A strong law of large numbers and a central limit theorem for fuzzy random variables

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Abstract

In this paper we give a strong law of large numbers and a central limit theorem for fuzzy random variables. To do this, we use the embedding of the space of compact fuzzy sets with continuous levels applications into a Banach space, via support functions.

Key words: fuzzy sets with continuous levels, fuzzy random variables, Minkowski embedding theorem, support functions, strong law of large numbers, central limit theorem.

1 Introduction

In many real situations uncertainty of data come from two sources: from the random mechanism generating it and from the vagueness of outcomes. Fuzzy random variables, introduced by [13], are suitable tools for modelling such situations. Fuzzy random variables (frv's) generalize the concept of random variables as well as that of random sets.

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To make this tool useful for statistical analysis of inexact data, several authors have extended some classical results on random variables to frv's. Of particular concern are those describing the asymptotic behaviour of suitably normalized sums of frv's. In this setting, the classical strong law of large numbers (SLLN) has been extended by [3] for compact random sets and by [7] for frv's. Although, to our knowledge, the result in [7] is the most general SLLN for frv's, here we give a SLLN which is a weaker version of that in [7]. The reason is that our proof is much simpler. It basically follows the same lines that the one in [3] for compact random sets.

Another classical result of great importance is the central limit theorem (CLT). [18] has proved a CLT for compact random sets. Extensions for frv's can be found in [9] and [11]. In this paper we prove a CLT that extends that in [9] by assuming different conditions than those in [11].

The CLT in this paper extend that in [9] in two important aspects: while these authors work with the space of convex compact fuzzy sets with Lipschitzian levels, which is a separable space but it is not complete (see [14]), our purpose is to show that it suffices to work with compact fuzzy sets with continuous levels to get CLT for frv's.

We want also to remark that the condition we assume on the frv's to get our CLT, the continuity of the level sets, is different from that assumed in [11], the convexity of the level sets. None of these assumptions entail the other. The difference in the assumed conditions is due to the way each result face the non-separability of involved metric spaces of fuzzy sets. To solve this difficulty, in [11] the authors identify isometrically each frv with convex and compact level sets with an empirical process and then apply the results in [17]. Our approach is similar to that used in [18].

Within the arguments used in the papers [3] and [18], there are two which turn out to be fundamental: the Theorem of Shapley and Folkman, that we generalize to the fuzzy context, and the results in Mourier [10], that we apply directly by using the Minkowski embedding Theorem given in [14]. This embedding Theorem is our main tool.

We have also extended the results stated here to the context of separable Banach spaces, by using a generalization of the Minkowski embedding Theorem in [14] given in [15]. These extensions will be published in a forthcoming paper [6].

The paper is organized as follows. In Section 2, after giving some definitions a basic results on fuzzy sets, we discuss some useful consequences of the embedding Theorem in [14] and we also deal with frv's, their expectations and their support functions. The main results are exposed in last Section.

2 Preliminaries

2.1 Fuzzy sets

Let $\mathcal{K}(\mathbb{R}^m)$ and $\mathcal{K}_c(\mathbb{R}^m)$ denote the set of the nonempty compact subsets of $\mathbb{R}^m$ and the set of the nonempty compact convex subsets of $\mathbb{R}^m$, respectively. The Hausdorff metric H over $\mathcal{K}(\mathbb{R}^m)$ is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

A linear structure is defined in $\mathcal{K}(\mathbb{R}^m)$ by the operations

$$A + B = \{a + b \ / \ a \in A, b \in B\} \ \text{ and } \ \lambda A = \{\lambda a \ / \ a \in A\},$$

where $A, B \in \mathcal{K}(\mathbb{R}^m)$, $\lambda \in \mathbb{R}$.

Let $\mathbb{F}^m$ be the space of fuzzy compact sets, that is, $\mathbb{F}^m$ is the set of $u : \mathbb{R}^m \to [0,1]$ with the following properties

(i) u is normal, i.e., there exists $x_0 \in \mathbb{R}^m$ such that $u(x_0) = 1$,
(ii) u is upper semicontinuous, and
(iii) $[u]^0 = \text{ supp } (u) = \overline{\{x \in \mathbb{R}^m \ / \ u(x) > 0\}} \in \mathcal{K}(\mathbb{R}^m)$.

For each $0 < \alpha \le 1$, let $[u]^\alpha = \{x \ / \ u(x) \ge \alpha\}$ denote the α-level set of u. From (i)-(iii), it follows that $[u]^\alpha \in \mathcal{K}(\mathbb{R}^m)$, $\forall\, \alpha \in [0,1]$.

Let $\mathbb{F}^m_k = \{u \in \mathbb{F}^m \ / \ [u]^\alpha \in \mathcal{K}_c(\mathbb{R}^m),\ \forall \alpha \in [0,1]\}$. For any $u \in \mathbb{F}^m_k$, the support function of u, $S_u(\cdot,\cdot) : [0,1] \times S^{m-1} \to \mathbb{R}$, is defined by

$$S_u(y,\alpha) = \sigma_{[u]^\alpha}(y),$$

where $\sigma_A(y) = \sup_{a\in A} \langle y, a\rangle$ is the support function of the set $A \subset \mathbb{R}^m$, $\langle\cdot,\cdot\rangle$ denotes usual inner product in $\mathbb{R}^m$, $S^{m-1} = \{x \in \mathbb{R}^m \ / \ \|x\| = 1\}$ and $\|.\|$ is the Euclidean norm.

The linear structure in $\mathbb{F}^m$ is defined by the operations

$$(u+v)(x) = \sup_{y\in X} \min\{u(y), v(x-y)\}, \quad (\lambda u)(x) = \begin{cases} u(x\lambda^{-1}) & \text{if } \lambda \neq 0, \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in \mathbb{F}^m$, $\lambda \in \mathbb{R}$ and χ_A denotes the characteristic function of $A \subseteq \mathbb{R}^m$. Note that $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall\, u, v \in \mathbb{F}^m$, $\forall\, \alpha \in [0,1]$, $\forall\, \lambda \in \mathbb{R}$.

We can endow $\mathbb{F}^m$ with several metrics. Some usual distances between fuzzy sets are

$$D_p(u,v) = \begin{cases} \left(\int_0^1 H([u]^\alpha, [v]^\alpha)^p d\alpha \right)^{1/p} & \text{if } p \in [1, \infty), \\ \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha) & \text{if } p = \infty. \end{cases}$$

With each distance, we can also define the norm of a fuzzy set u by

$$\|u\|_p = D_p(\chi_{\{0\}}, u).$$

It is well known (see for example [8]) that the metric space $(\mathbb{F}^m, D_p)$ is complete for each $1 \le p \le \infty$, and that $(\mathbb{F}^m, D_p)$ is separable for each $1 \le p < \infty$, but $(\mathbb{F}^m, D_\infty)$ is not.

We say that the fuzzy set u is *level continuous* if the mapping $\alpha \to [u]^\alpha$ is H-continuous, that is, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\alpha - \beta| < \delta \Rightarrow H([u]^\alpha, [u]^\beta) < \epsilon.$$

We say that u is *Lipschitz* if there is a constant $L \ge 0$, such that

$$H([u]^\alpha, [u]^\beta) < L\, |\alpha - \beta|$$

$\forall\, \alpha, \beta \in [0,1]$.

From now on we will work with the metric D_∞. To simplify notation, we will suppress the subindex $_\infty$ in the distance and in the induced norm. Note that this way we use the same notation for the Euclidean norm of a point $x \in \mathbb{R}^m$ and for the norm of a fuzzy set.

In this paper we will consider the following subspaces of $(\mathbb{F}^m, D)$

$\mathbb{F}^m_k = \{u \in \mathbb{F}^m \;/\; [u]^\alpha \in \mathcal{K}_c(\mathbb{R}^m)\ \forall \alpha \in [0,1]\}$,
$\mathbb{F}^m_c = \{u \in \mathbb{F}^m \;/\; u \text{ is level continuous}\}$,
$\mathbb{F}^m_{ck} = \mathbb{F}^m_k \cap \mathbb{F}^m_c$ and
$\mathbb{F}^m_{Lk} = \{u \in \mathbb{F}^m_k \;/\; u \text{ is Lipschitz}\}$.

Note that $\mathcal{K}_c(\mathbb{R}^m) \hookrightarrow \mathbb{F}^m_{Lk} \hookrightarrow \mathbb{F}^m_{ck}$.

2.2 The embedding Theorem

As we said in the introduction, the main tool for proving the our SLLN and CLT for frv's is the extension of the Minkowski embedding Theorem given in [14]. For completeness, we include here this Theorem and discuss some of its consequences. Before stating it, we first give some useful results.

For each $A \in \mathcal{K}(\mathbb{R}^m)$ we denote by $\mathrm{co}A$ the convex hull of A.

Proposition 2.1 *Let $u \in \mathbb{F}^m$. Then the family $(\mathrm{co}[u]^\alpha)_{\alpha\in[0,1]}$ satisfies*

(*a*) $\mathrm{co}[u]^\alpha \in \mathcal{K}(\mathbb{R}^m)$, $\forall\, \alpha \in [0,1]$,
(*b*) *if* $\alpha \leq \beta$ *then* $\mathrm{co}[u]^\alpha \supseteq \mathrm{co}[u]^\beta$, *and*
(*c*) *for all* $\alpha_1 \leq ... \leq \alpha_n, ...$ *such that* $\alpha_n \uparrow \alpha$, $\mathrm{co}[u]^\alpha = \bigcap_{n=1}^\infty \mathrm{co}[u]^{\alpha_n}$.

PROOF. The statements in (a) and (b) are immediate. To show (c), let $\alpha_1 \leq ... \leq \alpha_n, ...$ such that $\alpha_n \uparrow \alpha$. Then, $[u]^\alpha = \bigcap_{n=1}^\infty [u]^{\alpha_n}$ and hence $\mathrm{co}[u]^\alpha = \mathrm{co}\bigcap_{n=1}^\infty [u]^{\alpha_n} = \bigcap_{n=1}^\infty \mathrm{co}[u]^{\alpha_n}$. This completes the proof. □

From Proposition 2.1 is follows that the family $(\mathrm{co}[u]^\alpha)_{\alpha\in[0,1]}$ satisfies the conditions in the representation Theorem of Negoita and Ralescu (see [8]), and therefore there exists a fuzzy set, $\mathrm{co}(u)$, such that

$$[\mathrm{co}(u)]^\alpha = \mathrm{co}[u]^\alpha, \qquad \forall \alpha \in [0,1].$$

Note that if $u \in \mathbb{F}_c^m$, then $\mathrm{co}(u) \in \mathbb{F}_{ck}^m$, because $H(\mathrm{co}A, \mathrm{co}B) \leq H(A,B)$, $\forall\, A, B \in \mathcal{K}(\mathbb{R}^m)$.

Next we give another further useful immediate consequence of Proposition 2.1.

Corollary 2.2 *Let $u, v \in \mathbb{F}^m$, then $\mathrm{co}(u+v) = \mathrm{co}(u) + \mathrm{co}(v)$.*

Let $C([0,1]\times S^{m-1})$ denote the set of continuous real functions defined on $[0,1]\times S^{m-1}$ endowed with the usual metric d_∞,

$$d_\infty(f,g) = \max_{z\in[0,1]\times S^{m-1}} |f(z) - g(z)|\,,$$

and let $\|\cdot\|_\infty$ denote the associated norm.

Puri and Ralescu [12] showed that there is an embedding $j : \mathbb{F}_{Lk}^m \to C([0,1]\times S^{m-1})$. This fact is very important since $(\mathbb{F}_k^m, D)$ is not separable, which is an interesting property in integration theory, and $(\mathbb{F}^m_{Lk}, D)$ is. Unfortunately, $(\mathbb{F}^m_{Lk}, D)$ is not a complete subspace of $(\mathbb{F}^m_k, D)$.

Since $\mathbb{F}^m_{Lk} \hookrightarrow \mathbb{F}^m_{ck}$ and $(\mathbb{F}^m_{ck}, D)$ is a closed subspace of $(\mathbb{F}^m_k, D)$ and therefore complete, the question that arises is if the application j can be extended to the class $\mathbb{F}^m_{ck}$. The answer to this question is affirmative and was given in [14] as follows.

Theorem 2.3 *The application* $j : \mathbb{F}^m_{ck} \to C([0,1]\times S^{m-1})$ *defined by* $j(u) = S_u$ *is positively homogeneous, additive and it is also an isometry.*

As an immediate consequence of Theorem 2.3, we have that the metric space $(\mathbb{F}^m_{ck}, D)$ is separable. In [14] the authors also show that $\mathbb{F}^m_{ck}$ is the maximal complete and separable subspace of $\mathbb{F}^m$ that can be embedded in $C([0,1] \times S^{m-1})$ via the isometry j.

Let $u, v \in \mathbb{F}^m_{ck}$, from Theorem 2.3 it follows that

$$\begin{aligned} D(u,v) &= \sup\{|S_u(\alpha,z) - S_v(\alpha,z)| \ / \ (\alpha,z) \in [0,1] \times S^{m-1}\} \\ &= \|S_u - S_v\|_\infty \,, \end{aligned}$$

and hence,

$$\|u\| = \sup\{|S_u(\alpha,z)| \ / \ (\alpha,z) \in [0,1] \times S^{m-1}\} = \|S_u\|_\infty \,.$$

2.3 Fuzzy random variables

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A frv X is a Borel measurable function, $X : \Omega \to \mathbb{F}^m$, in the sense that $X^{-1}(\mathcal{S}) \subseteq \mathcal{A}$, where $\mathcal{S}$ is the σ-field generated by the open sets of the metric space $(\mathbb{F}^m, D)$, that is, X is a random element in $\mathbb{F}^m$. Note that if X is a frv, then $\|X\|$ is a random variable.

The expectation of X is the fuzzy set $\mathbb{E}X$ whose level sets satisfy

$$[\mathbb{E}X]^\alpha = \mathbb{E}X_\alpha$$

where $X_\alpha : \Omega \to \mathcal{K}(\mathbb{R}^m)$, defined by $X_\alpha(w) = [X(w)]^\alpha$, is a random compact set and $\mathbb{E}X_\alpha$ is the Aumann expectation,

$$\mathbb{E}X_\alpha = \{EZ \ / \ Z \in L_1(\Omega, \mathcal{A}, P) \text{ and } Z(w) \in X_\alpha(w) \text{ a.s.}\},$$

where EZ is the expectation of the random vector Z. Each random vector Z in the definition of $\mathbb{E}X_\alpha$ is called a selection of X_α. Aumann [4] showed that such selections exist, under some conditions on X_α.

As a consequence of the results in Aumann (see also [16]) we have that if P is nonatomic, then $\mathbb{E}X = \mathbb{E}(\text{co}X)$.

Note that if X is a random vector, then $\mathbb{E}X = EX$. To simplify notation, from now on we will use the same symbol $\mathbb{E}$ to denote both expectations, the expectation of a frv and the expectation of a random vector.

If X is a frv in $\mathbb{F}^m_{ck}$, then

$$S_X(\alpha, z) = \sigma_{[X]^\alpha}(z)$$

is a stochastic process in $C([0,1] \times S^{m-1})$ with continuous parameter $(\alpha, z) \in [0,1] \times S^{m-1}$. From Theorem 2.3, the correspondence between X and $S_X(\cdot,\cdot)$ is isometric. This isometry allow us to define concepts for the frv X by using the corresponding concept for the random functions $S_X(\alpha, z)$. This way, the concept of independence for frv's follows from the well defined independence of random functions.

3 Main results

As we said in the introduction, to prove our SLLN and CLT for frv's we need, in addition of Theorem 2.3, to generalize to the fuzzy context the Theorem of Shapley and Folkman (see [2]). So, before stating our main results, we give this generalization.

Proposition 3.1 *Let $u_1, u_2, ..., u_n \in \mathbb{F}^m$ with $\|u_i\| < M$, $1 \le i \le n$, for some positive finite constant $M \in \mathbb{R}$. If $a_n \in \mathbb{R}$, $a_n > 0$, then*

$$D\left(\sum_{i=1}^{n} \frac{u_i}{a_n}, \text{co}\left(\sum_{i=1}^{n} \frac{u_i}{a_n}\right)\right) \le a_n^{-1}\sqrt{m}M, \qquad \forall n.$$

In particular, if $a_n^{-1} \to 0$, then

$$\lim_{n\to\infty} D\left(\sum_{i=1}^{n} \frac{u_i}{a_n}, \text{co}\left(\sum_{i=1}^{n} \frac{u_i}{a_n}\right)\right) = 0.$$

PROOF. Since $[u_i]^\alpha \in \mathcal{K}(\mathbb{R}^m)$ and $\|[u_i]^\alpha\| \le \|u_i\| < M$, $i = 1, 2, ...n$, from the Proposition in [3] we have that

$$H\left(\sum_{i=1}^{n}\frac{[u_i]^\alpha}{a_n}, \mathrm{co}\left(\sum_{i=1}^{n}\frac{[u_i]^\alpha}{a_n}\right)\right) \le a_n^{-1} H\left(\sum_{i=1}^{n}[u_i]^\alpha, \mathrm{co}\left(\sum_{i=1}^{n}[u_i]^\alpha\right)\right)$$
$$\le a_n^{-1}\sqrt{m}M,$$

and hence

$$D\left(\sum_{i=1}^{n}\frac{u_i}{a_n}, \mathrm{co}\left(\sum_{i=1}^{n}\frac{u_i}{a_n}\right)\right) \le a_n^{-1}\sqrt{m}M.$$

□

3.1 Strong law of large numbers

Theorem 3.2 *Let X_i, $i = 1, 2, ...$ be independent and identically distributed rfv's in $\mathbb{F}_c^m$ with $\mathbb{E}\|X_1\| < \infty$ and $T_n = X_1 + ... + X_n$. Then*

$$\lim_{n\to\infty} D\left(\frac{T_n}{n}, \mathbb{E}\mathrm{co}X_1\right) = 0 \quad a.s.$$

PROOF. Let $Y_i = \mathrm{co}X_i$, $1 \le i \le n$, and $R_n = Y_1 + ... + Y_n$. Since $\mathbb{E}\|\mathrm{co}X_1\| = \mathbb{E}\|Y_1\| = \mathbb{E}\|S_{Y_1}\|_\infty < \infty$, by the triangle inequality, we have

$$D\left(\frac{T_n}{n}, \mathbb{E}X_1\right) \le D\left(\frac{T_n}{n}, \frac{R_n}{n}\right) + D\left(\frac{R_n}{n}, \mathbb{E}Y_1\right). \tag{1}$$

From Proposition 3.1 and Lemma 1 in [5],

$$D\left(\frac{T_n}{n}, \frac{R_n}{n}\right) \le n^{-1}\sqrt{m}\max_{i\le n}\|X_i\| \to 0 \quad a.s., \tag{2}$$

as $n \to \infty$.

Since $\mathbb{E}S_{Y_i}(\alpha, z) = S_{\mathbb{E}Y_i}(\alpha, z)$, by using the the isometry between Y_i and its support process $S_{Y_i}(\alpha, z)$, we get

$$D\left(\frac{R_n(w)}{n}, \mathbb{E}Y_1\right) = \left\|S_{\frac{R_n(w)}{n}} - S_{\mathbb{E}Y_1}\right\|_\infty, \quad \forall\omega \in \Omega. \tag{3}$$

Now, from (3) and the SLLN in $C([0,1] \times S^{m-1})$ (see [10]),

$$D\left(\frac{R_n}{n}, \mathbb{E}Y_1\right) \to 0 \quad a.s., \tag{4}$$

as $n \to \infty$. Finally, the result follows from (1), (2) and (4). $\square$

3.2 Central limit theorem

For each frv X in $\mathbb{F}^m_{ck}$, let Ψ_X the covariance kernel of the stochastic process $S_X(\alpha, z)$,

$$\Psi_X\{(\alpha,z),(\beta,g)\} = \mathbb{E}\{S_X(\alpha,z) - S_{\mathbb{E}X}(\alpha,z)\}\{S_X(\beta,g) - S_{\mathbb{E}X}(\beta,g)\}.$$

Theorem 3.3 *Let X_i, $i = 1, 2, ...$ be independent and identically distributed frv's in $\mathbb{F}^m_c$ with $\mathbb{E}\,\|X_1\|^2 < \infty$ and $T_n = X_1 + ... + X_n$. Then*

$$\sqrt{n} D\left(\frac{T_n}{n}, \mathbb{E}\mathrm{co}X_1\right) \longrightarrow \|Z\|, \quad \textit{in distribution,}$$

as $n \to \infty$, where Z is a centered Gaussian variable in $C([0,1] \times S^{m-1})$ with covariance kernel $\Psi_{\mathrm{co}X_1}$.

PROOF. With the triangle inequality and following the notation in the proof of Theorem 3.2, we have

$$\sqrt{n} D\left(\frac{T_n}{n}, \mathbb{E}X_1\right) \le \sqrt{n} D\left(\frac{T_n}{n}, \frac{R_n}{n}\right) + \sqrt{n} D\left(\frac{R_n}{n}, \mathbb{E}Y_1\right). \tag{5}$$

From Proposition 3.1 and Lemma 1 in [5],

$$\sqrt{n} D\left(\frac{T_n}{n}, \frac{R_n}{n}\right) \le \sqrt{m}\left(\frac{\max_{1\le i\le n}\|X_i\|^2}{n}\right)^{1/2} \to 0 \quad a.s., \tag{6}$$

as $n \to \infty$.

From Theorem 2.3,

$$\sqrt{n} D\left(\frac{R_n}{n}, \mathbb{E}Y_1\right) = \sqrt{n}\left\|S_{\frac{R_n}{n}} - S_{\mathbb{E}Y_1}\right\|_\infty = \left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n (S_{Y_i} - \mathbb{E}S_{Y_1})\right\|_\infty. \tag{7}$$

Now, to apply the CLT in $C([0,1] \times S^{m-1})$ (Corollary 7.17 in [1]), we have to check two conditions. We first see that

$$\int_0^1 h^{\frac{1}{2}}(t)dt < \infty, \tag{8}$$

where

$$h(t) = \log N(t), \quad t > 0,$$

and $N(t)$ is the minimum number of spheres with radius t covering $[0,1]\times S^{m-1}$ with the metric e on $[0,1] \times S^{m-1}$, defined by

$$e\{(\alpha, z), (\beta, g)\} = |\alpha - \beta| + \|z - g\|\,.$$

Since a cube with side 2 can be covered with $(2k)^m$ cubes, all of them with side $1/k$, in the same way $[0,1] \times S^{m-1}$ can be covered with $(2k)^{m+1}$ spheres, each having radius $1/k$, and hence

$$N(t) \leq K_m t^{-1}, \quad t > 0$$

where K_m is a constant. Therefore, the integral (8) is finite.

Now we check the second condition,

$$|S_{Y_1}(\alpha, z) - S_{Y_1}(\beta, g)| \leq Me\{(\alpha, z), (\beta, g)\}.$$

where M is a nonnegative random variable having finite second moment. This condition holds since

$$|S_{Y_1}(\alpha, z) - S_{Y_1}(\beta, g)| = |S_{Y_1}\{(\alpha, z) - (\beta, g)\}| \leq \|S_{Y_1}\|\, e\{(\alpha, z), (\beta, g)\}.$$

Now, by the CLT in $C([0,1] \times S^{m-1})$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{S_{Y_i} - \mathbb{E}(S_{Y_1})\} \longrightarrow Z, \quad \text{in distribution}, \tag{9}$$

as $n \to \infty$, where Z is a centered Gaussian variable in $C([0,1] \times S^{m-1})$ with covariance kernel Ψ_{X_1}. Finally, as the mapping $(\alpha, z) \to \|(\alpha, z)\|$, $(\alpha, z) \in [0,1] \times S^{m-1}$, is continuous, the result follows from (5), (6), (7) and (9). □

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