MULTIPLE SOLUTIONS FOR ELLIPTIC PROBLEMS WITH ASYMMETRIC NONLINEARITY

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Abstract. In this paper we establish the existence of multiple solutions for the semilinear elliptic problem

$$\begin{array}{lll} -\Delta u = g(x,u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(x,0) = 0$ and which is asymptotically linear at infinity with jumping nonlinearities. We considered both cases resonant and nonresonant with respect to Fučik Spectrum. We use critical groups to distinguish the critical points.

1. Introduction

Let us consider the problem

$$\begin{array}{lll} -\Delta u = g(x,u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a open bounded domain with smooth boundary $\partial\Omega$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $g(x,0) = 0$, which implies that (1) possesses the trivial solution $u = 0$. We will be interested in nontrivial solutions. Assume that

$$\alpha_\pm = \lim_{t\to\pm\infty} \frac{g(x,t)}{t}, \quad \alpha_\pm \in \mathbb{R}, \quad \text{uniformly in } \ \Omega. \tag{2}$$

Without loss of generality, we assume $\alpha_- \le \alpha_+$. The classical solutions of the problem (1) correspond to critical points of the functional F defined on $H_0^1(\Omega)$, by

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \quad u \in H_0^1(\Omega), \tag{3}$$

where $G(x,t) = \int_0^t g(x,s)ds$. Under the above assumptions $F \in C^2$.

Denote by $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \dots$ the eigenvalues of $(-\Delta, H_0^1)$, where each λ_j occurs in the sequence as often as its multiplicity. We note that the strict inequalities $\lambda_{j-1} < \lambda_j < \lambda_{j+1}$ imply that λ_j is simple.

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It is know that the existence and multiplicity of solutions for (1) strongly rely on the position of the pair $(\alpha_-, \alpha_+) \in \mathbb{R}^2$ with respect to the so called Fučik spectrum of $(-\Delta, H_0^1(\Omega))$ (see [17], where this notion of spectrum was introduced). The latter is defined as

$$\Sigma := \{(\mu, \nu) \in \mathbb{R}^2 \ ; \ \exists\, u \in H_0^1(\Omega) \setminus \{0\}, -\Delta u = \mu u^+ - \nu^-\}, \tag{4}$$

where $u^+ = \max\{u, 0\}$, $u^- = u - u^+$. It is clear that Σ contains the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ and the points (λ_j, λ_j), $j \geq 1$. In the one dimensional case $N = 1$, the set Σ can be easily described (see e.g. [17]).

Let's assume $N \geq 2$ and that $\lambda_{j-1} < \alpha_- \leq \lambda_j = \lambda_{j+k} \leq \alpha_+ < \lambda_{j+k+1}$ for some $j \geq 2$ and $k \geq 0$. It is known that Σ contains at least two paths $c_{ji}(t)$, $i = 1, 2$, with image in $Q = [\lambda_j, \lambda_{j+k+1}[\times]\lambda_{j-1}, \lambda_j]$ and starting at the point (λ_j, λ_j). In fact, $\Sigma \cap Q = \text{range}(c_{j1}) \cup \text{range}(c_{j2})$ if λ_j is simple. We also recall that it may happen that $c_{j1} = c_{j2}$. Otherwise, say, the graph of c_{j1} lies below the graph of c_{j2}. For this and other properties of c_{j1} and c_{j2} we refer to [5], [19], [21] and [26].

Thus, with the above notations, we assume that

$$(\alpha_+, \alpha_-) \ \text{ lies in range}(c_{j1}) \ \text{ (or below it)}. \tag{5}$$

Moreover, we assume the following hypotheses

$$\begin{cases} \frac{g(x,t)}{t} \text{ is strictly increasing for } t \geq 0, \text{ a.e. in } \Omega, \quad \text{and} \\ \frac{g(x,t)}{t} \text{ is strictly decreasing for } t \leq 0, \text{ a.e. in } \Omega. \end{cases} \tag{6}$$

$$\lim_{|t| \to \infty} [tg(x, t) - 2G(x, t)] = \infty, \quad \text{for a.e. } x \in \Omega. \tag{7}$$

Theorem 1.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x, 0) = 0$, which satisfies* (2) *and* (6). *Suppose that there exist $k \geq 2$, $m \geq k + 2$ and r, $\alpha > 0$ such that*

$$\lambda_{k-1} \leq \frac{g(x, t)}{t} \leq \alpha < \lambda_k, \quad \forall\ |t| \leq r; \quad \textit{and} \quad \lambda_{m-1} < \alpha_- \leq \lambda_m \leq \alpha_+ < \lambda_{m+1}.$$

Assume that (5) *hold with $j = m$. Moreover, if $(\alpha_+, \alpha_-) \in c_{m1}$ assume* (7). *Then problem* (1) *has at least two nontrivial solutions.*

Theorem 1.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x, 0) = 0$, which satisfies* (2) *and* (6). *Suppose that there exists $m > 3$, $k \geq 0$ and r, $\alpha > 0$ such that*

$$\frac{g(x, t)}{t} \leq \alpha < \lambda_1, \quad \forall\ |t| \leq r; \quad \textit{and} \quad \lambda_{m-1} < \alpha_- \leq \lambda_m = \lambda_{m+k} \leq \alpha_+ < \lambda_{m+k+1}.$$

Assume that (5) *hold with $j = m$. Moreover, if $(\alpha_+, \alpha_-) \in c_{m1}$ assume* (7). *Then problem* (1) *has at least four nontrivial solutions, one of those changing sign, another one positive and a third one negative.*

Remark 1.1. With the above conditions the functional F, defined by (3), has a critical point of saddle point type, see [15]. Without the jumping nonlinearity the above results have been proved by the author in [24]. Dancer and Zhang, in [14], have proved that problem (1) has at least one sign changing solution, one positive solution, and one negative solution, under the hypotheses of Theorem 1.2.

Now consider the autonomous problem

$$\begin{aligned} -\Delta u = g(u) \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{8}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(0) = 0$. Assume that

$$\begin{cases} g(t) \text{ is convex if } t \geq 0, \quad \text{and} \\ g(t) \text{ is concave if } t \leq 0. \end{cases} \tag{9}$$

The latter conditions was assumed by several authors, e.g. [1], [8] and [23] for the case when $[\alpha_-, \alpha_+] \cap \{\lambda_j\} = \emptyset$. In [6] and [7], the authors has considered this problem when $]\alpha_-, \alpha_+[\cap \{\lambda_j\} \neq \emptyset$.

Theorem 1.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(0) = 0$, which satisfies* (9). *Suppose that there exist $k \geq 2$ and $m \geq k + 2$ such that $\lambda_{k-1} \leq g'(0) < \lambda_k$ and*

$$\lambda_{m-1} < \alpha_- = \lim_{t \to -\infty} g'(t) \leq \lambda_m \leq \alpha_+ = \lim_{t \to +\infty} g'(t) < \lambda_{m+1}.$$

Assume that either (α_+, α_-) lies in c_{m1} or below it. Then problem (8) *has at least two nontrivial solutions.*

In fact, the above hypothesis on the convexity of g implies that (see Proposition 2.1)

$$\lim_{|t| \to \infty} [tg(t) - 2G(t)] = \infty.$$

Hence the previous theorem is a corollary of Theorem 1.1.

Remark 1.2. In [4], Bartsch, Chang & Wang showed that if $g'(t) > g(t)/t \; \forall \; t \neq 0$ and

$$g'(0) < \lambda_1 < \lambda_2 \leq \lambda_k < \lim_{|t| \to \infty} g'(t) < \lambda_{k+1}, \;\; (k > 2),$$

then problem (8) has at least four nontrivial solution, two of these solutions changing sign, one is positive and another one is negative. In [7], the authors have proved that if $tg''(t) > 0$ and $\ell_\pm > \lambda_2$, then (8) has at least three nontrivial solutions, one of these solutions change sign. The next result is a corollary of Theorem 1.2.

Theorem 1.4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(0) = 0$, which satisfies* (9). *Suppose that $g'(0) < \lambda_1$ and that exist $m > 3$ and $k \geq 0$ such that*

$$\lambda_{m-1} < \alpha_- = \lim_{t \to -\infty} \frac{g(t)}{t} \leq \lambda_m = \lambda_{m+k} \leq \alpha_+ = \lim_{t \to +\infty} \frac{g(t)}{t} < \lambda_{m+k+1}.$$

Assume that either (α_+, α_-) lies in c_{m1} or below it. Then problem (8) *has at least four nontrivial solutions, one of those change sign, one is positive and another one is negative.*

Remark 1.3. The functional in the nonresonant case satisfies the Palais-Smale Condition, (PS) in short, and the difficulty in the resonant case is the lack of a (PS) condition. But if the function g satisfies

$$\lim_{|t| \to \infty} [tg(x,t) - 2G(x,t)] = \infty, \quad \text{uniformly in } \Omega,$$

then in [13], Costa & Cuesta showed that this condition is sufficient to obtain a weak version of the (PS) condition, namely the (C) condition, which was introduced by Cerami in [9]. The (C) condition was used by Bartolo, Benci & Fortunato in [2] to prove a general minimax theorem (see [25] for these results with the (PS) condition). The so called Second Deformation Lemma, proved by Chang (see [11]), has a version with the Cerami condition replacing the usual (PS) condition, as proved by Silva in [27].

In the section 4 we make some remarks on the one dimensional case, $N = 1$, under either Dirichlet or periodic boundary conditions. In [20] and [29], the authors study the asymptotically linear case (only the Dirichlet case). Our conditions are new in the asymptotically linear case.

2. Preliminary Lemmas

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $g(x, 0) = 0$. Suppose that there exist $k \geq 2$ and $m \geq k + 2$ such that

$$\begin{gathered} \lambda_{k-1} \leq g'(x,0) < \lambda_k \\ \lambda_{m-1} < \lim_{t\to-\infty} \frac{g(x,t)}{t} \leq \lambda_m \leq \lim_{t\to+\infty} \frac{g(x,t)}{t} < \lambda_{m+1}, \end{gathered} \tag{10}$$

where the limits are uniform for x in Ω.

Let $H = H_0^1(\Omega)$ and denote the norms in $H_0^1(\Omega)$ and $L^2(\Omega)$ by $||\cdot||$ and $|\cdot|_2$, respectively. Let H_1, H_2 and H_3 be the subspaces of H spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_1, \ldots, \lambda_{k-1}\}$, $\{\lambda_k, \ldots, \lambda_{m-1}\}$ and $\{\lambda_m, \ldots\}$, respectively ($\dim H_2 \geq 2$ provided $m \geq k + 2$).

The proof of the lemmas below can be found in [24]. Let F be defined as in (3).

Lemma 2.1. *Under the assumptions above and the hypothesis* (6)*, the following statements hold:*

(i) *There are $r > 0$ and $a > 0$ such that $F(u) \geq a$ for all $u \in H_2 \oplus H_3$ with $||u|| = r$;*
(ii) *$F(u) \to -\infty$, as $||u|| \to \infty$, for $u \in H_1 \oplus H_2$; and*
(iiii) *$F(u) \leq 0$ for all $u \in H_1$.*

Let u_0 be a critical point of F, defined by (3). The Morse index $\mu(u_0)$ of u_0 measures the dimension of the maximal subspace of $H = H_0^1(\Omega)$ on which $F''(u_0)$ is negative definite. We denote the dimension of the kernel of $F''(u_0)$ by $\nu(u_0)$. The next lemma evaluates $\nu(u_0)$ for a nonzero critical point of F. The proof can be found in [24].

Lemma 2.2. *Under the hypotheses of Lemma 2.1, $\nu(u_0) \leq m - k$ provided that u_0 is a nonzero critical point of F defined in* (3)*.*

Now we observe a compactness condition for the functional F defined by (3), in the resonant case.

Consider $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and $G(x,t) = \int_0^t g(x,s)ds$ such that

$$\lambda_{j-1} \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{j+1}, \quad \text{uniformly in } \Omega\,; \tag{11}$$

there exists $C(x) \in L^1(\Omega)$ such that

$$tg(x,t) - 2G(x,t) \geq C(x), \quad \forall\, t \in \mathbb{R}, \text{ a.e } x \in \Omega\,; \tag{12}$$

and

$$\lim_{|t|\to\infty} [tg(x,t) - 2G(x,t)] = \infty, \quad \text{a.e } x \in \Omega. \tag{13}$$

In [13], Lemma 2.2, it was shown that the assumptions (11), (12) and (13) are enough to prove that functional the F, defined by (3), satisfies the Cerami condition (see [18]). Note that the hypothesis (6) implies (12) with $C(x) = 0$.

In order to prove that Theorems 1.3 and 1.4 follow from Theorems 1.1 and 1.2, respectively, we have to prove that the function g satisfies (13).

Proposition 2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonlinear function of class C^1, $g(0) = 0$, which satisfies*

$$\begin{cases} g(t) \text{ is convex if } t \geq 0 \quad \text{and} \\ g(t) \text{ is concave if } t \leq 0. \end{cases}$$

Moreover, assume that $g(t)/t$ is bounded. Then

$$\lim_{|t|\to\infty} [tg(t) - 2G(t)] = \infty. \tag{14}$$

Proof. Fix $t > 0$, and note that

$$\frac{1}{2}[tg(t) - 2G(t)] = \int_0^t \Big(\frac{g(t)}{t}s - g(s)\Big)ds$$

The convexity of g gives that $(g(t)/t)s > g(s)$ for $s \in (0,t)$. Denote by A_t the region of the plane between the line $s \mapsto (g(t)/t)s$ and $s \mapsto g(s)$ in $(0,t)$. Let $s(t) \in (0,t)$ defined by

$$\frac{g(t)}{t}s(t) - g(s(t)) = \max_{s\in(0,t)} \Big(\frac{g(t)}{t}s - g(s)\Big),$$

and the triangle $\triangle_t$ with vertices $(0,0)$, $(s(t), g(s(t)))$ and $(t, g(t))$. We have $\triangle_t \subset A_t$ by convexity of g, hence

$$|\triangle_t| \leq \frac{1}{2}[tg(t) - 2G(t)].$$

Therefore the Proposition follows from

Claim: $|\triangle_t| \to \infty$, as $t \to \infty$.

In fact, the height of $\triangle_t$, with reference to the basis $b_t = [(0,0),(t,g(t))]$, is

$$h(t) = \Big[\frac{g(t)}{t}s(t) - g(s(t))\Big] \cos\Big(\arctan\Big(\frac{g(t)}{t}\Big)\Big).$$

Hence $\liminf_{t\to\infty} h(t) > 0$, since $g(t)/t$ is bounded; and $b_t \to \infty$ as $t \to \infty$. The claim is proved. The argument with $t < 0$ is entirely similar and the proof of proposition is complete. □

Lemma 2.3. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying: $g(x,t)/t$ is bounded, $g(x,t) = 0$ for all $t \leq 0$, and*

$$\lambda_j \leq L(x) = \lim_{t\to\infty} \frac{g(x,t)}{t} \leq \lambda_{j+1}, \quad j \geq 2. \tag{15}$$

Then the C^{2-0}-functional $F_+ : H_0^1 \to \mathbb{R}$ defined by

$$F_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,t)dx,$$

satisfies the (PS) condition.

Proof. Let $\{u_n\} \in H_0^1$ be a sequence such that $\{F_+(u_n)\}$ is bounded, and $||F'_+(u_n)|| \to 0$ as $n \to \infty$. It follows that for all $\varphi \in H_0^1$ we have

$$< F'_+(u_n), \varphi >= \int_\Omega \nabla u \nabla \varphi - \int_\Omega g(x,u_n)\varphi dx \to 0, \quad as\ n \to \infty. \tag{16}$$

Set $\varphi = u_n$; we have

$$||u_n||^2 \leq \int_\Omega g(x,u_n)u_n dx + O(||u_n||) \leq C|u_n|_2^2 + O(||u_n||).$$

Therefore, we need to show that $\{|u_n|_2\}$ is bounded, which implies that $\{||u_n||\}$ is bounded. Since Ω is bounded and g is subcritical, then if $\{||u_n||\}$ is bounded, by the compactness of Sobolev embedding and by the standard processes we know that there exists a subsequence of $\{u_n\}$ in H_0^1 which converges strongly, hence the Lemma is proved.

Assume by contradiction that $|u_n|_2 \to \infty$ as $n \to \infty$. Let $v_n = u_n/|u_n|_2$. Then $|v_n|_2 = 1$ and $\{||v_n||\}$ is bounded. We can assume that $v_n \to v$ weakly in H_0^1, strongly in L_2 and a.e. in Ω. Thus, $u_n(x) \to \infty$ a.e. in Ω. From (16) it follows that

$$\int_\Omega [\nabla v \nabla \varphi - L(x)v^+\varphi]dx, \qquad \forall\ \varphi \in H_0^1, \tag{17}$$

where $v^+(x) = \max\{0, v(x)\}$. By the regularity theory we have

$$-\Delta v = L(x)v^+ \quad \text{in } \Omega.$$

By the maximum principle and by the unique continuation property, $v = v^+ \geq 0$ and $L \equiv \lambda_j$ or $L \equiv \lambda_{j+1}$. Since, $j \geq 2$, $v \equiv 0$, which contradicts $|v|_2 = 1$. The proof is completed. □

3. Proofs of the main Theorems

It follows from [13] that the functional F, defined by (3), satisfies the (C) condition (or the (PS) condition on the nonresonant case, see Lemma 6.3 in [16]). In this section some classical definitions and results from Morse Theory are used, these results can be found in [11] and [22] (see [28] for some results with Cerami condition). Without loss of generality, we assume that F has only a finite number of critical points.

We first observe a result proved in [15], that says that problem (1) has a solution. Now we consider a hypothesis analogue to the condition (7).

$$\lim_{|t|\to\infty} [tg(x,t) - 2G(x,t)] = -\infty, \quad \text{for a.e. } x \in \Omega. \tag{18}$$

Theorem 3.1 (Theorems 2.2 and 3.5, [15])**.** *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which satisfies* (2)*. Suppose that there exists $j \geq 2$ and $k \geq 0$, such that*

$$\lambda_{j-1} < \alpha_- \leq \lambda_j = \lambda_{j+k} \leq \alpha_+ < \lambda_{j+k+1}.$$

Moreover, if either

(i) $(\alpha_+, \alpha_-) \in c_{j1}$*, or below it, suppose* (7) *and* $tg(x,t) - 2G(x,t) \geq C(x)$*,* $C(x) \in L^1(\Omega)$*; or*

(ii) $(\alpha_+, \alpha_-) \in c_{j2}$*, or above it, suppose (*18) and $tg(x,t) - 2G(x,t) \leq C(x)$, $C(x) \in L^1(\Omega)$.

Then problem (1) *has a solution.*

If $(\alpha_+, \alpha_-) \notin \Sigma$ the hypotheses (7) or (18) is not necessary.

Consider the decomposition

$$H_0^1(\Omega) = V_1 \oplus V_2,$$

where V_1 is the subspace of $H_0^1(\Omega)$ spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_1, ..., \lambda_l\}$, where $l = j-1$ in the case (i) or $l = j+k$ in the case (ii). For every $\epsilon > 0$ small enough there exists a homeomorphism γ_0 in $H_0^1(\Omega)$ such that if

$$S = \gamma_0(V_2), \quad A = R\gamma_0(B_1) \text{ and } \partial A = R\gamma_0(\partial B_1),$$

where B_1 denotes the unit closed ball in V_1 centered of 0, and large R (depending on ϵ); then there exist constants a and b such that

$$\sup_{\partial A} F_\epsilon^\pm < a \leq \inf_S \leq \sup_A F_\epsilon^\pm \leq b, \tag{19}$$

where $F_\epsilon^\pm = F(u) \pm 2\epsilon||u||^2$ ($+$ in the case (i) and $-$ in the case (ii)). It is observed in [15] that F_ϵ (here F_ϵ denote F_ϵ^+ or F_ϵ^-) also satisfies the Cerami condition, as long as $0 < \epsilon < 1/4$. This, together with (19) implies that F_ϵ has a critical point u_ϵ, which satisfies

$$a \leq F_\epsilon(u_\epsilon) \leq b \text{ and } F_\epsilon'(u_\epsilon) = 0.$$

As in [15] we have that $u_{\epsilon_n} \to u$ in $H_0^1(\Omega)$ along some sequence $\epsilon_n \to 0$. Clearly u is a critical point of F.

Remark 3.1. If g is C^1, then the functionals F_ϵ are of class C^2. Moreover, the sets ∂A and S homologically link and A is a l-topological ball ($l = j-1$ in the case (i) and $l = j+k$ in the case (ii)). Then by (19, the Theorems II 1.5 and II 1.5 in [11] implies that (see [28] for Cerami condition)

$$C_l(F_\epsilon, u_\epsilon) \neq 0.$$

Since $u_{\epsilon_n} \to u$ and $F_{\epsilon_n} \to F$ in $C^1(B_\rho(u), \mathbb{R})$ for some $\rho > 0$, by the continuity of the critical groups (see Theorem I 5.6 in [11]) we have

$$C_l(F, u) \neq 0.$$

Proof of Theorem 1.1. Let H_i, $i = 1,\ 2,\ 3$, be as in Lemma 2.1. Consider

$$S_1 = B_r \cap (H_2 \oplus H_3) \text{ and } D_1 = \{v + te\ ;\ v \in H_1,\ 0 \le t \le R,\ ||v + te|| \le R\},$$

where B_r denotes the closed ball with radius r centered of 0, and $e \in H_2$ is chosen such that

$$F(u) > 0 \ \ \forall\ u \in S_1 \ \text{ and } \ F(u) \le 0 \ \ \forall\ u \in \partial D_1, \tag{20}$$

this is possible by (i) and (iii) in Lemma 2.1. Since ∂D_1 and S_1 homologically link and D_1 is a k-topological ball, by (20) we have $H_k(F_b, F_0) \neq 0$, where $b > \max\{F(u) \mid u \in D\}$ (see Theorem II 1.1' in [11]). Hence we can conclude, by Theorem II 1.5 in [11], that there exists u_1 critical point of F, such that

$$C_k(F, u_1) \neq 0. \tag{21}$$

By the Remark 3.1 we have that there exist u_2 critical point of F, such that

$$C_{m-1}(F, u_2) \neq 0. \tag{22}$$

Now we have to prove that $u_1 \neq u_2$, and are nontrivial. Note that 0 is a critical point of F and $\mu(0) + \nu(0) \le k - 1$. By Shifting Theorem (see [11]), $C_p(F, 0) = 0$ for all $p \ge k$. So u_1 and u_2 are nontrivial, by (21) and (22). Again by Shifting Theorem we have, either

(i) $C_p(F, u_1) = \delta_{p\mu(u_1)}$, or
(ii) $C_p(F, u_1) = \delta_{p(\mu(u_1)+\nu(u_1))}$, or
(iii) $C_p(F, u_1) = 0$ if $p \notin (\mu(u_1), (\mu(u_1) + \nu(u_1))$.

If (i) or (ii) hold, then $C_{m-1}(F, u_1) = 0$ by (21) provided $m - 1 > k$. If (iii) hold then $k > \mu(u_1)$ by (21) and hence $m - 1 = k - 1 + m - k \ge \mu(u_1) + \nu(u_1)$ by Lemma 2.2, again $C_{m-1}(F, u_1) = 0$ by (iii). Therefore $u_1 \neq u_2$ by (22).
The proof of Theorem 1.1 is finished. □

Proof of Theorem 1.2. Set

$$g_+(x, t) = \begin{cases} g(x, t), & t \ge 0, \\ 0, & t \le 0, \end{cases}$$

and consider the problem

$$\begin{aligned} -\Delta u = g_+(x, u) \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{23}$$

Define

$$F_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G_+(x, u)dx, \quad u \in H_0^1(\Omega).$$

Then $F_+ \in C^{2-0}$ and, by Lemma 2.3, satisfies (PS) condition.

Since $g'(x, 0) < \lambda_1$, $u = 0$ is a strictly local minimum of F_+. Let $\varphi_1 > 0$ to be the first eigenfunction of (Δ, H_0^1), and consider $\gamma > \lambda_1$ such that $G_+(x, t) \ge (\gamma/2)t^2 - C$ for $t > 0$.

Then

$$\begin{aligned} F_+(s\varphi_1) &= \frac{s^2}{2}\int_\Omega |\nabla\varphi_1|^2 dx - \int_\Omega G_+(x, s\varphi_1)dx \\ &\le \frac{\lambda_1 s^2}{2}\int_\Omega \varphi_1^2 dx - \frac{\gamma s^2}{2}\int_\Omega \varphi_1^2 dx + C \\ &= \frac{s^2(\lambda_1-\gamma)}{2}\int_\Omega \varphi_1^2 dx + C \to -\infty, \quad as\ s\to\infty. \end{aligned}$$

By the mountain pass theorem, F_+ has a nontrivial critical point u_+. By the maximum principle, $u_+ > 0$. Therefore u_+ is a critical point of the functional F defined by (3). Similarly, we get a negative critical point u_- of F. Moreover, as in [12], we have

$$\text{rank } C_p(F_\pm|_{C_0^1}, u_\pm) = \delta_{p1}.$$

Thus,

$$\text{rank } C_p(F|_{C_0^1}, u_\pm) = \text{rank } C_p(F_\pm|_{C_0^1}, u_\pm) = \delta_{p1} \quad \forall\ p = 0,\ 1,\ 2, ...$$

Again by the Remark 3.1 there exists a nontrivial solution u such that

$$C_{m-1}(F, u) \neq 0, \quad \text{where } m > 3.$$

By Theorem 1 in [10], we have

$$C_{m-1}(F|_{C_0^1}, u) = C_{m-1}(F, u).$$

Therefore u is a third nontrivial solution.

So the Theorem follows from the next claim.

Claim: (1) has a sign changing solution w such that

$$C_p(F, w) = \delta_{p2}\mathbb{Z}.$$

Proof: We use the notation as in [4].

Let $P = \{u \in X = C_0^1(\Omega); u \geq 0\}$, $D = P \cup (-P)$, $\dot{D}$ and φ_i the normalized eigenfunction associated to λ_i, $i = 1, 2$; we have $\varphi_1 \in \overset{\circ}{P}$.

The main ingredient in the proof of the *Claim* is the negative gradient flow φ^t of F in H, that is,

$$\frac{d}{dt}\varphi^t = -\nabla F \circ \varphi^t, \quad \varphi^0 = \text{id}.$$

We have that $\varphi^t(u) \in X$ for $u \in X$ and φ^t induces a continuous (local) flow on X which we continue to denote by φ^t. The main order related property of φ^t is that P and $-P$ are positively invariant (by $g(x,t)t \geq 0$). F has the retracting property on X (see [14]).

Now the proof follows as in Theorem 3.6 in [4] (see [24]). We sketch it briefly for completeness. Here we denote by $F^a = \{u \in X;\ F(u) \leq a\}$.

As $k > 2$ by (ii) in Lemma 2.1 there exists $R > 0$ such that $F(u) < 0$ for any $u \in \text{span}\{\varphi_1, \varphi_2\}$ with $||u|| \geq R$. Now we set

$$B = \{s\varphi_1 + \varphi_2\ ;\ |s| \leq R,\ 0 \leq t \leq R\}$$

and

$$\partial B = \{s\varphi_1 + \varphi_2 \ ; \ |s| = R \text{ or } t \in \{0, R\}\}.$$

We have $\partial B \subset F^0 \cup D$. Let $\beta = \max F(B)$ so that $(B, \partial B) \hookrightarrow (F^\beta \cup D, F^0 \cup D)$. Let $\xi_\beta \in H_2(F^\beta \cup D, F^0 \cup D)$ be the image of $1 \in \mathbb{Z} = H_2(B, \partial B)$ under the homomorphism

$$\mathbb{Z} = H_2(B, \partial B) \to H_2(F^\beta \cup D, F^0 \cup D)$$

induced by the inclusion. For $\gamma \leq \beta$ let

$$j_\gamma : H_2(F^\gamma \cup D, F^0 \cup D) \to H_2(F^\beta \cup D, F^0 \cup D)$$

be also induced by the inclusion. Now we define

$$\Gamma = \{\gamma \leq \beta \ ; \ \xi_\beta \in \text{image } (j_\gamma)\}$$

and $c = \inf \Gamma$. It is a critical value by the next lemma and standard deformation arguments.

Lemma 3.1. $\xi_\beta \neq 0$.

In fact, let $e_1 \in \overset{\circ}{P}$ be the first eigenvalue of

$$\begin{aligned} -\Delta u - g'(x,0)u = \lambda u \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned}$$

and set $X_1 = \text{span}\{e_1\}$, $X_2 = X_1^\perp \cap X$. We have $\inf F(X_2 \cap \partial B_\rho) \geq \alpha > 0$ for some $\rho > 0$ small. This implies

$$(B, \partial B) \subset (F^\beta \cup D, F^0 \cup D) \subset (X, X \setminus X_2 \cap \partial B_\rho).$$

Therefore the lemma follows of that the homeomorphism

$$H_2(B, \partial B) \to H_2(X, X \setminus X_2 \cap \partial B_\rho)$$

induced by inclusion is nontrivial (it is showed in [4]).

As a consequence of previous lemma we have $0 \notin \Gamma$ because $j_0 = 0$. As $F^0 \cup D$ is a strong deformation retract of $F^\gamma \cup D$ for $\gamma > 0$ small enough, we have $c > 0$. Clearly $\beta \in \Gamma$, hence $c \in (0, \beta]$.

We choose $\epsilon > 0$ small enough. Consider the commutative diagram

$$\begin{array}{ccc} H_2(F^{c-\epsilon} \cup D, F^0 \cup D) & & \\ \Big\downarrow j & \overset{j_{c-\epsilon}}{\searrow} & \\ H_2(F^{c+\epsilon} \cup D, F^0 \cup D) & \overset{j_{c+\epsilon}}{\longrightarrow} & H_2(F^\beta \cup D, F^0 \cup D) \\ \Big\downarrow & & \\ H_2(F^{c+\epsilon} \cup D, F^{c-\epsilon} \cup D) & & \end{array}$$

Since $c + \epsilon \in \Gamma$ there exists $\xi_{c+\epsilon} \in H_2(F^{c+\epsilon} \cup D, F^0 \cup D)$ with $j_{c+\epsilon}(\xi_{c+\epsilon}) = \xi_\beta$. Now $\xi_{c+\epsilon} \notin$ image $(j_{c-\epsilon})$ because $c - \epsilon \notin \Gamma$. Therefore the exactness of the left column yields $H_2(F^{c+\epsilon} \cup D, F^{c-\epsilon} \cup D) \neq 0$. This implies that there exists a critical point w such that $w \notin D$ and $C_2(F, w) \neq 0$ (see [24]).

Let $w_+ = \max\{w, 0\}$ and $w_- = w_+ - w$. By (6) we have

$$\begin{aligned}\langle F''(w)w_+, w_+\rangle &= \int_\Omega (|\nabla w_+|^2 - g'(x,w)w_+^2) \\ &= \int_\Omega (w_+ g(x,w) - g'(x,w)w_+^2) \\ &= \int_\Omega w_+^2\Big(\frac{g(x,w)}{w_+} - g'(x,w)\Big) \\ &= \int_\Omega w_+^2\Big(\frac{g(x,w_+)}{w_+} - g'(x,w_+)\Big) < 0.\end{aligned}$$

Similarly $\langle F''(w)w_-, w_-\rangle < 0$. As w_+ and w_- are orthogonal, we have $\langle F''(w)u, u\rangle < 0$ for all $u \in \operatorname{span}\{w_+, w_-\}$, that is, the Morse index of w is 2. By the Shifting Theorem we have $C_p(F, w) = \delta_{p2}\mathbb{Z}$. □

4. Further Results

We now consider the one dimensional case $N = 1$ with, say, $\Omega =]0, \pi[$. In this case Σ can be computed explicitly (see [17]) and it is precisely the union of the (globally defined) curves c_{j1}, c_{j2} ($j \geq 2$), mentioned in the introduction, together with the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$.

Theorem 4.1. *Let* $g :]0, \pi[\times \mathbb{R} \to \mathbb{R}$ *be a function of class* C^1, $g(x, 0) = 0$, *which satisfies* (2). *Suppose that there exist* $k \geq 2$ *and* $r,\ \alpha > 0$ *such that*

$$\lambda_{k-1} \leq \inf_{t\neq 0} \frac{g(x,t)}{t} \leq \frac{g(x,t)}{t} \leq \alpha < \lambda_k \quad |t| \leq r, \quad \text{and} \quad \alpha_+ \geq \alpha_- > \lambda_{k+1}. \tag{24}$$

Assume that (α_-, α_+) *lies between the curves* $c_{(m-1)2}$ *and* c_{m1} *with* $m \geq k + 2$. *Moreover, if either*
(i) $(\alpha_-, \alpha_+) \in \operatorname{range} c_{m1}$, *suppose* (7) *and* $tg(x, t) - 2G(x, t) \geq C(x)$, $C(x) \in L^1$; *or*
(ii) $(\alpha_-, \alpha_+) \in \operatorname{range} c_{(m-1)2}$, *suppose* (18) *and* $tg(x, t) - 2G(x, t) \leq C(x)$, $C(x) \in L^1$; *or*
(ii)$(\alpha_-, \alpha_+) \notin \Sigma$, *suppose* $tg(x, t) - 2G(x, t) \geq C(x)$ *or* $tg(x, t) - 2G(x, t) \leq C(x)$, *with* $C(x) \in L^1$.
Then problem

$$\begin{array}{l} -\ddot{u} = g(x,u) \quad \text{in} \quad]0,\pi[\\ u(0) = u(\pi) = 0, \end{array} \tag{25}$$

has at least two nontrivial solutions.

Proof. The idea of the proof is the same of Theorem 1.1.

The hypotheses (24) implies the statements (i), (iii) of Lemma 2.1 and the statement (ii) with $H_1 \oplus \varphi_{k+1}$ (φ_{k+1} the eigenfunction of the linear problem). So the problem (25) have a nontrivial solutions u such that

$$C_k(F, u) \neq 0.$$

By Remarks 3.1, we have a nontrivial solution w such that

$$C_{m-1}(F, w) \neq 0.$$

Since

$$\mathrm{Ker}(F''(u_0)) = \{u \in H_0^1(]0, \phi[) \ ; \ -\ddot{u} = g'(x, u_0)u\},$$

we have

$$\nu(u_0) = \dim \mathrm{Ker}(F''(u_0)) \leq 1.$$

Thus, by the Shifting Theorem, we have

$$C_p(F, u) = \delta_{pk}\mathbb{Z} \ \text{ and } \ C_p(F, w) = \delta_{p(m-1)}\mathbb{Z}.$$

Therefore $w \neq u$ since $k < m - 1$, and the proof is finished. ☐

Now we consider the periodic problem

$$\begin{array}{l} -\ddot{u} = g(x, u) \ \text{ in } \ ]0, 2\pi[\\ u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi), \end{array} \tag{26}$$

In this case $\lambda_j = (j-1)^2$ for $j \geq 1$. The Fučik Spectrum Σ is defined as in (4) except that now we work in the space $H^1_{per}(]0, 2\pi[)$, consisting of 2π periodic functions of the space $H_0^1(]0, 2\pi[)$. It is well know and it can be easily verified that Σ is composed of two lines $\mathbb{R} \times \{0\}$, $\{0\} \times \mathbb{R}$ and the curves C_j, $j \geq 2$,

$$C_j = \left\{(\mu, \nu) \in \mathbb{R}^2_+ \ ; \ \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{j-1}\right\}, \quad j \geq 2.$$

We have an analogue theorem for this case. Again we can use the existence result in [15] (see also [13]) and the statements of the Lemma 2.1 holds too. Since

$$\mathrm{Ker}(F''(u_0)) = \{u \in H^1_{per}(]0, 2\pi[) \ ; \ -\ddot{u} = g'(x, u_0)u\},$$

we have

$$\nu(u_0) = \dim \mathrm{Ker}(F''(u_0)) \leq 2.$$

Thus we can apply the Shifting Theorem to obtain the same conclusion as the proof of Theorem 4.1 about the critical groups.

Theorem 4.2. *Let $g :]0, 2\pi[\times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x, 0) = 0$, which satisfies* (2). *Suppose that there exist $k \geq 2$, and r, $\alpha > 0$ such that* (24) *holds. Assume that (α_-, α_+) lies between the curves C_{m-1} and C_m with $m \geq k+2$. Moreover, if either*
(i) $(\alpha_-, \alpha_+) \in$ range C_m, *suppose* (7) *and $tg(x,t) - 2G(x,t) \geq C(x)$, $C(x) \in L^1$; or*
(ii) $(\alpha_-, \alpha_+) \in$ range $C_{(m-1)}$, *suppose* (18) *and $tg(x,t) - 2G(x,t) \leq C(x)$, $C(x) \in L^1$; or*
(ii) $(\alpha_-, \alpha_+) \notin \Sigma$, *suppose $tg(x,t) - 2G(x,t) \geq C(x)$ or $tg(x,t) - 2G(x,t) \leq C(x)$, with $C(x) \in L^1$.*
Then problem (26) *has at least two nontrivial solutions.*

References

[1] A. Ambrosetti & G. Mancini, *Sharp nonuniqueness results for some nolinear problems*, Nonlinear Anal. **5** (1979), 635-645.

[2] P. Bartolo, V. Benci & D. Fortunato, *Abstract Critical Point Theory and Applications to some Nonlinear Problems with Strong Resonance at Infinity*, Nonlinear Anal. **7** (1983), 981-1012.

[3] T. Bartsch & S.L. Li, *Critical point theory for asymptotically quadratic functionals and applications to problems with resonance*, Nonlinear Anal. **28** (1997), 419-441.

[4] T. Bartsch, K.C. Chang & Z-Q. Wang, *On the Morse indices of sign changing solutions of nonlinear elliptic problems*, Math. Z **233** (2000), 655-677.

[5] N.P. Các, *On Nontrivial Solutions of a Dirichlet Problem Whose Jumping Nonlinearity Crosses a Multiple Eingenvalue*, J. Differential Equations **80** (1989), 379-404.

[6] N.P. Các, *Sharp Multiplicity Results for a Semilinear Dirichlet Problem*, Comm. Partial Differential Equations **18** (1993), 557-582.

[7] A. Castro, J. Cossio & J. M. Neuberger, *A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems*, Eletronical J. of Differential Equations, Vol.**1998** (1998), No. 02, 1-18.

[8] A. Castro & A.C. Lazer, *Critical Point Theory and the Number of Solutions of a Nonlinear Dirichlet Problem*, Ann. Mat. Pura Appl. **120** (1979), 113-137.

[9] G. Cerami, *Un criterio de esistenza per i punti critic su varietà ilimitade*, Istit. Lombardo Accad. Sci. Lett. Rend. A **112** (1978), 332–336.

[10] K.C. Chang, *H^1 versus C^1 isolated critical points*, C.R. Acad. Sci. Paris Sér I Math. **319** (1994), 441-446.

[11] K.C. Chang, *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhäuser, Boston (1993).

[12] K.C. Chang, S.J. Li & J.Q. Liu, *Remarks on Multiple Solutions for Asymptotically Linear Elliptic Boundary Value Problems*, Topol. Methods Nonlinear Anal. **3** (1994), 179-187.

[13] D.G. Costa & M. Cuesta, *Existence Results for Resonant Pertubations Fučk Spectrum*, Topol. Methods Nonlinear Anal. **8** (1996), 295-314.

[14] D.N. Dancer & Z. Zhang, *Fucik Spectrum, Sign-Changing, and Multiple Solutions for Semilinear Elliptic Boundary Value Problems with Resonance at infity*, J. Math. Anal. Appl. **250** (2000), 449-464.

[15] A.N. Domingos & M. Ramos, *On the Solvability of a Resonant Elliptic Equations with Asymmetric Nonlinearity*, Topol. Methods Nonlinear Anal. **11** (1998), 45-57.

[16] D.G. de Figueiredo, *The Ekeland Variational Principle with Applications and Detours*, TATA, Bombay (1989)

[17] S. Fučik, *Boundary Value Problems with Jumping Nonlinearities*, Časopis pro Pěstóvani Matematiky **101** (1976), 69-87.

[18] M.F. Furtado & E.A.B. Silva, *Double resonant problems whitc are locally non-quadratic at infinity*, Proceedings of the USA-Chile Workshop on Nonlinear Analysis. Electron. J. Differential Equations. Conf. 06 (2001), 155-171.

[19] T. Gallouët & O. Kavian, *Résultats d'existence et de non-existence pour certains problèms demi-linéaires à l'infini*, Ann. Fac. Sci. Toulouse Math. **3** (1981), 129-136.

[20] S-J. Li, & J-B. Su *Existense of Multiple Solutions of a Two-Point Boundary Value Problems*, Topological Methods in Nonlinear Analysis **10** (1997), 123-135.

[21] C.A. Magalhães, *Multiplicity results for a semilinear elliptic problem with crossing of multiple eigenvalues*, Differential Integral Equations **4** (1991), 155-171.

[22] J. Mawhin & M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, 1989.

[23] N. Mizoguchi, *Multiple Nontrivial Solutions of Semilinear Elliptic Equations and their Homotopy Indices*, J. Differential Equations **108** (1994), 101-119.

[24] F.O.V. de Paiva, *Multiple Solutions for Asymptotically Linear Resonant Elliptic Problems*, preprint.
[25] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations* (1986), 65 AMS Conf. Ser. Math..
[26] M. Schechter, *The Fučik Spectrum*, Indiana Univ. Math. J. **43** (1994), 1139-1157.
[27] E.A.B. Silva, *Existence and multiplicity of solutions for semilinear elliptic systems*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), 339-363.
[28] E.A.B. Silva, *Critical point theorems and applications to a semilinear elliptic problem*, NoDEA Nonlinear Differential Equations Appl. **3** (1996), 245-261.
[29] W. Zou, *Multiple Solutions Results for Two-Point Boundary Value Problems with Resonance*, Discret and Continuous Dynamical Systems **4** (1998), 485-496.

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