Pre-invex functions and weak efficient solutions of the vectorial problem between Banach spaces

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Abstract

In this work we introduce the notion of pre-invex function for functions between Banach spaces. By using these functions, we obtain necessary and sufficient conditions of optimality for vectorial problems with restrictions of inequalities. Moreover, we will show that this class of problems has the property that all local optimal solution is in fact global.

Keywords: *Multiobjective optimization, pre-invexity, optimality conditions*

1. Introduction and formulation of the problem

In this work, we consider the following problem of optimization:

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ & \\ \text{subject to} & \begin{array}{c} -g(x) \in K \\ x \in S \subset E \end{array} \end{array}\right\} \tag{P}$$

where E, F, G are Banach spaces, $f : E \to F$, $g : E \to G$. We assume that the spaces F and G are ordered by cones $Q \subset F$, $K \subset G$ and that these cones are closed, convex and with nonempty interior.

We denote by $\mathcal{F} = \{x \in S : -g(x) \in K\}$ the feasible set of (P).

We can consider the following order partial in F :

$$y, z \in F,\ y \preceq_F z \Leftrightarrow z - y \in Q$$

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(analogously for G).

Also, we can consider the following relation:

$$y, z \in F,\ y \prec_G z \Leftrightarrow z - y \in \operatorname{int} Q$$

(where $\operatorname{int} Q$ is the interior of Q).

Then, we have two concepts of solution for (P):

Definition 1. *We say that $x_0 \in \mathcal{F}$ is an* ***efficient solution*** *for (P) if $x \in \mathcal{F}$, $f(x) \preceq_F f(x_0) \Rightarrow f(x) = f(x_0)$.*

Definition 2. *We say that $x_0 \in \mathcal{F}$ is a* ***weak efficient solution*** *for (P) if there is not $x \in \mathcal{F}$ such that $f(x) \prec_F f(x_0)$.*

This class of problem has many applications in mathematical economy and engineering. The problem finite dimensional (i.e., when in (P) we take $E = \mathbb{R}^n$, $F = \mathbb{R}^p$ and $G = \mathbb{R}^m$; $Q = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$) was studied by Osuna-Gómez [1] with relation to the optimality conditions. Our purpose in this work is extend these results for arbitray Banach spaces.

The notion of convexity is very important in the optimization theory. The following results are well known: if $\theta : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function denfined on S, where S is a nonempty, convex subset of $\mathbb{R}^n$, then

1. If $\bar{x} \in S$ is a local minimum of f on S, then $\bar{x}$ is a global minimum of f (on S);
2. If θ is differentiable on S and S is open set, $\nabla\theta(x_1)(x_2 - x_1) \leq \theta(x_2) - \theta(x_1), \forall x_1, x_2 \in S$ (and, in particular, if $\bar{x} \in S$, $\nabla\theta(\bar{x}) = 0$ then $\bar{x}$ is a global minimum of S).

These two properties of convex functions are very important in optimization theory.

In fact, there exist other class of functions that are not convex and that has properties analogous: are the so-called generalized convex functions ([14], [15], [16], [17].)

We suggest the following definition of pre-invex functions between Banach spaces. We note that this definition generalize the notion given early by Hanson and Mond [4], for the case scalar. We will prove that these class of functions satisfy properties analogous to (1) and (2) and this will be useful to obtain some optimality conditions for problem (P), (In the sense of weak efficiency).

The paper is organized as follows: in Section 2 we give the definition of pre-invexity and we prove some results. In Section 3 we study the optimality conditions. In the Section 4 we prove the global results for weak efficiency.

2. Pre-invex functions

In this Section, we define the pre-invexity for functions between Banach spaces and we study some properties. Also, we stress the alternative theorem of Gordan type for pre-invex functions. This results will be crucial to obtain the optimality conditions for problem (P).

Definition 3. *(Hanson & Mond, [4]) Let E be a Banach space. The function $\theta : \Omega \subset E \to \mathbb{R}$ is called **pre-invex** with respect to η on $S \subset \Omega$ if for all $x_1, x_2 \in S$ and for each $\lambda \in (0,1)$, there exists a vector function $\eta : S \times S \to E$ such that*

$$\theta(x_2 + \lambda\eta(x_1, x_2)) \leq \lambda\theta(x_1) + (1-\lambda)\theta(x_2)$$

If the set $S \subset E$ has the following property

$$x_2 + \lambda\eta(x_1, x_2) \in S, \forall x_1, x_2 \in S, \forall \lambda \in (0,1)$$

we will say that S is **invex** with respect to the vectorial function η.

Let $Q^* := \{\omega^* \in F^* : \langle \omega^*, x\rangle \geq 0 \quad \forall x \in Q\}$ the **dual cone** of Q and F^* the topological dual of F. We denote $\langle \cdot, \cdot \rangle$ the **canonical duality** in the $F^* \times F$ (that is, $\langle \omega^*, x\rangle = w^*(x), \forall w^* \in F^*, \forall x \in F$)

We generalize the Definition 3 to the functions betwen Banach spaces in the following way:

Definition 4. *Let E and F two Banach spaces. The function $f : \Omega \subset E \to F$ is called **pre-invex** with respect to η on $S \subset \Omega$ if for each $\omega^* \in Q^*$, the composition function $\omega^* \circ f$ is pre-invex with respect to η, in the sense of the Definition 3.*

Lemma 1. *The Definition 4 is equivalent to: for all $x_1, x_2 \in S$ and each $\lambda \in (0,1)$, there exists a vector $\eta : S \times S \to E$ such that*

$$f(x_2 + \lambda\eta(x_1, x_2)) \preceq_F \lambda f(x_1) + (1-\lambda)f(x_2) \tag{1}$$

To prove this result, we will need recall the following lemma (see [13], p. 215).

Lemma 2. *Let F be a Banach space ordered by the cone $Q \subset F$, with Q convex and closed. If there exists $y \in F$ such that $\langle y^*, y\rangle \geq 0$ $\forall y^* \in Q^*$ then $y \in Q$.*

The inverse affirmation is cleary true.

Proof: (Of the Lemma 1)
Let $x_1, x_2 \in S$ and $\lambda \in (0,1)$.
The following equivalences are true

$$f(x_2 + \lambda\eta(x_1, x_2)) \preceq_F \lambda f(x_1) + (1-\lambda)f(x_2)$$
$$\Longleftrightarrow \lambda f(x_1) + (1-\lambda)f(x_2) - f(x_2 + \lambda\eta(x_1, x_2)) \in Q$$

$$\Longleftrightarrow \omega^*(\lambda f(x_1) + (1-\lambda)f(x_2) - f(x_2 + \lambda\eta(x_1, x_2)) \geq 0, \forall \omega^* \in Q^*$$

$$\Longleftrightarrow \omega^* \circ f(x_2 + \lambda\eta(x_1, x_2)) \leq \lambda\omega^* \circ f(x_1) + (1-\lambda)\omega^* \circ f(x_2), \forall \omega^* \in Q^*$$

where the firts equivalence follows from the definition of $\preceq_F$, the second from Lemma 2 and the third from the linearity of ω^*.■

The following property of the directionally differentiable pre-invex functions will be extensively used in the rest of the paper.

Lemma 3. *Let $f : \Omega \subset E \to F$ be a pre-invex function on $S \subset \Omega$, directionally differentiable. Then*

$$(\omega^* \circ f)'(x, \eta(x, y)) \leq \omega^* \circ f(y) - \omega^* \circ f(x)$$

$\forall \omega^* \in Q^*, \forall x, y \in S.$

Proof: Assume that f is pre-invex on S. Then, by Definition 4 $\omega^* \circ f$ is pre-invex on S, for all $\omega^* \in Q^*$ (in the sense of Definition 3).

Then, by Definition 3,

$$\omega^* \circ f(x + \lambda\eta(x, y)) \leq \omega^*(\lambda f(y) + (1-\lambda)f(x)) \tag{2}$$

$\forall \lambda \in (0, 1)$.

From (2)

$$\omega^* \circ f(x + \lambda\eta(x, y)) - \omega^* \circ f(x) \leq \lambda\omega^*(f(y) - f(x)). \tag{3}$$

Dividing the inequality (3) by λ, and taking the limit when $\lambda \to 0^+$, we obtain

$$(\omega^* \circ f)'(x, \eta(x, y)) \leq \omega^* \circ f(y) - \omega^* \circ f(x)$$

$\forall \omega^* \in Q^*, \forall x, y \in S.$■

We will recall the following result, see [2], p.54.

Lemma 4. *If $Q \subset F$ is a convex cone, $\operatorname{int} Q \neq \varnothing$ and $0 \neq p \in Q^*$, then $p(s) > 0$ when $s \in \operatorname{int} Q$.*

Also, we will prove the following Alternative Theorem of Gordan's Type. This result will be useful in the next sections.

Theorem 5. *Let $f : E \to F$ be a pre-invex function with respect to η on $\mathcal{F} \subset E$, where $\mathcal{F}$ is an invex set with respect to η. Let $Q \subset F$ be a convex cone with nonempty interior. Then, either:*

(i) *there exists* $x \in \mathcal{F}$ *such that* $-f(x) \in \operatorname{int} Q$;

or

(ii) *there exists* $p \in Q^* \setminus \{0\}$ *such that* $(p \circ f)(\mathcal{F}) \subset \mathbb{R}_+$.

Proof: Firstly, we assume that the system (i) and (ii) have solutions $x \in \mathcal{F}$ and $p \in Q^* \setminus \{0\}$.

Then, from Lemma 4, we have that $p(f(x)) < 0$, with $x \in \mathcal{F}$, consequently we obtain one contradiction with (ii).

Now, we assume that the system (ii) has not solution. We will prove that the system (i) has solution.

We put

$$A := f(\mathcal{F}) + \operatorname{int} Q.$$

The set A is open: In fact, let $k \in A$. Then there exist $x \in \mathcal{F}$ and $s \in \operatorname{int} Q$ such that $k = f(x) + s$.

Since, $s \in \operatorname{int} Q$ there exist a ball N with center at zero such that $s + N \subset Q$.

But, $k + N = f(x) + (s + N) \subset A$ and, consequently A is open.

Now, we will prove that A is convex. Let $k_1, k_2 \in A$ and $\tau \in (0, 1)$.

Then, $k_1 = f(x_1) + s_1$, $k_2 = f(x_2) + s_2$, with $x_1, x_2 \in \mathcal{F}$ and $s_1, s_2 \in \operatorname{int} S$.

$$(1 - \tau)k_1 + \tau k_2 = [(1 - \tau)f(x_1) + \tau f(x_2)] + [(1 - \tau)s_1 + \tau s_2]. \tag{4}$$

But, since f is pre-invex, we have

$$(1 - \tau)f(x_1) + \tau f(x_2) \in f(x_2 + \tau\eta(x_1, x_2)) + Q \tag{5}$$

and

$$(1 - \tau)s_1 + \tau s_2 \in \operatorname{int} Q. \tag{6}$$

By hypothesis $\mathcal{F}$ is invex, that is,

$$x_2 + \tau\eta(x_1, x_2) \in \mathcal{F}, \tag{7}$$

is true.

From (4)-(7) we obtain $(1 - \tau)k_1 + \tau k_2 \in A$, that is, the set A is convex.

Since the system (1) has not solution, then $0 \notin A$. From Hahn-Banach Theorem, there exists $p \in F^* \setminus \{0\}$ such that

$$p(A) \subset \mathbb{R}_+. \tag{8}$$

We fix $s \in \operatorname{int} Q$. We would like to prove: $p(f(x)) \geq 0, \forall x \in \mathcal{F}$.

Since $s \in \operatorname{int} Q$, we have

$$s + N \subset \operatorname{int} Q \tag{9}$$

for some ball N.

For $\tau \in \mathbb{R}_+$ sufficiently big, we have $\frac{1}{\tau}f(x) \in N$ and from (9) we have $s - \frac{1}{\tau}f(x) \in \operatorname{int} Q$, and recalling that $\operatorname{int} Q$ is a cone, we obtain $\tau s - f(x) \in \operatorname{int} Q$, that is $\tau s \in f(x) + \operatorname{int} Q \subset A$, and, therefore, by (8) we have

$$p(s) \geq 0, \forall s \in \operatorname{int} Q. \tag{10}$$

But, for each $\varepsilon > 0$ sufficiently small such that $k = f(x) + \varepsilon s \in A$ and, therefore,

$$(p \circ f)(x) = p(k) - \varepsilon p(s) \geq -\varepsilon p(s) \to 0$$

as $\varepsilon \to 0^+$, consequently

$$(p \circ f)(x) \geq 0 \ \forall x \in \mathcal{F}. \tag{11}$$

For each $s_0 \in Q$, $p(s_0) = \frac{1}{\tau}p(\tau s_0)$ and for $\tau > 0$ small, $\tau s_0 \in \operatorname{int} Q$, therefore of (10) we have $p(s_0) \geq 0 \ \forall s_0 \in \operatorname{int} Q$, that is,

$$p \in Q^* \backslash \{0\} \tag{12}$$

and, (11) and (12) imply that p is a solution of the system (2).■

3. Conditions of optimality

This Section is divided in three subsections. In Subsection 3.1, we study the scalar optimization problem, i.e. when the objective function is real-valued; in 3.2 we establish and prove the scalarization theorem, this theorem will be important because relationship the optimal solution of the scalar problem with the vectorial problem, and in 3.3 we use the above result to obtain optimality conditions for the vectorial problem (P).

3.1. Conditions of optimality for scalar problems

Now, we consider the following scalar optimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & \theta(x) \\ & \\ \text{subject to} & \begin{array}{l} -g(x) \in K \\ x \in S \subset E \end{array} \end{array}\right\} \tag{PM}$$

where E, G are Banach spaces, G is ordered by the closed convex cone with nonempty interior K, $\theta : E \to \mathbb{R}$, $g : E \to G$ are continuous and S is a nonempty open subset of E.

Theorem 6. *We assume that the functions in the problem (PM), θ and g are preinvex functions with respect to the same η and are directionally differentiable. Let $\overline{x}$ be a solution of (PM). Then, there exist $\alpha \geq 0$ and $\mu^* \in K^*$, not simultaneously zeros such that*

$$(\alpha\theta)'(\overline{x}, \eta(\overline{x}, y)) + (\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, y)) \geq 0, \forall y \in S$$

$$\langle \mu^*, g(\overline{x}) \rangle = 0.$$

Proof: From the hypotheses make, we have that the feasible set $\mathcal{F} := \{x \in S : -g(x) \in K\}$ is invex with respect to η.

Let $\overline{x}$ be the solution of (PM). In this case, the system

$$- \begin{bmatrix} \theta(x) - \theta(\overline{x}) \\ g(x) \end{bmatrix} \in \operatorname{int}(\mathbb{R}_+ \times K)$$

has no solution $x \in \mathcal{F}$.

From Theorem 5, we have that there exists $p = (\tau, v^*) \in (\mathbb{R}_+ \times K^*) \backslash \{(0,0)\}$ such that

$$\tau[\theta(x) - \theta(\overline{x})] + v^* \circ g(x) \geq 0, \forall x \in \mathcal{F} \tag{13}$$

consequently

$$v^* \circ g(\overline{x}) = 0. \tag{14}$$

We observe that for each $\varepsilon > 0$ sufficiently small, we have $\overline{x} + \varepsilon\eta(\overline{x}, y) \in \mathcal{F}, \forall y \in S$ since $\mathcal{F}$ is invex with respect to η.

From (13) and (14) we obtain

$$\lim_{\varepsilon \to 0^+} \frac{\tau\theta(\overline{x} + \varepsilon\eta(\overline{x}, y)) - \tau\theta(\overline{x}) + v^* \circ g(\overline{x} + \varepsilon\eta(\overline{x}, y)) - v^* \circ g(\overline{x})}{\varepsilon} = \tag{15}$$

$$= (\tau\theta)'(\overline{x}, \eta(\overline{x}, y)) + (v^* \circ g)'(\overline{x}, \eta(\overline{x}, y)) \geq 0.$$

Setting in (15) $\alpha = \tau$ e $\mu^* = v^*$, we obtain the desirable result.■

3.2. A theorem of scalarization

We will consider the following optimization problem

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ \\ \text{subject to} & x \in \Gamma \end{array}\right\} \tag{P1}$$

where $f : E \to F$, $\Gamma \subset E$, E and F are Banach spaces, F is ordered for the closed, convex cone Q with nonempty interior.

The following theorem of scalarization is true for the problem (P1):

Theorem 7. *Assume that in (P1) the function f is pre-invex with respect to η in the set Γ and that the feasible set Γ is invex with respect to η. If $x^* \in \Gamma$ is a weak efficient solution of (P1), then there exists $\omega^* \in Q^* \setminus \{0\}$ such that*

$$\omega^* \circ f(x^*) \leq \omega^* \circ f(x), \;\; \forall x \in \Gamma.$$

Proof: We consider the following sets

$$U := \{u \in F : 0 \prec_F u\};\ V := \{v \in F : v \preceq_F f(x^*) - f(x),\ \text{for some } x \in \Gamma\}.$$

Since x^* is a weak efficient solution of (P1), we obtain $U \cap V = \varnothing$.
In fact, assume the contrary, that is, that there exists $z \in U \cap V$.
In this case, there exists $x \in \Gamma$ such that $0 \prec_F z \preceq_F f(x^*) - f(x)$ and consequently,

$$f(x) \prec_F f(x^*),\ x \in \Gamma.$$

But this is a contradiction with the fact that x^* is a weak efficient solution of (P1). Then, $U \cap V = \varnothing$.

U is obviously open and convex (because $U = \operatorname{int} Q$ and Q is convex).

In view of the fact that the function f pre-invex and the set Γ is invex, we have that V is convex.

In fact, let $v_1, v_2 \in V$ and $\lambda \in (0, 1)$. Then, there exist $x_1, x_2 \in \Gamma$ such that

$$v_1 \preceq_F f(x^*) - f(x_1) \text{ e } v_2 \preceq_F f(x^*) - f(x_2).$$

It is easily see that

$$\lambda v_1 \preceq_F \lambda f(x^*) - \lambda f(x_1) \text{ e } (1-\lambda)v_2 \preceq_F (1-\lambda)f(x^*) - (1-\lambda)f(x_2)$$

and we deduce that

$$\begin{aligned} \lambda v_1 + (1-\lambda)v_2 &\preceq_F f(x^*) - [\lambda f(x_1) + (1-\lambda)f(x_2)] \\ &\preceq_F f(x^*) - f(x_2 + \lambda\eta(x_1, x_2)) \end{aligned}$$

where the last inequality is consequence of the pre-invexity of f. Since Γ is an invex set with respect to η, we have $x_2 + \lambda\eta(x_1, x_2) \in \Gamma$ and, therefore, V is convex.

From Hahn-Banach Theorem, there exists $w^* \in F^* \backslash \{0\}$ such that

$$\langle w^*, v\rangle \leq 0 \leq \langle w^*, u\rangle,\ \forall u \in U\ , \forall v \in V.$$

The second inequality implies $\langle w^*, u\rangle \geq 0, \forall u \in \operatorname{int} Q$.

But, Q is convex with nonempty interior, and then is verified that $\overline{\operatorname{int} Q} = Q$ (v. [7], p. 413) and this implies $w^* \in Q^*$.

We observe that for each $x \in \Gamma$, we have $f(x^*) - f(x) \in V$, moreover

$$\langle w^*, f(x^*) - f(x)\rangle \leq 0, \forall x \in \Gamma.$$

■

3.3. Conditions of optimality for vectorial problems

In this section, we obtain optimality conditions for the problem (P).

Observe that the weak efficient solutions of a vectorial pre-invex functions are completaly characterized by a estationary condition. In fact:

Theorem 8. *Let $f : \Omega \subset E \to F$ be a pre-invex function on $S \subset \Omega$ with respect to η and are directionally differentiable. Then, $\overline{x}$ is a weak efficient solution of f on the open set S if and only if*

$$(\omega^* \circ f)'(\overline{x}, \eta(\overline{x}, y)) \geq 0 \tag{16}$$

$\forall y \in S, \forall \omega^* \in Q^*$.

Proof: Firstly we show the implication ($\Rightarrow$). We assume that $\overline{x}$ is a weak efficient solution and that (16) is not true.

In this case, there exist $y \in S$ and $\omega^* \in Q^*$ such that

$$(\omega^* \circ f)'(\overline{x}, \eta(\overline{x}, y)) < 0. \tag{17}$$

Since S is open and $\overline{x} \in S$, we have that $\overline{x} + \lambda\eta(\overline{x}, y) \in S$, for $\lambda > 0$ sufficiently small.

From (17), we obtain

$$\lim_{\lambda \to 0^+} \frac{\omega^* \circ f(\overline{x} + \lambda\eta(\overline{x}, y)) - \omega^* \circ f(\overline{x})}{\lambda} < 0$$

and, therefore, for $\lambda > 0$ sufficient small, we get

$$\omega^*(f(\overline{x} + \lambda\eta(\overline{x}, y)) - f(\overline{x})) < 0.$$

Since $\omega^* \in Q^*$, $\omega^* \neq 0$, we have

$$f(\overline{x} + \lambda\eta(\overline{x}, y)) \prec_F f(\overline{x})$$

with $\overline{x} + \lambda\eta(\overline{x}, y) \in S$. This is a contradiction whit the fact that $\overline{x}$ is a weak efficient solution.

Now, we prove the reverse implication ($\Leftarrow$).

To done this, we assume that is true the condition (16) and that $\overline{x}$ is not weak efficient solution.

In this case, there exists $y \in S$ such that $f(y) \prec_F f(\overline{x})$.

Let $\omega^* \in Q^* \setminus \{0\}$ (it is possible to show that $Q^* \neq \{0\}$; see [9]) and we obtain

$$\omega^* \circ f(y) - \omega^* \circ f(\overline{x}) < 0. \tag{18}$$

Then,

$$0 \leq (\omega^* \circ f)'(\overline{x}, \eta(\overline{x}, y)) \leq \omega^* \circ f(y) - \omega^* \circ f(\overline{x}) < 0$$

(where the first inequality is obtained from the hypothesis done, the second follows of Lemma 3 and the third from (18)) and therefore this is absurd.■

The finite dimensional version of the Theorem 8 is in [1], Teorema 2.2, p. 24.

Next, we give some optimality conditions (necessary and sufficient conditions) for the problem (P).

Theorem 9. *(Necessary condition) Assume that in the problem (P) the functions f and g are pre-invex with respect to the same η, are directionally differentiable and the set S is invex with respect to η. If $\overline{x}$ is a weak efficient solution of (P), then there exist $\lambda^* \in Q^*$, $\mu^* \in K^*$, not all zeros such that*

$$(\lambda^* \circ f)'(\overline{x}, \eta(\overline{x}, y)) + (\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, y)) \geq 0, \forall y \in \mathcal{F}$$

$$\langle \mu^*, g(\overline{x}) \rangle = 0.$$

Proof: From the hypotheses done, we have that the feasible set $\mathcal{F}$ is invex with respect to η. By using Theorem 7 , there exists $\lambda^* \in Q^* \backslash \{0\}$ such that

$$\lambda^* \circ f(\overline{x}) \leq \lambda^* \circ f(x), \forall x \in \mathcal{F}.$$

Then, by applying Theorem 6 , there exist $\alpha \geq 0$ and $\mu^* \in K^*$ not all zeros such that

$$\alpha(\lambda^* \circ f)'(\overline{x}, \eta(\overline{x}, y)) + (\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, y)) \geq 0\,, y \in \mathcal{F},$$

$$\langle \mu^*, g(\overline{x}) \rangle = 0.$$

It is sufficient set $\lambda^* = \alpha\lambda^*$ and we obtain the desirable result.■

Theorem 10. *(Sufficient condition) Assume that in the problem (P) the functions f and g are pre-invex with respect to to the same function η, directionally differentiable and that the set S is invex with respect to η. If there exist $\overline{x} \in \mathcal{F}$ and $(\lambda^*, \mu^*) \in Q^* \times K^*$, with $\lambda^* \neq 0$ such that*

$$(\lambda^* \circ f)'(\overline{x}, \eta(\overline{x}, x)) + (\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, x)) \geq 0, \forall y \in \mathcal{F} \tag{19}$$

$$\langle \mu^*, g(\overline{x}) \rangle = 0. \tag{20}$$

Then $\overline{x}$ is a weak efficient solution of (P).

Proof: Assume the contrary, that is $\overline{x}$ is not a weak efficient solution of (P). Then, there exist $x \in \mathcal{F}$ such that $f(x) \prec_F f(\overline{x})$ and since $\lambda^* \in Q^*, \lambda^* \neq 0$, by using Lemma 4, we have $\lambda^*(f(x) - f(\overline{x})) < 0$ and using Lemma 3, we obtain

$$(\lambda^* \circ f)'(\overline{x}, \eta(\overline{x}, x)) < 0. \tag{21}$$

Also, we have

$$(\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, x)) \leq \mu^* \circ g(x) - \mu^* \circ g(\overline{x}) \leq 0$$

where the first inequality is obtained from Lemma 3 and the second by the feasibility of x and of (20). Consequently, we have

$$(\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, x)) \leq 0. \tag{22}$$

Adding the inequalities (21) and (22), we obtain

$$(\lambda^* \circ f)'(\overline{x}, \eta(\overline{x}, x)) + (\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, x)) < 0.$$

This is a contradiction with (19), because $x \in \mathcal{F}$.

Therefore, $\overline{x}$ is a weak efficient solution for (P). ■

Remark 1. *We observe that* λ^* *can be* 0, *in this case of the problem is called abnormal. To obtain* $\lambda^* \neq 0$ *is necessary some restrictions on the data, maybe the most popular is the following condition:*

Slater regularity condition: $\exists\, x_0 \in \mathcal{F}$ such that $g(x_0) \prec_F 0$.

Lemma 11. *On the hypotheses of the above Theorem, if the Slater regularity condition is verified, then* $\lambda^* \neq 0$.

Proof: In fact, supose that the hypotheses of the Theorem 10 is verified and the Slater regularity conditions is true. If we consider $\lambda^* = 0$, we will prove a contradiction. To done this, we observe that there exists $\mu^* \in K^* \backslash \{0\}$, such that

$$(\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, y)) \geq 0, \forall y \in \mathcal{F} \tag{23}$$

and

$$\langle \mu^*, g(\overline{x}) \rangle = 0. \tag{24}$$

But,

$$\begin{aligned}(\mu^* \circ g)'(\overline{x}, \eta(\overline{x}, x_0)) &\leq \mu^* \circ g(x_0) - \mu^* \circ g(\overline{x}) \\ &= \mu^* \circ g(x_0) < 0\end{aligned} \tag{25}$$

(where the first inequality is consequence of the Lemma 3, the equality is obtained (24) and the last inequality from $g(x_0) \prec_F 0$ e $\mu^* \neq 0$). This is a contradiction with (23). Consequently, $\lambda^* \neq 0$.■

4. Global weak efficiency

In this Section we will consider the following optimization problem

$$\left.\begin{array}{ll}\text{Minimize} & f(x) \\ \\ \text{subject to} & x \in \Gamma\end{array}\right\} \tag{P1}$$

where $f : E \to F$, $\Gamma \subset E$, E and F are Banach spaces, F is ordered by the cone, closed, convex, pointed and with interior nonempty Q.

We will call $x_0 \in \Gamma$ a **global weak efficient solution** for the problem (P1) if it is a weak efficient solution of f on the set Γ, in the sense of the Definition 2.

Also, we will say that $x_0 \in \Gamma$ is **local weak efficient solution** for the problem (P1) if there exists some neighborhood N of x_0 such that x_0 is a weak efficient solution of f on the set $\Gamma \cap N$.

Now, we will proof that if f is a pre-invex function in the problem (P1), then local efficiency implies global efficiency. In fact, we have

Theorem 12. *If f is pre-invex with respect to η and the set Γ is invex with respect to η, then all solution weakly efficient local of (P1) is one solution weakly efficient global of (P1).*

Proof: Assume that the function f is pre-invex on Γ and that $\overline{x} \in \Gamma$ is a local weak efficient solution of (P1) but that is not global.

Then, there exist $x' \in \Gamma$ such that

$$f(\overline{x}) - f(x') \in \operatorname{int} Q. \tag{26}$$

Since f is pre-invex and Γ is invex (with respect to η), there exists a function $\eta : E \times E \to E$ such that $\overline{x} + \alpha\eta(x', \overline{x}) \in \Gamma$ for each $\alpha \in (0, 1)$ and

$$f(\overline{x} + \alpha\eta(x', \overline{x})) \preceq_F \alpha f(x') + (1 - \alpha)f(\overline{x})$$

or equivalently,

$$\alpha f(x') + (1 - \alpha)f(\overline{x}) - f(\overline{x} + \alpha\eta(x', \overline{x})) \in Q$$

or

$$\alpha(f(x') - f(\overline{x})) + f(\overline{x}) - f(\overline{x} + \alpha\eta(x', \overline{x})) \in Q, \forall\alpha \in (0, 1). \tag{27}$$

Since Q is a pointed cone, from (26) and (27) we obtain $\eta(x', \overline{x}) \neq 0$.

We observe

$$\begin{aligned} f(\overline{x}) - f(\overline{x} + \alpha\eta(x', \overline{x})) &= [\alpha(f(x') - f(\overline{x})) + f(\overline{x}) - f(\overline{x} + \alpha\eta(x', \overline{x}))] \\ &\quad +\alpha(f(\overline{x}) - f(x')) \\ &\in Q + \operatorname{int} Q \subset \operatorname{int} Q, \forall\alpha \in (0, 1) \end{aligned}$$

this is contradiction with the optimality of the point $\overline{x}$.■

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