The Relation between Maxwell, Dirac and the Seiberg-Witten Equations*

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Abstract

In this paper we discuss some unusual and unsuspected relations between Maxwell, Dirac and the Seiberg-Witten equations. First we investigate what is now known as the Maxwell-Dirac equivalence (*MDE*) of the first kind. Crucial to that proposed equivalence is the possibility of solving for ψ (a representative on a given spinorial frame of a Dirac-Hestenes spinor field) the equation $F = \psi\gamma_{21}\tilde{\psi}$, where F is a given electromagnetic field. Such non trivial task is presented in this paper and it permits to clarify some possible objections to the *MDE* which claims that no *MDE* may exist, because F has six (real) degrees of freedom and ψ has eight (real) degrees of freedom. Also, we review the generalized Maxwell equation describing charges and monopoles. The enterprise is worth even if there is *no* evidence until now for magnetic monopoles, because there are at least two faithful field equations that have the form of the generalized Maxwell equations. One is the generalized Hertz potential field equation (which we discuss in detail) associated with Maxwell theory and the other is a (non linear) equation (of the generalized Maxwell type) satisfied by

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the 2-form field part of a Dirac-Hestenes spinor field that solves the Dirac-Hestenes equation for a free electron. This is a new and surprising result, which can also be called *MDE* of the second kind. It strongly suggests that the electron is a composed system with more elementary "charges" of the electric and magnetic types. This finding may eventually account for the recent claims that the electron has been splited into two electrinos. Finally, we use the *MDE* of the first kind together with a reasonable hypothesis to give a derivation of the famous Seiberg-Witten equations on Minkowski spacetime. A suggestive physical interpretation for those equations is also given.

Contents

1 Introduction

In ([1]-[5]) using standard covariant spinor fields Campolattaro proposed that Maxwell equations are equivalent to a *non* linear Dirac like equation. The subject has been further developed in ([6],[8]) using the Clifford bundle formalism[1]. The crucial point in proving the mentioned equivalence[2], starts once we observe that to any given *representative* of a Dirac-Hestenes spinor field[3] $\psi \in \sec[\bigwedge^0(M) + \bigwedge^2(M) + \bigwedge^4(M)] \subset \sec \mathcal{C}\ell(M,g)$ there is associated an electromagnetic field $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M,g)$, $(F^2 \neq 0)$ through the Rainich-Misner theorem ([20],[6]-[8])by[4]

$$F = \psi\gamma_{21}\tilde{\psi} \tag{1}$$

Now, since an electromagnetic field F satisfying Maxwell equation[5] has six degrees of freedom and a Dirac-Hestenes spinor field has eight (real) degrees of freedom some authors fell uncomfortable with the approach used in ([7],[8]) where some *gauge conditions* have been imposed on a *nonlinear* equation (equivalent to Maxwell equation), thereby transforming it in a usual *linear* Dirac equation (called the *Dirac-Hestenes equation* in the Clifford bundle formalism).The claim, e.g., in [21] is that the *MDE* found in ([7], [8]) cannot be general. The argument is that the imposition of *gauge conditions* implies that a ψ satisfying Eq.(1) can have only six (real) degrees of freedom, and this

[1]The Clifford bundle formalism and some of their applications has been presented in a series of papers, e.g., ([6]-[19]).

[2]The Maxwel-Dirac equivalence is abreviated as *MDE* when no confusion arises.

[3]For more information see section 2 and for details see ([12],[14]). Paper [12] unfortunately contains many misprints and some errors that are corrected in the recent ([16],[17]), which we claim to clarify in definitive the *ontology* of Dirac-Hestenes spinor fields.

[4]When $F^2 = 0$ the spinor associated with F through eq.(1) must be a Majorana spinor field [6]. In the Clifford bundle formalism, if ψ is a representative of a Dirac-Hestenes spinor, then $\psi_M^{\pm} = \psi\frac{1}{2}(1 \pm \gamma_{01})$ are representatives of Majorana spinors ([14],[15]).

[5]The singular here is appropriate, since we are going to see that in our formalism all the original Maxwell equations are represented with a single equation.

implies that the Dirac-Hestenes equation corresponding to Maxwell equation[6] can be only satisfied by a restricted class of Dirac-Hestenes spinor fields, namely the ones that have six degrees of freedom.

Incidentally, in [21] it is also claimed that the generalized Maxwell equation[7]

$$\partial F = J_e + \gamma_5 J_m \tag{2}$$

describing the electromagnetic field generated by charges and monopoles [9] cannot hold in the Clifford bundle formalism, because according to that author the formalism implies that $J_m = 0$.

In what follows we analyze these claims of [21] and prove that they are wrong (section 3). The reasons for our enterprise is that as will become clear in what follows, understanding of Eqs.(1) and (2) and some reasonable hypothesis permits a derivation and even a possible physical interpretation of the famous Seiberg-Witten monopole equations[22] [8]. So, our plan is the following: first we introduce in section 2 the mathematical formalism used in the paper, showing how to write Maxwell and Dirac equations using Clifford fields. We also introduce Weyl spinor fields and parity operators in the Clifford bundle formalism. In section 3 we prove that given F in Eq.(1) we can solve that equation for ψ, and we find that ψ has eight degrees of freedom, two of them being undetermined, the indetermination being related to the elements of the *stability group* of the spin plane γ_{21}. This is a non trivial and beautiful result which can called *inversion* formula. In section 4 we introduce a *generalized* Maxwell equation and in section 5 we introduce the *generalized* Hertz equation. In section 6 we prove a Dirac-Maxwell equivalence of the *first kind* ([1],[8]), thereby deriving a Dirac-Hestenes equation from the free Maxwell equations. In section 7 we introduce a new form of the Maxwell-Dirac equivalence (called *MDE* of the second kind) different from the one studied in section 6. This new *MDE* of the second kind suggests that the electron is a 'composit' system. To prove the Maxwell-Dirac equivalence of the *second kind* we decompose a Dirac-Hestenes spinor field satisfying a Dirac-Hestenes equation in such a way that it results in a non-linear generalized Maxwell (like) equation (Eq.(141)) satisfied by a certain

[6]There is no misprint here. In the Clifford bundle formalism the traditional Maxwell equations is represent by a unique equation.

[7]In Eq.(2) J_e, $J_m \in \sec \bigwedge^1(M)$.

[8]See also ([23]-[26]) for more details on Seiberg-Witten theory.

Hertz potential field[9,10]. What is nice concerning these ideas is that we are able to show in section 8 that (the analogous on Minkowski spacetime) of the famous Seiberg-Witten monopole equations arises naturally from the *MDE* of the first kind once a reasonable hypothesis is imposed. We also present a coherent interpretation of that equations. Indeed, we prove that when the Dirac-Hestenes spinor field satisfying the first of Seiberg-Witten equations is an eigenvector of the parity operator them that equation describe a pair of massless 'monopoles' of opposite 'magnetic' like charges, coupled together by its interaction electromagnetic field. Finally, in section 9 we present our conclusions.

2 Clifford and Spin-Clifford Bundles

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. (M, g) is a four dimensional time oriented and space oriented Lorentzian manifold, with $M \simeq \mathbb{R}^4$ and $g \in \sec T^{0,2}M$ being a Lorentzian metric of signature (1,3). T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T_x^*M$, $TM = \cup_{x\in M} T_xM$, and $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space . D is the Levi-Civita connection of g, *i.e.*, $Dg = 0$, $\mathbf{R}(D) = 0$. Also $\mathbf{T}(D) = 0$, $\mathbf{R}$ and $\mathbf{T}$ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}\ell(M)$ is the bundle of algebras[11] $\mathcal{C}\ell(M, g) = \cup_{x\in M}\mathcal{C}\ell(T_x^*M)$, where $\forall x \in M, \mathcal{C}\ell(T_x^*M) = \mathcal{C}\ell_{1,3}$, the so called *spacetime algebra*. For any $x \in M$, $\mathcal{C}\ell(T_x^*M)$ as a linear space over the real field $\mathbb{R}$. Moreover, $\mathcal{C}\ell(T_x^*M)$ is isomorphic to the Cartan algebra $\bigwedge(T_x^*M)$ of the cotangent space and $\bigwedge(T_x^*M) = \sum_{k=0}^4 \bigwedge^k(T_x^*M)$, where $\bigwedge^k(T_x^*M)$ is the $\binom{4}{k}$-dimensional space of k-forms. Then, sections of $\mathcal{C}\ell(M, g)$ can be represented as a sum of inhomogeneous differential forms. Let $\langle x^\mu \rangle$ be Lorentz coordinate functions for M and let $\{e_\mu\} \in \sec FM$ (the frame bundle) be an orthonormal basis for TM, i.e., $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Let $\gamma^\nu = dx^\nu \in$

[9]This new equivalence is very suggestive in view of the fact that there are recent (wild) speculations that the electron can be splitted in two components[27] (see also[28]). If this fantastic claim announced by Maris [27] is true, it is necessary to understand what is going on. The new Maxwell-Dirac equivalence presented in section 6 may eventuallly be usefull to understand the mechanism behind the "electron splitting" into electrinos.

[10]A Hertz potential field Π is an object of the same mathamtical nature as an electromagentic field, i.e., $\Pi \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$. .

[11]$\mathcal{C}\ell(M, g)$ is a vector bundle associated to the *orthonormal frame bundle*, i.e., $\mathcal{C}\ell(M, g)$ $= P_{SO_{+(1,3)}} \times_{ad} Cl_{1,3}$ ([16],[17]).

$\sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M,g)$ $(\nu = 0,1,2,3)$ such that the set $\{\gamma^\nu\}$ is the dual basis of $\{e_\mu\}$. Moreover, we denote by $\breve{g}$ the metric in the cotangent bundle.

2.1 Clifford Product

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ and if $\mathcal{C} \in \sec \mathcal{C}\ell(M,g)$ we have

$$\mathcal{C} = s + v_\mu\gamma^\mu + \frac{1}{2!}b_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{3!}a_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + p\gamma^5 \,, \tag{3}$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = dx^0dx^1dx^2dx^3$ is the volume element and s, v_μ, $b_{\mu v}$, $a_{\mu\nu\rho}$, $p \in \sec \bigwedge^0(M) \subset \sec \mathcal{C}\ell(M,g)$.

Let $A_r, \in \sec \bigwedge^r(M), B_s \in \sec \bigwedge^s(M)$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M,g).$,

$$a \cdot b = \frac{1}{2}(ab + ba) = \breve{g}(a,b). \tag{4}$$

We define also the *exterior product* $(\forall r, s = 0,1,2,3)$ by

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s\rangle_{r+s}, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r, \end{aligned} \tag{5}$$

where $\langle\rangle_k$ is the component in $\bigwedge^k(M)$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(M,g)$.

For $A_r = a_1 \wedge ... \wedge a_r$, $B_r = b_1 \wedge ... \wedge b_r$, the scalar product is defined as follows[12],

$$\begin{aligned} A_r \cdot B_r &= (a_1 \wedge ... \wedge a_r).(b_1 \wedge ... \wedge b_r) \\ &= \begin{vmatrix} a_1 \cdot b_1 & .... & a_1 \cdot b_r \\ .... & .... & .... \\ a_r \cdot b_1 & .... & a_r \cdot b_r \end{vmatrix} \end{aligned} \tag{6}$$

[12]Note that some authors define the scalar product of multivectors in such a way that it may differ form a signal from our definition.

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, then $A_r \cdot B_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\mathcal{C}\ell(M, g)$.

For $r \leq s, A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s$ we define the left contraction by

$$\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... b_r)^{\sim} b_{r+1} \wedge ... \wedge b_s \tag{7}$$

where $\sim$ is the reverse mapping (*reversion*) defined by

$$\sim : \sec \bigwedge^{p}(M) \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1 \tag{8}$$

and extended by linearity to all sections of $\mathcal{C}\ell(M, g)$. We agree that for $\alpha, \beta \in \sec \bigwedge^0(M)$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0(M)$, $A_r, \in \sec \bigwedge^r(M), B_s \in \sec \bigwedge^s(M)$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}\ell(M, g)$, i.e., for $A, B \in \sec \mathcal{C}\ell(M, g)$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, r \leq s \tag{9}$$

It is also necessary to introduce in the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r, \in \sec \bigwedge^r(M), B_s \in \sec \bigwedge^s(M)$ then $A_r \llcorner (\alpha B_s) = (\alpha A_r) \llcorner B_s$.

The main formulas used in the Clifford calculus can be obtained from the following ones (where $a \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M, g)$):

$$\begin{aligned}
aB_s &= a \lrcorner B_s + a \wedge B_s, B_s a = B_s \llcorner a + B_s \wedge a, \\
a \lrcorner B_s &= \frac{1}{2}(aB_s - (-)^s B_s a), \\
A_r \lrcorner B_s &= (-)^{r(s-1)} B_r \lrcorner A_s, \\
a \wedge B_s &= \frac{1}{2}(aB_s + (-)^s B_s a), \\
A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + ... + \langle A_r B_s \rangle_{|r+s|} \\
&= \sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}
\end{aligned} \tag{10}$$

2.1.1 Hodge Star Operator

Let $\star$ be the Hodge star operator, i.e., the mapping

$$\star : \bigwedge^{k}(M) \to \bigwedge^{4-k}(M), \; A_k \mapsto \star A_k$$

where for $A_k \in \sec \bigwedge^k(M) \subset \sec \mathcal{C}\ell(M,g)$

$$[B_k \cdot A_k]\tau_g = B_k \wedge \star A_k, \forall B_k \in \sec \bigwedge^k(M) \subset \sec \mathcal{C}\ell(M). \tag{11}$$

$\tau_g \in \bigwedge^4(M)$ is a *standard* volume element. Then we can verify that

$$\star A_k = \widetilde{A}_k \gamma^5. \tag{12}$$

2.1.2 Dirac Operator

Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\bigwedge(M)$. If $A_p \in \sec \bigwedge^p(M) \subset \sec \mathcal{C}\ell(M)$, then $\delta A_p = (-)^p \star^{-1} d \star A_p$, with $\star^{-1}\star =$ identity.

The Dirac operator acting on sections of $\mathcal{C}\ell(M,g)$ is the invariant first order differential operator

$$\partial = \gamma^a D_{e_a}, \tag{13}$$

where $\{e_a\}$ is an arbitrary *orthonormal basis* for $TU \subset TM$ and $\{\gamma_b\}$ is a basis for $T^*U \subset T^*M$ dual to the basis $\{e_a\}$, i.e., $\gamma^b(e_a) = \delta^a_b$, $a,b = 0,1,2,3$. The reciprocal basis of $\{\gamma^b\}$ is denoted $\{\gamma_a\}$ and we have $\gamma_a \cdot \gamma_b = \eta_{ab}$ ($\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$). Also,

$$D_{e_a}\gamma^b = -\omega_a^{bc}\gamma_c \tag{14}$$

Defining

$$\omega_a = \omega_a^{bc}\gamma_a \wedge \gamma_b, \tag{15}$$

we have that for any $A_p \in \sec \bigwedge^p(M)$, $p = 0,1,2,3,4$

$$D_{e_a}A = e_a + \frac{1}{2}[\omega_a, A]. \tag{16}$$

Using Eq.(16) we can show the very important result:

$$\begin{aligned} \partial A_p &= \partial \wedge A_p + \partial \lrcorner A_p = dA_p - \delta A_p, \\ \partial \wedge A_p &= dA_p, \quad \partial \lrcorner A_p = -\delta A_p, \end{aligned} \tag{17}$$

2.2 Dirac-Hestenes Spinor Fields

Now, as is well known, an electromagnetic field is represented by $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M,g)$. How to represent the Dirac spinor fields in this formalism ? We can show that *Dirac-Hestenes* spinor fields, do the job. We give here an elementary introduction to these objects[13] when living on Minkowski spacetime, using the notations of (27). There is a 2 : 1 mapping $\mathbf{L}' : \Theta \to \mathcal{B}$ between $\mathcal{B}$ be the set of all orthonormal ordered vector frames and Θ the set of all *spin frames*[14] of T^*M. Consider the set $\mathcal{S}$ of mappings[15]

$$M \ni x \mapsto u(x) \in \mathrm{Spin}_+(1,3) \tag{18}$$

Choose in a constant spin frame $\{\gamma_a\} \in \mathcal{B}$, $a = 0,1,2,3$ and choose $\Xi_0 \in \Theta$ corresponding to a constant mapping $u_0 \in \mathcal{S}$. By constant we mean that the equation $\gamma_\mu(x) = \gamma_\mu(y)$ (($\mu = 0,1,2,3$) and $u_0(x) = u_0(y)$, $\forall\, x, y \in M$) has meaning due to the usual *affine* structure that can be given to Minkowski spacetime. Given any other basis $\Xi_u \in \mathcal{B}$ we suppose that it is related to Ξ_0 by

$$u_0 \Xi_0 u_0^{-1} = u \Xi_u u^{-1} \tag{19}$$

¿From now on in order to simplify the notation we take $u_0 = 1$. The frame $\mathbf{L}'(\Xi_0) = \{\gamma a\}$ is called the *fiducial* vector frame and Ξ_0 the fiducial *spin* frame. We note that Eq.(19) is satisfied by *two* such u's differing by a signal, and of course, $\mathbf{L}'(\Xi_u) = \mathbf{L}'(\Xi_{-u})$.

Denote by $\hat{\Xi}$ the set of all spin frames. Let,

$$\mathfrak{T} = \{(\Xi_u, \Psi_{\Sigma_u}) \mid u \in \mathcal{S}, \Xi_u \in \Theta, \Psi_{\Xi_u} \in \sec \overset{+}{\bigwedge} M \subset \sec C\ell^+(M,g) \tag{20}$$

where $\overset{+}{\bigwedge} M = \bigwedge^0 M + \bigwedge^2 M + \bigwedge^4 M$

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[13] For the theory of these objects (using vector bundles) on a general Riemann-Cartan manifold see ([17]). Note that papers ([16],[17]) substitute ref.([12]), which must be considered outdated.

[14] As discussed at lenght in ([16],[17]) a spin frame can be thought as a basis of T^*M, such that two ordered basis even if consisting of the same vectors, but differing by a 2π rotation are considered *distinct* and two ordered basis even if consisting of the same vectors, but differing by a 4π rotation are identified.

[15] $Sl(2,\mathbb{C}) \simeq \mathrm{Spin}_+(1,3) \in C\ell^+(1,3)$ is the universal covering group of the homogeneous, restrict and orthocronous Lorentz group. There is a bijection between $\mathcal{S}$ and Θ.

We define an equivalence relation on $\mathfrak{T}$ by setting

$$(\Xi_u, \Psi_{\Xi_u}) \sim (\Xi_{u'}, \Psi_{\Xi_{u'}}) \tag{21}$$

if and only if

$$u\Xi_u u^{-1} = u'^{-1}\Xi_{u'}u',\ \ \Psi_{\Xi_{u'}} = \Psi_{\Xi_u} u u'^{-1}. \tag{22}$$

Definition: Any equivalence class $[\Xi_u, \Psi_{\Xi_u})]$ will be called a Dirac-Hestenes spinor field[16].

Note that the pairs (Ξ_u, Ψ_{Ξ_u}) and $(\Xi_{-u}, -\Psi_{\Sigma_{-u}})$ are equivalent, but the pairs (Ξ_u, Ψ_{Σ_u}) and $(\Xi_{-u}, \Psi_{\Xi_{-u}})$ are not. This distinction is essential in order to give a structure of *linear space* (over the real numbers) to the set $\mathcal{T}$. Indeed, we define a linear structure on $\mathcal{T}$ as follows

$$\begin{aligned} a[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})] + b[(\Xi_{u_2}, \Psi_{\Xi_{u_2}})] &= [(\Xi_{u_1}, a\Psi_{\Xi_{u_1}})] + [(\Xi_{u_2}, b\Psi_{\Xi_{u_2}})], \\ (a+b)[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})] &= a[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})] + b[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})]. \\ a, b &\in \mathbb{R} \end{aligned} \tag{23}$$

We can simplify the notation by recalling that every $u \in \mathcal{S}$ determines, of course, a unique spin frame Ξ_u . Taking this into account we consider the set of all pairs $(u, \Psi_{\Xi_u}) \in \mathcal{S}\times \sec C\ell^+(M, g)$

We define an equivalence relation $\mathcal{R}$ in $\mathcal{S}\times \sec C\ell^+(M, g)$ as follows. Two pairs (u, Ψ_{Ξ_u}), $(u', \Psi_{\Xi_{u'}}) \in \sec \mathcal{S}\times \sec C\ell^+(M, g)$ are equivalent if and only if

$$\Psi_{\Xi_{u'}} u' = \Psi_{\Xi_u} u \tag{24}$$

Of course, $\Xi_{u'} = (v)\Xi_u(v)^{-1}$with $v = (u')^{-1}u \in \mathcal{S}$. Note that the pairs (u, Ψ_{Ξ_u}) and $(-u, -\Psi_{\Xi_u})$ are equivalent but the pairs (u, Ψ_{Ξ_u}) and $(-u, \Psi_{\Xi_u})$ are not.

Denote by $\mathcal{S}\times \sec C\ell^+(M, g)\ /\mathcal{R}$ the quotient set of the equivalence classes generated by $\mathcal{R}$. Their elements will be called Dirac-Hestenes *spinors*. Of course, this is the same definition as above.

¿From now on we simplify even more our notation. In that way, if we take two orthonormal spin frames $\mathbf{L}'(\Xi) = \{\gamma^\mu\}$ and $\mathbf{L}'(\dot{\Xi}) = \{\dot{\gamma}^\mu = R\gamma^\mu\widetilde{R} = \Lambda^\mu_\nu\gamma^\nu\}$ with $\Lambda^\mu_\nu \in \mathrm{SO}^e(1,3)$ and $R(x) \in \mathrm{Spin}^e(1,3)\ \forall x \in M$, $R\widetilde{R} = \widetilde{R}R = 1$,

[16]A more rigorous definition of a *DHSF* as a section of a (right) spin-Clifford bundle is given in [17]. We will not need such a sofistication in the present paper.

then we simply write the relation (Eq.(24)) between representatives of a Dirac-Hestenes spinor field in the two spin frames as the sections ψ_{Ξ} and $\psi_{\dot{\Xi}}$ of $\mathcal{C}\ell^{+}(M,g)$ related by

$$\psi_{\dot{\Xi}} = \psi_{\Xi} R. \tag{25}$$

Recall that since $\psi_{\Xi} \in \sec\sec \overset{+}{\bigwedge} M \subset \sec C\ell^{+}(M,g)$, we have

$$\psi_{\Xi} = s + \frac{1}{2!} b_{\mu\nu}\gamma^{\mu}\gamma^{\nu} + p\gamma^{5}. \tag{26}$$

Note that ψ_{Ξ} has the correct number of degrees of freedom in order to represent a *Dirac* spinor field and recall that the specification of ψ_{Ξ} depends on the frame Σ. To simplify even more our notation, when it is clear which is the spin frame Ξ, and no possibility of *confusion* arises we write simply ψ instead of ψ_{Ξ}.

When $\psi\tilde{\psi} \neq 0$, where $\sim$ is the reversion operator, we can show that ψ has the following canonical decomposition:

$$\psi = \sqrt{\rho}\, e^{\beta\gamma_5/2} R\,, \tag{27}$$

where $\rho,\ \beta \in \sec \bigwedge^{0}(M) \subset \sec \mathcal{C}\ell(M,g)$ and $R(x) \in \mathrm{Spin}^{e}(1,3) \subset \mathcal{C}\ell^{+}_{1,3}$, $\forall x \in M$. β is called the Takabayasi angle. If we want to work in terms of the usual Dirac spinor field formalism, we can translate our results by choosing, for example, the standard matrix representation of the one forms $\{\gamma^{\mu}\}$ in $\mathbb{C}(4)$ (the algebra of the complex 4×4 matrices), and for ψ_{Σ} given by Eq.(15) we have the following (standard) matrix representation [12],[16]):

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^{*} & \psi_3 & \psi_4^{*} \\ \psi_2 & \psi_1^{*} & \psi_4 & -\psi_3^{*} \\ \psi_3 & \psi_4^{*} & \psi_1 & -\psi_2^{\star} \\ \psi_4 & -\psi_3^{*} & \psi_2 & \psi_1^{\star} \end{pmatrix}. \tag{28}$$

where $\psi_k(x) \in \mathbb{C}$, $k = 1,2,3,4$ and for all $x \in M$.

We recall that a *standard* Dirac spinor field is a section Ψ_D of the vector bundle $P_{\mathrm{Spin}^{e}(1,3)} \times_{\lambda} \mathbb{C}(4)$, where λ is the $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$ representation of $SL(2,\mathbb{C}) \sim \mathrm{Spin}^{e}(1,3)$. For details see, e.g.,([16][17]). The relation between

Ψ_D and ψ is given by

$$\Psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} s - ib_{12} \\ -b_{13} - ib_{23} \\ -b_{03} + ip \\ -b_{01} - ib_{02} \end{pmatrix} . \tag{29}$$

where s, $b_{12}, \ldots$ are the real functions in Eq.(26) and $*$ denotes the complex conjugation.

We recall that the even subbundle $\mathcal{C}\ell^+(M,g)$ of $\mathcal{C}\ell(M,g)$ is such that its typical fiber is the Pauli algebra $\mathcal{C}\ell_{3,0} \equiv \mathcal{C}\ell^+_{1,3}$ (which is isomorphic to $\mathbb{C}(2)$, the algebra of 2×2 complex matrices). Elements of $\mathcal{C}\ell^+_{1,3}$ are called *biquaternions* in the old literature. The isomorphism $\mathcal{C}\ell_{3,0} \equiv \mathcal{C}\ell^+_{1,3}$ is exhibited by putting $\vec{\sigma}_i = \gamma_i\gamma_0$, whence $\vec{\sigma}_i\vec{\sigma}_j + \vec{\sigma}_j\vec{\sigma}_i = 2\delta_{ij}$. We recall also that the Dirac algebra is $\mathcal{C}\ell_{4,1} \equiv \mathbb{C}(4)$ and $\mathcal{C}\ell_{4,1} \equiv \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.

Consider the *complexification* $\mathcal{C}\ell_C(M,g)$ of $\mathcal{C}\ell(g)$ called the *complex Clifford bundle*. Then $\mathcal{C}\ell_C(M,g) = \mathbb{C}\otimes\mathcal{C}\ell(M,g)$ and we can verify that the typical fiber of $\mathcal{C}\ell_C(M,g)$ is $\mathcal{C}\ell_{4,1} = \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$, the Dirac algebra. Now let $\{\Delta_0,\Delta_1,\Delta_2,\Delta_3,\Delta_4\} \subset \sec\mathcal{C}\ell_C(M,g)$ be for all $x\in M$ an orthonormal basis of $\mathcal{C}\ell_{4,1}$. We have,

$$\begin{aligned} \Delta_a\Delta_b + \Delta_b\Delta_a = 2g_{ab}\ , & \\ & g_{ab} = diag(+1,+1,+1,+1,-1)\ . \end{aligned} \tag{30}$$

Let us identify $\gamma_\mu = \Delta_\mu\Delta_4$ and call $I = \Delta_0\Delta_1\Delta_2\Delta_3\Delta_4$. Since $I^2 = -1$ and I commutes with all elements of $\mathcal{C}\ell_{4,1}$we identify I with $i = \sqrt{-1}$ and γ_μ with a fundamental set generating the local Clifford algebra of $\mathcal{C}\ell(M,g)$. Then if $\mathcal{A}\in\sec\mathcal{C}\ell_C(M,g)$ we have

$$\mathcal{A} = \Phi_s + A^\mu_C\gamma_\mu + \frac{1}{2}B^{\mu\nu}_C\gamma_\mu\gamma_\nu + \frac{1}{3!}\tau^{\mu\nu\rho}_C\gamma_\mu\gamma_\nu\gamma_\nu + \Phi_p\gamma_5, \tag{31}$$

where Φ_s, Φ_p, A^μ_C, $B^{\mu\nu}_C$, $\tau^{\mu\nu\rho}_C \in \sec\ \mathbb{C}\otimes\bigwedge^0(M) \subset \sec\mathcal{C}\ell_C(M,g)$, *i.e.*, $\forall x\in M$, $\Phi_s(x)$, $\Phi_p(x)$, $A^\mu_C(x)$, $B^{\mu\nu}_C(x)$, $\tau^{\mu\nu\rho}_C(x)$ are complex numbers.

Now, it can be verified that

$$f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_1\gamma_2)\,; \quad f^2 = f\,, \tag{32}$$

is a primitive idempotent field of $\mathcal{C}\ell_C(M,g)$. We can also verify without difficulty that $if = \gamma_2\gamma_1 f$.

Appropriate equivalence classes (see ([16],[17])) of $\mathcal{C}\ell_C(M,g)f$ are representatives of the standard Dirac spinor fields in $\mathcal{C}\ell_C(M,g)$. We can easily show that the representation of Ψ_D in $\mathcal{C}\ell_C(M,g)$ is given by

$$\Psi_D = \psi f \tag{33}$$

where ψ is the Dirac-Hestenes spinor field given by Eq.(26).

2.3 Weyl Spinors and Parity Operator

By definition, $\psi \in \sec \mathcal{C}\ell^+(M,g)$ is a *representative* of a Weyl spinor field ([14],[15]) if besides being a representative of a Dirac-Hestenes spinor field it satisfies

$$\gamma_5\psi = \pm\psi\gamma_{21}, \tag{34}$$

where $\gamma_{21} = \gamma_2\gamma_1$. The positive (negative) "eingestates" of γ_5 will be denoted ψ_+ (ψ_-). For a general $\psi \in \sec \mathcal{C}\ell^+(M,g)$ we can write

$$\psi_\pm = \frac{1}{2}\left[\psi \mp \gamma_5\psi\gamma_{21}\right]. \tag{35}$$

Then,

$$\psi = \psi_+ + \psi_-. \tag{36}$$

The parity operator $\mathbf{P}$ in our formalism is represented in such a way that for $\psi \in \sec \mathcal{C}\ell^+(M,g)$,

$$\mathbf{P}\psi = -\gamma_0\psi\gamma_0 \tag{37}$$

The following Dirac-Hestenes spinor fields are eingestates of the parity operator with eingenvalues ± 1:

$$\begin{aligned}\mathbf{P}\psi^\uparrow &= +\psi^\uparrow,\ \psi^\uparrow = \gamma_0\psi_-\gamma_0 - \psi_-,\\ \mathbf{P}\psi^\downarrow &= -\psi^\downarrow,\ \psi^\downarrow = \gamma_0\psi_+\gamma_0 + \psi_+\end{aligned} \tag{38}$$

2.4 The spin-Dirac Operator

Associated with the covariant derivative operator D_{e_a} (see Eq.(14)) acting on sections of the Clifford bundle there is a spin-covariant derivative operator[17] $D^s_{e_a}$ acting on sections of a right spin-Clifford bundle, such that its sections are Dirac-Hestenes spinor fields. Hopefully it will be not necessary to introduce this concept here. Enough is to say that $D^s_{e_a}$ has a representative om the Clifford bundle, called $D^{(s)}_{e_a}$, such that if ψ_Ξ is a *representative* of a Dirac-Hestenes spinor field we have

$$D^{(s)}_{e_a}\psi_\Xi = e_a(\psi_\Xi) + \frac{1}{2}\omega_a\psi_\Xi, \tag{39}$$

where ω_a has been defined by Eq.(15). The representative of the spin-Dirac operator acting on representatives of Dirac-Hestenes spinor fields is the invariant first order operator given by,

$$\partial^{(s)} = \gamma^a D^{(s)}_{e_a} \tag{40}$$

¿From the definition of spin-Dirac operator we see that if we restrict our considerations to orthonormal coordinate bases $\{\gamma^\mu = dx^\mu\}$ where $\{x^\mu\}$ are global Lorentz coordinates then $\omega_\mu = 0$ and the action of $\partial^{(s)}$ on Dirac-Hestenes spinor fields is the same as the action of ∂ on these fields. We are not going to use this operator in what follows.

2.5 Maxwell and Dirac-Hestenes Equations

With the mathematical tools presented above we have the following Maxwell equation,

$$\partial F = J_e \tag{41}$$

satisfied by an electromagnetic field $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M,g)$, and generated by a current $J_e \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M,g)$.

The Dirac-Hestenes equation in a spin frame Ξ satisfied by a Dirac-Hestenes spinor field $\psi \in \sec[\bigwedge^0(M) + \bigwedge^2(M) + \bigwedge^4(M)] \subset \sec \mathcal{C}\ell(M,g)$

[17] This operator is a *representative* in the Clifford bundle of the legitimate spin-Clifford operator that acts of sections of a vector bundle, called the right spin-Clifford bundle. The details of this theory is given in [17]

is

$$\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0 + \frac{1}{2}\gamma^a\psi\omega_a\gamma^2\gamma^1 = 0. \tag{42}$$

For what follows we restrict our considerations only for the case of orthonormal coordinate basis, in which case the Dirac-Hestenes equation reads

$$\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0 = 0 \tag{43}$$

3 Solution of $\psi\gamma_{21}\tilde{\psi} = F$

We now want solve[18] Eq.(1) for ψ. We are going to show that contrary to the claims of [21] a general solution for ψ has indeed eight degrees of freedom, although two of them are *arbitrary*, i.e., not fixed by F alone. Once we give a solution of Eq.(1) for ψ, the reason for the indetermination of two of the degrees of freedom will become clear. This involves the Fierz identities, boomerangs ([12],[14][31])and the general theorem permitting the reconstruction of spinors from is bilinear covariants.

Let us start by observing that from Eq.(1) and Eq.(27) we can write

$$F = \rho e^{\beta\gamma_5} R\gamma_{21}\tilde{R} \tag{44}$$

Then, defining $f = F/\rho e^{\beta\gamma_5}$ it follows that

$$f = R\gamma_{21}\tilde{R} \tag{45}$$

$$f^2 = -1 \tag{46}$$

Now, since all objects in Eq.(44) and Eq.(45) are even we can take advantage of the isomorphism $\mathcal{C}\ell_{3,0} \equiv \mathcal{C}\ell_{1,3}^{+}$ and making the calculations when convenient in the Pauli algebra. To this end we first write:

We have

$$F = \frac{1}{2}F^{\mu\nu}\gamma_\mu\gamma_\nu,\ F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{47}$$

[18] On euclidian spacetime this equation has been solved using Clifford algebra methods in [29]. On Minkowski spacetime a *particular* solution of an equivalent equation (written in terms of biquaternions) appear in [30].

where (E^1, E^2, E^3) and (B^1, B^2, B^3) are respectively the Cartesian components of the electric and magnetic fields.

We now write F in $\mathcal{C}\ell^+(M,g)$, the even sub-algebra of $\mathcal{C}\ell(M,g)$. The typical fiber of $\mathcal{C}\ell^+(M,g)$(which is also a vector bundle) is isomorphic to the Pauli algebra. We put

$$\vec{\sigma}_i = \gamma_i\gamma_0,\ \mathbf{i} = \vec{\sigma}_1\vec{\sigma}_2\vec{\sigma}_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5. \tag{48}$$

Recall that $\mathbf{i}$ commutes with bivectors and since $\mathbf{i}^2 = -1$ it acts like the imaginary unit $i = \sqrt{-1}$ in $\mathcal{C}\ell^+(M,g)$. From Eq.(47) and Eq.(48) (taking into account our previous discussion) we can write

$$F = \vec{E} + \mathbf{i}\vec{B}, \tag{49}$$

with $\vec{E} = E^i\vec{\sigma}_i$, $\vec{B} = B^j\vec{\sigma}_j$, $i,j = 1,2,3$. We can write an analogous equation for f,

$$f = \vec{e} + \mathbf{i}\vec{b} \tag{50}$$

Now, since $F^2 \neq 0$ and

$$\begin{aligned} F^2 &= F\cdot F + F\wedge F \\ &= -(\vec{E}^2 - \vec{B}^2) + 2\mathbf{i}(\vec{E}\cdot\vec{B}) \end{aligned} \tag{51}$$

the above equations give (in the more general case where both $I_1 = (\vec{E}^2 - \vec{B}^2) \neq 0$ and $I_2 = (\vec{E}\cdot\vec{B}) \neq 0$):

$$\rho = \frac{\sqrt{\vec{E}^2 - \vec{B}^2}}{\cos[arctg2\beta]}, \qquad \beta = \frac{1}{2}\arctan\left(\frac{2(\vec{E}\cdot\vec{B})}{\vec{E}^2 - \vec{B}^2}\right) \tag{52}$$

Also,

$$\vec{e} = \frac{1}{\rho}[(\vec{E}\cos\beta + \vec{B}\sin\beta)], \qquad \vec{b} = (\vec{B}\cos\beta - \vec{E}\sin\beta)] \tag{53}$$

3.1 A Particular Solution

Now, we can verify that[19]

$$L=\frac{\gamma_{21}+f}{\sqrt{2(1-\gamma_5\mathfrak{I})}}=\frac{\vec{\sigma}_3-\mathbf{i}\vec{f}}{\mathbf{i}\sqrt{2(1-\mathbf{i}(\vec{f}\cdot\vec{\sigma}_3)}}, \tag{54}$$

$$\mathfrak{I}=f^{03}-\gamma_5 f^{12}\equiv\vec{f}\cdot\vec{\sigma}_3 \tag{55}$$

is a Lorentz transformation, i.e., $L\tilde{L}=\tilde{L}L=1$. Moreover, L is a particular solution of Eq.(45). Indeed,

$$\frac{\gamma_{21}+f}{\sqrt{2(1-\gamma_5\mathfrak{I})}}\gamma_{21}\frac{\gamma_{12}-f}{\sqrt{2(1-\gamma_5\mathfrak{I})}}=\frac{f[2(1-\gamma_5\mathfrak{I})]}{2(1-\gamma_5\mathfrak{I})}=f \tag{56}$$

Of course, since $f^2=-1$, $\vec{e}^2=\vec{b}^2-1$and $\vec{e}\cdot\vec{b}=0$ and there are only four real degrees of freedom in the Lorentz transformation L. From this result in [21] it is concluded that the solution of the Eq.(1) is the Dirac-Hestenes spinor field

$$\phi=\sqrt{\rho}e^{\gamma_5\beta}L, \tag{57}$$

which has only *six* degrees of freedom and thus is not equivalent to a general Dirac-Hestenes spinor field (the spinor field that must appears in the Dirac-Hestenes equation), which has *eight* degrees of freedom. In this way it is stated in [21] that a the *MDE* of first kind proposed in ([6],[8]) cannot hold[20]. Well, although it is *true* that Eq.(57) is a solution of Eq.(1) it is not a *general* solution, it is only a *particular* solution.

3.2 The General Solution

The general solution R of Eq.(1) is trivially found. It is

$$R=LS, \tag{58}$$

where L is the particular solution just found and S is any member of the *stability group* of γ_{21}, i.e.,

$$S\gamma_{21}\tilde{S}=\gamma_{21},\ S\tilde{S}=\tilde{S}S=1. \tag{59}$$

[19] Observe that this formula is similar, but not equal to Eq.(6) of [21].

[20] There are many other Dirac-like forms of the Maxwell equations published in the literature. All are trivially related in a very simple way and in principle have nothing to do with the two kinds of *MDE* discussed in the present paper. See [31].

It is trivial to find that we can parametrize the elements of the stability group as

$$S = \exp(\gamma_{03}\nu)\exp(\gamma_{21}\varphi), \tag{60}$$

with $0 \leq \nu < \infty$ and $0 \leq \varphi < \infty$. This shows that the most *general* Dirac-Hestenes spinor field that solves Eq.(1) has indeed eight degrees of freedom (as it must be the case, if the claims of ([6],[8]) are to make sense), although two degrees of freedom are arbitrary, i.e., they are like *hidden variables*!

Now, the reason for the *indetermination* of two degrees of freedom has to do with a fundamental mathematical result: the fact that a spinor can only be reconstruct through the knowledge of its bilinear covariants and the Fierz identities. Explicitly, to reconstruct a Dirac-Hestenes spinor field ψ, it is necessary to know also, besides the bilinear covariant given by Eq.(1), the following bilinear covariants,

$$J = \psi\gamma_0\tilde{\psi} \text{ and } K = \psi\gamma_3\tilde{\psi}. \tag{61}$$

Now, J, K and F are related trough the so called Fierz identities,

$$\begin{gathered} J^2 = -K^2 = -\sigma^2 - \omega^2, \\ J \cdot K = 0,\ J \wedge K = -(\omega + \gamma_5)F, \\ \sigma = \rho\cos\beta,\ \omega = \rho\sin\beta. \end{gathered} \tag{62}$$

In the most general case when both σ, ω are not 0 we also have the notable identities first found by Crawford [31][21],

$$\begin{array}{cc} F \llcorner J = \omega K & F \llcorner K = \omega J \\ (\gamma_5 F)\llcorner J = \sigma K & (\gamma_5 F)\llcorner K = \sigma J \\ F \cdot F = \omega^2 - \sigma^2 & (\gamma_5 F)\cdot F = 2\sigma\omega \end{array} \tag{63}$$

$$\begin{array}{ll} JF = -(\omega + \gamma_5\sigma)K, & KF = -(\omega + \gamma_5\sigma)J \\ F^2 = \omega^2 - \sigma^2 - 2\gamma_5\sigma\omega, & F^{-1} = KFK/(\omega^2 + \sigma^2) \end{array} \tag{64}$$

[21] The original derivation given by Crawford uses standard Dirac spinor fields and is a very long one indeed. In the Clifford bundle formalism the derivation is an almost trivial exercise.

Once we know ω, σ, J, K and F we can recover the Dirac-Hestenes spinor field as follows. First, introduce a *boomerang* ([12],[14],[15]) $Z \in \mathcal{C}\ell_C(M,g)$ given by

$$Z = \sigma + J + iF - iK\omega \tag{65}$$

Then, we can construct $\Psi = Zf \in \mathcal{C}\ell_C(M,g)f$ which has the following matrix representation (once the standard representation of the Dirac gamma matrices are used)

$$\hat{\Psi} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \tag{66}$$

Now, it can be easily verified that[22] $\Psi = Zf$ determines the same bilinear covariants as the ones determined by ψ.

Recalling that (a representative) of a Dirac-Hestenes spinor field determines a unique element of $\Phi \in \mathcal{C}\ell_C(M)f$ by $\Phi = \psi f$, then it follows (from Eq.(66) and Eq.(28) that gives the matrix representation of ψ) that we can trivially reconstruct a ψ that solves our problem.

4 The Generalized Maxwell Equation

To comment on the basic error in [21] concerning the Clifford bundle formulation of the generalized Maxwell equation we recall the following.

The generalized Maxwell equation ([9],[31]) which describes the electromagnetic field generated by charges and monopoles, can be written in the Cartan bundle as

$$dF = K_m, \qquad dG = K_e \tag{67}$$

where $F, G \in \bigwedge^2(M)$ and $K_m, K_e \in \bigwedge^3(M)$.

[22] This spinor is not unique. In fact, Z determines a class of elements $Z\eta$ where η is an arbitrary element of $\mathcal{C}\ell_C(M,g)f$ which differs one from the other by a complex phase factor. See ([12], [14],[15]) for details.

These equations are independent of *any* metric structure defined on the world manifold. When a metric is given and the Hodge dual operator $\star$ is introduced it is supposed that in vacuum we have $G = \star F$. In this case putting $K_e = -\star J_e$ and $K_m = \star J_m$, with $J_e, J_m \in \sec \bigwedge^1(M)$, we can write the following equivalent set of equations

$$
\begin{aligned}
dF &= -\star J_m,\ d\star F = -\star J_e, && (68)\\
\delta(\star F) &= J_m,\ \delta F = -J_e && (69)\\
\delta(\star F) &= J_m,\ \delta F = -J_e && (70)\\
dF &= -\star J_m,\ \delta F = -J_e. && (71)
\end{aligned}
$$

Now, supposing that any $\sec \bigwedge^j(M) \subset \sec \mathcal{C}\ell(M,g)$ $(j = 0,1,2,3,4)$ and taking into account Eqs.(13-17) we get Eq.(2) by summing the two equations in (71), i.e.,

$$
(d-\delta)F = J_e + K_m \text{ or } \qquad (d-\delta)\star F = -J_m + K_e, \tag{72}
$$

or equivalently

$$
\partial F = J_e + \gamma_5 J_m \text{ or } \qquad \partial(-\gamma_5 F) = -J_m + \gamma_5 J_e. \tag{73}
$$

Now, writing with the conventions[23] of section 2 ,

$$
F = \frac{1}{2} F^{\mu\nu}\gamma_\mu\gamma_\nu,\ \star F = \frac{1}{2}({}^\star F^{\mu\nu})\gamma_\mu\gamma_\nu, \tag{74}
$$

then generalized Maxwell equations in the form given by Eq.(69) can be written in components (in Lorentz coordinates) as

$$
\partial_\mu F^{\mu\nu} = J_e^\mu,\ \partial_\mu({}^\star F^{\mu\nu}) = -J_m^\mu \tag{75}
$$

Now, assuming as in Eq.(1) that $F = \psi\gamma_{21}\tilde{\psi}$ and taking into account the relation between ψ and the representation of the standard Dirac spinor Ψ_D given by Eq.(29), we can write Eq.(75) as

$$
\begin{gathered}
\partial_\mu \bar{\Psi}_D \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu\right] \Psi_D = 2J_e^\mu, \qquad \partial_\mu \bar{\Psi}_D \hat{\gamma}_5 \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu\right] \Psi_D = -2J_m^\mu, \\
F^{\mu\nu} = \frac{1}{2}\bar{\Psi}_D \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu\right] \Psi,\ ({}^\star F^{\mu\nu}) = \frac{1}{2}\bar{\Psi}_D \hat{\gamma}_5 \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu\right] \Psi_D
\end{gathered} \tag{76}
$$

[23] In Eq.(74) ${}^\star F^{\mu\nu}$ are the components of $\star F$.

The reverse of the first of Eqs.(73) equation reads

$$\widetilde{(\partial F)} = J_e - K_m. \tag{77}$$

First summing, and then subtracting Eq.(2) with E.(67) we get the following equations for $F = \psi\gamma_{21}\tilde{\psi}$,

$$\partial\psi\gamma_{21}\tilde{\psi} + \widetilde{(\partial\psi\gamma_{21}\tilde{\psi})} = 2J_e, \qquad \partial\psi\gamma_{21}\tilde{\psi} - \widetilde{(\partial\psi\gamma_{21}\tilde{\psi})} = 2K_m \tag{78}$$

which is equivalent to Eq.(13) in [21][24]. There, it is observed that J_e is even under reversion and K_m is odd. Then, it is claimed that "since reversion is a purely algebraic operation without any particular physical meaning, the monopolar current K_m is necessarily zero if the Clifford formalism is assumed to provide a representation of Maxwell's equations equation where the source currents J_e and K_m correspond to fundamental physical fields." It is also stated that Eq.(76) and Eq.(78) imposes different constrains on the monopolar currents J_e and K_m.

It is clear that these arguments are fallacious. Indeed, it is obvious that if any *comparison* is to be made, it must be done between J_e and J_m or between K_e and K_m. In this case, it is obvious that both pairs of currents have the same behavior under reversion. This kind of confusion is widespread in the literature, mainly by people that works with the generalized Maxwell equation(s) in component form (Eqs.(75)).

It seems that experimentally $J_m = 0$ and the following question suggests itself: is there any real physical field governed by a equation of the type of the generalized Maxwell equation (Eq.(2)). The answer is *yes*.

5 The Generalized Hertz Potential Equation

In what follows we accept that $J_m = 0$ and take Maxwell equations for the electromagnetic field $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M,g)$ and a current $J_e \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M,g)$ as

$$\partial F = J_e. \tag{79}$$

[24] In [21] $\mathcal{G}$ is used for the three form of monopolar current.

Let $\Pi = \frac{1}{2}\Pi^{\mu\nu}\gamma_\mu\gamma_\nu = \vec{\Pi}_e + \mathbf{i}\vec{\Pi}_m \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M,g)$ be the so called *Hertz potential* ([33],[34]). We write

$$[\Pi^{\mu\nu}] = \begin{bmatrix} 0 & -\Pi_e^1 & -\Pi_e^2 & -\Pi_e^3 \\ \Pi_e^1 & 0 & -\Pi_m^3 & \Pi_m^2 \\ \Pi_e^2 & \Pi_m^3 & 0 & -\Pi_m^1 \\ \Pi_e^3 & -\Pi_m^2 & \Pi_m^1 & 0 \end{bmatrix}. \tag{80}$$

and define the *electromagnetic potential* by

$$A = -\delta\Pi \in \sec \Lambda^1(T^\star M) \subset \sec \mathcal{C}\ell(M,g), \tag{81}$$

Since $\delta^2 = 0$ it is clear that A satisfies the Lorenz gauge condition, i.e.,

$$\delta A = 0. \tag{82}$$

Also, let

$$\gamma^5 S = d\Pi \in \sec \bigwedge{}^3(M) \subset \sec \mathcal{C}\ell(M,g), \tag{83}$$

and call S, the *Stratton potential.* It follows also that

$$d\left(\gamma^5 S\right) = d^2\Pi = 0. \tag{84}$$

But $d(\gamma^5 S) = \gamma^5 \delta S$ from which we get, taking into account Eq.(76),

$$\delta S = 0 \tag{85}$$

We can put Eq.(81) and Eq.(83) in the form of a *single* generalized Maxwell like equation, i.e.,

$$\partial\Pi = (d - \delta)\Pi = A + \gamma^5 S = \mathcal{A}. \tag{86}$$

Eq.(86) is the equation we were looking for. It is a legitimate physical equation. We also have,

$$\Box\Pi = (d - \delta)^2\Pi = dA + \gamma_5 dS. \tag{87}$$

Next, we define the electromagnetic field by

$$F = \partial\mathcal{A} = \Box\Pi = dA + \gamma_5 dS = F_e + \gamma_5 F_m. \tag{88}$$

We observe that,

$$\Box \Pi = 0 \Rightarrow F_e = -\gamma_5 F_m. \tag{89}$$

Now, let us calculate ∂F. We have,

$$\begin{aligned} \partial F &= (d-\delta)F \\ &= d^2 A + d(\gamma^5 dS) - \delta(dA) - \delta(\gamma^5 dS). \end{aligned} \tag{90}$$

The first and last terms in the second line of Eq.(87) are obviously null. Writing,

$$J_e = -\delta dA, \text{and } \gamma^5 J_m = -d(\gamma^5 dS), \tag{91}$$

we get Maxwell equation

$$\partial F = (d-\delta)F = J_e, \tag{92}$$

if and only if the magnetic current $\gamma^5 J_m = 0$, i.e.,

$$\delta dS = 0. \tag{93}$$

a condition that we suppose to be satisfied in what follows. Then,

$$\begin{aligned} \Box A &= J_e = -\delta dA, \\ \Box S &= 0. \end{aligned} \tag{94}$$

Now, we define,

$$F_e = dA = \vec{E}_e + \mathbf{i}\vec{B}_e, \tag{95}$$

$$F_m = dS = \vec{B}_m + \mathbf{i}\vec{E}_m. \tag{96}$$

and also

$$F = F_e + \gamma_5 F_m = \vec{E} + \mathbf{i}\vec{B} = (\vec{E}_e - \vec{E}_m) + \mathbf{i}(\vec{B}_e + \vec{B}_m). \tag{97}$$

Then, we get

$$\Box \vec{\Pi}_e = \vec{E}, \qquad \Box \vec{\Pi}_m = \vec{B}. \tag{98}$$

It is important to keep in mind that:

$$\Box\Pi = 0 \Rightarrow \vec{E} = 0, \text{ and } \vec{B} = 0. \tag{99}$$

Nevertheless, despite this result we have,

Hertz Theorem [25]

$$\Box\Pi = 0 \text{ leads to } \partial F_e = 0 \tag{100}$$

Proof. We have immediately from the above equations that

$$\partial F_e = -\partial(\gamma_5 F_m) = -d(\gamma_5 dS) + \delta(\gamma_5 dS) = \gamma_5 d^2 S - \gamma_5 \delta dS = 0.\blacksquare \tag{101}$$

6 Maxwell Dirac Equivalence of First Kind

Let us consider a *generalized* Maxwell equation

$$\partial F = \mathcal{J}\,, \tag{102}$$

where $\partial = \gamma^\mu \partial_\mu$ is the Dirac operator and $\mathcal{J}$ is the electromagnetic current (an electric current J_e plus a magnetic monopole current $-\gamma_5 J_m$, where J_e, $J_m \in \sec \bigwedge^1 M \subset \mathcal{C}\!\ell(M,g)$). We proved in section 2 that if $F^2 \neq 0$, then we can write

$$F = \psi\gamma_{21}\tilde{\psi}\,, \tag{103}$$

where $\psi \in \sec \mathcal{C}\!\ell^+(M,g)$ is a representative of a Dirac-Hestenes field. If we use Eq.(103) in Eq.(102) we get

$$\partial(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu\partial_\mu(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu(\partial_\mu\psi\gamma_{21}\tilde{\psi} + \psi\gamma_{21}\partial_\mu\tilde{\psi}) = \mathcal{J}. \tag{104}$$

from where it follows that

$$2\gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_2 = \mathcal{J}, \tag{105}$$

[25] Eq.(100) has been called the Hertz theorem in (54,[35]) and it has been used there and and also in [36]-[42] in order to find nontrivial *superluminal* solutions of Maxwell equations.

Consider the identity

$$\gamma^\mu \langle \partial_\mu \psi \gamma_{21} \tilde{\psi} \rangle_2 = \partial \psi \gamma_{21} \tilde{\psi} - \gamma^\mu \langle \partial_\mu \psi \gamma_{21} \tilde{\psi} \rangle_0 - \gamma^\mu \langle \partial_\mu \psi \gamma_{21} \tilde{\psi} \rangle_4, \tag{106}$$

and define moreover the vectors

$$j = \gamma^\mu \langle \partial_\mu \psi \gamma_{21} \tilde{\psi} \rangle_0, \tag{107}$$

$$g = \gamma^\mu \langle \partial_\mu \psi \gamma_5 \gamma_{21} \tilde{\psi} \rangle_0. \tag{108}$$

Taking into account Eqs.(104)-(108), we can rewrite Eq.(104) as

$$\partial \psi \gamma_{21} \tilde{\psi} = \left[\frac{1}{2} \mathcal{J} + (j + \gamma_5 g) \right]. \tag{109}$$

Eq.(109) is a *spinorial representation* of Maxwell equation. In the case where ψ is non-singular (which corresponds to non-null electromagnetic fields) we have

$$\partial \psi \gamma_{21} = \frac{e^{\gamma_5 \beta}}{\rho} \left[\frac{1}{2} \mathcal{J} + (j + \gamma_5 g) \right] \psi. \tag{110}$$

The Eq.(110) representing Maxwell equation, written in that form, does not appear to have any relationship with the Dirac-Hestenes equation (Eq.(43)). However, we shall make some *algebraic* modifications on it in such a way as to put it in a form that suggest a very interesting and *intriguing relationship* between them, and consequently a possible (?) connection between electromagnetism and quantum mechanics.

Since ψ is supposed to be non-singular ($F \neq 0$) we can use the canonical decomposition of ψ and write $\psi = \rho e^{\beta \gamma_5 / 2} R$, with ρ, $\beta \in \sec \bigwedge^0 M \subset \sec \mathcal{C}\ell(M, g)$ and $R \in \mathrm{Spin}_+(1,3)$, $\forall x \in M$. Then

$$\partial_\mu \psi = \frac{1}{2} (\partial_\mu \ln \rho + \gamma_5 \partial_\mu \beta + \Omega_\mu) \psi, \tag{111}$$

where we define the 2-form

$$\Omega_\mu = 2(\partial_\mu R) \tilde{R}. \tag{112}$$

Using this expression for $\partial_\mu \psi$ into the definitions of the vectors j and g (Eqs.(107,108)) we obtain that

$$j = \gamma^\mu (\Omega_\mu \cdot S) \rho \cos \beta + \gamma_\mu [\Omega_\mu \cdot (\gamma_5 S)] \rho \sin \beta, \tag{113}$$

$$g = [\Omega_\mu \cdot (\gamma_5 S)]\rho \cos\beta - \gamma_\mu(\Omega_\mu \cdot S)\rho \sin\beta, \tag{114}$$

where we define the *spin* 2-form S by

$$S = \frac{1}{2}\psi\gamma_{21}\psi^{-1} = \frac{1}{2}R\gamma_{21}\tilde{R}. \tag{115}$$

We now define

$$J = \psi\gamma_0\tilde{\psi} = \rho v = \rho R\gamma^0 R^{-1}, \tag{116}$$

where v is the *velocity field* of the system. To continue, we define the 2-form $\Omega = v^\mu\Omega_\mu$ and the scalars Λ and K by

$$\Lambda = \Omega \cdot S, \tag{117}$$

$$K = \Omega \cdot (\gamma_5 S). \tag{118}$$

Using these definition we have that

$$\Omega_\mu \cdot S = \Lambda v_\mu, \tag{119}$$

$$\Omega_\mu \cdot (\gamma_5 S) = K v_\mu, \tag{120}$$

and for the vectors j and g can be written as

$$j = \Lambda v\rho \cos\beta + K v\rho \sin\beta = \lambda\rho v, \tag{121}$$

$$g = K v\rho \cos\beta - \Lambda v\rho \sin\beta = \kappa\rho v, \tag{122}$$

where we defined

$$\lambda = \Lambda \cos\beta + K \sin\beta, \tag{123}$$

$$\kappa = K \cos\beta - \Lambda \sin\beta. \tag{124}$$

The spinorial representation of Maxwell equation is written now as

$$\partial\psi\gamma_{21} = \frac{e^{\gamma_5\beta}}{2\rho}\mathcal{J}\psi + \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0. \tag{125}$$

If $\mathcal{J} = 0$ (free case)[26] we have that

$$\partial\psi\gamma_{21} = \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0, \tag{126}$$

which is *very* similar to the Dirac-Hestenes equation.

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has *six* degrees of freedom, while a Dirac-Hestenes spinor field has *eight* degrees of freedom and that we proved in section 2 that two of these degrees of freedom are *hidden* variables. We are free therefore to impose two constraints on ψ if it is to represent an electromagnetic field. We choose these two constraints as

$$\partial \cdot j = 0 \text{ and } \partial \cdot g = 0. \tag{127}$$

Using Eqs.(121,122) these two constraints become

$$\partial \cdot j = \rho\dot{\lambda} + \lambda\partial \cdot J = 0, \tag{128}$$

$$\partial \cdot g = \rho\dot{\kappa} + k\partial \cdot J = 0, \tag{129}$$

where $J = \rho v$ and $\dot{\lambda} = (v \cdot \partial)\lambda$, $\dot{k} = (v \cdot \partial)k$. These conditions imply that

$$\kappa\lambda = \lambda\kappa \tag{130}$$

which gives ($\lambda \neq 0$):

$$\frac{\kappa}{\lambda} = const. = -\tan\beta_0, \tag{131}$$

or from Eqs.(123,124):

$$\frac{K}{\Lambda} = \tan(\beta - \beta_0). \tag{132}$$

Now we observe that β is the angle of the duality rotation from F to $F' = e^{\gamma_5\beta}F$. If we perform another duality rotation by β_0 we have $F \mapsto e^{\gamma_5(\beta+\beta_0)}F$, and for the Takabayasi angle $\beta \mapsto \beta + \beta_0$. If we work therefore

[26]There are ([33]-[41]) infinite families of non trivial solutions of Maxwell equations such that $F^2 \neq 0$. These solutions correspond to *subluminal* and *superluminal* solutions of Maxwell equation.

with an electromagnetic field duality rotated by an additional angle β_0, the above relationship becomes

$$\frac{K}{\Lambda} = \tan\beta. \tag{133}$$

This is, of course, just a way to say that we can choose the constant β_0 in Eq.(131) to be zero. Now, this expression gives

$$\lambda = \Lambda\cos\beta + \Lambda\tan\beta\sin\beta = \frac{\Lambda}{\cos\beta}, \tag{134}$$

$$\kappa = \Lambda\tan\beta\cos\beta - \Lambda\sin\beta = 0, \tag{135}$$

and the spinorial representation 126 of the Maxwell equations becomes

$$\partial\psi\gamma_{21} - \lambda\psi\gamma_0 = 0 \tag{136}$$

Note that λ is such that

$$\rho\dot{\lambda} = -\lambda\partial\cdot J. \tag{137}$$

The current $J = \psi\gamma_0\tilde{\psi}$ is not conserved unless λ is constant. If we suppose also that

$$\partial\cdot J = 0 \tag{138}$$

we must have

$$\lambda = \text{const.}$$

Now, throughout these calculations we have assumed $\hbar = c = 1$. We observe that in Eq.(136) λ has the units of $(\text{length})^{-1}$, and if we introduce the constants $\hbar$ and c we have to introduce another constant with unit of mass. If we denote this constant by m such that

$$\lambda = \frac{mc}{\hbar}, \tag{139}$$

then Eq.(136) assumes a form which is identical to Dirac-Hestenes equation:

$$\partial\psi\gamma_{21} - \frac{mc}{\hbar}\psi\gamma_0 = 0. \tag{140}$$

It is true that we didn't prove that Eq.(140) is really Dirac equation since the constant m has to be identified in this case with the electron's mass, and we do not have *any* good physical argument to make that identification, until now. Of course, it is to earlier to know if these results are of some physical value or only a mathematical curiosity. Let us wait...

7 Maxwell-Dirac Equivalence of Second Kind

We now look for a Hertz potential field $\Pi \in \sec \bigwedge^2(M)$ satisfying the following (*non linear*) equation

$$\partial\Pi = (\partial\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_1) + \gamma_5(\partial\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3) \quad (141)$$

where $\mathfrak{G}, \mathfrak{P} \in \sec \bigwedge^0(M)$, and m is a constant. According to section 5 the electromagnetic and Stratton potentials are

$$A = (\partial\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_{1}, \quad (142)$$

$$\gamma_5 S = \gamma_5(\partial\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3), \quad (143)$$

and must satisfy the following subsidiary conditions,

$$\Box(\partial\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_1) = J_e \quad (144)$$

$$\Box(\gamma_5(\partial\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3) = 0, \quad (145)$$

$$\Box\mathfrak{G} + m\partial\cdot\langle\Pi\gamma_{012}\rangle_1 = 0, \quad (146)$$

$$\Box\mathfrak{P} - m\partial\cdot(\gamma_5\langle\Pi\gamma_{012}\rangle_3) = 0. \quad (147)$$

Now, in the Clifford bundle formalism, as we already explained above, the following sum is a legitimate operation

$$\psi = -\mathfrak{G} + \Pi + \gamma_5\mathfrak{P} \quad (148)$$

and according to the results of section 2 defines ψ as a (representative) of some Dirac-Hestenes spinor field. Now, we can verify that ψ satisfies the equation

$$\partial\psi\gamma_{21} - m\psi\gamma_0 = 0 \quad (149)$$

which is as we already know a *representative* of the standard Dirac equation (for a free electron) in the Clifford bundle, which is a Dirac-Hestenes equation (Eq.(43)), written in an orthonormal coordinate spin frame.

The above developments suggest (consistently with the spirit of the generalized Hertz potential theory developed in section 5) the following interpretation. The Hertz potential field Π generates the real electromagnetic field of the electron[27]. Moreover, the above developments suggest that the electron is "composed" of two "fundamental" currents, one of *electric* type and the *other* of magnetic type circulating at the ultra microscopic level, which generate the observed electric charge and magnetic moment of the electron. Then, it may be the case, as speculated by Maris [27], that the electromagnetic field of the electron can be spliced into two parts, each corresponding to a new kind of subelectron type particle, the *electrino*. Of course, the above developments leaves open the possibility to generate electrinos of fractional charges. We still study more properties of the above system in another paper.

8 Seiberg-Witten Equations

The famous Seiberg-Witten monopole equations *read* in the Clifford bundle formalism and on Minkowski spacetime[28] as [27]

$$\begin{cases} \partial\psi\gamma_{21} - A\psi = 0 \\ F = \frac{1}{2}\psi\gamma_{21}\tilde{\psi} \\ F = dA \end{cases} \tag{150}$$

where $\psi \in \sec C\ell^{+}(M,g)$ is a Dirac-Hestenes spinor field, $A \in \sec \bigwedge^1(M) \subset \sec C\ell(M,g)$ is an electromagnetic vector potential and $F \in \sec \bigwedge^2(M) \subset \sec C\ell(M,g)$ is an electromagnetic field.

Our intention in this section is:

(a) To use the Maxwell Dirac-Equivalence of the first kind (proved in section 7) and an additional hypothesis to be discussed below to derive the Seiberg-Witten equations.

(b) to give a (possible) physical interpretation for that equations.

[27] The question of the physical dimension of the Dirac-Hestenes and Maxwell fields is discussed in [8].

[28] The original Seiberg-Witten (monopole) equations have been written in euclidian "spacetime".

8.1 Derivation of Seiberg-Witten Equations

Step 1. We assume that the electromagnetic field F appearing in the second of the Seiberg-Witten equations satisfy the free Maxwell equation, i.e., $\partial F = 0$.

Step 2. We use the Maxwell-Dirac equivalence of the first kind proved in section 6 to obtain Eq.(136),

$$\partial\psi\gamma_{21} - \lambda\psi\gamma_0 = 0 \tag{151}$$

Step 3. We introduce the *ansatz*

$$A = \lambda\psi\gamma_0\psi^{-1}. \tag{152}$$

This means that the electromagnetic potential (in our geometrical units) is identified with a multiply of the velocity field defined through Eq.(116). Under this condition Eq.(151) becomes

$$\partial\psi\gamma_{21} - A\psi = 0, \tag{153}$$

which is the first Seiberg-Witten equation!

8.2 A Possible Interpretation of the Seiberg-Witten Equations

Well, it is time to find an interpretation for Eq.(153). In order to do that we recall from section 2.5 that if $\psi_\pm$ are Weyl spinor fields (as defined through Eq.(34), then $\psi_\pm$ satisfy a Weyl equation, i.e.,

$$\partial\psi_\pm = 0. \tag{154}$$

Consider now, the equation for ψ_+ coupled with an electromagnetic field $A = gB \in \sec \bigwedge^1(M) \subset \sec C\ell(M, g)$, i.e.,

$$\partial\psi_+\gamma_{21} + gB\psi_+ = 0. \tag{155}$$

This equation is invariant under the gauge transformations

$$\psi_+ \mapsto \psi_+ e^{g\gamma_5\theta}; B \mapsto B + \partial\theta. \tag{156}$$

Also, the equation for ψ_- coupled with an electromagnetic field $gB \in \sec \bigwedge^1(M)$ is

$$\partial\psi_-\gamma_{21} + gB\psi_- = 0. \tag{157}$$

which is invariant under the gauge transformations

$$\psi_- \mapsto \psi_- e^{g\gamma_5\theta}; B \mapsto B + \partial\theta. \tag{158}$$

showing clearly that the fields ψ_+ and ψ_- carry *opposite* 'charges'. Consider now the Dirac-Hestenes spinor fields $\psi^\uparrow, \psi^\downarrow$ given by Eq.(38) which are eigenvectors of the *parity* operator and look for solutions of Eq.(153) such that $\psi = \psi^\uparrow$. We have,

$$\partial\psi^\uparrow\gamma_{21} + gB\psi^\uparrow = 0 \tag{159}$$

which decouples in two equations,

$$\partial\psi_+^\uparrow\gamma_{21} + g\gamma_5 B\psi_+^\uparrow = 0;\ \partial\psi_-^\uparrow\gamma_{21} + g\gamma_5 B\psi_-^\uparrow = 0. \tag{160}$$

These results show that when a Dirac-Hestenes spinor field associated with the first of the Seiberg-Witten equations is in an eigenstate of the parity operator, that spinor field describes a *pair* of particles with opposite 'charges'. We interpret these particles[29] as being *massless* 'monopoles' in *auto-interaction*. Observe that our proposed interaction is also consistent with the third of Seiberg-Witten equations, for $F = dA$ implies a *null* magnetic current.

It is now well known that Seiberg-Witten equations have non trivial solutions on Minkowski manifolds (see [25]). From the above results, in particular, taking into account the inversion formula (Eq.(56)) it seems to be possible to find whole family of solutions for the Seiberg-Witten equations, which has been here derived from a Maxwell-Dirac equivalence of first kind (proved in section 6) with the additional hypothesis that electromagnetic potential A is parallel to the velocity field v (Eq.(152)) of the system described by Eq.(116). We conclude that a consistent set of Seiberg-Witten equations on Minkowski spacetime must be

$$\begin{cases} \partial\psi\gamma_{21} - A\psi = 0 \\ F = \frac{1}{2}\psi\gamma_{21}\tilde{\psi} \\ F = dA \\ A = \lambda\psi\gamma_0\psi^{-1} \end{cases} \tag{161}$$

[29] Lochack [42] suggest that an equation equivalent to Eq.(160) describe massless monopoles of opposite 'charges'.

9 Conclusions

In this paper we exhibit two different kinds of possible Maxwell-Dirac equivalences (*MDE*). Although many will find the ideas presented above speculative from the physical point of view, we hope that they may become important, at least from a mathematical point of view. Indeed, not to long ago, researching solutions of the free Maxwell equation ($\partial F = 0$) satisfying the constraint $F^2 \neq 0$ (a necessary condition for derivation of a *MDE* of the first kind) conduced to the *discovery* of families of *superluminal* solutions of Maxwell equations and also of all the main linear relativistic equations of theoretical Physics ([34],[42]). The study of the *MDE* of the second kind reveal an unsuspected interpretation of the Dirac equation, namely that the electron seems to be a composed system build up from the self interaction of two currents of 'electrical' and 'magnetic' types. Of course, it is to earlier to say if this discovery has any physical significance. We showed also, that by using the *MDE* of the first kind together with a reasonable hypothesis we can shed light on the meaning of Seiberg-Witten monopole equations on Minkowski spacetime. We hope that the results just found may be an indication that Seiberg-Witten equations (which are a fundamental key in the study of the topology of four manifolds), is of some importance to Physics.

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