FULLY SUMMING MULTILINEAR AND HOLOMORPHIC MAPPINGS INTO HILBERT SPACES

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Abstract. It is known that whenever E_1,...,E_n are infinite dimensional $\mathcal{L}_\infty$-spaces and F is any infinite dimensional Banach space, there exists a bounded n-linear mapping that fails to be absolutely $(1;2)$-summing. In this paper we obtain a sufficient condition in order to assure that a given n-linear mapping T from infinite dimensional $\mathcal{L}_\infty$-spaces into an infinite dimensional Hilbert space is absolutely $(1;2)$-summing. Besides, we also give a sufficient condition in order to obtain a fully $(1;1)$-summing multilinear mapping from $l_1 \times ... \times l_1 \times l_2$ into an infinite dimensional Hilbert space. In the last section we introduce the concept of fully summing holomorphic mappings and give the first examples of this kind of maps.

1. Introduction and notation

The search for a convenient multilinear version for the concept of absolutely summing operators lead to the investigation of innumerous different classes of multilinear mappings between Banach spaces. The first attempts in this direction were made by A. Pietsch [19] in 1983. Since then, several related classes of multilinear mappings and polynomials have been studied (we mention Botelho [3], Matos [8], Floret-Matos [5], Meléndez-Tonge [12] among many others).

Throughout $E_1, ..., E_n, E, F$ will stand for Banach spaces. The scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$.

The Banach spaces of all continuous n-linear mappings from $E_1 \times ... \times E_n$ into F endowed with sup norm will be denoted by $\mathcal{L}(E_1, ..., E_n;F)$. To denote the Banach space of all continuous n-homogeneous polynomials P from E into F with the sup norm we use $\mathcal{P}(^nE, F)$.

If $p > 0$, the linear space of all sequences $(x_j)_{j=1}^\infty$ in E such that

$$\|(x_j)_{j=1}^\infty\|_p = (\sum_{j=1}^\infty \|x_j\|^p)^{\frac{1}{p}} < \infty$$

will be denoted by $l_p(E)$. We will also denote by $l_p^w(E)$ the linear space formed by the sequences $(x_j)_{j=1}^\infty$ in E such that $(< \varphi, x_j >)_{j=1}^\infty \in l_p$ for every continuous linear functional $\varphi \in E'$. We define $\|.\|_{w,p}$ in $l_p^w(E)$ by

$$\|(x_j)_{j=1}^\infty\|_{w,p} = \sup_{\varphi \in B_{E'}} \|(< \varphi, x_j >)_{j=1}^\infty\|_p.$$

In the case $p = \infty$ we use the sup norm. One can see that $\|.\|_p$ ($\|.\|_{w,p}$) is a p-norm in $l_p(E)$($l_p^w(E)$) for $p < 1$ and a norm in $l_p(E)$($l_p^w(E)$) for $p \geq 1$. In any case, they are complete metrizable linear spaces.

One of the possible natural definitions of absolutely summing multilinear mapping is the following

Definition 1. *(Matos [10]) A continuous n-linear mapping $T : E_1 \times ... \times E_n \to F$ is* ***absolutely*** *$(p; q_1, ..., q_n)$**-summing** (or $(p; q_1, ..., q_n)$**-summing**)* ***at*** $(a_1, ..., a_n) \in E_1 \times$

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$... \times E_n$ if

$$(T(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - T(a_1, ..., a_n))_{j=1}^{\infty} \in l_p(F)$$

for every $(x_j^{(s)})_{j=1}^{\infty} \in l_{q_s}^w(E_s)$, $s = 1, ..., n$.

The space of all absolutely $(p; q_1, ..., q_n)$-summing n-linear mappings (at every point) from $E_1 \times ... \times E_n$ into F is denoted by $\mathcal{L}_{as(p;q_1,...,q_n)}^{ev}(E_1, ..., E_n; F)$. When $q_1 = ... = q_n = q$, we write $\mathcal{L}_{as(p;q)}^{ev}(E_1, ..., E_n; F)$. The development of this concept for polynomials, multilinear and analytic mappings can be found in [14] and some other important results will appear in Matos [10].

When we restrict ourselves to the origin, we have the following definition (characterization):

Definition 2. *(Matos [8]) A continuous multilinear mapping*

$$T : E_1 \times ... \times E_n \to F$$

*is **absolutely** $(p; q_1, ..., q_n)$**-summing** (or $(p; q_1, ..., q_n)$**-summing**) if there exists $C > 0$ such that*

$$(\sum_{j=1}^{\infty} \|T(x_j^{(1)}, ..., x_j^{(n)})\|^p)^{\frac{1}{p}} \leq C \prod_{r=1}^{n} \|(x_j^{(r)})_{j=1}^{\infty}\|_{w,q_r} \tag{1.1}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in l_{q_k}^w(E_k)$, $k = 1, ..., n$.

Henceforth we will denote the space of all absolutely $(p; q_1, ..., q_n)$-summing n-linear mappings (at the origin) from $E_1 \times ... \times E_n$ into F by $\mathcal{L}_{as(p;q_1,...,q_n)}(E_1, ..., E_n; F)$.

The infimum of the $C > 0$ for which inequality (1.1) always holds defines a norm for the space of all absolutely $(p; q_1, ..., q_n)$-summing multilinear mappings. This norm is denoted by $\|.\|_{as(p;q_1,...,q_n)}$. Under this norm, $\mathcal{L}_{as(p;q_1,...,q_n)}(E_1, ..., E_n; F)$ is a Banach space. When $q_1 = ... = q_n = q$, we write $\mathcal{L}_{as(p;q)}(E_1, ..., E_n; F)$, $\|.\|_{as(p;q)}$ and whenever $T \in \mathcal{L}_{as(p;q)}(E_1, ..., E_n; F)$ we say that T is absolutely $(p; q)$-summing. A particular and very important case is obtained when we deal with $\mathcal{L}_{as(\frac{p}{n};p,...,p)}(E_1, ..., E_n; F)$. If $T \in \mathcal{L}_{as(\frac{p}{n};p,...,p)}(E_1, ..., E_n; F)$ we say that T is p-dominated. This terminology was introduced by Matos [8] and is motivated by a multilinear version of the Grothendieck-Pietsch Domination Theorem for p-dominated mappings.

Another natural generalization of the linear concept of absolutely summing operator is the following

Definition 3. *(Matos [9]) A continuous n-linear mapping $T : E_1 \times ... \times E_n \to F$ is said to be **fully** $(p; q_1, ..., q_n)$**-summing** if there exists $C \geq 0$ such that*

$$\left(\sum_{j_1,...,j_n=1}^{\infty} \left\| T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{r=1}^{n} \|(x_j^{(r)})_{j=1}^{\infty}\|_{w,q_r} \tag{1.2}$$

whenever $(x_k^{(l)})_{k=1}^{\infty} \in l_{q_l}^w(E_l), l = 1, ..., n$. In this case we will write

$$T \in \mathcal{L}_{fas(p;q_1,...,q_n)}(E_1, ..., E_n; F).$$

The infimum of the $C > 0$ for which inequality (1.2) always holds defines a norm for the space of all fully $(p; q_1, ..., q_n)$-summing multilinear mappings and this norm is denoted by $\|.\|_{fas(p;q_1,...,q_n)}$. Under this norm, $\mathcal{L}_{fas(p;q_1,...,q_n)}(E_1, ..., E_n; F)$ is a Banach space. If $q_1 = ... = q_n = q$ we write $\|.\|_{fas(p;q)}$, $\mathcal{L}_{fas(p;q)}(E_1, ..., E_n; F)$.

Several coincidence results for fully summing mappings can be seen in Souza [20] and connections with tensor products and Hilbert-Schmidt operators will appear in [9]. A recent result due to Bombal, Pérez-García and Villanueva [2] asserts that every n-linear mapping from $\mathcal{L}_1$-spaces into any Hilbert space is fully $(1; 1)$-summing (multiple 1-summing in their terminology).

As in the linear case, there are innumerous other interesting coincidence and non-coincidence theorems for the different classes of multilinear mappings related to summability. In this note we firstly deal with a different problem. We first focalize in investigating situations in which we do have

$$\mathcal{L}_{as(1;q_1...,q_n)}(E_1,...,E_n;F) \neq \mathcal{L}(E_1,...,E_n;F)$$

and/or

$$\mathcal{L}_{fas(1;1)}(E_1,...,E_n;F) \neq \mathcal{L}(E_1,...,E_n;F)$$

in order to obtain sufficient conditions to yield that a particular multilinear mapping $T : E_1 \times ... \times E_n \to F$ is absolutely $(1; q_1,...,q_n)$-summing and/or fully $(1;1)$-summing. In the last section we define the concept of fully summing holomorphic mappings and give some examples of this kind of maps.

2. Absolutely $(1;2)$-summing multilinear mappings from $\mathcal{L}_\infty$-spaces into infinite dimensional Hilbert spaces

The great importance of $\mathcal{L}_\infty$-spaces in the theory of absolutely summing linear operators can be seen in the seminal paper of Maurey and Pisier [11]. For absolutely summing multilinear mappings $\mathcal{L}_\infty$-spaces are not less crucial. Recently, several authors have been investigating absolutely summing multilinear mappings defined on $\mathcal{L}_\infty$-spaces (see Meléndez-Tonge [12], Botelho [3] among others). Besides, since every $\mathcal{L}_\infty$-space has only infinite cotype, there are several interesting nontrivial questions concerning coincidence results for absolutely summing multilinear mappings defined on such spaces. In this direction, using a generalized Grothendieck's Inequality, D. Pérez-García [18] showed that if $E_1,...,E_n$ are $\mathcal{L}_\infty$-spaces then every bounded scalar valued multilinear mapping $T : E_1 \times .... \times E_n \to \mathbb{K}$ is $(1;2)$-summing. So, naturally, if H is a finite dimensional Banach space, we can also prove that every bounded multilinear mapping $T : E_1 \times .... \times E_n \to H$ is $(1;2)$-summing. In [15], among other negative results, it is indicated how to prove that if $E_1,...,E_n$ are infinite dimensional $\mathcal{L}_\infty$-spaces and F is any infinite dimensional Banach space, then

$$\mathcal{L}(E_1,...,E_n;F) \neq \mathcal{L}_{as(1;2,...,2)}(E_1,...,E_n;F). \tag{2.1}$$

Thus, if $E_1,...,E_n$ are $\mathcal{L}_\infty$-spaces and H is an infinite dimensional Hilbert space, a natural question is to give sufficient conditions for a bounded multilinear mapping $T : E_1 \times .... \times E_n \to H$ be absolutely $(1;2)$-summing. This will be one of the goals of this note.

Firstly, for completeness, we will give a proof for (2.1). The crucial result is the following:

Theorem 1. *(Pellegrino [15]) Let F be an infinite dimensional Banach space and E_1,..., E_m denote infinite dimensional Banach spaces with unconditional Schauder basis. If q is so that $\frac{1}{m} \leq q < 2$ and $\mathcal{L}_{as(q;1)}(E_1,...,E_m;F) = \mathcal{L}(E_1,...,E_m;F)$ we conclude that for any normalized unconditional Schauder basis $\{x_j^{(1)}\}_{j=1}^\infty$,..., $\{x_j^{(m)}\}_{j=1}^\infty$ for E_1,..., E_m, respectively, the natural mapping*

$$\psi : E_1 \times ... \times E_m \to l_\infty : (\sum_{i=1}^{\infty} a_i^{(1)} x_i^{(1)}, ..., \sum_{i=1}^{\infty} a_i^{(m)} x_i^{(m)}) \to (a_i^{(1)}...a_i^{(m)})_{i=1}^\infty$$

is such that $\psi(E_1 \times ... \times E_m) \subset l_{\frac{2q}{2-q}}$.

Proof. By hypothesis there exists $K > 0$ so that $\|T\|_{as(q;1)} \leq K\|T\|$ for all continuous m-linear mappings $T : E_1 \times ... \times E_m \to F$.

By the main Lemma of the well known Dvoretzky-Rogers Theorem (see [7, Theorem 4.2]), for every n, there exist normalized $y_1, ..., y_n$ in F such that

$$\|\sum_{j=1}^{n} \lambda_j y_j\| \leq 2(\sum_{j=1}^{n} \mid \lambda_j \mid^2)^{1/2} \tag{2.2}$$

regardless of the scalars $\lambda_1, ..., \lambda_n$.

For each $k = 1, ..., m$, consider $z_k = \sum\limits_{i=1}^{\infty} a_i^{(k)} x_i^{(k)} \in E_k$ and for each natural n, let $\{\mu_i\}_{i=1}^{n}$be such that $\sum\limits_{j=1}^{n} \mid \mu_j \mid^s = 1$ with $s = \frac{2}{q}$. Define $T : E_1 \times ... \times E_m \to F$ by

$$T(z_1, ..., z_m) = \sum_{j=1}^{n} |\mu_j|^{\frac{1}{q}} \, a_j^{(1)} ... a_j^{(m)} y_j,$$

where we chose y_j satisfying (2.2).

Since each $\{x_j^{(k)}\}_{j=1}^{\infty}$ is an unconditional basis, there exists $\rho_k > 0$ such that

$$\|\sum_{j=1}^{\infty} \varepsilon_j a_j^{(k)} x_j^{(k)}\| \leq \rho_k \|\sum_{j=1}^{\infty} a_j^{(k)} x_j^{(k)}\| = \rho_k \|z_k\|$$

for all $\varepsilon_j \in \{1, -1\}$ and $z_k = \sum\limits_{j=1}^{\infty} a_j^{(k)} x_j^{(k)} \in E_k$. Hence

$$\|\sum_{j=1}^{\infty} \varepsilon_j a_j^{(k)} x_j^{(k)}\| \leq \rho_k \|z_k\|$$

and

$$\|\sum_{j=1}^{r} \varepsilon_j a_j^{(k)} x_j^{(k)}\| \leq \rho_k \|z_k\|$$

for all natural r and any $\varepsilon_j = 1$ or -1. We thus have

$$\begin{aligned}
\|T(z_1, ..., z_m)\| &= \|\sum_{j=1}^{n} |\mu_j|^{\frac{1}{q}} \, a_j^{(1)} ... a_j^{(m)} y_j\| \\
&\leq 2 \left(\sum_{j=1}^{n} |\mu_j|^{\frac{2}{q}} \mid a_j^{(1)} ... a_j^{(m)} \mid^2 \right)^{1/2} \\
&\leq 2 \left(\sum_{j=1}^{n} \mid \mu_j \mid^{2/q} \rho_1^2 ... \rho_m^2 \right)^{1/2} \|z_1\| ... \|z_m\| \\
&\leq 2 \rho_1 ... \rho_m \|z_1\| ... \|z_m\| (\sum_{j=1}^{n} \mid \mu_j \mid^{2/q})^{1/2} \\
&\leq 2 \rho_1 ... \rho_m \|z_1\| ... \|z_m\|.
\end{aligned}$$

Then $\|T\| \leq 2\rho_1...\rho_m$ and $\|T\|_{as(q;1)} \leq 2K\rho_1...\rho_m$. Therefore

$$\begin{aligned}
\left[\sum_{j=1}^{n}\left(|\mu_j|^{\frac{1}{q}}\left|a_j^{(1)}...a_j^{(m)}\right|\right)^q\right]^{1/q} &= \left(\sum_{j=1}^{n}\left\|T(a_j^{(1)}x_j^{(1)},...,a_j^{(m)}x_j^{(m)})\right\|^q\right)^{1/q} \\
&\leq \|T\|_{as(q;1)}\prod_{k=1}^{m}\|(a_j^{(k)}x_j^{(k)})_{j=1}^{n}\|_{w,1} \\
&= \|T\|_{as(q;1)}\prod_{k=1}^{m}\max_{\varepsilon_j\in\{1,-1\}}\{\|\sum_{j=1}^{n}\varepsilon_j a_j^{(k)}x_j^{(k)}\|\} \\
&\leq \|T\|_{as(q;1)}\prod_{k=1}^{m}(\rho_k\|z_k\|) \\
&\leq 2K\rho_1^2...\rho_m^2\|z_1\|...\|z_m\|.
\end{aligned}$$

Recall that the last inequality holds whenever $\sum\limits_{j=1}^{n} \mid \mu_j \mid^s = 1$. Hence

$$\begin{aligned}
\left[\sum_{j=1}^{n}(\mid a_j^{(1)}...a_j^{(m)}\mid^{\frac{s}{s-1}q})\right]^{1/(\frac{s}{s-1})} &= \left\|\left(\left|a_j^{(1)}...a_j^{(m)}\right|^q\right)_{j=1}^{n}\right\|_{\frac{s}{s-1}} \\
&= \sup\left\{\left|\sum_{j=1}^{n}\mu_j\left|a_j^{(1)}...a_j^{(m)}\right|^q\right|; \sum_{j=1}^{n}\mid\mu_j\mid^s = 1\right\} \\
&\leq \sup\left\{\sum_{j=1}^{n}(\mid\mu_j\mid\left|a_j^{(1)}...a_j^{(m)}\right|^q; \sum_{j=1}^{n}\mid\mu_j\mid^s = 1\right\}
\end{aligned}$$

and thus

$$[\sum_{j=1}^{n}(\mid a_j^{(1)}...a_j^{(m)}\mid^{\frac{s}{s-1}q})]^{1/(\frac{s}{s-1})} \leq (2K\rho_1^2...\rho_m^2\|z_1\|...\|z_m\|)^q.$$

Since $\frac{s}{s-1}q = \frac{2q}{2-q}$, and n is arbitrary, the proof is done. □

Assuming $E_1 = ... = E_m = c_0$ and using a standard localization argument, Theorem 1 furnishes a proof for (2.1). Now, let us state a natural definition of adjoint of an n-linear mapping.

Definition 4. *Let $E_1,...,\ E_n$ and F be Banach spaces. If $T \in \mathcal{L}(E_1,...,\ E_n;F)$, we define the* ***adjoint*** *of T by*

$$\begin{aligned}
T^* : F^* &\longrightarrow \mathcal{L}(E_1,...,\ E_n;\mathbb{K}) \\
\varphi &\longrightarrow T^*\varphi : E_1 \times ... \times E_n \longrightarrow \mathbb{K}
\end{aligned}$$

with $(T^*\varphi)(x_1,...,x_n) = \varphi(T(x_1,...,x_n))$.

In a recent result [17] we give a sufficient condition in order to prove that a particular n-linear mapping into a Hilbert space is absolutely $(1;1)$-summing, generalizing a result due to S. Kwapien [6]. The next result explores the essential idea of the proof and give us a stronger result.

Theorem 2. *If H is a Hilbert space and $E_1,...,\ E_N$ are Banach spaces such that*

$$\mathcal{L}(E_1,...,E_N;\mathbb{K}) = \mathcal{L}_{as(1;p_1,...,p_N)}(E_1,...,E_N;\mathbb{K})$$

and

$$T \in \mathcal{L}(E_1,...,E_N;H)$$

is such that T^* is almost summing, then T is absolutely $(1;p_1,...,p_N)$-summing

Proof. Consider, for each $n \in \mathbb{N}$, a continuous multilinear operator $T : E_1 \times ... \times E_N \longrightarrow l_2^n$ $(n \in \mathbb{N})$.

If $x^{(k,1)}, ..., x^{(k,m)} \in E_k, 1 \leq k \leq N$, using Khinchin's Inequality (see [4, Theorem 1.10]), we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\| \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle T(x^{(1,j)}, ..., x^{(N,j)}), e_k \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle (x^{(1,j)}, ..., x^{(N,j)}), T^* e_k \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{j=1}^{m} \left[A_1^{-1} \left(\int_0^1 \left| \sum_{k=1}^{n} \left\langle (x^{(1,j)}, ..., x^{(N,j)}), T^* e_k \right\rangle r_k(t) \right| dt \right) \right] \\
&= A_1^{-1} \int_0^1 \sum_{j=1}^{m} \left| \left\langle (x^{(1,j)}, ..., x^{(N,j)}), \sum_{k=1}^{n} r_k(t) T^* e_k \right\rangle \right| dt \\
&\leq A_1^{-1} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|_{as(1;p_1,...,p_N)} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^m \right\|_{w,1} dt.
\end{aligned}$$

Thus, since $\mathcal{L}(E_1, ..., E_N; \mathbb{K}) = \mathcal{L}_{as(1;p_1,...,p_N)}(E_1, ..., E_N; \mathbb{K})$ there exists $C > 0$ such that

$$\left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|_{as(1;p_1,...,p_N)} \leq C \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|$$

and thus

$$\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\| \leq C A_1^{-1} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^m \right\|_{w,p_i} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\| dt. \tag{2.3}$$

Since T^* is almost summing we obtain

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|^2 dt \right)^{\frac{1}{2}} \leq \| T^* \|_{al,2} \, \| (e_k)_{k=1}^n \|_{w,2} = \| T^* \|_{al,2} \, . \tag{2.4}$$

We complete the proof by considering an operator $T \in \mathcal{L}(E_1, ..., E_N; H)$ which adjoint $T^* : H \longrightarrow \mathcal{L}(E_1, ..., E_N; \mathbb{K})$ is almost summing.

If $x^{(k,1)}, ..., x^{(k,m)} \in E_k$, with $1 \leq k \leq N$, identify the span of the $T(x^{(1,j)}, ..., x^{(N,j)})'s$, $j = 1, ..., m$, with l_2^n for an appropriate n and set by Ψ this map. This is possible, since such span is a finite dimensional Hilbert space. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto this span. We have $P^* = P$ and using (2.3) and (2.4), we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\| \\
&= \sum_{j=1}^{m} \left\| \Psi \circ P \circ T(x^{(1,j)}, ..., x^{(N,j)}) \right\|
\end{aligned}$$

$$\leq CA_1^{-1}\left\|T^*\circ P^*\circ\Psi^*\right\|_{al,2}\prod_{i=1}^{N}\left\|(x^{(i,j)})_{j=1}^m\right\|_{w,p_i}$$
$$\leq A_1^{-1}\left\|T^*\right\|_{al,2}\left\|P^*\right\|\left\|\Psi^*\right\|\prod_{i=1}^{N}\left\|(x^{(i,j)})_{j=1}^m\right\|_{w,p_i}$$
$$\leq A_1^{-1}\left\|T^*\right\|_{al,2}\left\|P\right\|\left\|\Psi\right\|\prod_{i=1}^{N}\left\|(x^{(i,j)})_{j=1}^m\right\|_{w,p_i}$$
$$= A_1^{-1}\left\|T^*\right\|_{al,2}\prod_{i=1}^{N}\left\|(x^{(i,j)})_{j=1}^m\right\|_{w,p_i}.$$

Therefore, T is absolutely $(1;p_1,...,p_N)$-summing .□

Corollary 1. *If H is an infinite dimensional Hilbert space, E_1,..., E_m are $\mathcal{L}_\infty$-spaces and*
$$T\in\mathcal{L}(E_1,...,E_m;H)$$
is such that T^* is almost summing, then T is absolutely $(1;2)$-summing.

3. Fully $(1;1)$-summing multilinear mappings from $l_1\times...\times l_1\times l_2$ into infinite dimensional Hilbert spaces

In this section we first intend to prove that if each E_j is an $\mathcal{L}_1$-space and H is a Hilbert space, then every n-linear form defined on $E_1\times...\times E_n\times H$ is fully $(1;1)$-summing whereas if F is an infinite dimensional Banach space, there is always an n-linear mapping from $E_1\times...\times E_n\times H$ into F which fails to be fully $(1;1)$-summing. Our first step is the following straightforward result:

Lemma 1. *If $T\in\mathcal{L}(E_1,...,E_n;F)$ is such that $T_1:E_1\times...\times E_{n-1}\to\mathcal{L}_{as(1;1)}(E_n;F)$ defined by $T_1(x_1,...,x_{n-1})(x_n)=T(x_1,...,x_n)$ is fully $(1;1)$-summing, then T is fully $(1;1)$-summing.*

So, we have the following theorem:

Theorem 3. *If each E_j is an $\mathcal{L}_1$-space and H is a Hilbert space, then*
$$\mathcal{L}(E_1,...,E_{n-1},H;\mathbb{K})=\mathcal{L}_{fas(1;1)}(E_1,...,E_{n-1},H;\mathbb{K}).$$

Proof. It suffices to use that $\mathcal{L}(E_1,...,E_{n-1};H)=\mathcal{L}_{fas(1;1)}(E_1,...,E_{n-1};H)$ and $H=\mathcal{L}(H;\mathbb{K})=\mathcal{L}_{as(1;1)}(H;\mathbb{K})$ and apply Lemma 1.□

Now, we must observe that $\mathcal{L}(E_1,...,E_{n-1},H;F)\neq\mathcal{L}_{fas(1;1)}(E_1,...,E_{n-1},H;F)$ for any infinite dimensional Banach space F. To achieve this result we state a simple but useful "descending" result which proof we omit.

Proposition 1. *If $\mathcal{L}^{ev}_{as(p;p_1,...,p_n)}(E_1,...,E_n;F)=\mathcal{L}(E_1,...,E_n;F)$ then*
$$\mathcal{L}(E_j;F)=\mathcal{L}_{as(p;p_j)}(E_j;F)$$
for every j.

Corollary 2. *If $\mathcal{L}_{fas(p;p_1,...,p_n)}(E_1,...,E_n;F)=\mathcal{L}(E_1,...,E_n;F)$ then*
$$\mathcal{L}(E_j;F)=\mathcal{L}_{as(p;p_j)}(E_j;F)$$
for every j.

Proof. It is not hard to prove that every fully $(p;p_1,...,p_n)$-summing is absolutely $(p;p_1,...,p_n)$-summing at every point. Thus the descending property furnishes the proof. □

Now, we can assert the aforementioned non coincidence result.

Corollary 3. *If F is an infinite dimensional Banach space, H is a Hilbert space and each E_j is an $\mathcal{L}_1$-space, then*

$$\mathcal{L}(E_1, ..., E_{n-1}, H; F) \neq \mathcal{L}_{fas(1;1)}(E_1, ..., E_{n-1}, H; F).$$

Proof. If the result did not hold we would have $\mathcal{L}(H;F) = \mathcal{L}_{as(1;1)}(H;F)$, which is impossible (see [7, Theorem 4.2]).

Finally, if H is an infinite dimensional Hilbert space, it makes sense to ask when a mapping T from $l_1 \times ... \times l_1 \times H$ into H is fully $(1;1)$-summing.

Using the same reasoning of the proof of Theorem 2, one can obtain a version of Theorem 2 for fully summing mappings. Precisely,

Theorem 4. *If H is an infinite dimensional Hilbert space, E_1,..., E_{n-1} are $\mathcal{L}_1$-spaces and*

$$T \in \mathcal{L}(E_1, ..., E_{n-1}, H; H)$$

is such that T^* is almost summing, then T is fully $(1;1)$-summing.

4. Fully summing holomorphic mappings

The theory of absolutely summing holomorphic mappings was introduced by Matos [8] (see also Floret-Matos [5]). Further work of the first named author [14] showed several examples of this kind of mappings. In order to find an adequate concept of "fully summing holomorphic mapping" we first need to introduce the concept of fully summing homogeneous polynomials. Naturally, we will say that an n-homogeneous polynomial is fully $(r;s)$-summing if its associated symmetric n-linear mapping $\overset{\vee}{P}$ is fully $(r;s)$-summing. The fully summing norm for polynomials will be the one induced by the fully summing norm of its associated n-linear map, i.e., $\|P\|_{fas(r;s)} = \|\overset{\vee}{P}\|_{fas(r;s)}$. One can prove that $(\mathcal{P}_{fas(r;s)}(^nE;F), \|.\|_{fas(r;s)})$ is a Banach space. Detailed results involving fully summing polynomials will appear in the doctoral thesis of the second named author, under supervision of M. C. Matos.

In this section our initial step is to prove that $(\mathcal{P}_{fas(r;s)}(^nE;F), \|.\|_{fas(r;s)})_{n=0}^{\infty}$ is a holomorphy type in the sense of Nachbin.

Definition 5. *(Nachbin [13]) A **holomorphy type** θ from E to F is a sequence of Banach spaces*

$$(\mathcal{P}_\theta(^mE;F))_{m=0}^{\infty},$$

the norm of each of them denoted by $\|.\|_\theta$, such that the following conditions hold true

(1) *Each $\mathcal{P}_\theta(^mE;F)$ is a linear subspace of $\mathcal{P}(^mE;F)$;*

(2) *$\mathcal{P}_\theta(^0E;F) = F$, as a normed linear space;*

(3) *There exists a real number $\sigma \geq 1$ for which the following is true. Given any $l, m \in \mathbb{N}$, $l \leq m$, $x \in E$ and $P \in \mathcal{P}_\theta(^mE;F)$, we have*

$$\hat{d}{}^l P(x) \in \mathcal{P}_\theta(^lE;F) \text{ and}$$

$$\left\| \frac{1}{l!} \hat{d}{}^l P(x) \right\|_\theta \leq \sigma^m \|P\|_\theta \|x\|^{m-l}.$$

A related definition, also due to Nachbin, give the concept of θ-holomorphy type for holomorphic mappings

Definition 6. *If $U \subset E$ is an open set, a given $f \in \mathcal{H}(E;F)$ is said to be of θ-**holomorphy type** at $\xi \in U$ if*

(1) $\hat{d}{}^m f(\xi) \in \mathcal{P}_\theta(^mE;F)$ *for every natural m*

(2) *There exist real numbers $C \geq 0$ and $c \geq 0$ such that*

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\|_\theta \leq Cc^m \text{ for every natural } m.$$

Moreover, if f is of θ-holomorphy type at every point of U we say that f is of θ-holomorphy type on U. We denote by $\mathcal{H}_\theta(U;F)$ the linear subspace of $\mathcal{H}(U;F)$ composed by all such f of holomorphy type on U.

Theorem 5. $(\mathcal{P}_{fas(r;s)}(^mE;F), \|.\|_{fas(r;s)})_{m=0}^\infty$ *is a holomorphy type $fas(r;s)$.*

Proof. Consider $k, n \in \mathbb{N}$ so that $k \leq n$ and $P \in \mathcal{P}_{fas(r;s)}(^nE;F)$. Let us consider $(x_j^{(l)})_{j=1}^\infty \in l_s^w(E)$, $l = 1, ..., n$. Thus, defining $(x_j^{(l)})_{j=1}^\infty = (a, 0, 0, ...)$ for each $k+1 \leq l \leq n$ we obtain

$$\begin{aligned}
&\left(\sum_{j_1,\dots,j_k=1}^\infty \left\| \frac{1}{k!} d^k P(a)(x_{j_1}^{(1)}, ..., x_{j_k}^{(k)}) \right\|^r \right)^{\frac{1}{r}} \\
&= \binom{n}{k} \left(\sum_{j_1,\dots,j_k=1}^\infty \left\| \overset{\vee}{P}(x_{j_1}^{(1)} ... x_{j_k}^{(k)}, a, ..., a) \right\|^r \right)^{\frac{1}{r}} \\
&= \binom{n}{k} \left(\sum_{j_1,\dots,j_n=1}^\infty \left\| \overset{\vee}{P}(x_{j_1}^{(1)} ... x_{j_n}^{(n)}) \right\|^r \right)^{\frac{1}{r}} \\
&\leq \binom{n}{k} \left\| \overset{\vee}{P} \right\|_{fas(r;s)} \prod_{l=1}^n \left\| (x_j^{(l)})_{j=1}^\infty \right\|_{w,s} \\
&= \binom{n}{k} \|P\|_{fas(r;s)} \|a\|^{n-k} \prod_{l=1}^k \left\| (x_j^{(l)})_{j=1}^\infty \right\|_{w,s}
\end{aligned}$$

Thus $d^k P(a) \in \mathcal{L}_{fas(r;s)}(^kE;F)$ for each k. Finally,

$$\left\| \frac{1}{k!} \hat{d}^k P(a) \right\|_{fas(r;s)} \leq \binom{n}{k} \|P\|_{fas(r;s)} \|a\|^{n-k} \leq 2^n \|P\|_{fas(r;s)} \|a\|^{n-k}.$$

□

From now on we will denote the space of holomorphic mappings from E into F by $\mathcal{H}(E;F)$.

Definition 7. *We say that an holomorphic mapping $f : E \to F$ is* ***fully*** $(r;s)$***-summing*** *if f is of $fas(r;s)$-holomorphy type on E.*

If f is fully $(r;s)$-summing, we will write $f \in \mathcal{H}_{fas(r;s)}(E;F)$.

Now, we will give the first examples in which we have $\mathcal{H}(E;F) = \mathcal{H}_{fas(r;s)}(E;F)$ for some r, s.

Based on the ideas of [2] we have the following result.

Lemma 2. *Suppose that*

$$\mathcal{L}(E;H) = \mathcal{L}_{as(2;2)}(E;H) \tag{4.1}$$

for every Hilbert space H. Then for each H there exists $C_H > 0$ such that

$$\mathcal{L}(^nE;H) = \mathcal{L}_{fas(2;2)}(^nE;H)$$

and

$$\|T\|_{fas(2;2)} \leq C_H^n \|T\|$$

for every $T \in \mathcal{L}(^nE;H)$.