On the differentiability of fuzzy-valued mappings and the stability of a fuzzy differential inclusion

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Abstract

We introduce a new concept of differentiability for fuzzy-valued mapping and we study some of its properties. Using this concept, we give a result on stability of the Lyapunov type for fuzzy differential inclusions and a simple application in Biology.

Key words: Fuzzy sets, differentiability, fuzzy-valued mappings, stability of fuzzy differential inclusions.

1 Introduction

The concept of differentiability for fuzzy valued mappings has been considered by many authors from different points of view. For instance, the concept of $H-$differentiability due to Puri and Ralescu [17] has been studied and applied by several mathematicians in the context of fuzzy differential equations, including Ding and Kandel [7], Kaleva [12,13] and Seikkala [21]. Goetschel and Voxman [9] have introduced the notion of a derivative for fuzzy mappings

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of one variable. Basically, they viewed fuzzy numbers in a topological vector space and then they defined differentiation of fuzzy mappings of one variable in ways paralleling the definition of real-valued functions. Syau [22] extends such definition for fuzzy mappings of several variables. Others concepts of differentiability were introduced by Diamond and Kloeden [8] and Román-Flores and Rojas-Medar [20], which extend to the fuzzy context, the concepts of Fréchet differentiability (see De Blasi [6]) and Gâteaux differentiability (see Ibrahim [10]) for set-valued mappings respectively.

As we know, the main idea of the classic differential calculus consists in local approximation of a mapping by a linear operator. In this article we propose a new notion of differentiability for fuzzy mappings, where the role of linear operators is played, in the fuzzy context, by fuzzy quasilinear operators. The theory of fuzzy quasilinear spaces and fuzzy quasilinear operators have been introduced by the authors in [11], inspired in the concept of quasilinear spaces and quasilinear operators given by Assev in [1].

This new concept of differentiability is followed by some properties, examples and rules of calculus. As an application of our results we prove a theorem on stability of a fuzzy differential inclusion. Zhu and Rao [23] have introduced a notion of fuzzy differential inclusion and stated some results on existence of solution. This work has motivated us to develop some ideas concerning stability of fuzzy differential inclusion by using our new notion of differentiability of fuzzy mappings.

The structure of this paper is as follows. In Section 2 we give the definitions and previous results that will be used in this article. In Section 3 we introduce the concepts of Fréchet and Gâteaux differentiability for fuzzy valued mappings and we give some properties. In Section 4 we present some rules of calculus and in the last Section we give some applications on stability of fuzzy differential inclusions.

2 Preliminaries

Let Y be a real separable Banach space with norm $\|\cdot\|$ and dual Y^*. Let $\mathcal{K}(Y)$ and $\mathcal{K}_{\mathcal{C}}(Y)$ be respectively, the class of all nonempty and compact subsets of Y and the class of all nonempty compact and convex subsets of Y.

The Hausdorff metric H on $\mathcal{K}(Y)$ is defined by

$$H(A,B) = \inf\{r \geq 0 : \ A \subset B + rS_1(\theta), \ B \subset A + rS_1(\theta)\},$$

where $S_1(\theta)$ is the closed ball of radius 1 about $\theta \in Y$. It is known that

$(\mathcal{K}(Y), H)$ is a complete and separable metric space and $\mathcal{K}_{\mathcal{C}}(Y)$ is a closed subspace of $\mathcal{K}(Y)$(see [2], [4]). A linear structure is defined in $\mathcal{K}(Y)$ by the operations

$$A + B = \{a + b/\ a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a/a \in A\},$$

where $A, B \in \mathcal{K}(Y)$, $\lambda \in \mathbb{R}$. We observe that $\mathcal{K}(Y)$ is not linear space, but he is a quasilinear space [1].

Let $\mathbb{F}(Y)$ be the space of fuzzy compact sets, that is, $\mathbb{F}(Y)$ is the set of $u : Y \to [0, 1]$ with the following properties

(i) u is normal, i.e., there exists $y_0 \in Y$ such that $u(y_0) = 1$,
(ii) u is upper semicontinuous, and
(iii) $[u]^0 = \text{ supp } (u) = \overline{\{y \in Y\ /\ u(y) > 0\}} \in \mathcal{K}(Y)$.

For each $0 < \alpha \leq 1$, let $[u]^\alpha = \{y\ /\ u(y) \geq \alpha\}$ denote the α-level set of u. From (i)-(iii), it follows that $[u]^\alpha \in \mathcal{K}(Y)$, $\forall\, \alpha \in [0, 1]$.

Let $\mathbb{F}_C(Y) = \{u \in \mathbb{F}(Y)\ /\ [u]^\alpha \in \mathcal{K}_C(Y),\ \forall \alpha \in [0, 1]\}$. For any $u \in \mathbb{F}_C(Y)$, the support function of u, $S_u(\cdot, \cdot) : [0, 1] \times Y^* \to \mathbb{R}$, is defined by

$$S_u(y, \alpha) = \sigma_{[u]^\alpha}(y),$$

where $\sigma_A(y^*) = \sup_{a\in A} \langle y^*, a\rangle$ is the support function of the set $A \subset Y$, $\langle\cdot, \cdot\rangle$ denotes the duality between Y^* and Y.

The linear structure in $\mathbb{F}(Y)$ is defined by the operations

$$(u + v)(x) = \sup_{y\in X} \min\{u(y), v(x - y)\}, \quad (\lambda u)(x) = \begin{cases} u(x\lambda^{-1}) \text{ if } \lambda \neq 0, \\ \chi_{\{0\}}(x) \text{ if } \lambda = 0, \end{cases}$$

where $u, v \in \mathbb{F}(Y)$, $\lambda \in \mathbb{R}$ and χ_A denotes the characteristic function of $A \subseteq Y$. Note that $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda[u]^\alpha$, $\forall\, u, v \in \mathbb{F}(Y)$, $\forall\, \alpha \in [0, 1]$, $\forall\, \lambda \in \mathbb{R}$.

We can endow $\mathbb{F}(Y)$ with several metrics. Some usual distances between fuzzy sets are

$$D_p(u, v) = \begin{cases} \left(\int_0^1 H([u]^\alpha, [v]^\alpha)^p d\alpha\right)^{1/p} & \text{if } p \in [1, \infty), \\ \sup_{\alpha\in[0,1]} H([u]^\alpha, [v]^\alpha) & \text{if } p = \infty. \end{cases}$$

With each distance, we can also define the norm (in fact, a quasinorm) of a fuzzy set u by

$$\|u\|_p = D_p(\chi_{\{0\}}, u).$$

It is well known (see for example [8]) that the metric space $(\mathbb{F}(Y), D_p)$ is complete for each $1 \leq p \leq \infty$, and that $(\mathbb{F}(Y), D_p)$ is separable for each $1 \leq p < \infty$, but $(\mathbb{F}(Y), D_\infty)$ is not.

From now on we will work with the metric D_∞. To simplify notation, we will suppress the subindex $_\infty$ in the distance and in the induced norm. Note that this way we use the same notation for the norm of a point $y \in Y$ and for the norm of a fuzzy set.

Also, we recall the following properties of fuzzy sets.

Remark 2.1 *If $u \in \mathbb{F}(Y)$, then the family $\{[u]^\alpha \;/\; \alpha \in [0,1]\}$ satisfies the following properties:*

(*a*) $[u]^0 \supseteq [u]^\alpha \supseteq [u]^\beta \quad \forall\ 0 \leq \alpha \leq \beta.$
(*b*) *Se* $\alpha_n \uparrow \alpha \Rightarrow [u]^\alpha = \bigcap_{n=1}^{\infty} [u]^{\alpha_n}$
(i.e., the level-application is left-continuous).
(*c*) $u = v \Leftrightarrow [u]^\alpha = [v]^\alpha \quad \forall \alpha \in [0,1].$
(*d*) $[u]^\alpha \neq \emptyset \quad \forall \alpha \in [0,1]$, *is equivalent to* $u(y) = 1$ *for some* $y \in Y$.
(*e*) *We can define a partial order* $\subseteq$ *on* $\mathbb{F}(Y)$ *by setting*

$$u \subseteq v \Leftrightarrow u(y) \leq v(y) \;\; \forall y \in Y \Leftrightarrow [u]^\alpha \subseteq [u]^\beta; \; \forall \alpha \in [0,1].$$

(*e*) *With the operations of addition, scalar multiplication and partial order* $\subseteq$ *defined above,* $\mathbb{F}(Y)$ *and* $\mathbb{F}_C(Y)$ *are normed quasilinear spaces (see [1],[11]).*

Proposition 2.2 *If* $u, v, w, u_1, v_1 \in \mathbb{F}(Y)$. *Then*

(*a*) $D(\lambda u, \lambda v) = \lambda D(u,v)$, *for all* $\lambda \geq 0$.
(*b*) $D(u+v, u_1+v_1) \leq D(u+u_1, v+v_1)$.
If $u, v \in \mathbb{F}_C(Y)$ *we have*
(*c*) $D(u+w, v+w) = D(u,v)$.

An application $F : X \rightarrow \mathbb{F}(Y)$ is called a fuzzy valued mapping.

Definition 2.3 *A fuzzy valued mapping* $F : X \rightarrow \mathbb{F}(Y)$ *will be called a* ***quasi-linear operator*** *if it satisfies the following conditions:*

$$F(\lambda x) = \lambda F(x) \quad \forall x \in X\ , \forall \lambda \in \mathbb{R} \tag{1}$$
$$F(x_1 + x_2) \subseteq F(x_1) + F(x_2) \quad \forall x_1, x_2 \in X. \tag{2}$$

A fuzzy valued mapping $F : X \to \mathbb{F}(Y)$ is said to be **bounded** if there exists a number $k > 0$ such that $\|F(x)\| \leq k\|x\|$ for any $x \in X$.

The importance of the quasilinear operator is well see in the next Theorem.

Theorem 2.4 *([11])The quasilinear operator $F : X \to \mathbb{F}(Y)$ is bounded if and only if it is continuous at the point $\theta \in X$. The continuity of F at θ implies that it is uniformly continuous on X.*

Denote by $L(X, \mathbb{F}(Y))$ the space of all bounded quasilinear operators from X to $\mathbb{F}(Y)$. We write $F_1 \leq F_2$ if $F_1(x) \leq F_2(x)$ for any $x \in X$. Multiplication by real numbers is defined on $L(X, \mathbb{F}(Y))$ by the equality $(\lambda F)(x) = \lambda\Gamma(x)$. Moreover, it is assumed that the operation of algebraic sum is defined on $L(X, \mathbb{F}(Y))$ by the equality $(F_1+F_2)(x) = F_1(x)+F_2(x)$. The space $L(X, \mathbb{F}(Y))$ is closed under these operation of algebraic sum and multiplication by real numbers. Then $L(X, \mathbb{F}(Y))$ is a quasilinear space.

The norm on $L(X, \mathbb{F}(Y))$ is defined by

$$\|F\|_L = \sup_{\|x\|=1} \|F(x)\|.$$

Thus, $L(X, \mathbb{F}(Y))$ is a normed quasilinear space.

3 Differentiability of fuzzy mappings

In this Section we extend the notion of Fréchet differentiability to the fuzzy-valued context, by using the concept of bounded quasilinear operator.

Definition 3.1 *A fuzzy valued mapping $F : X \to \mathbb{F}(Y)$is said to be **Fréchet differentiable** at $x_0 \in X$ if exists a bounded quasilinear operator $\mathcal{D}^F_{x_0}(F) : X \to \mathbb{F}_C(Y)$ such that*

$$D(F(x), F(x_0) + \mathcal{D}^F_{x_0}(F)(x - x_0)) = o(\|x - x_0\|).$$

The quasilinear operator $\mathcal{D}^F_{x_0}(F)$ is called the Fréchet differential of F at x_0.

Example 3.2 *Let $X = \mathbb{R}$ be given. Consider the fuzzy valued mapping F :*

$\mathbb{R} \to \mathbb{F}(\mathbb{R})$ *defined by*

$$F(x)(y) = \begin{cases} -\frac{1}{x^2}(y - x^2) & \text{if } 0 \leq y \leq x^2 \\ \frac{1}{x^2}(y + x^2) & \text{if } -x^2 \leq y \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

if $x \neq 0$, *and* $F(0) = \chi_{\{0\}}$. *It is easily seen that*

$$[F(x)]^\alpha = \left[-(1-\alpha)x^2, (1-\alpha)x^2\right]$$

for each $\alpha \in [0,1]$. *Now,*

$$\begin{aligned} D(F(t), F(0) + \chi_{\{0\}}) &= \sup_{\alpha \in [0,1]} H\left(\left[-(1-\alpha)x^2, (1-\alpha)x^2\right], \{0\} + \{0\}\right) \\ &= \sup_{\alpha \in [0,1]} \left|(1-\alpha)x^2\right| = \left|x^2\right|. \end{aligned}$$

It follows that F *is Fréchet differentiable at* $x = 0$ *and* $\mathcal{D}_0^F(F)(x) = \chi_{\{0\}}$.

We shall now establish the uniqueness of the Fréchet derivative.

Theorem 3.3 *The fuzzy valued mapping* F *has at most one Fréchet derivative at a point.*

Proof: Let $\mathcal{D}_{x_0}^F(F)$ and $\overline{\mathcal{D}_{x_0}^F(F)}$ be two differentials of F at x_0. Then, by Proposition 2.2, we have that

$$\begin{aligned} & D(\mathcal{D}_{x_0}^F(F)(x - x_0), \overline{\mathcal{D}_{x_0}^F(F)}(x - x_0)) \\ &= D(F(x_0) + \mathcal{D}_{x_0}^F(F)(x - x_0), F(x_0) + \overline{\mathcal{D}_{x_0}^F(F)}(x - x_0)) \\ &\leq D(F(x), F(x_0) + \mathcal{D}_{x_0}^F(F)(x - x_0)) \\ &\quad + D(F(x), F(x_0) + \overline{\mathcal{D}_{x_0}^F(F)}(x - x_0)) \\ &= o(\|x - x_0\|). \end{aligned}$$

Thus, $D(\mathcal{D}_{x_0}^F(F)(x - x_0), \overline{\mathcal{D}_{x_0}^F(F)}(x - x_0)) = o(\|x - x_0\|)$ for all $x \in X$. This prove que $\mathcal{D}_{x_0}^F(F) = \overline{\mathcal{D}_{x_0}^F(F)}$.

Proposition 3.4 *A fuzzy valued application* $F : X \to \mathbb{F}(Y)$ *is constant if and only if, for every* $x_0 \in X$, $\mathcal{D}_{x_0}^F(F)(x) = \chi_{\{\theta\}}$ $\forall x \in X$.

Proof: First we assume that F is constant i.e. $F(x) = K$ $\forall x$. Then for any $x \in X$, we have

$$
\begin{aligned}
D\left(F(x), F(x_0)+\mathcal{D}^F_{x_0}(F)(x-x_0)\right) &= D\left((K, K+\mathcal{D}^F_{x_0}(F)(x-x_0)\right) \\
&= D\left(\chi_{\{\theta\}}, \mathcal{D}^F_{x_0}(F)(x-x_0)\right).
\end{aligned}
$$

Therefore, $\mathcal{D}^F_{x_0}(F)(x-x_0)=\chi_{\{\theta\}}\ \forall x\in X$. Conversely, if

$$\mathcal{D}^F_{x_0}(F)(x-x_0)=\chi_{\{\theta\}}\forall x\in X$$

then for any $x\in X$

$$D(F(x), F(x_0)) = o(\|x-x_0\|),$$

thus $F(x)=F(x_0)\equiv K$.

Proposition 3.5 *Let $F : X \to \mathbb{F}_C(Y)$ be a bounded quasilinear operator. Then F is Fréchet differentiable at $\theta\in X$ and $\mathcal{D}^F_\theta(F)=F$.*

Proof: It is sufficient to observe that $F(\theta)=\chi_{\{\theta\}}$ when F is quasilinear.

Theorem 3.6 *If $F : X\to\mathbb{F}(Y)$ is differentiable at x_0, then F is continuous at x_0.*

Proof: Suppose that $x_i\to x_0$, then

$$
\begin{aligned}
D(F(x_i), F(x_0)) &\le D(F(x_i), F(x_0)+\mathcal{D}^F_{x_0}(F)(x_i-x_0)) \\
&\quad + D(F(x_0), F(x_0)+\mathcal{D}^F_{x_0}(F)(x_i-x_0)) \\
&= o(\|x_i-x_0\|)+\|\mathcal{D}^F_{x_0}(F)\|_{\mathcal{F}}\|x_i-x_0\|\to 0
\end{aligned}
$$

as $i\to\infty$. The Theorem is proved.■

Let $F : X\to\mathbb{F}(Y)$ be a fuzzy-valued mapping. The level set-valued mapping $F_\alpha : X\to\mathcal{K}(Y)$, with $\alpha\in[0,1]$, is defined by

$$F_\alpha(x)=[F(x)]^\alpha.$$

Now, we study the relation between the derivative of F and their associated level set-valued mapping.

Proposition 3.7 *If F is differentiable at x_0, then the level set-valued mapping F_α is differential at x_0 for each $\alpha\in[0,1]$ and*

$$\mathcal{D}^F_{x_0}(F_\alpha)=\left[\mathcal{D}^F_{x_0}(F)\right]^\alpha.$$

Proof: Let $\alpha \in [0,1]$ be arbitrary. Then

$$\begin{aligned}
&H\left([F(x)]^\alpha, [F(x_0)]^\alpha + \left[\mathcal{D}^F_{x_0}(F)(x-x_0)\right]^\alpha\right)\\
=&H\left([F(x)]^\alpha, \left[F(x_0) + \mathcal{D}^F_{x_0}(F)(x-x_0)\right]^\alpha\right)\\
\leq& D(F(x), F(x_0) + \mathcal{D}^F_{x_0}(F)(x-x_0)\\
=& o(\|x-x_0\|).
\end{aligned}$$

Consequently, the Proposition is proved.

We observe that the converse of the Proposition is false without supplementary hypotheses. The following example show this fact.

Example 3.8 *We consider the fuzzy valued mapping* $F : (-1/2, 1/2) \to \mathbb{F}_C(\mathbb{R})$ *defined through its levels,*

$$F_\alpha(t) = \begin{cases} [-t, 1+t] \ \textit{if}\ 0 < \alpha \leq 1 \\ [-1/2, 1] \ \ \textit{if} \quad \alpha = 0 \end{cases}$$

for $-1/2 < t < 0$ *and* $F(t) = \chi_{\{[0,1]\}}$ *for* $0 \leq t < 1/2$. *Each level set-valued mapping* F_α *is differentiable at* $t = 0$ *with derivative*

$$\mathcal{D}^F_0(F_\alpha)(t) = \begin{cases} [-1,1]t \ \textit{if}\ 0 < \alpha \leq 1 \\ \quad \{t\} \quad \textit{if} \quad \alpha = 0 \end{cases}$$

We observe that F *is not Fréchet differentiable, since the following inclusion would be true*

$$\mathcal{D}^F_0(F_\beta)(1) \subset \mathcal{D}^F_0(F_\alpha)(1)$$

$\forall\ 0 \leq \alpha \leq \beta \leq 1.$

The Gateâux derivative has the following generalization.

Definition 3.9 *A fuzzy valued mapping* $F : X \to \mathbb{F}(Y)$ *is* ***Gâteaux differentiable*** *at* $x_0 \in X$ *if exists an bounded quasilinear operator* $\mathcal{D}^G_{x_0}(F) : X \to \mathbb{F}_C(Y)$ *such that, for all* $z \in X$

$$D(F(x_0 + tz), F(x_0) + t\mathcal{D}^G_{x_0}(F)(z)) = o(t) \quad as \quad t \to +0.$$

$\mathcal{D}^G_{x_0}(F)(z)$ *is called the* ***Gâteaux derivative*** *of* F *at* x_0.

Theorem 3.10 *The Gâteaux derivative is unique if it exists.*

Proof: Let $\mathcal{D}^G_{x_0}(F)$ and $\overline{\mathcal{D}^G_{x_0}(F)}$ two Gateâux derivatives of F at x_0. Then, from Proposition 2.2 follows that

$$\begin{aligned} D(\mathcal{D}^G_{x_0}(\Gamma)(tz), \overline{\mathcal{D}^G_{x_0}(F)}(tz)) &= D(F(x_0) + \mathcal{D}^G_{x_0}(F)(tz), F(x_0) + \overline{\mathcal{D}^G_{x_0}(F)}(tz)) \\ &\leq D(F(x_0+tz), F(x_0) + \mathcal{D}^G_{x_0}(F)(tz)) \\ &\quad + D(F(x_0+tz), F(x_0) + \overline{\mathcal{D}^G_{x_0}(F)}(tz)) \\ &= o(t) \ \text{ as } \ t \to +0 \end{aligned}$$

Consequently,

$$D(\mathcal{D}^G_{x_0}(F)(z), \overline{\mathcal{D}^G_{x_0}(F)}(z)) = \frac{o(t)}{t} \ \text{ as } \ t \to +0.$$

Therefore, $\mathcal{D}^G_{x_0}(F)(z) = \overline{\mathcal{D}^G_{x_0}(F)}(z)$ for all $z \in X$.

A relation between Fréchet derivative and Gâteaux derivative is given in the next Theorem.

Theorem 3.11 *Suppose that a fuzzy valued mapping* $F : X \to \mathbb{F}(Y)$ *is Fréchet differentiability at* $x_0 \in X$. *Then,* F *is Gâteaux differentiable and*

$$\mathcal{D}^G_{x_0}(F) = \mathcal{D}^F_{x_0}(F).$$

Proof: In fact, we have

$$\begin{aligned} & D(F(x_0+tz), F(x_0) + t\mathcal{D}^F_{x_0}(F)(z)) \\ &= D(F(x_0+tz), F(x_0) + \mathcal{D}^F_{x_0}(F)(tz)) \\ &= o(t\|z\|) = o(t) \ \text{ as } \ t \to +\infty. \quad \blacksquare \end{aligned}$$

We recall that a fuzzy valued mapping $F : X \to \mathbb{F}(Y)$ is upper semicontinuous at x_0 if $\forall \epsilon > 0, \exists \delta = \delta(x_0, \epsilon) > 0$ such that

$$\sup_{\alpha \in [0,1]} D^*(F(x), F(x_0)) < \epsilon$$

when $\|x - x_0\| < \delta$, where

$$D^*(u,v) = \sup_{\alpha \in [0,1]} h^*([u]^\alpha, [v]^\alpha),$$

and

$$h^*(A,B) = \inf_{r\geq 0} \{r \geq 0 \ / \ A \subset B + rS_1(\theta)\}.$$

Therefore, F is said to be homogeneous if $F(\lambda x) = \lambda F(x)$ for $\lambda \geq 0, x \in X$. To more details see [20].

Definition 3.12 *A fuzzy-valued mapping $F : X \to \mathbb{F}(Y)$ is De Blasi differentiable at $x_0 \in X$ if exists an upper semicontinuous, positive homogeneous mapping $\mathbb{D}^F_{x_0}(F) : X \to \mathbb{F}_C(Y)$ such that*

$$D(F(x_0 + x), F(x_0) + \mathbb{D}^F_{x_0}(F)(x)) = o(\|x\|).$$

The mapping $\mathbb{D}^F_{x_0}(F)$ is called the De Blasi differential of F at x_0.

Definition 3.13 *A fuzzy-valued mapping $F : X \to \mathbb{F}(Y)$ is Ibrahim-Gâteaux differentiable at $x_0 \in X$ if exists an upper semicontinuous, positive homogeneous mapping $\mathbb{D}^G_{x_0}(F) : X \to \mathbb{F}_C(Y)$ such that, for all $z \in X$*

$$D(F(x_0 + tz), F(x_0) + t\mathbb{D}^G_{x_0}(F)(z)) = o(t) \ \ as \ \ t \to +0.$$

$\mathbb{D}^F_{x_0}(F)$ is called the Ibrahim-Gâteaux differential of F at x_0.

It is clear that if $F : X \to \mathbb{F}(Y)$ is Fréchet differentiable at x_0 (Gâteaux differential) then F is De Blasi differentiable at x_0 (Ibrahim-Gâteaux differential, respectively) and $\mathcal{D}^F_{x_0}(F)(x) = \mathbb{D}^F_{x_0}(F)(x)$ ($\mathcal{D}^G_{x_0}(F)(x) = \mathbb{D}^G_{x_0}(F)(x)$ respectively).

4 Rules of calculus

We shall first establish the basic algebraic relations concerning the derivative.

Theorem 4.1 *Let F_1 and F_2 be two fuzzy valued mapping from X to $\mathbb{F}(Y)$. If F_1 and F_2 are Fréchet differentiable at $x_0 \in X$, then the mapping $F = \lambda F_1 + \beta F_2$ with $\lambda, \beta \in \mathbb{R}$, is Fréchet differentiable at x_0, and*

$$\mathcal{D}^F_{x_0}(\lambda F_1 + \beta F_2) = \lambda \mathcal{D}^F_{x_0}(F_1) + \beta \mathcal{D}^F_{x_0}(F_2).$$

Proof: We observe

$$D(F(x), F(x_0) + (\lambda \mathcal{D}^F_{x_0}(F_1) + \beta \mathcal{D}^F_{x_0}(F_2))(x - x_0))$$

$$
\begin{aligned}
&\le D(\lambda F_1(x), \lambda F_1(x_0) + \lambda \mathcal{D}^F_{x_0}(F_1)(x - x_0)) \\
&\quad + D(\lambda F_2(x), \lambda F_2(x_0) + \lambda \mathcal{D}^F_{x_0}(F_2)(x - x_0)) \\
&\le |\lambda| o(\|x - x_0\|) + |\beta| o(\|x - x_0\|) \\
&= o(\|x - x_0\|).
\end{aligned}
$$

This prove the Theorem.

Remark 4.2 *Theorem 4.1 still holds if we suppose that F_1 and F_2 are Gâteaux differentiable at x_0.*

Theorem 4.3 *A fuzzy valued mapping $F : X \to \mathbb{F}_C(Y)$ is Gâteaux differentiable at x_0 if and only if, the support function $S_{F(x)}(\alpha, \psi)$ is Gâteaux differentiable at x_0 and $\mathcal{D}^G_{x_0}(S_{F(x)})$ is a support function. Moreover, in this case*

$$
\mathcal{D}^G_{x_0}(S_{F(x)})(\alpha, \psi) = S_{\mathcal{D}^G_{x_0}(F)(x)}(\alpha, \psi).
$$

Proof: Suppose that F is differentiable at x_0 and $z \in X$. Then

$$
\begin{aligned}
&\frac{1}{t}\|S_{F(x_0+t.z)}(\alpha, \psi) - S_{F(x_0)}(\alpha, \psi) - t.S_{\mathcal{D}^G_{x_0}(F)(z)}(\alpha, \psi)\| \\
&= \frac{1}{t}\|S_{F(x_0+t.z)}(\alpha, \psi) - S_{F(x_0)+t.\mathcal{D}^G_{x_0}(F)(z)}(\alpha, \psi)\| \\
&\le \frac{1}{t} D(F(x_0 + t.z), F(x_0) + t.\mathcal{D}^G_{x_0}(F)(z))\|(\alpha, \psi)\| \\
&= \frac{o(t)}{t} \to 0 \quad \text{as} \quad t \to +0.
\end{aligned}
$$

Thus, a support function $S_{F(x)}(\alpha, \psi)$ is Gâteaux differentiable at x_0 and

$$
\mathcal{D}^G_{x_0}(S_{F(x)}(\alpha, \psi)) = S_{\mathcal{D}^G_{x_0}(F(x))}(\alpha, \psi).
$$

Conversely, suppose that $S_{F(x)}(\alpha, \psi)$ is differentiable at x_0 and $\mathcal{D}^G_{x_0}(S_{F(x)})(\alpha, \psi) = S_\Lambda(\alpha, \psi)$. Then, for any $z \in X$

$$
\begin{aligned}
&D(F(x_0 + t.z), F(x_0) + t.\Lambda) \\
&= \max_{\|(\alpha,\psi)\|=1} \|S_{F(x_0+t.z)}(\alpha, \psi) - S_{F(x_0)+t.\Lambda}(\alpha, \psi)\| \\
&= \max_{\|(\alpha,\psi)\|=1} \|S_{F(x_0+t.z)}(\alpha, \psi) - S_{F(x_0)}(\alpha, \psi) - t.S_\Lambda(\alpha, \psi)\| \\
&= o(t)
\end{aligned}
$$

as $t \to +0$ and the Theorem is proved.

5 Stability of Fuzzy Differential Inclusion

Let $F : X \to \mathbb{F}(X)$ be a fuzzy-valued mapping. Let $\alpha : X \to [0,1]$ be a function and J a interval in $\mathbb{R}$. The problem (see [23]): find $x \in C(J, X)$ such that

$$x'(t) \in [F(x(t))]^{\alpha(x(t))} \tag{3}$$

is said a fuzzy differential inclusion.

A quasilinear differential inclusion is defined by a differential inclusion

$$x^{'} \in F(x),$$

where $F : X \to \mathcal{K}_C(X)$ is a quasilinear operator.

Consider the fuzzy differential inclusion (3), assuming the condition $F(\theta) = \chi_{\{\theta\}}$. We say that the equilibrium position $x = \theta$ of (3) is **Lyapunov-stable** if the following conditions hold;

(a) There is a $\delta_0 > 0$ such that if $\|x(t_0)\| < \delta_0$, then there exists a solution $x(t)$ with the initial condition $x(t_0)$, and it is defined for any $t > t_0$.
(b) For any $\epsilon > 0$ there exists a $0 < \delta_1 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \epsilon$ for any $t \geq t_0$.

A Lyapunov-stable equilibrium position $x = \theta$ is said to be **asymptotically stable** is there exists a positive number $\delta_2 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_2$, then $\lim_{t\to\infty} \|x(t)\| = 0$.

The next result was proved in [15].

Theorem 5.1 *Suppose that the set-valued mapping $F : X \to \mathcal{K}_C(X)$ is positive-homogeneous and upper semicontinuous. Assume that any solution $x(t)$ of the differential inclusions $x^{'} \in F(x)$ tends to θ as $t \to \infty$. Let $G : X \to \mathcal{K}_C(X)$ be an upper semicontinuous set-valued mapping, with $\|G(x)\| = o(\|x\|)$ as $\|x\| \to 0$. Then there exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of the differential inclusion $x^{'} \in F(x) + G(x)$ with $\|x(0)\| < \delta$ satisfies the inequality*

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$.

By using the above Theorem, we can prove the stability for Problem (3). In fact, we have

Theorem 5.2 *Suppose that the point θ is an equilibrium position of the fuzzy differential inclusion (3). Moreover, suppose that the fuzzy-valued mapping $F : X \to \mathbb{F}(X)$ is differentiable at θ and that there exists a number $\delta_0 > 0$ such that any solution $x(t)$ of (3) exists on the whole interval $[0, +\infty)$ if $\|x(0)\| \leq \delta_0$. If for some $\alpha \in [0,1]$ the equilibrium position $x = \theta$ of the quasilinear differential inclusion*

$$x' \in [\mathcal{D}_\theta^F(F)(x)]^\alpha \tag{4}$$

is asymptotically stable, then this point is an asymptotically stable equilibrium position of the fuzzy differential inclusion (3), that is, there exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of (3) satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$ if $\|x(0)\| < \delta$.

Proof: Since F is differentiable at θ, then the application $\widetilde{\mathcal{D}_\theta^F(F)} : X \to \mathcal{K}_C(X)$ defined by

$$\widetilde{\mathcal{D}_\theta^F(F)}(x) = L_\alpha \mathcal{D}_\theta^F(F)(x),$$

exists for all $x \in X$, is homogeneous and uniformly continuous. Also, since the equilibrium position $x = \theta$ of the quasilinear differential inclusion (4) is asymptotically stable, then any solution $x(t)$ of (4) tends to θ as $t \to \infty$.

Now, we denote $\bar{F}(x) = [F(x)]^{\alpha(x)}$

$$\begin{aligned}
\|\bar{F}(x)\| &= H([F(x)]^{\alpha(x)}, \theta) \\
&\leq D(F(x), \chi_{\{0\}}) \\
&\leq D(F(x), \mathcal{D}_\theta^F(F)(x)) + D(\mathcal{D}_\theta^F(F)(x), \chi_{\{0\}}) \\
&\leq o(\|x\|) + \|\mathcal{D}_\theta^F(F)\|_{\mathcal{F}}\|x\| \\
&= o(\|x\|) \ \text{ as } \|x\| \to 0.
\end{aligned}$$

Thus, due to Theorem 5.1, exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of (3) satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$ if $\|x(0)\| < \delta$ and the theorem is proved.

As a simple application of ours results we give an example from Biology, to more details see [5].

Example 1 (life expectation)

Let us suppose that A is the set of the workers with $x(t)$ individuals in instant t and consider the problem of life expectation of the elements of A, supposing that the poverty is a factor that contributes to the increase of the mortality individuals rate.

To model the level of poverty of a person, we use any poverty indicator, for example, consumption of vitamins, basic cleaning up, income, etc. In [3] the authors made a complete study of the differential model for the life expectation of a group of workers, using the salary (income) as factor of the uncertainty in the mortality rate. In this case, the fuzzy set that evaluates the pertinence degree of the poverty was defined as follows

$$u(r) = \begin{cases} \left[1 - \left(\frac{r}{r_0}\right)^2\right]^k & if\ 0 < r < r_0 \\ 0 & if\quad r \geq r_0 \end{cases}$$

where, k is a parameter that gives some characteristic of the group, r is a parameter proportional to the individual's income and r_0 is the minimum income starting from which the individuals are not more differentiated with relationship to the poverty and therefore, not more influence in the mortality rate.

We define $\alpha : \mathbb{R} \to [0, 1]$ by

$$\alpha(x) = \begin{cases} 0 \ \ if \quad x < 0 \\ x^k \ if\ 0 \leq x \leq 1 \\ 1 \ \ if \quad x > 1 \end{cases}$$

We are considering the normalized model, that is, $x = 1$ is the total population of the individuals.

Considering (3), we have the following differential inclusion

$$x' \in -[(\lambda_1 + \lambda_2 \cdot u)x]^{\alpha(x)}, \tag{5}$$

where

λ_1 is the rate of natural mortality (obtained in a group that has satisfactory conditions of survival);
$\lambda_2 \cdot u$ indicates the influence of the poverty in the increase of the rate of mortality of the group;
u is the fuzzy set of the poor individuals in agreement with the income r.

Let us notice that if $r \geq r_0$, then $u(r) = 0$ and (5) is reduced to deterministic model

$$x' = -\lambda_1 x.$$

Now for $r \leq r_0$, we have the following differential inclusions

$$x' \in -\lambda_1 x - \lambda_2 r_0 x \sqrt{1-x}\,[0,1] \tag{6}$$

We have that $x = 0$ is stable asymptotically for the problem (5), because:

(i) In this fuzzy differential inclusion we have that $F(x) = -(\lambda_1 + \lambda_2 u)x$ and therefore $x = 0$ is a solution of equilibrium ($F(0) = \chi_{\{0\}}$);

(ii) F is a bounded quasilinear operator. It proceeds of the Proposition 3.5 that F is Fréchet differentiable and $\mathcal{D}_0^F(F)(x) = -(\lambda_1 + \lambda_2 u)x$.

We will prove that $x = 0$ is stable asymptotically, for some $\alpha \in [0;1]$ of fuzzy quasilinear differential inclusion

$$x' \in [\mathcal{D}_0^F(F)(x)]^\alpha. \tag{7}$$

Let us take $\alpha = (\frac{1}{2})^k$, then (3) is given for

$$x' \in -\left(\lambda_1 + \lambda_2 r_0 \left[0, \frac{1}{\sqrt{2}}\right]\right) x \tag{8}$$

We have that $x = 0$ is stable asymptotically for the differential inclusion (8). Therefore, by Theorem 5.2 we have that $x = 0$ is stable asymptotically for the fuzzy differential inclusion (3).

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