Theoretical Evaluation of Elliptic Integrals Based on Computer Graphics

Valério Ramos Batista
IMECC, UNICAMP, Cx. P. 6065
13083-970 Campinas, SP, Brazil
e-mail: valerio@ime.unicamp.br

Abstract

We use numerical simulations to build theoretical estimates for some elliptic integrals. The procedures explained herein can be generalized to a larger class of integrals and transcendental functions.

1. Introduction

Elliptic integrals come out naturally in Complex Analysis, particularly in the Theory of Riemann Surfaces. Special functions can be used in order to handle these integrals, but they succeed only for the most particular cases. A large reference of handbooks and tables of integrals provide prompt equalities, which saves a valuable amount of time. Nevertheless, no literature can be complete in this subject, and it happens frequently that one finds no formal way to evaluate an integral, or a transcendental function.

The variety of such functions and integrals is infinitely large. Therefore, cases not included in the literature are handled with the help of numerical computation. Nevertheless, numerical integration of a function turns out to be reliable if the function satisfies some special conditions, which depend on the integration method. For instance, the Simpson integration method on a closed real interval $[a, b]$, equally partitioned by an even natural n, produces an absolute error not bigger than $(b-a)^5 M/n^4$, where $M := \max|f^{(4)}|/180$. So, the error depends on the fourth derivative of the integrand in $[a, b]$. In general, one does not compute this error.

Most of times, the integral is considered to be approaching its actual value if an increase in n produces a small change in the previous result. But this procedure is not reliable, as illustrated in Figure 1. In some cases, the function is considered to be "well behaved" if a picture of its graph does not present too much oscillations. Of course, this concept is imprecise and subject to a particular sample of points.

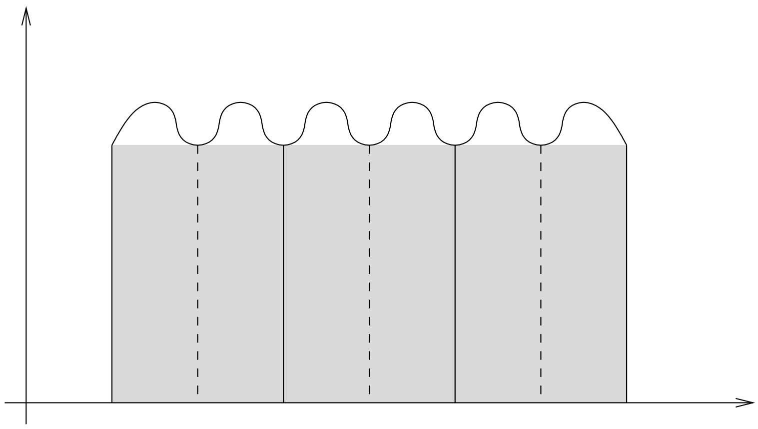


Figure 1: An increase in n (dashed lines) with no difference in the area.

This report has the purpose of illustrating how to prove theoretical estimates of elliptic integrals with the help of computer graphs. The numerical data are used to decide the way we direct our formal demonstrations. This procedure can be used for a larger class of integrals and transcendental functions. Each specific case must be handled differently, but the method presented here is structurally general.

2. First Approach

We shall begin with an example with practical application in the solution of period problems for minimal surfaces (see [3,p.70-108]). Any complete minimal surface with finite total curvature in a flat space is conformal to a compact Riemann surface minus a finite set of points (see [1] and [2]). Because of that, elliptic integrals come out naturally in the study of these surfaces.

The following result is part of the existence proof of triply periodic minimal surfaces which cannot be constructed by the *Conjugate Plateau Method* (see [3,p.70-108]):

Proposition 2.1. *Let λ be a real variable, $0 < \lambda < 1$, and consider the variable y such that $2\lambda - 1 < y < \lambda$. Define the functions $X(\lambda, y) := \lambda^{-1}(1 + 2y\lambda - y^2)$ and $0 < x(\lambda, y) < 1$ such that $x + x^{-1} = X$. Hence, for a certain positive $\varepsilon < 0.4$ one has that every $\lambda \in (0; 0.6 + \varepsilon)$ admits a unique $y_\lambda \in (2\lambda - 1, \lambda)$ such that, if X_λ is the corresponding value of $x = x(\lambda, y)$, then*

$$\int_0^1 \frac{(t + y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 + X_\lambda t + 1)(t + \lambda)}} = \int_0^\lambda \frac{(t - y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 - X_\lambda t + 1)(\lambda - t)}}.$$

Proof:

We shall simplify notations by defining

$$II_1(\lambda, y) := \int_0^1 \frac{(t + y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 + X_\lambda t + 1)(t + \lambda)}}$$

and

$$II_2(\lambda, y) := \int_0^\lambda \frac{(t - y_\lambda)dt}{\sqrt{(1 - t^2)(t^2 - X_\lambda t + 1)(\lambda - t)}}.$$

Both are continuous functions defined on $(0, 1) \times (2\lambda - 1, \lambda)$. We use the intermediate value theorem to prove Proposition 2.1. We do this in three steps:

Step I. $II_1(\lambda, \lambda) > II_2(\lambda, \lambda)$:

This is trivial because $y = \lambda$ implies $x = \lambda$. Then

$$II_1(\lambda, \lambda) = \int_0^1 \frac{(t + \lambda)dt}{(t + \lambda)\sqrt{(t + \lambda^{-1})(1 - t^2)}}$$

and

$$II_2(\lambda, \lambda) = -\int_0^\lambda \frac{dt}{\sqrt{(\lambda^{-1} - t)(1 - t^2)}}.$$

It is clear that $II_1(\lambda, \lambda)$ is positive and $II_2(\lambda, \lambda)$ is negative for any λ in the interval $(0, 1)$.

Step II. For $0 < \lambda \leq 0.6$, one has $\Pi_1(\lambda, 2\lambda - 1) < \Pi_2(\lambda, 2\lambda - 1)$.

To simplify notations, let us call $I_j = I_j(\lambda) := \Pi_j(\lambda, 2\lambda - 1), j = 1, 2$. We must show that, for $0 < \lambda \leq 0.6$:

$$I_2 = \int_0^\lambda \frac{(t - 2\lambda + 1)dt}{(1-t)\sqrt{(1-t^2)(\lambda - t)}} > I_1 = \int_0^1 \frac{(t + 2\lambda - 1)dt}{(t+1)\sqrt{(1-t^2)(t+\lambda)}}.$$

Since it is too complicated to get the exact values of the integrals, we are going to make use of the following strategy: first we make some changes of variables in such a way that the integrands will turn out to be bounded and defined on the same interval of integration. Then, we compare these new integrands. If an inequality holds for them on the whole interval of integration, them the same inequality holds for their integral values. Nevertheless, the change of variables can lead to integrands whose expressions are more complicated to deal with. For this reason we are going to simplify the comparison by means of some additional inequalities. However, this will lead to a certain loss of information. For example, the inequality

$$\frac{1}{\sqrt{a-b}} > \frac{1}{\sqrt{a}} + \frac{b}{2a^{3/2}} \tag{1}$$

holds for every $a, b \in \mathbb{R}$ with $a > b > 0$ (the right-hand side corresponds to the first two terms of the Taylor series of the left-hand side). This inequality takes part in our proof and is used in such a way that we get another integrand, much simpler to integrate than the former one. Of course, with this we lose information about the exact value of the former integral, but this is always done in such a way that we minimize this loss. That is, we studied some graphic comparisons and they indicate a good approximation between the integrands. Examples of such comparisons will be shown along our demonstration of this second step to prove Proposition 2.1.

We begin with by putting the two integrands on the same interval and making them bounded. For I_1 the substitution $t = 1 - u^2$ leads to

$$I_1 = 2\int_0^1 \frac{(2\lambda - u^2)du}{(2-u^2)^{3/2}(1+\lambda-u^2)^{1/2}} \tag{2}$$

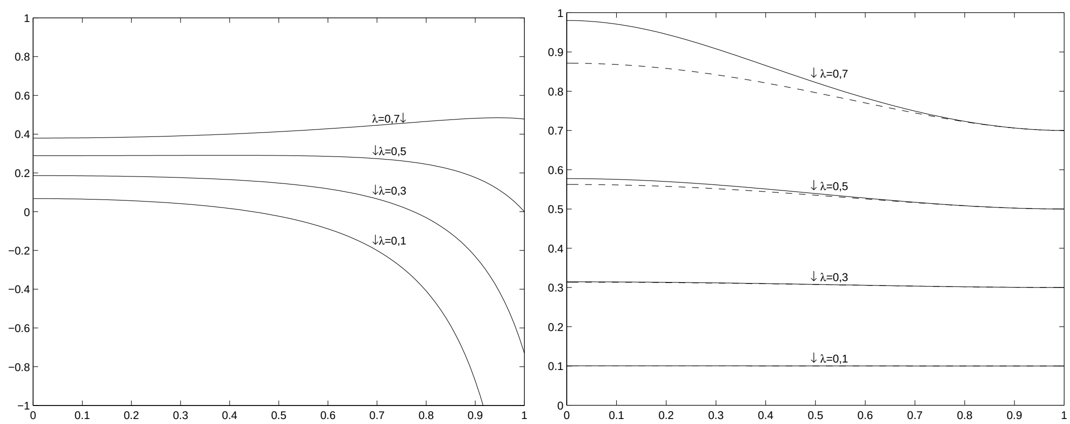
and for I_2 we have that $t = \lambda - \lambda u^2$ implies

$$I_2 = 2\int_0^1 \frac{\sqrt{\lambda}(1-\lambda-\lambda u^2)du}{(1-t(u))\sqrt{1-t^2(u)}} = \frac{2}{\sqrt{\lambda}}\int_0^1 \frac{(1-\lambda-\lambda u^2)du}{(1-\lambda+\lambda u^2)\sqrt{\lambda^{-2}-(1-u^2)^2}}.$$

Now we use (1) for the integrand of I_2, namely

$$\frac{1}{\sqrt{\lambda^{-2}-(1-u^2)^2}} > \lambda + \frac{\lambda^3}{2}(1-u^2)^2, \ \forall\ u, \lambda \in (0,1). \tag{3}$$

Figure 2(b) shows how close both sides of the Inequality (3) are. The function on the left-hand side of this inequality is plotted in bold line style and the function on the right-hand side is plotted in dashed line style. Notice that there is practically no difference for both sides of (3) if $\lambda \leq 0.3$.



(a) (b)

Figure 2: (a) Graphic comparisons for the integrand of (2); (b) graphic comparisons for (3).

Before applying (3) to the integrand of I_2 we remark that this integrand **is not** always positive for $\lambda > 0.5$ and $u \in (0,1)$. It changes sign at $u = \sqrt{\lambda^{-1}-1}$, and because of this we cannot directly use (3). Nevertheless, for $\lambda > 0.5$ we introduce the factor

$$S(\lambda) = \frac{1}{2}\left(\sqrt{\frac{\lambda}{1-\lambda}} - \lambda(3-4\lambda+4\lambda^2)\right),$$

which is just the difference from the left- to the right-hand side of (3), calculated at $u = \sqrt{\lambda^{-1}-1}$. Thus, we obtain:

$$\begin{aligned} \frac{1}{\sqrt{\lambda^{-2}-(1-u^2)^2}} &\geq \lambda + \frac{\lambda^3}{2}(1-u^2)^2 + S(\lambda), u \in [0, \sqrt{\lambda^{-1}-1}], \\ \frac{1}{\sqrt{\lambda^{-2}-(1-u^2)^2}} &\leq \lambda + \frac{\lambda^3}{2}(1-u^2)^2 + S(\lambda), u \in [\sqrt{\lambda^{-1}-1}, 1]. \end{aligned} \tag{4}$$

To prove Inequalities (4), one can verify that the U-derivative of the function $\frac{1}{\sqrt{\lambda^{-2}-U}} - \frac{\lambda^3}{2}U$ is always positive for every $(\lambda, U) \in (0,1)\times(0,1)$.

Now we are ready to use the above inequalities. From (3) it follows

$$I_2 > \sqrt{\lambda}\int_0^1 \frac{1-\lambda-\lambda u^2}{1-\lambda+\lambda u^2}(2+\lambda^2(1-u^2)^2)du = F(u)\Big|_{u=0}^{u=1}, \lambda \in (0, 0.5], \quad (5)$$

where

$$F(u) = \sqrt{\lambda}\left((\lambda^2-4)u + \frac{2\lambda u^3}{3} - \frac{\lambda^2 u^5}{5}\right) + 6\sqrt{1-\lambda}\,\arctan\frac{u}{\sqrt{\lambda^{-1}-1}}.$$

And from (4) it follows

$$I_2 > F(u)\Big|_{u=0}^{u=1} + S_2(\lambda), \lambda \in [0.5, 1), \quad \text{where} \quad (6)$$

$$S_2(\lambda) = \frac{2S(\lambda)}{\lambda}\left(-\sqrt{\lambda} + 2\sqrt{1-\lambda}\,\arctan\sqrt{\frac{\lambda}{1-\lambda}}\right).$$

As mentioned before, the factor $S(\lambda)$ is the difference between the left- and right-hand side of (3), calculated at $u = \sqrt{\lambda^{-1}-1}$. Therefore, $S(\lambda)$ is positive for every $u, \lambda \in (0,1)$. Consequently, $S_2(\lambda)$ is also positive for $\lambda \in (0.5, 0.8)$. This means that (5) is in fact valid for $\lambda \in (0, 0.8)$.

Although Inequality (6) gives us a better evaluation of I_2, the term $S_2(\lambda)$ is too complicated to deal with and numeric computations show that $S_2(\lambda) < 0.015$ for $\lambda \in (0.5, 0.8)$. For these reasons we are going to simplify our work and use just Inequality (5) for $\lambda \in (0, 0.8)$. At this point, we explicitly calculate the integral on the right-hand side of (5). By taking $F(1) - F(0)$ we reach:

$$I_2 > 2\sqrt{\lambda}(\frac{2\lambda^2}{5} + \frac{\lambda}{3} - 2) + 6\sqrt{1-\lambda}\,\arctan\sqrt{\frac{\lambda}{1-\lambda}}, \lambda \in (0, 0.8). \quad (7)$$

For the other integral I_1 we shall have a little bit more of work. Graphics show that the function in the integrand of (2) is mostly negative for $\lambda \cong 0$ and "almost" constant for $\lambda \cong 0.5$ (see Figure 2(a)). For this second region it is then reasonable to estimate the integrand of I_1 from above by its maximum in the interval (0,1). The following calculations show that just for a limited range of λ we have a maximum in (0,1). Therefore we split the analysis of

I_1 in two parts. To simplify matters, take $U = u^2$. Then we consider the derivative with respect to U:

$$\left[\frac{2\lambda - U}{(2-U)^{3/2}(1+\lambda-U)^{1/2}}\right]' = \frac{E(U)}{(2-U)^3(1+\lambda-U)} \tag{8}$$

where

$$\begin{aligned} &E(U) = -(2-U)^{3/2}(1+\lambda-U)^{1/2} + \\ &(\lambda - \frac{U}{2})\left[3(2-U)^{1/2}(1+\lambda-U)^{1/2} + \frac{(2-U)^{3/2}}{(1+\lambda-U)^{1/2}}\right]. \end{aligned} \tag{9}$$

For $E(U)$ to be zero on $U \in (0,1)$, we must have

$$U = \frac{9\lambda - 1 \pm (1-\lambda)\sqrt{33}}{4}.$$

But $U \in (0,1)$ implies $U = [9\lambda - 1 - (1-\lambda)\sqrt{33}]/4$, and in this case we must have

$$m := \frac{1+\sqrt{33}}{9+\sqrt{33}} < \lambda < M := \frac{5+\sqrt{33}}{9+\sqrt{33}}. \tag{10}$$

Then, the comparison with the maximum will be useful from $\lambda = m \cong 0.457$ to $\lambda = M \cong 0.729$. To see that the zero-derivative really corresponds to a maximum just calculate $E(U)$. We obtain

$$E(1) = \frac{1}{2\sqrt{\lambda}}(6\lambda^2 - 3\lambda - 1) \quad \text{and} \quad E(0) = \frac{2}{\sqrt{2+2\lambda}}(3\lambda^2 + 3\lambda - 2).$$

Then, the derivative in (8) will be negative at $U = 1$ whenever $E(1)$ is negative and this occurs for λ varying from 0 until $M = (3+\sqrt{33})/12$. The derivative in (8) will be positive at $U = 0$ whenever $E(0)$ is positive and this occurs for λ varying from $m = (\sqrt{33}-3)/6$ to 1. This means, for $\lambda \leq m$ the maximum of the integrand (2) in [0,1] is always at $U = 0$, where the function takes the value $\lambda/\sqrt{2+2\lambda}$. For λ inside the limits (10) we use the maximum value, namely

$$\left.\frac{2\lambda - U}{(2-U)^{3/2}(1+\lambda-U)^{1/2}}\right|_{U=\frac{9\lambda-1-(1-\lambda)\sqrt{33}}{4}} = \frac{A}{1-\lambda}, \quad \text{where} \tag{11}$$

$$A = \frac{4(1+\sqrt{33})}{(9+\sqrt{33})^{3/2}(5+\sqrt{33})^{1/2}} \cong 0.145.$$

Remark: The inequality $\dfrac{A}{1-\lambda} \geq \dfrac{\lambda}{\sqrt{2+2\lambda}}$ holds for every $\lambda \in (0,1)$.

Now we are ready to make the final comparisons. Our problem is reduced to the proof of the following two inequalities, with some eventual further restriction of the λ-intervals where they are supposed to hold. For this reason, we have made use of the above remark and settled $m/2$ as the lower value of the λ-range for one of the inequalities. We then reduce our demonstration of $I_2 > I_1$ to the proof of the following (recall (2), (7), (10) and (11)):

$$2\sqrt{\lambda}(\frac{2\lambda^2}{5}+\frac{\lambda}{3}-2)+6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > \frac{2\lambda}{\sqrt{2+2\lambda}},\ 0<\lambda\leq m,$$

and

$$2\sqrt{\lambda}(\frac{2\lambda^2}{5}+\frac{\lambda}{3}-2)+6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > \frac{2A}{1-\lambda},\ \frac{m}{2}<\lambda<M.$$

These inequalities involve transcendental functions. To simplify this matter, we are going to use the three following results, numbered (12), (13) and (14):

$$\frac{\lambda}{\sqrt{2+2\lambda}} < \frac{\lambda}{2}[(1-\sqrt{2})\lambda+\sqrt{2}] \text{ for } 0<\lambda<1. \tag{12}$$

Inequality (12) is equivalent to a 2^{nd}-degree polynomial inequality.

$$6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > \frac{2\sqrt{\lambda}(3-4\lambda)}{1-\lambda} \text{ for } 0<\lambda<0.5. \tag{13}$$

Inequality (13) comes from the Taylor series: $\arctan\tau=\tau-\tau^3/3+\tau^5/5-$ ..., which converges for $\tau\in(-1,1)$. Take $\tau=\sqrt{\lambda/(1-\lambda)}$ for $\lambda\in(0,0.5)$.

The following inequality considers $\lambda\in[0.4,0.65]$. This is because we do not expect $I_2>I_1$ to hold for $\lambda>0.65$ (see Figure 3).

$$6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > \sqrt{\lambda}(r-s\lambda) \text{ for } 0.4<\lambda<0.65, \tag{14}$$

where $s\cong 3.831$ and $r\cong 6.564$. Inequality (14) will be proved later.

By using (12), (13) and (14) we simplify our work and reduce once more the demonstration of $I_2>I_1$ to the proof of the following two inequalities:

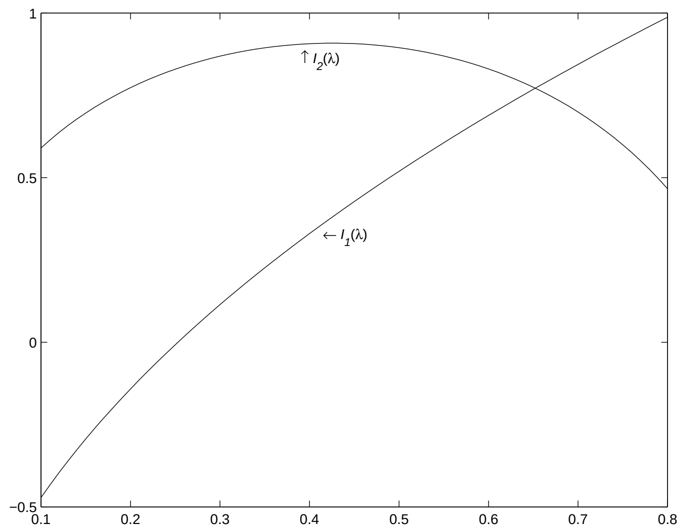


Figure 3: Numeric values for I_1 and I_2.

$$2(\frac{2\lambda^2}{5}+\frac{\lambda}{3}-2)+\frac{2(3-4\lambda)}{1-\lambda}>(1-\sqrt{2})\lambda^{3/2}+\sqrt{2}\lambda^{1/2},\ 0<\lambda\le m,$$

and

$$2\sqrt{\lambda}(\frac{2\lambda^2}{5}+\frac{\lambda}{3}-2)+\sqrt{\lambda}(r-s\lambda)>\frac{2A}{1-\lambda},\ \frac{m}{2}<\lambda\le M.$$

Rearranging the terms by decreasing powers of λ we get:

$\frac{4}{5}\lambda^3+(\sqrt{2}-1)\lambda^{5/2}-\frac{2}{15}\lambda^2+(1-2\sqrt{2})\lambda^{3/2}+\frac{10}{3}\lambda+\sqrt{2}\lambda^{1/2}-2<0, \lambda\in(0,m)$,
and
$\frac{4}{5}\lambda^{7/2}-(s+\frac{2}{15})\lambda^{5/2}+(r+s-\frac{14}{3})\lambda^{3/2}+(4-r)\lambda^{1/2}+2A<0, \lambda\in(m/2,M)$.

By making the substitution $\Lambda=\sqrt{\lambda}$ we get:

$$\frac{4}{5}\Lambda^6+(\sqrt{2}-1)\Lambda^5-\frac{2}{15}\Lambda^4+(1-2\sqrt{2})\Lambda^3+\frac{10}{3}\Lambda^2+\sqrt{2}\Lambda-2<0,\ 0<\Lambda\le\sqrt{m},$$

and

$$\frac{4}{5}\Lambda^7-(s+\frac{2}{15})\Lambda^5+(r+s-\frac{14}{3})\Lambda^3+(4-r)\Lambda+2A<0,\ \sqrt{m/2}<\Lambda<\sqrt{M}.$$

Thus, we have reduced our problem to the analysis of two polynomial inequalities in certain intervals. The first polynomial is negative at $\Lambda=0$, positive at $\Lambda=1$ and has positive derivative for $\Lambda\in(0,1)$. A way of proving this is to analyse separately the terms $\frac{4}{5}\Lambda^6 - \frac{2}{15}\Lambda^4$ and $(1-2\sqrt{2})\Lambda^3 + \frac{10}{3}\Lambda^2 + \sqrt{2}\Lambda$. The positive derivative implies that the polynomial has a unique root Λ_0 in the interval $(0,1)$, and whenever we calculate the value of the polynomial at a point Λ, we can say if $\Lambda \geq \Lambda_0$ or $\Lambda \leq \Lambda_0$. At $\Lambda = 0.66^2 = 0.4356$ the polynomial takes a negative value (approximately -0.05), but at $\Lambda = \sqrt{m}$ it takes a positive one (approximately 0.023). Although this value is smaller than m we can still make use of the second inequality. In any case, the only conclusion we can take for now is that $I_1 < I_2$ for $\lambda \in (0, 0.435]$.

The second polynomial is negative at $\Lambda = 0.5$, positive at $\Lambda = 1$ and its derivative is always positive in the interval $[0.5, 1]$. A way of proving this is to observe that the first derivative is a 3^{rd}-degree polynomial in Λ^2, which is clearly negative for $\Lambda \leq 0$, and has a unique real root. Since the derivative is positive for $\Lambda = 0.5$, it will then be positive for any $\Lambda \in [0.5, \infty)$. Once again, this implies that the second polynomial has a unique root Λ_0 in the interval $[0.5, 1]$ and whenever we calculate the value of the polynomial at a point Λ we can say if $\Lambda \geq \Lambda_0$ or $\Lambda \leq \Lambda_0$. At $\Lambda_0 = \sqrt{0.6} = 0.7746$ the polynomial takes a negative value, which is approximately -0.005. Then, the inequality holds for $0.5 < \Lambda \leq 0.7746$, or equivalently, for $0.25 < \lambda \leq 0.6$. Our final conclusion is that $I_1 < I_2$ for $\lambda \in (0, 0.6 + \varepsilon)$, for some $\varepsilon > 0$. We show in the next section that $\varepsilon < 0.2$. Graphic comparisons indicate that ε is small, namely $\varepsilon \cong 0.05$ (see Figure 3).

To complete our proof, we outline the arguments for Inequality (14). We want to analyse what happens for $\lambda \in [0.4, 0.65]$. So take the limit values $k = \sqrt{0.4/(1-0.4)} \cong 0.817$ and $K = \sqrt{0.65/(1-0.65)} \cong 1.363$. Then, for $\kappa \in [k, K]$ we have

$$\frac{\arctan \kappa}{\kappa} \leq p(\kappa - k) + q \tag{15}$$

where $p = (\kappa^{-1} \arctan \kappa)'|_{\kappa=k} \cong -0.292$ and $q = (\kappa^{-1} \arctan \kappa)|_{\kappa=k} \cong 0.839$. To prove (15) just take the derivatives of $\arctan \kappa$ and $\kappa(p(\kappa - k) + q)$ and compare them for $\kappa \in [k, K]$.

Now consider $\kappa = \sqrt{\lambda/(1-\lambda)}$, $\lambda \in [0.4, 0.65]$. We can write our primary inequality as follows:

$$6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > 6\sqrt{\lambda}\left[p(\sqrt{\frac{\lambda}{1-\lambda}} - k) + q\right] =$$

$$6|p|\sqrt{\lambda}\left(-\sqrt{\frac{\lambda}{1-\lambda}} + k + q/|p|\right).$$

Now we take a linear approximation of $\sqrt{\lambda/(1-\lambda)}$ for the required λ-interval. It is

$$\sqrt{\frac{\lambda}{1-\lambda}} \leq (b-a)\frac{\lambda-0.4}{0.65-0.4}+a$$

where $a = \sqrt{\lambda/(1-\lambda)}\Big|_{\lambda=0.4} \cong 0.817$ and $b = \sqrt{\lambda/(1-\lambda)}\Big|_{\lambda=0.65} \cong 1.363$.

This last inequality is easy to be proved: one reaches a 3$^{\text{rd}}$-degree polynomial equation with negative 2$^{\text{nd}}$ derivative. Since the functions on both sides agree at the extremes, the inequality (in this case) must hold in the whole interval. By joining the last two inequalities we finally get

$$6\sqrt{1-\lambda}\arctan\sqrt{\frac{\lambda}{1-\lambda}} > \sqrt{\lambda}(r-s\lambda), \lambda \in [0.4, 0.65],$$

where $s = 24|p|(b-a) \cong 3.831$ and $r = 6|p|(-1.6(a-b)-a+k+q/|p|) \cong 6.564$.

Step III. The solution of is unique.

This last step proves monotonicity and this is important in the study of limit members of a continuous family of surfaces.

We find a y_λ such that $I\!I_1(\lambda, y_\lambda) = I\!I_2(\lambda, y_\lambda)$, where

$$I\!I_1(\lambda, y) := \int_0^1 \frac{(t+y)dt}{\sqrt{(1-t^2)(t^2+Xt+1)(t+\lambda)}},$$

and

$$I\!I_2(\lambda, y) := \int_0^\lambda \frac{(t-y)dt}{\sqrt{(1-t^2)(t^2-Xt+1)(\lambda-t)}}.$$

The variable X is defined in (4) with $y \in (2\lambda-1, \lambda)$. For every fixed $\lambda \in (0,1)$, denote X_y the derivative of X with respect to y:

$$X_y = 2\left(1-\frac{y}{\lambda}\right). \tag{16}$$

Because of $y < \lambda$, from (16) it follows that $X_y > 0$. On one hand, for any fixed $t \in (0,1)$ we can prove that the function

$$\frac{t+y}{\sqrt{t^2+Xt+1}} \tag{17}$$

is increasing if y varies from $2\lambda - 1$ to λ. On the other hand, for any fixed $t \in (0, \lambda)$ we can prove that the function

$$\frac{t - y}{\sqrt{t^2 - Xt + 1}} \tag{18}$$

is decreasing if y varies from $2\lambda - 1$ to λ. Because of that, the integral values II_1 and II_2 are increasing and decreasing, respectively, for this variation of y. Then, we conclude that the solution of the period problem is unique. To prove what was asserted about the function at (17), just verify that any of the following steps is a consequence of the previous one.

(a) The y-derivative of $\dfrac{t + y}{\sqrt{t^2 + Xt + 1}}$ is positive;

(b) $t^2 + Xt + 1 > \left(1 - \frac{y}{\lambda}\right) t(t + y), 0 < t < 1;$

(c) $X + 2 > \left(1 - \frac{y}{\lambda}\right)(1 + y);$

(d) $\frac{1}{\lambda} + y > -1 - \frac{y}{\lambda};$

(e) $(2\lambda - 1)(1 + \lambda) > -1 - \lambda;$

(f) $0 < \lambda$.

To prove what was asserted about the function at (18), just verify that $t^2 - Xt + 1 > \left(1 - \frac{y}{\lambda}\right) t(t - y), 0 < t < \lambda$. This implies that the y-derivative of $\dfrac{t - y}{\sqrt{t^2 - Xt + 1}}$ is positive.

q.e.d.

The next section is devoted to the refinement of Proposition 2.1, in the sense that we find an upper bound for λ. This will imply that $\varepsilon < 0.2$.

3. Second Approach

At Step III of the above demonstration we showed that, for every fixed $\lambda \in (0, 1)$, the integral values $II_1(\lambda, y)$ and $II_2(\lambda, y)$ are increasing and decreasing with y, respectively. Therefore,

$$II_2(\lambda, y) < II_2(\lambda, 2\lambda - 1) \text{ and } II_1(\lambda, 2\lambda - 1) < II_1(\lambda, y).$$

To simplify notations, let us define $I_j := II_j(\lambda, 2\lambda - 1), j = 1, 2$. Hence, the non-existence of solutions will be proved if we show that $I_2 < I_1$ for every

$\lambda > 0.8$. The integrals I_1 and I_2 can be written as follows:

$$I_1 = 2\int_0^1 \frac{(2\lambda - u^2)du}{(2-u^2)^{3/2}(1+\lambda-u^2)^{1/2}} \tag{19}$$

and

$$I_2 = \int_0^\lambda \frac{(t-2\lambda+1)dt}{(1-t)\sqrt{(1-t^2)(\lambda-t)}} = \frac{2}{\sqrt{\lambda}}\int_0^1 \frac{(1-\lambda-\lambda u^2)du}{(1-\lambda+\lambda u^2)\sqrt{\lambda^{-2}-(1-u^2)^2}}. \tag{20}$$

We shall make use of the following proposition, which will be proved in the next section:

Proposition 3.1. *The integral $I_1(\lambda)$ is increasing with λ. For $\lambda > \frac{1}{2}$, the integral $I_2(\lambda)$ is decreasing with λ.*

We want to show that $II_1(\lambda, y)$ and $II_2(\lambda, y)$ cannot be equal for $\lambda > 0.8$. By using Proposition 3.1, it is sufficient to prove that $I_2(0.8) < I_1(0.8)$. To do this, we must work with inequalities valid for a smaller value of λ. Our analysis considers $\lambda > 0.6$, hence $2\lambda - u^2 > 0$ for $u \in (0,1)$. Because of this, for the integrand of I_1 in (19) we use the following inequality, which is valid for any positive reals a and b with $a > b$:

$$\frac{1}{\sqrt{a-b}} > \frac{1}{\sqrt{a}} + \frac{b}{2a^{3/2}}. \tag{21}$$

By applying (21) to (19) we get:

$$\frac{(2\lambda-u^2)}{(2-u^2)\sqrt{2-u^2}\sqrt{1+\lambda-u^2}} > \frac{2\lambda-u^2}{2-u^2}\left(\frac{1}{\sqrt{2}}+\frac{u^2}{4\sqrt{2}}\right)\left(\frac{1}{\sqrt{1+\lambda}}+\frac{u^2}{2(1+\lambda)^{3/2}}\right)$$

$$= \frac{1}{8\sqrt{2}(1+\lambda)^{3/2}}\left(u^4+8u^2-4(\lambda^2+\lambda-6)+\frac{24(\lambda^2+\lambda-2)}{2-u^2}\right).$$

Therefore,

$$I_1 > \frac{43/15 - 4(\lambda^2+\lambda-6)}{4\sqrt{2}(1+\lambda)^{3/2}} + 3\ln(1+\sqrt{2})\frac{\lambda^2+\lambda-2}{(1+\lambda)^{3/2}}. \tag{22}$$

For the integrand of I_2 in (20) we use the following two inequalities:

$$\frac{1-\lambda-\lambda u^2}{1-\lambda+\lambda u^2} < \frac{1-\lambda-\lambda u^2}{1-\lambda}, u \in (0, \sqrt{\lambda^{-1}-1}), \lambda \in (\frac{1}{2}, 1), \tag{23}$$

and

$$\frac{1-\lambda-\lambda u^2}{1-\lambda+\lambda u^2} < 1-\lambda-\lambda u^2, u\in(\sqrt{\lambda^{-1}-1},1), \lambda\in(\frac{1}{2},1). \tag{24}$$

Now we apply (23) and (24) to I_2 and obtain

$$I_2 < \frac{2}{\sqrt{\lambda}}\int_0^1 \frac{du}{\sqrt{\lambda^{-2}-(1-u^2)^2}} - 2\sqrt{\lambda}\int_{\sqrt{\lambda^{-1}-1}}^1 \frac{du}{\sqrt{\lambda^{-2}-(1-u^2)^2}}$$

$$-\frac{2\sqrt{\lambda}}{1-\lambda}\int_0^{\sqrt{\lambda^{-1}-1}} \frac{u^2du}{\sqrt{\lambda^{-2}-(1-u^2)^2}} - 2\sqrt{\lambda}\int_{\sqrt{\lambda^{-1}-1}}^1 \frac{u^2du}{\sqrt{\lambda^{-2}-(1-u^2)^2}}.$$

On one hand,

$$\int_0^1 \frac{du}{\sqrt{\lambda^{-2}-(1-u^2)^2}} < \int_0^1 \frac{du}{\sqrt{\lambda^{-2}-(1-u^2)}} = \text{Asinh}\frac{\lambda}{\sqrt{1-\lambda^2}}. \tag{25}$$

On the other hand, for the negative coefficients we use (21) and get

$$\int \frac{u^2du}{\sqrt{\lambda^{-2}-(1-u^2)^2}} > \int\left(\lambda+\frac{\lambda^3}{2}(1-u^2)^2\right)u^2du = F_1(u) \tag{26}$$

and

$$\int \frac{du}{\sqrt{\lambda^{-2}-(1-u^2)^2}} > \int\left(\lambda+\frac{\lambda^3}{2}(1-u^2)^2\right)du = F_2(u), \tag{27}$$

where

$$F_1(u) = \lambda u^3\left[\frac{1}{3}+\frac{\lambda^2}{2}\left(\frac{1}{3}-\frac{2u^2}{5}+\frac{u^4}{7}\right)\right], \quad \text{and}$$

$$F_2(u) = \lambda u\left[1+\frac{\lambda^2}{2}\left(1-\frac{2u^2}{3}+\frac{u^4}{5}\right)\right].$$

Now we apply (25), (26) and (27) and obtain

$$I_2 < \frac{2}{\sqrt{\lambda}}\text{Asinh}\frac{\lambda}{\sqrt{1-\lambda^2}} - 2\sqrt{\lambda}(F_1(u)+F_2(u))\Big|_{\sqrt{\lambda^{-1}-1}}^1 - \frac{2\sqrt{\lambda}}{1-\lambda}F_1(\sqrt{\lambda^{-1}-1}). \tag{28}$$

We apply the value $\lambda = 0.8$ to (22) and (28) and get the following inequality: $I_2(0.8) < 0.9 < I_1(0.8)$.

q.e.d.

4. Enhancements

We have already proved the existence of an $\varepsilon > 0$ such that the equation $\mathit{\Pi}_1(\lambda, y) = \mathit{\Pi}_2(\lambda, y)$ is uniquely solvable for every $\lambda \in (0; 0.6 + \varepsilon)$. From the last section, it is clear that $\varepsilon < 0.8 - 0.6 = 0.2$. Moreover, Proposition 3.1 allows us to conclude the following: if ε_0 is the biggest value of ε, then the equation $\mathit{\Pi}_1(\lambda, y) = \mathit{\Pi}_2(\lambda, y)$ is unsolvable for every $\lambda > 0.6 + \varepsilon_0$. In other words, the family of solutions is unique in the sense that there are no other subintervals of (0,1), except $(0; 0.6+\varepsilon_0)$, in which one can find solutions.

This section is devoted to the proof of Proposition 3.1. From (19), it is easy to calculate the derivative $\frac{dI_1}{d\lambda}$ and verify that it is positive, for every $u, \lambda \in (0, 1)$. For the integral I_2, it is difficult to differentiate (20) with respect to λ. We could try to differentiate the original formula of I_2, namely:

$$I_2 = \int_0^{\lambda} \frac{(t - 2\lambda + 1)dt}{(1 - t)\sqrt{(1 - t^2)(\lambda - t)}}. \tag{29}$$

Nevertheless, it leads to a difference of two infinite terms. To overcome this difficulty, we are going to split up the integration interval of (29) in two parts. We recall that our analysis is been done for $\lambda > 0.5$. The two integration intervals will be $(0, 2\lambda - 1)$ and $(2\lambda - 1, 1)$. Notice that $2\lambda - 1$ is the value at which the integrand vanishes. Hence:

$$I_2 = \int_0^{2\lambda - 1} \frac{(t - 2\lambda + 1)dt}{(1 - t)\sqrt{(1 - t^2)(\lambda - t)}} + \int_{2\lambda - 1}^{\lambda} \frac{(t - 2\lambda + 1)dt}{(1 - t)\sqrt{(1 - t^2)(\lambda - t)}}. \tag{30}$$

For the second integral term of (30) we use the substitution $t = \lambda - v$ and obtain

$$I_2 = \int_0^{2\lambda - 1} \frac{(t - 2\lambda + 1)dt}{(1 - t)\sqrt{(1 - t^2)(\lambda - t)}} + \int_0^{1 - \lambda} \frac{(1 - \lambda - v)dv}{(1 - \lambda + v)\sqrt{(1 - (\lambda - v)^2)v}}. \tag{31}$$

Now define

$$f_1(t, \lambda) := \frac{(t - 2\lambda + 1)}{(1 - t)\sqrt{(1 - t^2)(\lambda - t)}}, \forall (t, \lambda) \in (0, 2\lambda - 1) \times (\frac{1}{2}, 1) \quad \text{and}$$

$$f_2(v, \lambda) := \frac{(1 - \lambda - v)}{(1 - \lambda + v)\sqrt{(1 - (\lambda - v)^2)v}}, \forall (v, \lambda) \in (0, 1 - \lambda) \times (\frac{1}{2}, 1).$$

The functions f_1 and f_2 are decreasing with respect to λ. An easy way to conclude it for f_2 is taking the λ-derivative of f_2^2. We reach the following expression for $\frac{dI_2}{d\lambda}$:

$$\frac{dI_2}{d\lambda} = \int_0^{2\lambda-1} \frac{\partial f_1}{\partial \lambda} dt + 2f_1(2\lambda-1,\lambda) + \int_0^{1-\lambda} \frac{\partial f_2}{\partial \lambda} dv - f_2(1-\lambda,\lambda). \qquad (32)$$

Since $f_1(2\lambda-1,\lambda) = 0$ and $f_2(1-\lambda,\lambda) = 0$, we have $\frac{dI_2}{d\lambda} < 0$.

q.e.d.

References

[**1**] W.H. Meeks III and H. Rosenberg, *The geometry of periodic minimal surfaces*, Comment. Math. Helv. (4), **68** (1993), 538-578.

[**2**] R. Osserman, *A survey of minimal surfaces*, Second edition, Dover, New York (1986).

[**3**] V. Ramos Batista, *Construction of new complete minimal surfaces in* $\mathbb{R}^3$ *based on the Costa surface*, Doctoral thesis, University of Bonn (2000).