar{x} \in X$, then F is pseudoinvex at $\bar{x}$.*

Proof. That is enough to consider $\bar{y}(t) = 0, \quad t \in I$, in definition 9. □

In relation to the concept of pseudoinvexity on the suggestion of [11], we propose the following result:

PROPOSITION 3 *Let $\bar{x} \in K$. If for all $\bar{y} : I \to R^m$ piecewise smooth function such that $(\bar{x}, \bar{y})$ verifies (5) and (6), the Lagrangian function $\phi(x, \bar{y}) = \int_a^b \{f(t, x, \dot{x}) + \bar{y}(t)^T g(t, x, \dot{x})\}dt$, with $x \in K$, is pseudoinvex at $\bar{x}$, then (CVP) is L-KT-pseudoinvex.*

Proof. Let $\bar{y}$ be such that $(\bar{x}, \bar{y})$ verifies (5) and (6), and let's suppose $x \in K$ such that $F(x) - F(\bar{x}) < 0$, i.e.,

$$\int_a^b f(t, x, \dot{x})dt < \int_a^b f(t, \bar{x}, \dot{\bar{x}})dt$$

Since x is feasible and (6), it happens $\bar{y}(t)^T g(t, x, \dot{x}) \leq 0$, $t \in I$; and moreover from (5), it follows

$$\int_a^b (f(t, x, \dot{x}) + \bar{y}(t)^T g(t, x, \dot{x}))dt < \int_a^b (f(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}))dt$$

Since $\phi(\cdot, \bar{y})$ is pseudoinvex at $\bar{x}$, there exists a differentiable function $\eta(t, x, \bar{x})$ such that

$$\int_a^b \Big\{(f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x})$$

$$+(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}) \ \Big\} \, dt < 0,$$

and therefore, (CVP) is L-KT-pseudoinvex at $\bar{x}$ with respect to $\eta(t, x, \bar{x}, \bar{y}) = \eta(t, x, \bar{x})$. □

Before optimality conditions, and on the lines of Courant ([3], Lemma 2, pp. 500-1), we have the following result.

LEMMA 1 *Let z_1, z_2 be scalar piecewise continuous functions such that $\int_a^b (z_1(t)\varphi(t) + z_2(t)\dot{\varphi}(t))dt = 0$, $\forall \varphi$ differentiable, $\varphi(a) = 0 = \varphi(b)$, then z_2 is piecewise smooth and $Dz_2 = z_1$ on I.*

From this lemma, it follows the next result.

PROPOSITION 4 *Let w_1, w_2 be vector piecewise continuous functions such that $\int_a^b (w_1^T(t)\psi(t) + w_2^T(t)\dot{\psi}(t))dt = 0$, $\forall \psi$ differentiable, $\psi(a) = 0 = \psi(b)$, then w_2 is piecewise smooth and $Dw_2 = w_1$ on I.*

Proof. Let $i \in \{1, \ldots, n\}$. Let $\psi = (\psi_1, \ldots, \psi_n)$, ψ_i differentiable, $\psi_i(a) = 0 = \psi_i(b)$, and $\psi_j = 0 \quad \forall j \neq i$. Then ψ is differenciable, $\psi(a) = 0 = \psi(b)$, and then

$$0 = \int_a^b (w_1^T(t)\psi(t) + w_2^T(t)\dot{\psi}(t))dt = \int_a^b (w_{1,i}(t)\psi_i(t) + w_{2,i}(t)\dot{\psi}_i(t))dt,$$

$\forall \psi_i$ differentiable, $\psi_i(a) = 0 = \psi_i(b)$. By lemma 1, $w_{2,i}$ is differentiable and $\dot{w}_{2,i} = w_{1,i}$ on I, $i = 1, \ldots, n$. Therefore, w_2 is differentiable and $\dot{w}_2 = w_1$ on I.
□

Let's prove that L-KT-pseudoinvexity of (CVP) is necessary for its critical points to be optimal solutions:

THEOREM 4 *If all Kuhn-Tucker critical points are optimal solutions for (CVP) then (CVP) is L-KT-pseudoinvex.*

Proof. Let $x, \bar{x} \in K$, $(\bar{x}, \bar{y})$ verifies (5) and (6), such that $F(x) - F(\bar{x}) < 0$. We have to find $\eta(t, x, \bar{x}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$, such that

$$P(\eta(\cdot, x, \bar{x}, \bar{y})) = \int_a^b \{ \ (f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y}) \\ +(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y}) \ \} \, dt < 0$$

And thus, suppose $P(\eta(\cdot, x, \bar{x}, \bar{y})) < 0$ has no solution $\eta(t, x, \bar{x}, \bar{y})$, and then $P(\eta(\cdot, x, \bar{x}, \bar{y})) > 0$ has no solution too, since we could consider $-\eta(t, x, \bar{x}, \bar{y})$. Therefore,

$$P(\eta(\cdot, x, \bar{x}, \bar{y})) = \int_a^b \{ \ (f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y}) \\ +(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y}) \ \} \, dt = 0$$

$\forall \eta(t, x, \bar{x}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$. From proposition 4,

$$f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})$$

is piecewise smooth and

$$f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}) = \frac{d}{dt}\left\{ f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) \right\},$$

and therefore, $(\bar{x}, \bar{y})$ verifies (4), (5) and (6), i.e., $\bar{x}$ is a Kuhn-Tucker critical point,and then $\bar{x}$ is an optimal solution for (CVP), which stands in contradiction to $F(x) - F(\bar{x}) < 0$. So, there exists $\eta(t, x, \bar{x}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$, such that $P(\eta(\cdot, x, \bar{x}, \bar{y})) < 0$, and then, (CVP) is L-KT-pseudoinvex. □

We have proved that L-KT-pseudoinvexity is a necessary condition, and now, we are going to prove that it is a sufficient condition.

THEOREM 5 *If (CVP) is L-KT-pseudoinvex then all Kuhn-Tucker critical points are optimal solutions.*

Proof. Let $\bar{x}$ be a Kuhn-Tucker critical point, $\bar{x} \in K$, i. e., there exists $\bar{y}$ piecewise smooth function such that $(\bar{x}, \bar{y})$ verifies (4), (5) and (6). Let $x \in K$. Since (CVP) is L-KT-pseudoinvex, there exists $\eta(t, x, \bar{x}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$ which verifies the definition. It follows,

$$\begin{aligned}
&\int_a^b \{(f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y}) \\
&\qquad +(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y})\}dt \\
&= \int_a^b \Big\{(f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y}) \\
&\qquad -\frac{d}{dt}\,(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y})\Big\}\,dt \\
&\qquad +(f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{y})|_{t=a}^{t=b} \\
&\qquad \text{(by integration by parts)} \\
&= \int_a^b \Big\{f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}) \\
&\qquad -\frac{d}{dt}\,(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\Big\}\,\eta(t, x, \bar{x}, \bar{y})dt = 0 \\
&\qquad \text{(by (4))}
\end{aligned}$$

Since (CVP) is L-KT-pseudoinvex it follows $F(x) - F(\bar{x}) \geq 0$, i. e., $F(x) \geq F(\bar{x})$, $\forall x \in K$. Therefore, $\bar{x}$ is an optimal solution of (CVP). □

Therefore, we have proved that L-KT-pseudoinvexity of (CVP) is both sufficient and necessary condition in order that a Kuhn-Tucker critical point is an optimal solution of (CVP).

In the same way, let's prove that L-FJ-pseudoinvexity of (CVP) is necessary for its critical points to be optimal solutions.

THEOREM 6 *If all Fritz-John critical points are optimal solutions for (CVP) then (CVP) is L-FJ-pseudoinvex.*

Proof. Let $x, \bar{x} \in K$, $(\bar{x}, \bar{\tau}, \bar{y})$ verifies (2) and (3), such that $F(x) - F(\bar{x}) < 0$. We need to find a differentiable function $\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$, with $\eta(a, x, \bar{x}, \bar{\tau}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$, such that (CVP) verifies L-FJ-pseudoinvexity. If there exists no function η, and proceeding in the same way as theorem 4, it follows

$$\int_a^b \{ \ (\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$$
$$+(\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) \ \} \, dt = 0$$

$\forall \eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{\tau}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$. From proposition 4,

$$\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})$$

is piecewise smooth and

$$\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}) = \frac{d}{dt}\left\{\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right\},$$

and therefore, $(\bar{x}, \bar{\tau}, \bar{y})$ verifies (1), (2) and (3), i.e., $\bar{x}$ is a Fritz-John critical point, and then $\bar{x}$ is an optimal solution of (CVP), which stands in contradiction to $F(x) - F(\bar{x}) < 0$. Thus, there exists $\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{\tau}\bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$, such that

$$\int_a^b \{ \ (\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$$
$$+(\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) \ \} \, dt <, 0$$

and therefore, (CVP) is L-FJ-pseudoinvex. □

Moreover let's see that L-FJ-pseudoinvexity is a sufficient condition:

THEOREM 7 *If (CVP) is L-FJ-pseudoinvex the all Fritz-John critical points are optimal solutions.*

Proof. Let $\bar{x}$ be a Fritz-John critical point , $\bar{x} \in K$, i.e., there exists $\bar{y}$ piecewise smooth function and $\bar{\tau} \in R$ such that $(\bar{x}, \bar{\tau}, \bar{y})$ verifies (1), (2) and (3). Let $x \in K$. Since (CVP) is L-FJ-pseudinvex, there exists $\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{\tau}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$ which verifies the definition.
Then, and proceeding in the same way as theorem 5,

$$\int_a^b \{(\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$$

$$+(\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}))\frac{d}{dt}\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})\}dt$$

$$= \int_a^b \Big\{(\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$$

$$-\frac{d}{dt}\left(\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right)\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})\Big\}\, dt$$

$$+(\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}))\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})|_{t=a}^{t=b}$$

(by integration by parts)

$$= \int_a^b \Big\{\bar{\tau} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}})$$

$$-\frac{d}{dt}\left(\bar{\tau} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right)\Big\}\, \eta(t, x, \bar{x}, \bar{\tau}, \bar{y})dt = 0$$

(by (1))

Since (CVP) is L-FJ-pseudoinvex, it follows $F(x) - F(\bar{x}) \geq 0$, i.e., $F(x) \geq F(\bar{x})$, $\forall x \in K$. Therefore, $\bar{x}$ is an optimal solution of (CVP). □

So, we have proved that L-FJ-pseudoinvexity of (CVP) is both sufficient and necessary condition in order that a Fritz-John critical point is an optimal solution of (CVP).

3 Duality

We establish duality between (CVP) and the next dual problem (CVD1), which is a modified Mond-Weir type dual problem formulated by Bector, Chandra and Husain [1].

$$(CVD1) \qquad \text{Maximize} \int_a^b f(t,u,\dot{u})dt$$
$$\text{subject to:}$$

$$u(a) = \alpha, \quad u(b) = \beta \tag{8}$$

$$f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}) = \frac{d}{dt}\left\{ f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u}) \right\}, t \in I \tag{9}$$

$$y(t)^T g(t,u,\dot{u}) = 0 \tag{10}$$

$$y(t) \geq 0, \quad t \in I \tag{11}$$

Let H be the feasible set of (CVD1).

THEOREM 8 (weak duality) *Let $x \in K$, $(u,y) \in H$. If (F,G) is L-KT-pseudoinvex at u, then $\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,u,\dot{u})dt$.*

Proof. Suppose $\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,u,\dot{u})dt$ is not verified, i.e., $F(x) - F(u) < 0$. Since (u,y) verifies (10) y (11), and since (F,G) is L-KT-pseudoinvex, there exists a differentiable function $\eta(t,x,u,y)$, with $\eta(a,x,u,y) = 0 = \eta(b,x,u,y)$ such that

$$\int_a^b \Big\{ (f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}))\eta(t,x,u,y) + (f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u}))\frac{d}{dt}\eta(t,x,u,y) \Big\} dt < 0 \tag{12}$$

On the other hand,

$$\int_a^b \Big\{ (f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}))\eta(t,x,u,y) \qquad 1$$
$$+ (f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u}))\frac{d}{dt}\eta(t,x,u,y) \Big\} dt$$
$$= \int_a^b \Big\{ (f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u}))\eta(t,x,u,y)$$

$$- \left(\frac{d}{dt}(f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u})) \right) \eta(t,x,u,y) \Big\} dt$$

$$+(f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u}))\eta(t,x,u,y)|_{t=a}^{t=b}$$

(by integration by parts)

$$= \int_a^b \Big\{ f_x(t,u,\dot{u}) + y(t)^T g_x(t,u,\dot{u})$$

$$- \left(\frac{d}{dt}(f_{\dot{x}}(t,u,\dot{u}) + y(t)^T g_{\dot{x}}(t,u,\dot{u})) \right) \Big\} \eta(t,x,u,y)dt = 0,$$

(by (9))

which stands in contradiction to (12). Therefore,

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,u,\dot{u})dt. \qquad \square$$

As a consequence of this theorem, if (F,G) is L-KT-pseudoinvex, then

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,u,\dot{u})dt, \qquad \forall x \in K, \quad \forall (u,y) \in H.$$

Once weak duality has been established, strong and converse duality follows:

THEOREM 9 (Strong duality) *Let $\bar{x}$ be an optimal normal solution of (CVP). If (F,G) is L-KT-pseudoinvex, then there exists $\bar{y} : I \to R^m$ piecewise smooth function such that $(\bar{x}, \bar{y})$ is an optimal solution of (CVD1), and their objective function values are equal at these points.*

Proof. Since $\bar{x}$ is an optimal normal solution of (CVP), from Valentine necessary condition [15] there exists $\bar{y} : I \to R^m$ piecewise smooth function such that $(\bar{x}, \bar{y})$ verifies

$$f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}}) = \frac{d}{dt} \Big\{ f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) \Big\}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0$$

$$\bar{y}(t) \geq 0, \quad i \in I$$

Therefore, $(\bar{x}, \bar{y}) \in H$. From theorem 8, $(\bar{x}, \bar{y})$ is an optimal solution of (CVD1), and obviously, the objective function values of (CVD1) and (CVP) are equal. $\square$

We now consider the converse dual problem, that is, of finding conditions under which the existence of optimal solution to problem (CVD1) implies the existence of an optimal solution to problem (CVP).

THEOREM 10 (Converse duality) *Let $(\bar{u}, \bar{y})$ be an optimal solution of (CVD1). If $\bar{u} \in K$ and (F,G) is L-KT-pseudoinvex, then $\bar{u}$ is an optimal solution of (CVP) and their objective function values are equal at these points.*

Proof. Since (F, G) is L-KT-pseudoinvex, and from theorem 8, it follows that

$$\int_a^b f(t, x, \dot{x})dt \geq \int_a^b f(t, \bar{u}, \dot{\bar{u}})dt,$$

$\forall x \in K$. And since $\bar{u} \in K$, it follows that $\bar{u}$ is an optimal solution of (CVP), and their objective function values are equal at this point. $\square$

We continue our duality study with the dual problem (CVD2), as follows.

$$(CVD2) \qquad \text{Maximize} \int_a^b f(t, u, \dot{u})dt$$

subject to:

$$u(a) = \alpha, \quad u(b) = \beta \tag{13}$$

$$\tau f_x(t, u, \dot{u}) + y(t)^T g_x(t, u, \dot{u}) = \frac{d}{dt}\left\{\tau f_{\dot{x}}(t, u, \dot{u}) + y(t)^T g_{\dot{x}}(t, u, \dot{u})\right\}, t \in I \tag{14}$$

$$y(t)^T g(t, u, \dot{u}) = 0 \tag{15}$$

$$(\tau, y(t)) \geq 0, \quad t \in I \tag{16}$$

Again, let H be the feasible set of (CVD2). Proceeding in the same way as proofs of theorem 8, 9 and 10, but under L-FJ-pseudoinvexity, we state the following duality results between (CVP) and (CVD2).

THEOREM 11 (Weak duality) *Let $x \in K$, $(u, \tau, y) \in H$. If (F, G) is L-FJ-pseudoinvex at u, then $\int_a^b f(t, x, \dot{x})dt \geq \int_a^b f(t, u, \dot{u})dt$.*

As a consequence of this theorem, if (F, G) is L-FJ-pseudoinvex then

$$\int_a^b f(t, x, \dot{x})dt \geq \int_a^b f(t, u, \dot{u})dt, \qquad \forall x \in K, \quad \forall (u, \tau, y) \in H.$$

THEOREM 12 (Strong duality) *Let $\bar{x}$ be an optimal solution of (CVP). If (F,G) is L-FJ-pseudoinvex, then there exists $\bar{\tau} \in R$, $\bar{y} : I \to R^m$ piecewise smooth function such that $(\bar{x}, \bar{\tau}, \bar{y})$ is an optimal solution of (CVD2), and their objective function values are equal at this point.*

THEOREM 13 (Converse duality) *Let $(\bar{u}, \bar{\tau}, \bar{y})$ be an optimal solution of (CVD2). If $\bar{u} \in K$ and (F,G) is L-FJ-pseudoinvex, then $\bar{u}$ is an optimal solution of (CVP) and their objective function values are equal.*

4 Conclusion

In this paper, we have studied the properties of the functions of a constrained variational problem, such that from a critical point it follows an optimal solution. These properties are L-KT-pseudoinvexity and L-KT-pseudoinvexity, and we have proved that these are necessary and sufficient conditions for a critical point to be an optimal solution of the variational problem: a characterization.

Also, we have proved that the problems (CVP) and (CVD1) are a dual pair subject to L-KT-pseudoinvexity conditions; and (CVP) and (CVD2), under L-FJ-pseudoinvexity conditions.

References

[1] C. R. Bector, S. Chandra and I. Husain, Generalized Concavity and Duality in Continuous Programming, Utilitas Mathematica 25 (1984), 171-190.

[2] S. Chandra, B. D. Craven and I. Husain, A Class of Nondifferenciable Continuous Programming Problems, J. Math. Anal. Appl. 107 (1985), 122-131.

[3] R. Courant, "Differential and Integral Calculus", Vol 2, Blackie, London/Edinburgh (1936)/(1948).

[4] B. D. Craven, Mathematical Programming and Control Theory, Chapman & Hall, London (1978).

[5] B. D. Craven amd B. M. Glover, Invex Functions and Duality, J. Austral. Math. Soc. Ser. A 39 (1985), 97-99.

[6] B. D. Craven and B. Mond, Lagrangean Conditions for Quasidifferential Optimization, in "Proceedings, 9th Intern. Conf. on Mathematical Programming", Budapest (1976), in "Survey of Mathematical Programming" (A. Prékopa, Ed.), Vol. 1, pp. 177-192, Akad. Kiadó, Budapest, and North-Holland, Amsterdam (1979).

[7] M. A. Hanson, On sufficiency of the Kuhn-Tucker Conditions, J. Math. Anal. Appl. 80 (1981), 545-550.

[8] D. H. Martin, The Essence of Invexity, J. Optim. Theory Appl. 47 (1985), 65-76.

[9] B. Mond, S. Chandra and I. Husain, Duality for Variational Problems with Invexity, J. Math. Anal. Appl. 134 (1988), 322-328.

[10] B. Mond and M. A. Hanson, Duality for Variational Problems, J. Math. Anal. Appl. 18 (1967), 355-364.

[11] B. Mond and I. Husain, Sufficient Optimality Criteria and Duality for Variational Problems with Generalized Invexity, J. Austral. Math. Soc. Ser. B 31 (1989), 108-121.

[12] B. Mond and I. Smart, Duality and Sufficiency in Control Problems with Invexity, J. Math. Anal. Appl. 136 (1988), 325-333.

[13] C. Nahak and S. Nanda, Duality for Variational Problems with Pseudo-invexity, Optimization 34 (1995), 365-371.

[14] C. Nahak and S. Nanda, Symmetric Duality with Pseudo-invexity in Variational Problems, Europ. J. Oper. Res. 122 (2000), 145-150.

[15] F. A. Valentine, The Problem of Lagrange with Diferential Inequalities as Added Side Conditions, in "Contribution to the Calculus of Variational 1933-1937", pp. 407-448, Univ. of Chicago Press, Chicago (1937).