Population Dynamics by Fuzzy Differential Inclusions

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Abstract

We introduced another point of view of population dynamics using the theory of fuzzy differential inclusions. We give an application example as well as a study of stability for this example.

Keywords— Fuzzy differential inclusions, fuzzy valued mappings, dynamics population.

1 Introduction

The deterministics models formulated for the study of the populational dynamic consider, invariably, constant or temporals parameters, obtained as averages of analyzed situations. Such models don't contain types of subjectivities that are inherent to the process of the populational variation. The individuals are considered homogeneous and everybody possesses the same characteristics of the evolution. However, in fact, when we analyze each element of a community, we verify that the individual or a group of individuals possesses differentiated characteristics of the remaining that can influence in the population dynamic. In this case, we should consider differentiated states variables according to the pertinence of these characteristics. On the other hand, the populational dynamic can also be influenced by independent characteristics of the state variables: habitation, amusement, wage, atmosphere of work, violence, etc . The specific value of these characteristics not always can be evaluated or measured in the traditional sense,which are "uncertains"

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that we can only conjecture intuitively. Hence, we can affirm, always with some uncertainty, that exist uncertainties in the dynamic due to noises in the demography or in the environment.

Like this being, when we make analysis of more realists biological models we should contain the own uncertainties of the studied phenomenon.

Let us consider the deterministic model described by differential equation

$$x' = f(t, x). \tag{1}$$

Given (1), we can insert the uncertainty or noise, introducing one parameter u in the dynamics, that is,

$$x' = f(t, x, u) \tag{2}$$

There are two distinct approaches for (2).

1) If the nature of those uncertainties is aleatory, then the deterministic problem takes us to an stochastic differential equation . In this case, due to the complexity of the resulting equations to the stochastic models, generally, it is done the insertion of noises in a linear way in u, that is, assuming that the noise enters in the dynamic linearly,with a probabilistic distribution

$$x' = f(t, x) + g(t, x)u \tag{3}$$

In this case, u is denominated white noise, derivation of the stochastic differential of the Brownian motion.

2) If the noise doesn't possess probabilistic structure, or such structure cannot be evaluated , then it can be more appropriated the use of the fuzzy variational systems or of the fuzzy differential inclusions for the formulation of the mathematical models.

Let us suppose that U is a compact set of functions sufficiently regular, then (2) can be written as the following differential inclusion

$$x' \in F(t, x) = \{f(t, x, u)/u \in U\} \tag{4}$$

Let us observe that in the deterministic model (1) , the speed is known for each (t, x), while in the differential inclusion (4) the speed is not given, but we know that it is in the set $F(t, x)$, generating the uncertainty.

In [6], Krivan considers noise u bounded unknown, possessing deterministic nature , that is,

$$F(t, x) = h(t, x, c[-1, 1])$$

and takes the metric of " likelihood " to evaluate the as a solution is better than another.

Analyzing the proposal methodologies by May for the noise of the aleatory nature [7], the theory of differential inclusions and the proposal by Krivan [6], we consider that a reasonable generalization of the problem (1), to model dynamic systems with uncertainties, is to substitute, in the model (4), the set-valued mapping F by a fuzzy set-valued mapping , that is, $F(t,x)$ is a fuzzy set for each (t,x). This took us to use of the concept of the fuzzy differential inclusion formulated by Zhu and Rhao [8] who consider the differential inclusions given by the levels which depend on the state variable x.

In this work, we study a model with proportional variation, using the theory of fuzzy differential inclusions and we analyzed the stability of the equilibrium states, using the concept of differenciability of the fuzzy set-valued mappings[4].

2 Preliminaries

We denote by $K(\mathbb{R}^n)$ ($K_C(\mathbb{R}^n)$) the family of all non empty compact subsets of $\mathbb{R}^n$ (compact and convex). For $A, B \in K(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$ we define the operations of sum and scalar product as

$$A + B = \{a + b/a \in A, b \in B\} \qquad \lambda A = \{\lambda a/a \in A\}$$

The space $K(\mathbb{R}^n)$ with the operations defined above and the inclusion relationship, is a quasilinear space with neutral element $\{0\}$ (see [5]). The metric of Hausdorff defined on $K(\mathbb{R}^n)$ is given by

$$H(A,B) = \inf\{r \geq 0 \ \ / \ \ A \subset B + rS_1(0) \ \ B \subset A + rS_1(0)\},$$

where $S_1(0)$ is the closed ball of ratio 1 and center 0.

A fuzzy set on $\mathbb{R}^n$ is a function $u : \mathbb{R}^n \to [0;1]$. For $0 < \alpha \leq 1$ we will denote by $[u]^\alpha = \{x \in \mathbb{R}^n \ / \ u(x) \geq \alpha\}$ the $\alpha-$level of u and $[u]^0 = \text{supp} u = \overline{\{x \in \mathbb{R}^n / u(x) > 0\}}$, is called the support of u.

A fuzzy set u is called compact (compact, convex) if $[u]^\alpha \in K(\mathbb{R}^n)$, $\forall \alpha \in [0;1]$ ($[u]^\alpha \in K_C(\mathbb{R}^n)$, $\forall \alpha \in [0;1]$).

We denoted by $\mathcal{F}(\mathbb{R}^n)$ ($\mathcal{F}_C(\mathbb{R}^n)$) the space of all the fuzzy compact sets (fuzzy compact, convex sets). We can define the partial order inclusion $\subset$ on

$\mathcal{F}(X)$ as being

$$u \subset v \Leftrightarrow u(x) \leq v(x) \;\; \forall x \in X \Leftrightarrow [u]^\alpha \subseteq [v]^\alpha \quad \forall \alpha \in [0,1].$$

The operations of the sum and scalar product on $\mathcal{F}(\mathbb{R}^n)$ are defined as being

$$(u+v)(x) = \sup_{y \in X} \min\{u(y), v(x-y)\} \quad and \quad (\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & if \quad \lambda \neq 0 \\ \chi_{\{0\}}(x) & if \quad \lambda = 0 \end{cases}$$

With the previous definitions we obtain that $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda[u] \;\; \forall \alpha \in [0,1]$. The space $\mathcal{F}(\mathbb{R}^n)$ with the operations defined above and order partial $\subset$ is a quasilinear space with neutral element $\chi_{\{0\}}$ ($\chi_{\{0\}}$ denote the characteristic function of the set $\{0\}$) (ver [5]). We can also define the metric on $\mathcal{F}(\mathbb{R}^n)$

$$D(u,v) = \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha).$$

A fuzzy valued function $F : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ is called quasilinear (see [4], [5]) if

$$\begin{aligned} F(\lambda x) &= \lambda F(x), & \forall x \in \mathbb{R}^n \;\; \forall \lambda \in \mathbb{R} \\ F(x_1 + x_2) &\subset F(x_1) + F(x_2) & \forall x_{1,} x_2 \in \mathbb{R}^n \end{aligned}$$

F is **bounded** if there exists $K > 0$ such that $D(F(x), \{0\}) \leq K||x||$ for any $t \in \mathbb{R}$.

Let $F : U \subset \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy valued function, let $\alpha : \mathbb{R}^n \to [0;1]$ be a function and J a interval in $\mathbb{R}$. We considere the following problem of differential inclusion for fuzzy valued function (see [8]): determine $x \in C(J, \mathbb{R}^n)$ such that

$$x'(t) \in [F(x(t))]^{\alpha(x(t))}, \tag{5}$$

We said that (5) is a **fuzzy differential inclusion .** If F is a quasilinear operator, then (5) is a quasilinear fuzzy differential inclusion.

Next we will give the concepts of differentiability and stability and, we will enunciate the result of stability for fuzzy differential inclusions (see [4]).

Definition 1 *A fuzzy valued function $F : U \subset \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ is called Fréchet differentiable in $x_0 \in U$ if there exists a bounded quasilinear operator $\mathcal{D}_{x_0}(F) : \mathbb{R}^n \to \mathcal{F}_C(\mathbb{R}^n)$ such that*

$$D(F(x), F(x_0) + \mathcal{D}_{x_0}(F)(x - x_0)) = o(\|x - x_0\|).$$

The quasilinear operator $\mathcal{D}_{x_0}(F)$ is called the Fréchet differential of F at x_0.

Proposition 1 *Let $F : U \subset \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R}^n)$ be a quasilinear and bounded operator. Then F is Fréchet differentiable at $x = 0$ and $\mathcal{D}_0(F) = F$.*

We considere the fuzzy differential inclusion (5), assuming the condition $F(0) = \chi_{\{0\}}$. We say that the equilibrium position $x = 0$ of (5) is **Lyapunov-stable** if the following conditions hold:

1. If $\|x(t_0)\| < \delta_0$ for some $\delta_0 > 0$, then there exists a solution $x(t)$ with the initial condition $x(t_0)$ which is defined for any $t \geq t_0$;
2. For any $\epsilon > 0$ there exists $0 < \delta_1 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \epsilon$ for any $t \geq t_0$.

A Lyapunov-stable equilibrium position $x = 0$ is said to be **asymptotically stable** if there exists a positive number $\delta_2 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_2$, then $\lim_{t\to\infty} \|x(t)\| = 0$.

Theorem 1 *[4] Let 0 be the equilibrium position of the fuzzy differential inclusion (5). Let us suppose that fuzzy valued function $F : X \rightarrow \mathcal{F}(X)$ is differentiable at 0 and that there exist $\delta_0 > 0$ such that if $\|x(0)\| \leq \delta_0$, then there always exists solution $x(t)$ of (5) in $[0, +\infty)$. Then if for some $\alpha \in [0; 1]$ the equilibrium position $x = 0$ of quasilinear differential inclusion*

$$x' \in [\mathcal{D}_0(F)(x)]^\alpha$$

is asymptotically stable, then this point is a stable asymptotically equilibrium position of the fuzzy differential inclusion (5), that is , there exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of (5) satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

$\forall\ t \geq 0$ if $\|x(0)\| < \delta$.

3 Population Dynamic with noise

Let $x(t)$ be the density of the population in the time t and we considere the classic models of the exponential growth and the logistic, that is,

$$f(x) = rx \qquad , \qquad f(x) = rx\left(1 - \frac{x}{k}\right).$$

In theoritical populational biology there are two sources of perturbations of the type (3), which are environmental (abiotic variations) and demographic (distinct individual ability) noise . Nisbet and Gurney proposed the following approach for the demographic noise

$$g(x) = \sqrt{(b(x) + d(x))\, x}$$

where $b(x)$ and $d(x)$ are the reasons of the birth and death, respectively.

In populations with big density the demographic noise is not important and in this case it is more natural to consider just the noise in the parameters. We consider that only the growth rate r is affected. Hence for the exponential model we have:

$$x' = rx + xu = x(r + u), \tag{6}$$

and for the logistic model

$$x' = rx\left(1 - \frac{x}{k}\right) + x\left(1 - \frac{x}{k}\right) = x\left(1 - \frac{x}{k}\right)(r + u)$$

Let us suppose still the noise is bounded by a constant $c > 0$, then we can consider the following differential inclusion

$$x' \in f(x) + cg(x)[-1, 1]. \tag{7}$$

A detailed study of the problem (7) is done in [6].

Based on the previous concepts and on the naturalness of uncertainty of the constant r, we introduce a new model for the exponential problem, where we will take the constant r as a fuzzy set u. This fuzzy set should represent the fuzziness of some characteristic of the population that disturbs its variation.

Let $x(t)$ be the density of the population in the instant t and $\alpha : \mathbb{R} \to [0, 1]$ be a function. We will considere the differential inclusions of type

$$x' \in [u \cdot x]^{\alpha(x)} \tag{8}$$

where

u is a fuzzy set, and

$u \cdot x$ is scalar product in the space $\mathcal{F}(\mathbb{R})$.

In the fuzzy differential inclusion (8) we have that fuzzy valued mapping $F : \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R})$ is given by $F(x) = u \cdot x$.

F is the quasilinear operator and bounded and therefore differentiable in $x = 0$. We also have that $x = 0$ is a equilibrium point ($F(0) = \chi_{\{0\}}$) of fuzzy differential inclusion (8).

In the next we will give a relative application to (8) and the regarding results on the stability of Lyapunov.

Example 1 (life expectation)

Let us suppose that A be a set of the workers with $x(t)$ individuals in the instant t. We will consider the problem of life expectation of the elements of the A, supposing that the poverty be a factor that contributes to the increase of the rate of the individuals mortality.

To model the "poverty ", we could use any indicator for the same, for example, consumption of vitamins, income, etc. In [2] it is made a complete study of the differential model for the life expectation of a group of workers, using the salary (income) as factor of the uncertainty in the mortality rate

$$x'(t) = -(\lambda_1 + \lambda_2.u(r))x(t).$$

In this case, the fuzzy set that evaluates the pertinence degree of the poverty was defined by

$$u(r) = \begin{cases} \left[1 - \left(\frac{r}{r_0}\right)^2\right]^k & se \quad 0 < r < r_0 \\ 0 & se \quad r \geq r_0 \end{cases}$$

where, k is a parameter that gives some characteristic of the group, r is a proportional parameter to the individual's income and r_0 is a minimum income starting from which the individuals are not more differentiated with relationship to the poverty and therefore, not more influence in the mortality rate.

We define $\alpha : \mathbb{R} \rightarrow [0, 1]$ by

$$\alpha(x) = \begin{cases} 0 & se \quad x < 0 \\ x^k & se \quad 0 \leq x \leq 1 \\ 1 & se \quad x > 1. \end{cases}$$

We are considering the normalized model, that is, $x = 1$ is a total population of the individuals.

Considering (8), we have the following differential inclusion

$$x' \in -[(\lambda_1 + \lambda_2 \cdot u)x]^{\alpha(x)}, \tag{9}$$

where

λ_1 is rate of natural mortality (obtained in a group that has satisfactory conditions of survival);

$\lambda_2 \cdot u$ indicates the influence of the poverty in the increase of the rate of mortality of the group;

u is the fuzzy set of the poor individuals in agreement with the income r.

Let us notice that if $r \geq r_0$, then $u(r) = 0$ and (9) is reduced to deterministic model

$$x' = -\lambda_1 x.$$

Now for $r \leq r_0$

$$\begin{aligned} [u]^{\alpha(x)} &= \{r \,/\quad u(r) \geq \alpha(x)\} \\ &= \left\{ r \,/\quad \left[1 - \left(\frac{r}{r_0}\right)^2\right]^k \geq x^k \right\} \\ &= r_0[0, \sqrt{1-x}]. \end{aligned}$$

Therefore, the fuzzy differential inclusion (9), for $0 < x \leq 1$, is equivalent to differential inclusion

$$x' \in -\lambda_1 x - \lambda_2 r_0 x \left[0, \sqrt{1-x}\right]$$

or

$$x' \in -\lambda_1 x - \lambda_2 r_0 x \sqrt{1-x}\,[0,1] \tag{10}$$

Remark 1 *We can see that the differential inclusion (10) is similar to the problem (7). For this reason, this new idea of focusing the problems of population dynamic, using the fuzzy differential inclusion, is a good generalization of the previous studies.*

To find solutions of (10), one of the techniques is to find selections of the set-valued mappings, that is, to obtain functions f such that $f(x) \in G(x)$ $\forall x$. Imediately the solutions of (10) are those which solve the differential equation (see [1])

$$x' = f(x).$$

Hence for set-valued mapping $G(x) = -\lambda_1 x - \lambda_2 r_0 x \left[0, \sqrt{1-x}\right]$ in (10), we have that:

$$f_1(x) = \min_{m\in[0,1]} \{-\lambda_1 x - \left(\lambda_2 r_0 x\sqrt{1-x}\right) m/\ m \in [0;1]\} = -\lambda_1 x - \lambda_2 r_0 x\sqrt{1-x}$$

$$f_2(x) = \max_{m\in[0,1]} \{-\lambda_1 x - \left(\lambda_2 r_0 x\sqrt{1-x}\right) m/\ m \in [0,1]\} = -\lambda_1 x,$$

and any $f(x) \in G(x)$ is such that $f_1(x) \le f(x) \le f_2(x)$. Therefore, for example,

$$\begin{aligned} f_3(x) &= -\lambda_1 x - \lambda_2 r_0 x(1-x) \\ f_4(x) &= -\lambda_1 x - \lambda_2 r_0 x\sqrt{1-x}\left|sin(1/x^2)\right| \end{aligned}$$

are elements of $G(x)$.

For each $f(x) \in G(x)$ we have a solution of Cauchy problem

$$\begin{aligned} x'(t) &= f(x(t)) \\ x(0) &= x_0. \end{aligned}$$

In this case, the attainability set (ver [6]) is given by

$$R(t) = [x_1(t), x_2(t)]\,.$$

where

$$\begin{aligned} x_1(t) &= \frac{(\lambda_1 + \lambda_2 r_0)x_0}{[(\lambda_1 + \lambda_2 r_0) - \lambda_2 r_0 x_0]e^{(\lambda_1+\lambda_2 r_0)t} + \lambda_2 r_0 x_0}; \\ x_2(t) &= x_0 e^{-\lambda_1 t}; \end{aligned}$$

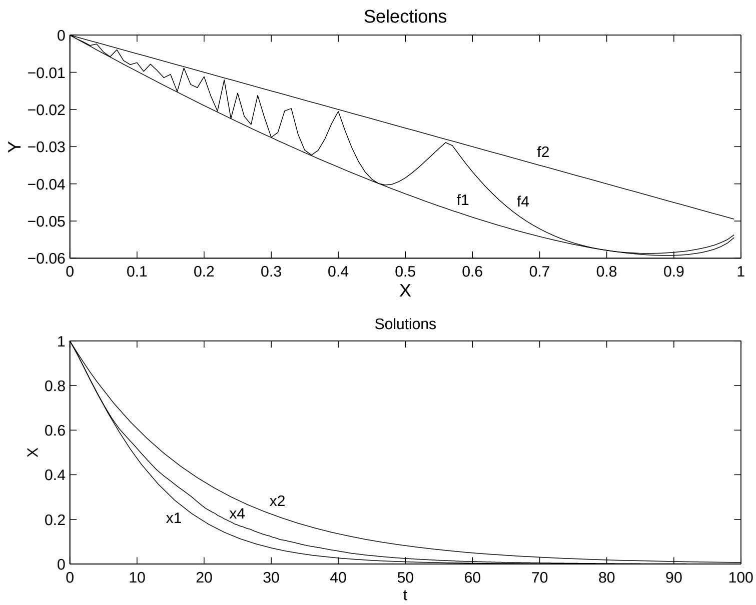

Figure 1: graph of the selections and of the attainability set for $\lambda_1 = 0.05$, $\lambda_2 = 0.001$ and $r_0 = 50$.

We have that $x = 0$ is asymptotically stable for the problem (9), because:

(1) In this fuzzy differential inclusion we have that $F(x) = -(\lambda_1 + \lambda_2 u)x$ and therefore $x = 0$ is a solution of equilibrium ($F(0) = \chi_{\{0\}}$);

(2) F is a quasilinear and bounded operator. It follows from the Proposition 1 that F is Fréchet differentiable and $\mathcal{D}_0(F)(x) = -(\lambda_1 + \lambda_2 u)x$.

We will prove that $x = 0$ is asymptotically stable, for some $\alpha \in [0, 1]$. From fuzzy quasilinear differential inclusion

$$x' \in [\mathcal{D}_0(F)(x)]^\alpha. \tag{11}$$

Let us take $\alpha = (\frac{1}{2})^k$, then (11) is given by

$$x' \in -\left(\lambda_1 + \lambda_2 r_0 \left[0, \frac{1}{\sqrt{2}}\right]\right) x \tag{12}$$

1. The solutions of (12) are of the type $x(t) = x(0)\exp(-(\lambda_1 + a\lambda_2 r_0)t)$, with $a \in \left[0, \frac{1}{\sqrt{2}}\right]$ and there exist for any $t \geq 0$.

2. Given $\epsilon > 0$ there exists $\delta_1 = \epsilon$ such that

$$\|x(t)\| = \|x(0)\exp(-(\lambda_1 + a\lambda_2 r_0)t)\| \leq \|x(0)\| \ \ \forall t > 0.$$

3. $\lim_{t\to\infty} \|x(t)\| = 0$.

It follows from (1), (2) and (3) that $x = 0$ is asintotically stable for the inclusion (11). Soon, by Theorem 1 we have that $x = 0$ is stable asintoticaly for the fuzzy differential inclusion (9), that is, there exists $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of (9) satisfies the inequality

$$\|x(t)\| \leq k\,\|x(0)\|\exp(-\sigma t)$$

$\forall\ t \geq 0$ if $\|x(0)\| < \delta$.

Conclusion : Given $f(x) = rx$, let us suposed that r it is perturbated by a fuzzy set U, then the fuzzy valued mapping is given by $F(x) = (r + U)x$. In this way, if $0 \in [U]^1$, we have that $f(x) \in [F(x)]^\alpha, \forall\alpha \in [0, 1]$. Therefore, for any $\alpha(x)$ we have that

$$\begin{cases} x' & \in & rx + [U]^{\alpha(x)} \\ X(0) & = & X_0, \end{cases}$$

that is, the deterministic solution always is in solution set of differential inclusion.

References

[1] Aubin, J.P. and Cellina, A.(1984), Differential Inclusions, Springer-Verleg, New york Tokyo.

[2] Bassanezi,R.C. and Barros, L.C.(1995), "A simples model of life expectancy with subjetive parameters", Kibernetes: Inter. Journal of Systems and Cybernetics, vol. 9, No 24, pp. 91-98.

[3] Barros, L.C., Bassanezi, R.C. and Tonelli, P.A.(2000), "Fuzzy modelling in population dynamics", Ecological Modelling, vol. 128, No2, pp. 27-33.

[4] Chalco-Cano, Y., Rojas-Medar, M.A., and Brandão,A.J.V. (2001), "On the differentiability of fuzzy-valued mappings and the stability of a fuzzy differential inclusion". Submitted to publication.

[5] Chalco-Cano, Y., Rojas-Medar, M.A., and Brandão,A.J.V.(2001), “Fuzzy quasilinear spaces”, preprint.

[6] Krivan, V. and Colombo, G.(1998), “A non-stochastic approach for modelling uncertainty in population dinamics”, Bulletin of Mathematical Biology, vol.60, 721-751.

[7] May, R.(1973), “Stability in Ramdomly Fluctuating versus Deterministic Environments”, The American Naturalist, pp. 621-650.

[8] Zhu, Yuanguo and Rao, Ling (2000), “Differential Inclusions for Fuzzy Maps”, Fuzzy Sets Syst., vol. 112, pp. 257-261.