

The Bundles of Algebraic and Dirac-Hestenes Spinor Fields

Ricardo A. Mosna^{1,2*} and Waldyr A. Rodrigues, Jr.^{2†}

¹Institute of Physics Gleb Wataghin
UNICAMP CP 6165

13083-970 Campinas, SP Brazil

²Institute of Mathematics, Statistics and Scientific Computation

IMECC-UNICAMP CP 6065

13083-970 Campinas, SP Brazil

11/25/2003

last revised: 04/06/2004 (final version)

Abstract

The main objective of this paper is to clarify the *ontology* of Dirac-Hestenes spinor fields (*DHSF*) and its relationship with even multivector fields, on a Riemann-Cartan spacetime (*RCST*) $\mathfrak{M} = (M, g, \nabla, \tau_g, \uparrow)$ admitting a spin structure, and to give a mathematically rigorous derivation of the so called Dirac-Hestenes equation (*DHE*) in the case where \mathfrak{M} is a Lorentzian spacetime (the general case when \mathfrak{M} is a *RCST* will be discussed in another publication). To this aim we introduce the Clifford bundle of multivector fields ($\mathcal{C}\ell(M, g)$) and the *left* ($\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$) and *right* ($\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$) spin-Clifford bundles on the spin manifold (M, g) . The relation between *left ideal algebraic spinor fields* (*LIASF*) and Dirac-Hestenes *spinor* fields (both fields are sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$) is clarified. We study in details the theory of covariant derivatives of Clifford fields as well as that of left and right spin-Clifford fields. A consistent Dirac equation for a *DHSF* $\Psi \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ (denoted *DECL*^{*l*}) on a Lorentzian spacetime is found. We also obtain a *representation* of the *DECL*^{*l*} in the Clifford bundle $\mathcal{C}\ell(M, g)$. It is such equation that we call the *DHE* and it is satisfied by Clifford fields $\psi_{\Xi} \in \text{sec } \mathcal{C}\ell(M, g)$. This means that to each *DHSF* $\Psi \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\Xi \in \text{sec } P_{\text{Spin}_{1,3}^e}(M)$, there is a well defined sum of even multivector fields $\psi_{\Xi} \in \text{sec } \mathcal{C}\ell(M, g)$ (*EMFS*) associated with Ψ . Such a *EMFS* is called a *representative* of the *DHSF* on the given spin frame. And, of course, such a *EMFS* (the representative of the *DHSF*) is *not* a spinor field. With this crucial distinction between a *DHSF*

*e-mail: mosna@ifi.unicamp.br

†e-mail: walrod@mpc.com.br or walrod@ime.unicamp.br

and its *representatives* on the Clifford bundle, we provide a consistent theory for the covariant derivatives of Clifford and spinor fields of all kinds. We emphasize that the $DECl^\ell$ and the DHE , although related, are equations of different mathematical natures. We study also the local Lorentz invariance and the electromagnetic gauge invariance and show that only for the DHE such transformations are of the same mathematical nature, thus suggesting a possible link between them.

1 Introduction

The main objective of this paper is to clarify the *ontology* of $DHSF$ ¹ on general Riemann-Cartan spacetimes ($RCST$) and to give a mathematically justified account of the Dirac-Hestenes equation on Lorentzian spacetimes², subjects that have been a matter of many misunderstandings and controversies (as discussed in [34]). To motivate our enterprise, we recall that $DHSF$ on a Minkowski spacetime were defined in [34] as equivalence classes of Clifford fields. Here we follow a different approach, and define $DHSF$ as even sections of an appropriate spinorial Clifford bundle. The objects satisfying the Dirac-Hestenes equation are then even multivector fields which are *representatives* of $DHSF$ on the tensorial Clifford bundle. Moreover, such a representative is manifestly spin-frame dependent, so that no contradiction arises in representing spinors by Clifford fields.

To achieve our goals, we introduce in section 2 the Clifford bundle of multivector fields³ ($\mathcal{Cl}(M, g)$), and the *left* ($\mathcal{Cl}_{\text{Spin}_{1,3}^e}^\ell(M)$) and *right* ($\mathcal{Cl}_{\text{Spin}_{1,3}^e}^r(M)$) spin-Clifford bundles on the spin manifold (M, g) , and study in details how these bundles are related. Left algebraic spinor fields and Dirac-Hestenes spinor fields (both fields are sections of $\mathcal{Cl}_{\text{Spin}_{1,3}^e}^\ell(M)$) are defined and the relation between them is established. In section 4, a consistent Dirac equation for a $DHSF$ $\Psi \in \text{sec } \mathcal{Cl}_{\text{Spin}_{1,3}^e}^\ell(M)$ (denoted $DECl^\ell$) on a Lorentzian manifold is found. In section 5, we obtain a *representation* of the $DECl^\ell$ in the Clifford bundle, an equation we call the Dirac-Hestenes equation (DHE), which is satisfied by Clifford fields $\psi_\Xi \in \text{sec } \mathcal{Cl}(M, g)$. This means that to each $DHSF$ $\Psi \in \text{sec } \mathcal{Cl}_{\text{Spin}_{1,3}^e}^\ell(M)$ and to each *spin frame* $\Xi \in \text{sec } P_{\text{Spin}_{1,3}^e}(M)$ there is a well defined sum of even multivector fields $\psi_\Xi \in \text{sec } \mathcal{Cl}(M, g)$ ($EMFS$) associated with Ψ . Such a $EMFS$ is called a *representative* of the $DHSF$ on the given spin frame. And, of course, such a $EMFS$ (the representative of the $DHSF$) is *not* a spinor field. With this crucial distinction between a $DHSF$ and their $EMFS$ representatives, we present in section 5 an *effective* spinorial connection⁴ for the representatives of a $DHSF$ on $\mathcal{Cl}(M, g)$, thus providing a consistent theory for the covariant derivatives of Clifford and spinor fields of all kinds.

¹For the genesis of these objects we quote [19].

²The Dirac-Hestenes equation on a general $RCST$ will be discussed in another publication.

³Of course, all the results of the present paper could also be obtained in the case where $\mathcal{Cl}(M, g)$ is a Clifford bundle of nonhomogeneous differential forms.

⁴The same as that used in [34].

We emphasize that the $DEC\ell^l$ and the DHE , although related, are of different mathematical natures. This issue has been particularly scrutinized in sections 4 and 5. We study also the local Lorentz invariance and the electromagnetic gauge invariance and show that only for the DHE such transformations are of the same mathematical nature, thus suggesting a possible link between them. In a sequel paper we are going to investigate this issue and also (a) the formulation of the $DEC\ell$ and DHE in an arbitrary Riemann-Cartan spacetime through the use of a variational principle⁵; (b) the theory of the Lie derivative of the $LIASF$ and $DHSF$; and (c) the claim in [17] that existence of spinor fields in a Lorentzian manifold requires a minimum amount of curvature. This problem is important in view of the proposed teleparallel theories of the gravitational field.

Finally, in the Appendix we derive some formulas employed in the main text for the covariant derivative of Clifford and spinor fields, using the general theory of covariant derivatives on associated vector bundles. In general, our notation corresponds to that in [34].

Some few acronyms are used in the present paper (to avoid long sentences) and they are summarized below for the reader's convenience.

DHE - Dirac-Hestenes Equation

$DHSF$ - Dirac-Hestenes Spinor Field

$DEC\ell^l$ - Dirac equation for a $DHSF$ $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^c}^l(M)$

$EMFS$ - Even Multivector Fields

$LIASF$ - Left Ideal Algebraic Spinor Field

PFB - Principal Fiber Bundle

$RIASF$ - Right Ideal Algebraic Spinor Field

$RCST$ - Riemann-Cartan Spacetime

2 The Clifford Bundle of Spacetime and their Irreducible Module Representations

2.1 The Clifford Bundle of Spacetime

Let M be a four dimensional, real, connected, paracompact and non-compact manifold. Let TM [T^*M] be the tangent [cotangent] bundle of M .

Definition 1 *A Lorentzian manifold is a pair (M, g) , where $g \in \sec T^{2,0}M$ is a Lorentzian metric of signature $(1, 3)$, i.e., for all $x \in M$, $T_x M \simeq T_x^* M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.*

Definition 2 *A spacetime \mathfrak{M} is a pentuple $(M, g, \nabla, \tau_g, \uparrow)$ where (M, g, τ_g, \uparrow) is an oriented Lorentzian manifold (oriented by τ_g) and time oriented by an appropriated equivalence relation⁶ (denoted \uparrow) for the timelike vectors at the tangent space $T_x M$, $\forall x \in M$. ∇ is a linear connection for M such that $\nabla g = 0$.*

⁵We shall use in our approach to the subject the techniques of the multivector and extensor calculus developed in ([12],[13],[14],[25],[26],[27],[28]),

⁶See [35] for details

Definition 3 Let \mathbf{T} and \mathbf{R} be respectively the torsion and curvature tensors of ∇ . If in addition to the requirements of the previous definitions, $\mathbf{T}(\nabla) = 0$, then \mathfrak{M} is said to be a Lorentzian spacetime. The particular Lorentzian spacetime where $M \simeq \mathbb{R}^4$ and such that $\mathbf{R}(\nabla) = 0$ is called Minkowski spacetime and will be denoted by \mathcal{M} . When $\nabla g = 0$, $\mathbf{T}(\nabla) \neq 0$ \mathfrak{M} is said to be a Riemann-Cartan spacetime (RCST). The particular RCST such that $\mathbf{R}(\nabla) = 0$ is called a teleparallel spacetime.

In what follows $P_{\text{SO}_{1,3}^e}(M)$ denotes the principal bundle of oriented Lorentz tetrads.⁷

It is well known [32] that the natural operations on metric vector spaces, such as direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metrics. We have the

Definition 4 The Clifford bundle of the Lorentzian manifold (M, g) is the bundle of algebras

$$\mathcal{C}\ell(M, g) = \bigcup_{x \in M} \mathcal{C}\ell(T_x M, g_x), \quad (1)$$

where $\mathcal{C}\ell(T_x M, g_x)$ is the Clifford algebra associated with $(T_x M, g_x)$ (see, e.g. [34]).

As is well known ([4],[5],[10]) $\mathcal{C}\ell(M, g)$ is a quotient (or factor) bundle⁸, namely

$$\mathcal{C}\ell(M, g) = \frac{\tau M}{\mathcal{J}(M, g)} \quad (2)$$

where $\tau M = \bigoplus_{r=0}^{\infty} T^{0,r} M$ and $T^{(0,r)} M$ is the space of r -contravariant tensor fields, and $\mathcal{J}(M, g)$ is the bundle of ideals whose fibers are the two-sided ideals in τM generated by the elements of the form $a \otimes b + b \otimes a - 2g(a, b)$, with $a, b \in TM$. In what follows, we denote the real Clifford algebra associated to $\mathbb{R}^{p,q}$ by $\mathbb{R}_{p,q}$. The even subalgebra of $\mathbb{R}_{p,q}$ will be denoted by $\mathbb{R}_{p,q}^0$ (see, e.g., [34]).

Let $\pi_c : \mathcal{C}\ell(M, g) \rightarrow M$ be the canonical projection of $\mathcal{C}\ell(M, g)$ and let $\{U_\alpha\}$ be an open covering of M . There are trivialization mappings $\psi_i : \pi_c^{-1}(U_i) \rightarrow U_i \times \mathbb{R}_{1,3}$ of the form $\psi_i(p) = (\pi_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p))$. If $x \in U_i \cap U_j$ and $p \in \pi_c^{-1}(x)$, then

$$\psi_{i,x}(p) = h_{ij}(x) \psi_{j,x}(p) \quad (3)$$

for $h_{ij}(x) \in \text{Aut}(\mathbb{R}_{1,3})$, where $h_{ij} : U_i \cap U_j \rightarrow \text{Aut}(\mathbb{R}_{1,3})$ are the transition mappings of $\mathcal{C}\ell(M, g)$. We know that every automorphism of $\mathbb{R}_{1,3}$ is inner and it follows that

$$h_{ij}(x) \psi_{j,x}(p) = g_{ij}(x) \psi_{i,x}(p) g_{ij}(x)^{-1} \quad (4)$$

⁷We assume that the reader is acquainted with the structure of $P_{\text{SO}_{1,3}^e}(M)$, whose sections are the time oriented and oriented orthonormal frames, each one associated by a local trivialization to a unique element of $SO_{1,3}^e(M)$. See, e.g., ([16],[22],[29],[30]).

⁸Crumeyrolle [10] denotes $\mathcal{C}\ell(M, g)$ by $\mathcal{C}\ell(M)$ and call it the Clifford algebra of the manifold M , but we do not like this nomenclature, because a given manifold can support many non isomorphic Clifford algebras, depending on the signature of g .

for some $g_{ij}(x) \in \mathbb{R}_{1,3}^*$, the group of invertible elements of $\mathbb{R}_{1,3}$.

Now, the group $SO_{1,3}^e$ has as it is well known (see, e.g., [2],[3],[5],[21],[34]) a natural extension in the Clifford algebra $\mathbb{R}_{1,3}$. Indeed we know that $\mathbb{R}_{1,3}^*$ acts naturally on $\mathbb{R}_{1,3}$ as an algebra automorphism through its adjoint representation. A set of *lifts* of the transition functions of $\mathcal{C}\ell(M, g)$ is a set of elements $\{g_{ij}\} \subset \mathbb{R}_{1,3}^*$ such that if

$$\begin{aligned} \text{Ad} &: g \mapsto \text{Ad}_g, \\ \text{Ad}_g(a) &= gag^{-1}, \forall a \in \mathbb{R}_{1,3}, \end{aligned} \quad (5)$$

then $\text{Ad}_{g_{ij}} = h_{ij}$ in all intersections.

Also⁹ $\sigma = \text{Ad}|_{\text{Spin}_{1,3}^e}$ defines a group homeomorphism $\sigma : \text{Spin}_{1,3}^e \rightarrow SO_{1,3}^e$ which is onto with kernel \mathbb{Z}_2 . We have that $\text{Ad}_{-1} = \text{identity}$, and so $\text{Ad} : \text{Spin}_{1,3}^e \rightarrow \text{Aut}(\mathbb{R}_{1,3})$ descends to a representation of $SO_{1,3}^e$. Let us call Ad' this representation, i.e., $\text{Ad}' : SO_{1,3}^e \rightarrow \text{Aut}(\mathbb{R}_{1,3})$. Then we can write $\text{Ad}'_{\sigma(g)}a = \text{Ad}_g a = gag^{-1}$.

From this it is clear that the structure group of the Clifford bundle $\mathcal{C}\ell(M, g)$ is reducible from $\text{Aut}(\mathbb{R}_{1,3})$ to $SO_{1,3}^e$. This follows immediately from the existence of the Lorentzian structure (M, g) and the fact that $\mathcal{C}\ell(M, g)$ is the exterior bundle where the fibres are equipped with the Clifford product. Thus the transition maps of the principal bundle of oriented Lorentz tetrads $P_{SO_{1,3}^e}(M)$ can be (through Ad') taken as transition maps for the Clifford bundle. We then have [5]

$$\mathcal{C}\ell(M, g) = P_{SO_{1,3}^e}(M) \times_{\text{Ad}'} \mathbb{R}_{1,3}, \quad (6)$$

i.e., the Clifford bundle is an associated vector bundle to the principal bundle $P_{SO_{1,3}^e}(M)$ of orthonormal Lorentz frames.

Definition 5 *Sections of $\mathcal{C}\ell(M, g)$ are called Clifford fields.*¹⁰

2.2 Spinor Bundles

Definition 6 *A spin structure on M consists of a principal fibre bundle $\pi_s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow M$ (called the Spin Frame Bundle) with group $\text{Spin}_{1,3}^e$ and a map*

$$s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow P_{SO_{1,3}^e}(M) \quad (7)$$

satisfying the following conditions

(i) $\pi(s(p)) = \pi_s(p) \forall p \in P_{\text{Spin}_{1,3}^e}(M)$, π is the projection map of the bundle $P_{SO_{1,3}^e}(M)$.

(ii) $s(pu) = s(p)Ad_u$, $\forall p \in P_{\text{Spin}_{1,3}^e}(M)$ and $Ad : \text{Spin}_{1,3}^e \rightarrow \text{Aut}(\mathbb{R}_{1,3})$, $Ad_u : \mathbb{R}_{1,3} \ni x \mapsto uxu^{-1} \in \mathbb{R}_{1,3}$.

⁹Recall that $\text{Spin}_{1,3}^e = \{a \in \mathbb{R}_{1,3}^0 : a\bar{a} = 1\} \simeq SU(2, \mathbb{C})$ is the universal covering group of the restricted Lorentz group $SO_{1,3}^e$. See, e.g., [34].

¹⁰We note that the term Clifford fields was used in [34] for mappings from a Minkowski spacetime to the Clifford algebra $\mathbb{R}_{1,3}$.

Recall that minimal left (right) ideals of $\mathbb{R}_{p,q}$ are left (right) modules for $\mathbb{R}_{p,q}$ [34]. In [34], covariant, algebraic and Dirac-Hestenes spinors (when $(p, q) = (1, 3)$) were defined as certain equivalence classes in appropriate sets, and a *preliminary* definition for fields of these objects living on *Minkowski* spacetime was given. We are now interested in defining algebraic Dirac spinor fields and also Dirac-Hestenes spinor fields, on a general Riemann-Cartan spacetime (definition 3), as sections of appropriate vector bundles (spinor bundles) associated to $P_{\text{Spin}_{1,3}^e}(M)$. The compatibility between $P_{\text{Spin}_{1,3}^e}(M)$ and $P_{\text{SO}_{1,3}^e}(M)$, as captured in definition 6, is essential for that matter.

It is therefore natural to ask: When does a spin structure exist on an oriented manifold M ? The answer, which is a classical result ([1],[4],[5],[10],[15],[22],[29]-[31],[33],[32]), is that the necessary and sufficient conditions for the existence of a spin structure on M is that the second Stiefel-Whitney class $w_2(M)$ of M is trivial. Moreover, when a spin structure exists, one can show that it is unique (modulo isomorphisms) if and only if $H^1(M, \mathbb{Z}_2)$ is trivial.

Remark 7 For a spacetime \mathfrak{M} (definition 2), a spin structure exists if and only if $P_{\text{SO}_{1,3}^e}(M)$ is a trivial bundle. This was originally shown by Geroch [16].

Definition 8 We call global sections $\xi \in \text{sec } P_{\text{SO}_{1,3}^e}(M)$ Lorentz frames and global sections $\Xi \in \text{sec } P_{\text{Spin}_{1,3}^e}(M)$ spin frames.

Remark 9 Recall that a principal bundle is trivial if and only if it admits a global section. Therefore, Geroch's result says that a (non-compact) spacetime admits a spin structure if and only if it admits a (globally defined) Lorentz frame. In fact, it is possible to replace $P_{\text{SO}_{1,3}^e}(M)$ by $P_{\text{Spin}_{1,3}^e}(M)$ in remark 7 (see [16], footnote 25). In this way, when a (non-compact) spacetime admits a spin structure, the bundle $P_{\text{Spin}_{1,3}^e}(M)$ is trivial and, therefore, every bundle associated to it is also trivial.

Definition 10 An oriented manifold endowed with a spin structure will be called a spin manifold.

We now present the most usual definitions of spinor bundles appearing in the literature¹¹ and next we find appropriate vector bundles such that particular sections are *LIASF* or *DHSF*.

Definition 11 A real spinor bundle for M is a vector bundle

$$S(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_1} \mathbf{M} \quad (8)$$

where \mathbf{M} is a left module for $\mathbb{R}_{1,3}$ and μ_1 is a representation of $\text{Spin}_{1,3}^e$ on $\text{End}(\mathbf{M})$ given by left multiplication by elements of $\text{Spin}_{1,3}^e$.

¹¹We recall that there are some other (equivalent) definitions of spinor bundles that we are not going to introduce in this paper as, e.g., the one given in [6] in terms of mappings from $P_{\text{Spin}_{1,3}^e}$ to some appropriate vector space.

Definition 12 *The dual bundle $S^*(M)$ is a real spinor bundle*

$$S^*(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_r} \mathbf{M}^* \quad (9)$$

where \mathbf{M}^* is a right module for $\mathbb{R}_{1,3}$ and μ_r is a representation $\text{Spin}_{1,3}^e$ in $\text{End}(\mathbf{M})$ given by right multiplication¹² by (inverse) elements of $\text{Spin}_{1,3}^e$.

Definition 13 *A complex spinor bundle for M is a vector bundle*

$$S_c(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_c} \mathbf{M}_c \quad (10)$$

where \mathbf{M}_c is a complex left module for $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\text{Spin}_{1,3}^e$ in $\text{End}(\mathbf{M}_c)$ given by left multiplication by elements of $\text{Spin}_{1,3}^e$.

Definition 14 *The dual complex spinor bundle for M is a vector bundle*

$$S_c^*(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_c} \mathbf{M}_c^* \quad (11)$$

where \mathbf{M}_c^* is a complex right module for $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\text{Spin}_{1,3}^e$ in $\text{End}(\mathbf{M}_c)$ given by right multiplication¹² by (inverse) elements of $\text{Spin}_{1,3}^e$.

Taking, e.g., $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{Spin}_{1,3}^e \cong \text{Sl}(2, \mathbb{C})$ in $\text{End}(\mathbb{C}^4)$, we immediately recognize the usual definition of the covariant spinor bundle of M as given, e.g., in ([8],[9],[15],[29],[30]).

2.3 Left Spin-Clifford Bundle

As shown in [34], besides the ideal $I = \mathbb{R}_{1,3} \frac{1}{2}(1 + E_0)$, other ideals exist in $\mathbb{R}_{1,3}$ that are only *algebraically* equivalent to this one.¹³ In order to capture all possibilities we recall that $\mathbb{R}_{1,3}$ can be considered as a module over itself by left (or right) multiplication. We are thus led to the

Definition 15 *The left real spin-Clifford bundle of M is the vector bundle*

$$\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M) = P_{\text{Spin}_{1,3}^e}(M) \times_l \mathbb{R}_{1,3} \quad (12)$$

where l is the representation of $\text{Spin}_{1,3}^e$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$. Sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ are called *left spin-Clifford fields*.

Remark 16 $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ is a “principal $\mathbb{R}_{1,3}$ - bundle”, i.e., it admits a free action of $\mathbb{R}_{1,3}$ on the right [5], which is denoted by R_g , $g \in \mathbb{R}_{1,3}$. This will be considered in section 5.

¹²More precisely, this means that given $u \in \text{Spin}_{1,3}^e$, $a \in \mathbf{M}^*$, $\mu_r(u)a = au^{-1}$, so that $\mu_r(uu)a = a(uu)^{-1} = au^{-1}u^{-1} = \mu_r(u)\mu_r(u)a$.

¹³This fact gives rise to a large class of multivector Dirac equations in flat spacetime, generalizing the Dirac-Hestenes equation [23, 24].

Remark 17 There is a natural embedding¹⁴ $P_{\text{Spin}_{1,3}^e}(M) \hookrightarrow \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ which comes from the embedding $\text{Spin}_{1,3}^e \hookrightarrow \mathbb{R}_{1,3}^0$. Hence (as we shall see in more details below), every real left spinor bundle (see definition 15) for M can be captured from $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$, which is a vector bundle very different from $\mathcal{C}\ell(M, g)$. Their relation is presented below, but before that we give the

Definition 18 Let $I(M)$ be a subbundle of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ such that there exists a primitive idempotent \mathbf{e} of $\mathbb{R}_{1,3}$ with

$$R_{\mathbf{e}}\Psi = \Psi \quad (13)$$

for all $\Psi \in \text{sec } I(M) \subset \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$. Then, $I(M)$ is called a subbundle of left ideal algebraic spinor fields. Any $\Psi \in \text{sec } I(M) \subset \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ is called a left ideal algebraic spinor field (LIASF). $I(M)$ can be thought of as a real spinor bundle for M such that \mathbf{M} in Eq.(8) is a minimal left ideal of $\mathbb{R}_{1,3}$.

Definition 19 Two subbundles $I(M)$ and $I'(M)$ of LIASF are said to be geometrically equivalent if the idempotents $e, e' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \text{Spin}_{1,3}^e$, i.e., $e' = ueu^{-1}$.

Definition 20 The right real spin-Clifford bundle of M is the vector bundle

$$\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M) = P_{\text{Spin}_{1,3}^e}(M) \times_r \mathbb{R}_{1,3}. \quad (14)$$

Sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ are called right spin-Clifford fields

In Eq.(14) r refers to a representation of $\text{Spin}_{1,3}^e$ on $\mathbb{R}_{1,3}$, given by $r(a)x = xa^{-1}$. As in the case for the left real spin-Clifford bundle, there is a natural embedding $P_{\text{Spin}_{1,3}^e}(M) \hookrightarrow \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ which comes from the embedding $\text{Spin}_{1,3}^e \hookrightarrow \mathbb{R}_{1,3}^0$. There exists also a natural left L_a action of $a \in \mathbb{R}_{1,3}$ on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$. This will be proved in section 5.

Definition 21 Let $I^*(M)$ be a subbundle of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ such that there exists a primitive idempotent element \mathbf{e} of $\mathbb{R}_{1,3}$ with

$$L_{\mathbf{e}}\Psi = \Psi \quad (15)$$

for any $\Psi \in \text{sec } I^*(M) \subset \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$. Then, $I^*(M)$ is called a subbundle of right ideal algebraic spinor fields. Any $\Psi \in \text{sec } I^*(M) \subset \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ is called a right ideal algebraic spinor field (RIASF). $I^*(M)$ can be thought of as a real spinor bundle for M such that \mathbf{M}^* in Eq.(9) is a minimal right ideal of $\mathbb{R}_{1,3}$.

Definition 22 Two subbundles $I^*(M)$ and $I'^*(M)$ of RIASF are said to be geometrically equivalent if the idempotents $e, e' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \text{Spin}_{1,3}^e$, i.e., $e' = ueu^{-1}$.

¹⁴The symbol $A \hookrightarrow B$ means that A is embedded in B and $A \subseteq B$.

Proposition 23 *In a spin manifold, we have*

$$\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}.$$

Proof. Remember once again that the representation

$$\text{Ad} : \text{Spin}_{1,3}^e \rightarrow \text{Aut}(\mathbb{R}_{1,3}) \quad \text{Ad}_u a = uau^{-1} \quad u \in \text{Spin}_{1,3}^e$$

is such that $\text{Ad}_{-1} = \text{identity}$ and so Ad descends to a representation Ad' of $\text{SO}_{1,3}^e$ which we considered above. It follows that when $P_{\text{Spin}_{1,3}^e}(M)$ exists $\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}$. ■

2.4 Bundle of Modules over a Bundle of Algebras

Proposition 24 *$S(M)$ (or $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$) is a bundle of (left) modules over the bundle of algebras $\mathcal{C}\ell(M, g)$. In particular, the sections of the spinor bundle ($S(M)$ or $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$) are a module over the sections of the Clifford bundle.*

For the proof, see [5], page 97.

Corollary 25 *Let $\Phi, \Psi \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\Psi \neq 0$. Then there exists $\psi \in \text{sec } \mathcal{C}\ell(M, g)$ such that*

$$\Psi = \psi\Phi. \tag{16}$$

Proof. It is an immediate consequence of proposition (24). ■

So, the corollary allows us to identify a *correspondence* between some sections of $\mathcal{C}\ell(M, g)$ and some sections of $I(M)$ or $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ once we fix a section on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$. This and other correspondences will be essential for the theory of section 5. Once we clarified what is the meaning of a bundle of modules $S(M)$ over a bundle of algebras $\mathcal{C}\ell(M, g)$, we can give the following

Definition 26 *Two real left spinor bundles (see definition 15) are equivalent if and only if they are equivalent as bundles of $\mathcal{C}\ell(M, g)$ modules.*

Remark 27 *Of course, geometrically equivalent real left spinor bundles are equivalent.*

Remark 28 *In what follows we denote the complexified left spin Clifford bundle by $\mathbb{C}\ell_{\text{Spin}_{1,3}^e}^l(M) = P_{\text{Spin}_{1,3}^e}(M) \times_l \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\text{Spin}_{1,3}^e}(M) \times_r \mathbb{R}_{4,1}$ and the complexified right spin Clifford bundle by $\mathbb{C}\ell_{\text{Spin}_{1,3}^e}^r(M) = P_{\text{Spin}_{1,3}^e}(M) \times_r \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\text{Spin}_{1,3}^e}(M) \times_r \mathbb{R}_{4,1}$.*

3 Dirac-Hestenes Spinor Fields

Let \mathbf{E}^μ , $\mu = 0, 1, 2, 3$ be the canonical basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ which generates the algebra $\mathbb{R}_{1,3}$. They satisfy the basic relation $\mathbf{E}^\mu \mathbf{E}^\nu + \mathbf{E}^\nu \mathbf{E}^\mu = 2\eta^{\mu\nu}$. As shown, e.g., in [34],

$$\mathbf{e} = \frac{1}{2}(1 + \mathbf{E}^0) \in \mathbb{R}_{1,3} \quad (17)$$

is a primitive idempotent of $\mathbb{R}_{1,3}$ and

$$\mathbf{f} = \frac{1}{2}(1 + \mathbf{E}^0) \frac{1}{2}(1 + i\mathbf{E}^2 \mathbf{E}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3} \quad (18)$$

is a primitive idempotent of $\mathbb{C} \otimes \mathbb{R}_{1,3}$. Now, let $\mathbf{I} = \mathbb{R}_{1,3} \mathbf{e}$ and $\mathbf{I}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}_{1,3} \mathbf{f}$ be respectively the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C} \otimes \mathbb{R}_{1,3}$ generated by \mathbf{e} and \mathbf{f} . Let $\phi = \phi \mathbf{e} \in \mathbf{I}$ and $\Psi = \Psi \mathbf{f} \in \mathbf{I}_{\mathbb{C}}$. Then, any $\phi \in \mathbf{I}$ can be written as

$$\phi = \psi \mathbf{e} \quad (19)$$

with $\psi \in \mathbb{R}_{1,3}^0$. Analogously, any $\Psi \in \mathbf{I}_{\mathbb{C}}$ can be written as

$$\Psi = \psi \mathbf{e} \frac{1}{2}(1 + i\mathbf{E}^2 \mathbf{E}^1), \quad (20)$$

with $\psi \in \mathbb{R}_{1,3}^0$.

Now, $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, where $\mathbb{C}(4)$ is the algebra of the 4×4 complex matrices. We can verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of \mathbf{f} . In this way we can prove (as shown, e.g., in [34]) that there is a bijection between column spinors, i.e., elements of \mathbb{C}^4 (the complex 4-dimensional vector space) and the elements of $\mathbf{I}_{\mathbb{C}}$. All that, plus the definitions of the left real and complex spin bundles and the subbundle $I(M)$ suggests the

Definition 29 Let $\Phi \in \sec I(M) \subset \sec \mathcal{C}\ell_{\text{Spin}_{1,3}}^l(M)$ be as in definition 18, i.e.,

$$R_{\mathbf{e}} \Phi = \Phi \mathbf{e} = \Phi, \mathbf{e}^2 = \mathbf{e} = \frac{1}{2}(1 + \mathbf{E}^0) \in \mathbb{R}_{1,3}. \quad (22)$$

A Dirac-Hestenes Spinor field (DHSF) associated with Φ is an even section¹⁵ ψ of $\mathcal{C}\ell_{\text{Spin}_{1,3}}^l(M)$ such that

$$\Phi = \psi \mathbf{e}. \quad (23)$$

¹⁵Note that it is meaningful to speak about even (or odd) elements in $\mathcal{C}\ell_{\text{Spin}_{1,3}}^l(M)$ since $\text{Spin}_{1,3}^e \subseteq \mathbb{R}_{1,3}^0$.

Remark 30 An equivalent definition of a DHSF is the following. Let $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ be such that

$$R_{\mathbf{f}}\Psi = \Psi\mathbf{f} = \Psi, \mathbf{f}^2 = \mathbf{f} = \frac{1}{2}(\mathbf{1} + \mathbf{E}^0)\frac{1}{2}(\mathbf{1} + i\mathbf{E}^2\mathbf{E}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3}. \quad (24)$$

Then, a DHSF associated to Ψ is an even section ψ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M) \subset \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ such that

$$\Psi = \psi\mathbf{f}. \quad (25)$$

Remark 31 In what follows, when we refer to a Dirac-Hestenes spinor field ψ we omit for simplicity the wording associated with Φ (or Ψ). It is very important to observe that DHSF are not sums of even multivector fields although, under a local trivialization, $\psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ is mapped on an even element of $\mathbb{R}_{1,3}$. We emphasize that DHSF are particular sections of a spinor bundle, not of the Clifford bundle. However, we show in section 5 how these objects have representatives in the Clifford bundle.

4 The Many Faces of the Dirac Equation

4.1 Dirac Equation for Covariant Dirac Fields

As well known [8], a *covariant* Dirac spinor field is a section $\Psi \in \sec S_c(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_l} \mathbb{C}^4$. Let $(U = M, \Phi), \Phi(\Psi) = (x, |\Psi(x)\rangle)$ be a global trivialization corresponding to a spin frame Ξ (definition (8)), such that

$$\begin{aligned} s(\Xi) &= \{e_a\} \in P_{\text{SO}_{1,3}^e}(M), \quad e^a \in \sec \mathcal{C}\ell(M, g), \\ e^a e^b + e^b e^a &= 2\eta^{ab}, \quad a, b = 0, 1, 2, 3. \end{aligned} \quad (26)$$

The usual Dirac equation in a Lorentzian spacetime for the spinor field Ψ — in interaction with an electromagnetic field¹⁶ $A \in \sec \wedge^1(M) \subset \sec \mathcal{C}\ell(M, g)$ — is then

$$i\gamma^a(\nabla_{e_a}^s + iqA_a)|\Psi(x)\rangle - m|\Psi(x)\rangle = 0, \quad (27)$$

where $\gamma^a \in \mathbb{C}(4)$, $a = 0, 1, 2, 3$ is a set of *constant* Dirac *matrices* satisfying

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \quad (28)$$

4.2 Dirac Equation in $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$

Due to the one-to-one correspondence between *ideal* sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$, $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and of $S_c(M)$ as explained in section 3, we can *translate* the Dirac Eq. (27) for a covariant spinor field into an equation for a spinor field, which is a section of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$, and finally write an equivalent equation for a DHSF $\psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$. In order to do that we introduce the spin-Dirac operator.

¹⁶We denote the space of sections of p -vectors by $\sec \wedge^p(M)$.

Definition 32 *The (spin) Dirac operator acting on sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$ is the first order differential operator [5]*

$$D^s = e^a \nabla_{e_a}^s. \quad (29)$$

where $\{e^a\}$ is a basis as defined in Eq.(26).

Now we give the details of the inverse translation. We start with the following equation which we call Dirac equation in $\mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$, denoted $DEC\ell^{\ell}$

$$D^s \psi \mathbf{E}^{21} - m \psi \mathbf{E}^0 - q A \psi = 0 \quad (30)$$

where $\psi \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$ is a *DHSF* and the $\mathbf{E}^a \in \mathbb{R}_{1,3}$ are such that $\mathbf{E}^a \mathbf{E}^b + \mathbf{E}^b \mathbf{E}^a = 2\eta^{ab}$. Multiplying Eq.(30) on the right by the idempotent $\mathbf{f} = \frac{1}{2}(1 + \mathbf{E}^0) \frac{1}{2}(1 + i\mathbf{E}^2 \mathbf{E}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$ we get after some simple algebraic manipulations the following equation for the (complex) left ideal spin-Clifford field $\Psi = \psi \mathbf{f} \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$

$$iD^s \Psi - m \Psi - q A \Psi = 0. \quad (31)$$

Now we can easily show, using the methods of [34], that given any global trivializations $(U = M, \Theta)$ and $(U = M, \Phi)$, of $\mathcal{C}\ell(M, g)$ and $\mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$, there exists matrix representations of the $\{e^a\}$ that are equal to the Dirac matrices γ^a (appearing in Eq.(27)). In that way the correspondence between Eqs.(27), (30) and (31) is proved.

Remark 33 *We emphasize at this point that we call Eq.(30) the $DEC\ell^{\ell}$. It looks similar to the Dirac-Hestenes equation (on Minkowski spacetime) discussed in [34], but it is indeed very different regarding its mathematical nature. It is an intrinsic equation satisfied by a legitimate spinor field, namely a *DHSF* $\psi \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M)$. The question naturally arises: May we write an equation with the same mathematical information of Eq.(30) but satisfied by objects living on the Clifford bundle of an arbitrary Lorentzian spacetime, admitting a spin structure? In the next section we show that the answer to that question is yes.*

4.3 Electromagnetic Gauge Invariance of the $DEC\ell^{\ell}$

Proposition 34 *The $DEC\ell^{\ell}$ is invariant under electromagnetic gauge transformations*

$$\psi \mapsto \psi' = \psi e^{q\mathbf{E}^{21}\chi}, \quad (32)$$

$$A \mapsto A + \partial\chi, \quad (33)$$

$$\omega_{e_a} \mapsto \omega_{e_a} \quad (34)$$

$$\psi, \psi' \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}}^{\ell}(M) \quad (35)$$

$$A \in \text{sec } \bigwedge^1(M) \subset \text{sec } \mathcal{C}\ell(M, g) \quad (36)$$

with ψ, ψ' distinct *DHSF*, and where $\chi : M \rightarrow \mathbb{R} \subset \mathbb{R}_{1,3}$ is a gauge function.

Proof. The proof is obtained by direct verification. ■

Remark 35 We note that local rotations and electromagnetic gauge transformations are very different mathematical transformations, without any obvious geometrical link between them, differently of what seems to be the case for the Dirac-Hestenes equation, which is studied in the next section.

5 The Dirac-Hestenes Equation (DHE)

We obtained above a Dirac equation, which we called DEC^{ℓ} , describing the motion of spinor fields represented by sections Ψ of $C\ell_{\text{Spin}_{1,3}^e}^{\ell}(M, g)$ in interaction with an electromagnetic field $A \in \text{sec } C\ell(M, g)$,

$$D^s \Psi \mathbf{E}^{21} - qA\Psi = m\Psi \mathbf{E}^0, \quad (37)$$

where $D^s = e^a \nabla_{e_a}^s$, $\{e^a\}$ is given by Eq.(26), $\nabla_{e_a}^s$ is the natural spinor covariant derivative acting on $\text{sec } C\ell_{\text{Spin}_{1,3}^e}^{\ell}(M, g)$ (see appendix), and $\{\mathbf{E}^a\} \in \mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$ is such that $\mathbf{E}^a \mathbf{E}^b + \mathbf{E}^b \mathbf{E}^a = 2\eta^{ab}$. As we already mentioned, although Eq. (37) is written in a kind of Clifford bundle (i.e. $C\ell_{\text{Spin}_{1,3}^e}^{\ell}(M, g)$), it does not suffer from the inconsistency of representing spinors as pure differential forms and, in fact, the object Ψ behaves as it should under Lorentz transformations.

As a matter of fact, Eq.(37) can be thought of as a mere *rewriting* of the usual Dirac equation, where the role of the constant gamma matrices is undertaken by the constant elements $\{\mathbf{E}^a\}$ in $\mathbb{R}_{1,3}$ and by the set $\{e^a\}$. In this way, Eq.(37) is *not* a kind of Dirac-Hestenes equation as discussed, e.g., in [34]. It suffices to say that (i) the state of the electron, represented by Ψ , is not a *Clifford field* and (ii) the \mathbf{E}^a 's are just *constant* elements of $\mathbb{R}_{1,3}$ and not sections of vectors in $\mathcal{C}\ell(M, g)$. Nevertheless, as we show in the following, Eq. (37) does lead to a multivector Dirac equation once we carefully employ the theory of right and left actions on the various Clifford bundles introduced earlier. It is the multivector equation¹⁷ to be derived below that we call the *DHE*. We shall need several preliminary results that we collect in the next two subsections.

5.1 The Various Natural Actions on the Vector Bundles Associated to $P_{\text{Spin}_{1,3}^e}(M)$

Recall that, when M is a spin manifold:

(i) The elements of $\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{Ad} \mathbb{R}_{1,3}$ are equivalence classes $[(p, a)]$ of pairs (p, a) , where $p \in P_{\text{Spin}_{1,3}^e}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1}$, $a' = uau^{-1}$, for some $u \in \text{Spin}_{1,3}^e$;

(ii) The elements of $C\ell_{\text{Spin}_{1,3}^e}^{\ell}(M)$ are equivalence classes of pairs (p, a) , where $p \in P_{\text{Spin}_{1,3}^e}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1}$, $a' = ua$, for some $u \in \text{Spin}_{1,3}^e$;

¹⁷Of course, we can write an equivalent multiform equation.

(iii) The elements of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ are equivalence classes of pairs (p, a) , where $p \in P_{\text{Spin}_{1,3}^e}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1}$, $a' = au^{-1}$, for some $u \in \text{Spin}_{1,3}^e$.

In this way, it is possible to define the following natural actions on these associated bundles.

Proposition 36 *There is a natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and a natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$.*

Proof. Given $b \in \mathbb{R}_{1,3}$ and $\alpha \in \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$, select a representative (p, a) for α and define $\alpha b := [(p, ab)]$. If another representative (pu^{-1}, ua) is chosen for α , we have $(pu^{-1}, uab) \sim (p, ab)$ and thus αb is a well defined element of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$. ■

Let us denote the space of $\mathbb{R}_{1,3}$ -valued smooth functions on M by $\mathcal{F}(M, \mathbb{R}_{1,3})$. Then, the above proposition immediately yields the following.

Corollary 37 *There is a natural right action of $\mathcal{F}(M, \mathbb{R}_{1,3})$ on $\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and a natural left action of $\mathcal{F}(M, \mathbb{R}_{1,3})$ on $\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$.*

Proposition 38 *There is a natural left action of $\text{sec } \mathcal{C}\ell(M, g)$ on $\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and a natural right action of $\text{sec } \mathcal{C}\ell(M, g)$ on $\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$.*

Proof. Given $\alpha \in \text{sec } \mathcal{C}\ell(M, g)$ and $\beta \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p, ab)] \in \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$. If alternative representatives (pu^{-1}, uau^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have

$$(pu^{-1}, uau^{-1}ub) = (pu^{-1}, uab) \sim (p, ab)$$

and thus $(\alpha\beta)(x)$ is a well defined element of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$. ■

Proposition 39 *There is a natural pairing*

$$\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M) \times \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M) \rightarrow \text{sec } \mathcal{C}\ell(M, g).$$

Proof. Given $\alpha \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\beta \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p, ab)] \in \mathcal{C}\ell(M, g)$. If alternative representatives (pu^{-1}, ua) and (pu^{-1}, bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$, we have $(pu^{-1}, uabu^{-1}) \sim (p, ab)$ and thus $(\alpha\beta)(x)$ is a well defined element of $\mathcal{C}\ell(M, g)$. ■

Proposition 40 *There is a natural pairing*

$$\text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M) \times \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M) \rightarrow \mathcal{F}(M, \mathbb{R}_{1,3}).$$

Proof. Given $\alpha \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ and $\beta \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := ab \in \mathbb{R}_{1,3}$. If alternative representatives (pu^{-1}, au^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have $au^{-1}ub = ab$ and thus $(\alpha\beta)(x)$ is a well defined element of $\mathbb{R}_{1,3}$. ■

5.2 Fiducial Sections Associated to a Spin Frame

We start by exploring the possibility of defining “unit sections” on the various vector bundles associated to the principal bundle $P_{\text{Spin}_{1,3}^e}(M)$. It immediately follows from the definition given by Eq. (1) that the unit section $\mathbf{1} \in \sec \mathcal{C}\ell(M, g)$, given by $x \mapsto 1 \in \mathcal{C}\ell(T_x M, g_x)$, is certainly well defined. For future reference, let us consider how this can also be seen from the associated bundle structure of $P_{\text{Spin}_{1,3}^e}(M) \times_{ad} \mathbb{R}_{1,3}$.

Let

$$\Phi_i : \pi^{-1}(U_i) \rightarrow U_i \times \text{Spin}_{1,3}^e, \quad \Phi_j : \pi^{-1}(U_j) \rightarrow U_j \times \text{Spin}_{1,3}^e$$

be two local trivializations for $P_{\text{Spin}_{1,3}^e}(M)$, with

$$\Phi_i(u) = (\pi(u) = x, \phi_{i,x}(u)), \quad \Phi_j(u) = (\pi(u) = x, \phi_{j,x}(u)).$$

Recall that the transition function on $g_{ij} : U_i \cap U_j \rightarrow \text{Spin}_{1,3}^e$ is then given by

$$g_{ij}(x) = \phi_{i,x}(u) \circ \phi_{j,x}(u)^{-1},$$

which does not depend on u .

Proposition 41 $\mathcal{C}\ell(M, g)$ has a naturally defined global unit section.

Proof. For the associated bundle $\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}$, the transition functions corresponding to local trivializations

$$\Psi_i : \pi_c^{-1}(U_i) \rightarrow U_i \times \mathbb{R}_{1,3}, \quad \Psi_j : \pi_c^{-1}(U_j) \rightarrow U_j \times \mathbb{R}_{1,3}, \quad (38)$$

are given by $h_{ij}(x) = \text{Ad}_{g_{ij}(x)}$. Define the local sections

$$\mathbf{1}_i(x) = \Psi_i^{-1}(x, 1), \quad \mathbf{1}_j(x) = \Psi_j^{-1}(x, 1), \quad (39)$$

where 1 is the unit element of $\mathbb{R}_{1,3}$. Since $h_{ij}(x) \cdot \mathbf{1} = \text{Ad}_{g_{ij}(x)}(\mathbf{1}) = g_{ij}(x) \mathbf{1} g_{ij}(x)^{-1} = \mathbf{1}$, we see that the expressions above uniquely define a global section $\mathbf{1} \in \mathcal{C}\ell(M, g)$ with $\mathbf{1}|_{U_i} = \mathbf{1}_i$. This proves the proposition. ■

It is clear that such a result can be immediately generalized for the Clifford bundle $\mathcal{C}\ell_{p,q}(M, g)$, of any n -dimensional manifold endowed with a metric of arbitrary signature (p, q) (where $n = p + q$). Now, we observe also that the left (and also the right) spin-Clifford bundle can be generalized in an obvious way for any spin manifold of arbitrary finite dimension $n = p + q$, with a metric of arbitrary signature (p, q) . However, another important difference between $\mathcal{C}\ell(M, g)$ and $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^l(M)$ or $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$ is that these latter bundles only admit a global unit section if they are *trivial*.

Proposition 42 *There exists an unit section on $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^r(M)$ (and also on $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^l(M)$) if and only if $P_{\text{Spin}_{p,q}^e}(M)$ is trivial.*

Proof. We show the necessity for the case of $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^r(M)$,¹⁸ the sufficiency is trivial. For $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^r(M)$, the transition functions corresponding to local trivializations

$$\Omega_i : \pi_{sc}^{-1}(U_i) \rightarrow U_i \times \mathbb{R}_{p,q}, \quad \Omega_j : \pi_{sc}^{-1}(U_j) \rightarrow U_j \times \mathbb{R}_{p,q}, \quad (40)$$

are given by $k_{ij}(x) = R_{g_{ij}(x)}$, with $R_a : \mathbb{R}_{p,q} \rightarrow \mathbb{R}_{p,q}, x \mapsto xa^{-1}$. Let 1 be the unit element of $\mathbb{R}_{1,3}$. An unit section in $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^r(M)$ – if it exists – is written in terms of these two local trivializations as

$$\mathbf{1}_i^r(x) = \Omega_i^{-1}(x, 1), \quad \mathbf{1}_j^r(x) = \Omega_j^{-1}(x, 1), \quad (41)$$

and we must have $\mathbf{1}_i^r(x) = \mathbf{1}_j^r(x) \forall x \in U_i \cap U_j$. As $\Omega_i(\mathbf{1}_i^r(x)) = (x, 1) = \Omega_j(\mathbf{1}_j^r(x))$, we have $\mathbf{1}_i^r(x) = \mathbf{1}_j^r(x) \Leftrightarrow 1 = k_{ij}(x) \cdot 1 \Leftrightarrow 1 = 1g_{ij}(x)^{-1} \Leftrightarrow g_{ij}(x) = 1$. This proves the proposition. ■

Remark 43 *For general spin manifolds, the bundle $P_{\text{Spin}_{p,q}^e}(M)$ is not necessarily trivial for arbitrary (p, q) , but Geroch's theorem (remark 9) warrants that, for the special case $(p, q) = (1, 3)$ with M non-compact, $P_{\text{Spin}_{1,3}^e}(M)$ is trivial. By the above proposition, we then see that $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ and also $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ have global “unit sections”. It is most important to note, however, that each different choice of a (global) trivialization Ω_i on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ (respectively $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$) induces a different global unit section $\mathbf{1}_i^r$ (respectively $\mathbf{1}_i^l$). Therefore, even in this case there is no canonical unit section on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$ (respectively on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$).*

By remark 9, when the (non-compact) spacetime M is a spin manifold, the bundle $P_{\text{Spin}_{1,3}^e}(M)$ admits global sections. With this in mind, let us fix a spin frame Ξ for M . This induces a global trivialization for $P_{\text{Spin}_{1,3}^e}(M)$, which we denote by $\Phi_\Xi : P_{\text{Spin}_{1,3}^e}(M) \rightarrow M \times \text{Spin}_{1,3}^e$, with $\Phi_\Xi^{-1}(x, 1) = \Xi(x)$. As we show in the following, the spin frame Ξ can also be used to induce certain fiducial global sections on the various vector bundles associated to $P_{\text{Spin}_{1,3}^e}(M)$:

(i) $\mathcal{C}\ell(M, g)$ Let $\{\mathbf{E}^a\}$ be a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$ (which can be thought of as the *canonical* basis of $\mathbb{R}^{1,3}$). We define basis sections in $\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{Ad} \mathbb{R}_{1,3}$ by $e_a(x) = [(\Xi(x), \mathbf{E}_a)]$. Of course, this induces a multivector basis $\{e_I(x)\}$ for each $x \in M$. Note that a more precise notation for e_a would be, for instance, $e_a^{(\Xi)}$.

¹⁸The proof for the case of $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^l(M)$ is analogous.

(ii) $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ Let $\mathbf{1}_{\Xi}^l \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ be defined by $\mathbf{1}_{\Xi}^l(x) = [(\Xi(x), 1)]$. Then the natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ leads to $\mathbf{1}_{\Xi}^l(x)a = [(\Xi(x), a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from corollary 37 that an arbitrary section $\alpha \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ can be written as $\alpha = \mathbf{1}_{\Xi}^l f$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

(iii) $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$ Let $\mathbf{1}_{\Xi}^r \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, g)$ be defined by $\mathbf{1}_{\Xi}^r(x) = [(\Xi(x), 1)]$. Then the natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ leads to $a\mathbf{1}_{\Xi}^r(x) = [(\Xi(x), a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from corollary 37 that an arbitrary section $\alpha \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ can be written as $\alpha = f\mathbf{1}_{\Xi}^r$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

Now recall (definition 6) that a spin structure on M is a 2-1 bundle map $s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow P_{\text{SO}_{1,3}^e}(M)$ such that $s(pu) = s(p)Ad_u$, $\forall p \in P_{\text{Spin}_{1,3}^e}(M)$, $u \in \text{Spin}_{1,3}^e$, where $Ad : \text{Spin}_{1,3}^e \rightarrow \text{SO}_{1,3}^e$, $Ad_u : x \mapsto uxu^{-1}$. We see that the specification of the global section in the case (i) above is compatible with the Lorentz frame $\{e_a\} = s(\Xi)$ assigned by s . More precisely, for each $x \in M$, the element $s(\Xi(x)) \in P_{\text{SO}_{1,3}^e}(M)$ is to be regarded as a proper isometry $s(\Xi(x)) : \mathbb{R}^{1,3} \rightarrow T_x M$, so that $e_a(x) := s(p) \cdot \mathbf{E}_a$ yields a Lorentz frame $\{e_a\}$ on M , which we denoted by $s(\Xi)$. On the other hand, $\mathcal{C}\ell(M, g)$ is isomorphic to $P_{\text{Spin}_{1,3}^e}(M) \times_{Ad} \mathbb{R}_{1,3}$, and we can always arrange things so that $e_a(x)$ is represented in this bundle as $e_a(x) = [(\Xi(x), \mathbf{E}_a)]$. In fact, all we have to do is to verify that this identification is covariant under a change of frames. To see that, let $\Xi' \in \text{sec}(P_{\text{Spin}_{1,3}^e}(M))$ be another spin frame on M . From the principal bundle structure of $P_{\text{Spin}_{1,3}^e}(M)$, we know that, for each $x \in M$, there exists (an unique) $u(x) \in \text{Spin}_{1,3}^e$ such that $\Xi'(x) = \Xi(x)u(x)$. If we define, as above, $e'_a(x) = s(\Xi'(x)) \cdot \mathbf{E}_a$, then $e'_a(x) = s(\Xi(x)u(x)) \cdot \mathbf{E}_a = s(\Xi(x))Ad_{u(x)} \cdot \mathbf{E}_a = [(\Xi(x), Ad_{u(x)} \cdot \mathbf{E}_a)] = [(\Xi(x)u(x), \mathbf{E}_a)] = [(\Xi'(x), \mathbf{E}_a)]$, which proves our claim.

Proposition 44

- (i) $\mathbf{E}_a = \mathbf{1}_{\Xi}^r(x)e_a(x)\mathbf{1}_{\Xi}^l(x)$, $\forall x \in M$,
- (ii) $\mathbf{1}_{\Xi}^l\mathbf{1}_{\Xi}^r = 1 \in \mathcal{C}\ell(M, g)$,
- (iii) $\mathbf{1}_{\Xi}^r\mathbf{1}_{\Xi}^l = 1 \in \mathbb{R}_{1,3}$.

Proof. This follows from the form of the various actions defined in propositions 36-40. For example, for each $x \in M$, we have $\mathbf{1}_{\Xi}^r(x)e_a(x) = [(\Xi(x), \mathbf{1E}_a)] = [(\Xi(x), \mathbf{E}_a)] \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ (from proposition 38). Then, it follows from proposition 40 that $\mathbf{1}_{\Xi}^r(x)e_a(x)\mathbf{1}_{\Xi}^l(x) = \mathbf{E}_a1 = \mathbf{E}_a \forall x \in M$. ■

Let us now consider how the various global sections defined above transform when the spin frame Ξ is changed. Let $\Xi' \in \text{sec } P_{\text{Spin}_{1,3}^e}(M)$ be another spin frame with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \text{Spin}_{1,3}^e$. Let e_a , $\mathbf{1}_{\Xi}^r$, $\mathbf{1}_{\Xi}^l$ and e'_a , $\mathbf{1}_{\Xi'}^r$, $\mathbf{1}_{\Xi'}^l$ be the global sections respectively defined by Ξ and Ξ' (as above). We then have

Proposition 45 Let Ξ, Ξ' be two spin frames related by $\Xi' = \Xi u$, where $u : M \rightarrow \text{Spin}_{1,3}^e$. Then

$$\begin{aligned} (i) \quad e'_a &= U e_a U^{-1} \\ (ii) \quad \mathbf{1}_{\Xi'}^l &= \mathbf{1}_{\Xi}^l u = U \mathbf{1}_{\Xi}^l, \\ (iii) \quad \mathbf{1}_{\Xi'}^r &= u^{-1} \mathbf{1}_{\Xi}^r = \mathbf{1}_{\Xi}^r U^{-1}, \end{aligned} \quad (42)$$

where $U \in \text{sec } \mathcal{C}\ell(M, g)$ is the Clifford field associated to u by $U(x) = [(\Xi(x), u(x))]$. Also, in (ii) and (iii), u and u^{-1} respectively act on $\mathbf{1}_{\Xi}^l \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\mathbf{1}_{\Xi}^r \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ according to proposition 37.

Proof. (i) We have

$$\begin{aligned} e'_a(x) &= [(\Xi'(x), \mathbf{E}_a)] = [(\Xi(x)u(x), \mathbf{E}_a)] \\ &= [(\Xi(x), u(x)\mathbf{E}_a u(x)^{-1})] \\ &= [(\Xi(x), u(x))][(\Xi(x), \mathbf{E}_a)][(\Xi(x), u(x)^{-1})] \\ &= U(x)e_a(x)U(x)^{-1}. \end{aligned} \quad (43)$$

(iii) It follows from proposition 38 that

$$\begin{aligned} \mathbf{1}_{\Xi'}^r(x) &= [(\Xi'(x), 1)] = [(\Xi(x)u(x), 1)] \\ &= [(\Xi(x), 1u(x)^{-1})] = [(\Xi(x), u(x)^{-1})] = u(x)^{-1} \mathbf{1}_{\Xi}^r(x), \end{aligned} \quad (44)$$

where in the last step we used proposition 37 and the fact that $\mathbf{1}_{\Xi}^r(x) = [(\Xi(x), 1)]$. To demonstrate the second part, note that

$$\begin{aligned} u^{-1}(x) \mathbf{1}_{\Xi}^r(x) &= [(\Xi(x), u(x)^{-1})] \\ &= [(\Xi(x), 1u(x)^{-1})] = [(\Xi(x), 1)] [(\Xi(x), u(x)^{-1})] \\ &= \mathbf{1}_{\Xi}^r(x) U^{-1}(x), \end{aligned} \quad (45)$$

for all $x \in M$. It is important to note that in the last step we have a product between an element of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ (i.e. $[(\Xi(x), 1)]$) and an element of $\mathcal{C}\ell(M, g)$ (i.e. $[(\Xi(x), u(x)^{-1})]$). ■

We emphasize that the right unit sections associated with spin frames are *not* constant in any covariant way. In fact, we have the

Proposition 46 Let $\mathbf{1}_{\Xi}^r \in \text{sec } \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ be the right unit section associated to the spin frame Ξ . Then

$$\nabla_{e_a}^s \mathbf{1}_{\Xi}^r = -\frac{1}{2} \mathbf{1}_{\Xi}^r \omega_{e_a}, \quad (46)$$

where ω_{e_a} is the connection 1-form (proposition 54) written in the basis $\{e_a\}$.

Proof. It follows from Eq. (71) of the appendix. ■

5.3 Representatives of *DHSF* on the Clifford Bundle

Let $\{\mathbf{E}^a\}$ be, as before, a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$. Remember that these objects are fundamental to the Dirac equation (37) in terms of sections Ψ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^c}^l(M, g)$:

$$D^s \Psi \mathbf{E}^{21} - qA\Psi = m\Psi \mathbf{E}^0.$$

Let $\Xi \in \text{sec } P_{\text{Spin}_{1,3}^c}(M)$ be a spin frame on M and define the sections $\mathbf{1}_{\Xi}^l, \mathbf{1}_{\Xi}^r$ and e_a , respectively on $\mathcal{C}\ell_{\text{Spin}_{1,3}^c}^l(M), \mathcal{C}\ell_{\text{Spin}_{1,3}^c}^r(M)$ and $\mathcal{C}\ell(M, g)$, as above. Now we can use proposition 44 to write the above equation in terms of sections of $\mathcal{C}\ell(M, g)$:

$$(D^s \Psi) \mathbf{1}_{\Xi}^r e^{21} \mathbf{1}_{\Xi}^l - qA\Psi = m\Psi \mathbf{1}_{\Xi}^r e^0 \mathbf{1}_{\Xi}^l. \quad (47)$$

Right-multiplying by $\mathbf{1}_{\Xi}^r$ yields, using proposition 44,

$$e^a (\nabla_{e_a}^s \Psi) \mathbf{1}_{\Xi}^r e^{21} - qA\Psi \mathbf{1}_{\Xi}^r = m\Psi \mathbf{1}_{\Xi}^r e^0. \quad (48)$$

It follows from proposition 59 that

$$\begin{aligned} (\nabla_{e_a}^s \Psi) \mathbf{1}_{\Xi}^r &= \nabla_{e_a} (\Psi \mathbf{1}_{\Xi}^r) - \Psi \nabla_{e_a} (\mathbf{1}_{\Xi}^r) \\ &= \nabla_{e_a} (\Psi \mathbf{1}_{\Xi}^r) + \frac{1}{2} \Psi \mathbf{1}_{\Xi}^r \omega_a, \end{aligned} \quad (49)$$

where proposition 46 was employed in the last step. Therefore

$$e^a \left[\nabla_{e_a} (\Psi \mathbf{1}_{\Xi}^r) + \frac{1}{2} \Psi \mathbf{1}_{\Xi}^r \omega_a \right] e^{21} - qA(\Psi \mathbf{1}_{\Xi}^r) = m(\Psi \mathbf{1}_{\Xi}^r) e^0. \quad (50)$$

Thus it is natural to define, for each spin frame Ξ , the Clifford field $\psi_{\Xi} \in \text{sec } \mathcal{C}\ell(M, g)$ (see proposition 39) by

$$\psi_{\Xi} := \Psi \mathbf{1}_{\Xi}^r. \quad (51)$$

We then have

$$e^a \left[\nabla_{e_a} \psi_{\Xi} + \frac{1}{2} \psi_{\Xi} \omega_a \right] e^{21} - qA\psi_{\Xi} = m\psi_{\Xi} e^0. \quad (52)$$

A comment about the nature of spinors is in order. As we repeatedly said in the previous sections, spinor fields should not be ultimately regarded as fields of multivectors (or multiforms), for their behavior under Lorentz transformations is not tensorial (they are able to distinguished between 2π and 4π rotations). So, how can the identification above be correct? The answer is that the definition in Eq. (51) is intrinsically spin-frame dependent. Clearly, this is the price one ought to pay if one wants to make sense of the procedure of representing spinors by differential forms.

Note also that the covariant derivative acting on ψ_{Ξ} in Eq. (52) is the tensorial covariant derivative ∇_V on $\mathcal{C}\ell(M, g)$, as it should be. However, we see from the expression above that ∇_V acts on ψ_{Ξ} together with the term $\frac{1}{2} \psi_{\Xi} \omega_a$.

Therefore, it is natural to define an “effective covariant derivative” $\nabla_V^{(s)}$ acting on ψ_Ξ by

$$\nabla_{e_a}^{(s)} \psi_\Xi := \nabla_a \psi_\Xi + \frac{1}{2} \psi_\Xi \omega_a. \quad (53)$$

Then, proposition 54 yields

$$\nabla_{e_a}^{(s)} \psi_\Xi = \partial_{e_a} (\psi_\Xi) + \frac{1}{2} \omega_a \psi_\Xi, \quad (54)$$

which emulates the spinorial covariant derivative¹⁹, as it should. We observe moreover that if $U \in \sec C\ell(M, g)$ and if $\psi_\Xi \in \sec C\ell(M, g)$ is a representative of a Dirac-Hestenes spinor field then

$$\nabla_{e_a}^{(s)} (U \psi_\Xi) = (\nabla_{e_a} U) \psi_\Xi + U \nabla_{e_a}^{(s)} \psi_\Xi \quad (55)$$

With this notation, we finally have the Dirac-Hestenes equation for the *representative* Clifford field $\psi_\Xi \in \sec C\ell(M, g)$, on a Lorentzian spacetime²⁰:

$$e^a \nabla_{e_a}^{(s)} \psi_\Xi e^{21} - q A \psi_\Xi = m \psi_\Xi e^0, \quad (56)$$

where ψ_Ξ is the representative of a *DHSF* Ψ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, g)$, relative to the spin frame Ξ .

Let us finally show that this formulation recovers the usual transformation properties characteristic of the Hestenes’s formalism as described, e.g., in [34]. For that matter, consider two spin frames $\Xi, \Xi' \in \sec P_{\text{Spin}_{1,3}^e}(M)$, with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \text{Spin}_{1,3}^e$. It follows from proposition 45 that $\psi_{\Xi'} = \Psi \mathbf{1}_{\Xi'}^r = \Psi u^{-1} \mathbf{1}_\Xi^r = \Psi \mathbf{1}_\Xi^r U^{-1} = \psi_\Xi U^{-1}$. Therefore, the various spin frame dependent Clifford fields from Eq. (56) transform as

$$\begin{aligned} e'_a &= U e_a U^{-1}, \\ \psi_{\Xi'} &= \psi_\Xi U^{-1}. \end{aligned} \quad (57)$$

These are exactly the transformation rules one expects from fields satisfying the Dirac-Hestenes equation.

5.4 Bilinear Covariants

5.4.1 Bilinear Covariants Associated to a *DHSF*

We are now in position to give a precise definition of the bilinear covariants of the Dirac theory, associated with a given *DHSF*.

Definition 47 *Recalling that $\bigwedge^p(M) \hookrightarrow C\ell(M, g)$, $p = 0, 1, 2, 3, 4$, and recalling propositions 39 and 40, the bilinear covariants associated to a *DHSF**

¹⁹This is the derivative used in [34], there introduced in an ad hoc way.

²⁰The *DHE* on a Riemann-Cartan spacetime will be discussed in another publication.

$\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}}^l(M)$ and ($\tilde{\Psi} \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}}^r(M)$) are the following sections of $\mathcal{C}\ell(M, g)$:

$$\begin{aligned} S &= \Psi \tilde{\Psi} = \sigma + e_5 \omega \in \sec(\bigwedge^0(M) + \bigwedge^4(M)), \\ J &= \Psi \mathbf{E}_0 \tilde{\Psi} \in \sec \bigwedge^1(M), \quad K = \Psi \mathbf{E}_3 \tilde{\Psi} \in \sec \bigwedge^1(M), \\ M &= \Psi \mathbf{E}_{12} \tilde{\Psi} \in \sec \bigwedge^2(M), \end{aligned} \tag{58}$$

where $\Psi = \Psi \frac{1}{2}(1 + \mathbf{E}_0)$, and $e_5 = e_0 e_1 e_2 e_3$.

Remark 48 Of course, since all bilinear invariants in Eq.(58) are sections of $\mathcal{C}\ell(M, g)$, they have the right transformations properties under arbitrary local Lorentz transformations, as required. As showed, e.g., in [21] these bilinear covariants and their Hodge duals satisfy a set of identities, called the Fierz identities (see e.g. [34]) that are crucial for the physical interpretation of the Dirac equation (in first and second quantizations).

Remark 49 Crumeyrolle [10] gives the name of amorphous spinor fields to ideal sections of the Clifford bundle $\mathcal{C}\ell(M, g)$. Thus an amorphous spinor field ϕ is a section of $\mathcal{C}\ell(M, g)$ such that $\phi P = \phi$, where $P = P^2$ is an idempotent section of $\mathcal{C}\ell(M, g)$. However, these fields and also the so-called Dirac-Kähler ([18], [20]) fields, which are also sections of $\mathcal{C}\ell(M, g)$, cannot be used in a physical theory of fermion fields since they do not have the correct transformation law under a Lorentz rotation of the local spin frame.

5.4.2 Bilinear Covariants Associated to a representative of a *DHSF*

We note that the bilinear covariants, when written in terms of $\psi_{\Xi} := \Psi \mathbf{1}_{\Xi}^r$, read (from proposition 44):

$$\begin{aligned} S &= \psi_{\Xi} \tilde{\psi}_{\Xi} = \sigma + e_5 \omega \in \sec(\bigwedge^0(M) + \bigwedge^4(M)), \\ J &= \psi_{\Xi} e_0 \tilde{\psi}_{\Xi} \in \sec \bigwedge^1(M), \quad K = \psi_{\Xi} e_3 \tilde{\psi}_{\Xi} \in \sec \bigwedge^1(M), \\ M &= \psi_{\Xi} e_1 e_2 \tilde{\psi}_{\Xi} \in \sec \bigwedge^2(M), \end{aligned}$$

where $e_5 = e_0 e_1 e_2 e_3$. These are all intrinsic quantities, as they should be. For a discussion on the Fierz identities satisfied by these bilinear covariants, written with the representatives of Dirac-Hestenes spinor fields, see [21, 34].

5.5 Electromagnetic Gauge Invariance of the *DHE*

Proposition 50 *The DHE is invariant under electromagnetic gauge transformations*

$$\psi_{\Xi} \mapsto \psi'_{\Xi} = \psi_{\Xi} e^{qe_{21}\chi}, \tag{59}$$

$$A \mapsto A + \partial\chi, \tag{60}$$

$$\omega_{e_a} \mapsto \omega_{e_a} \tag{61}$$

where $\psi_{\Xi}, \psi'_{\Xi} \in \sec C\ell^+(M, g)$, $A \in \sec \bigwedge^1(M) \subset \sec C\ell(M, g)$ and where $\chi \in \sec \bigwedge^0(M) \subset \sec C\ell(M, g)$ is a gauge function.

Proof. It is a direct calculation. ■

But, what are the meaning of these transformations? Eq.(59) looks similar to Eq.(57) defining the change of a representative of a *DHSF* once we change spin frame, but here we have an active transformation, since we did *not* change the spin frame. On the other hand Eq.(60) does not correspond either to a passive (no transformation at all) or active local Lorentz transformation for A . Nevertheless, writing $\chi = \theta/2$ yields

$$\begin{aligned} e^{-qe^{21}\theta/2} e^0 e^{qe^{21}\theta/2} &= e'^0 = e^0 \\ e^{-qe^{21}\theta/2} e^1 e^{qe^{21}\theta/2} &= e'^1 = \cos q\theta e^1 + \sin q\theta e^2, \\ e^{-qe^{21}\theta/2} e^2 e^{qe^{21}\theta/2} &= e'^2 = -\sin q\theta e^1 + \cos q\theta e^2, \\ e^{-qe^{21}\theta/2} e^3 e^{qe^{21}\theta/2} &= e'^3 = e^3. \end{aligned} \tag{62}$$

We see that Eqs.(62) define a spin frame Ξ' to which corresponds, as we already know, a basis $\{e'^0, e'^1, e'^2, e'^3\}$ for $\bigwedge^1(M) \hookrightarrow C\ell(M, g)$. We can then think of the electromagnetic gauge transformation as a rotation in the spin plane e^{21} by identifying ψ'_{Ξ} in Eq.(59) with $\psi_{\Xi'}$, the representative of the *DHSF* in the spin frame Ξ' and by supposing that instead of transforming the spin connection ω_{e_a} as in Eq.(69) it is taken as fixed and instead of maintaining the electromagnetic potential A fixed it is transformed as in Eq.(60).

We observe that, since in the theory of the gravitational field ω_{e_a} is associated with some aspects of that field, our interpretation for the electromagnetic gauge transformation suggests a possible non trivial coupling between electromagnetism and gravitation, *if* the Dirac-Hestenes equation is taken as a more fundamental representation of fermionic matter than the usual Dirac equation. We will explore this possibility in another publication.

6 Conclusions

In this paper, we hope to have clarified the ontology of Dirac-Hestenes spinor fields (on a general spacetime $\mathfrak{M}=(M, g, \nabla, \tau_g, \uparrow)$ of the Riemann-Cartan type admitting a spin structure and its relationship with sums of even multivector fields (or differential forms). This has been achieved through the introduction of the Clifford bundle of multivector fields ($C\ell(M, g)$) and the *left* ($C\ell_{\text{Spin}_{1,3}^e}^l(M)$) and *right* ($C\ell_{\text{Spin}_{1,3}^e}^r(M)$) spin-Clifford bundles on a spin manifold (M, g) , as well as a study of the relations among these bundles. Left algebraic spinor fields and Dirac-Hestenes spinor fields (both fields are sections of $C\ell_{\text{Spin}_{1,3}^e}^l(M)$) have been defined and the relation between them has been established. Moreover, a consistent Dirac equation for a *DHSF* $\Psi \in \sec C\ell_{\text{Spin}_{1,3}^e}^l(M)$ (denoted $DEC\ell^l$) on a Lorentzian spacetime was found. We succeeded also in obtaining in a consistent way a *representation* of the $DEC\ell^l$ in the Clifford bundle. It is such equation

satisfied by Clifford fields $\psi_{\Xi} \in \sec \mathcal{C}\ell(M, g)$ that we called the Dirac-Hestenes equation (*DHE*). This means that to each *DHSF* $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ and $\Xi \in \sec P_{\text{Spin}_{1,3}^e}(M)$ there is a well-defined even nonhomogeneous multivector field $\psi_{\Xi} \in \sec \mathcal{C}\ell(M, g)$ (*EMFS*) associated with Ψ . Such a *EMFS* is called a *representative* of the *DHSF* on the given spin frame. And, of course, such a *EMFS* (the representative of the *DHSF*) is *not* a spinor field. With this crucial distinction between a *DHSF* and their *EMFS* representatives we presented a consistent theory for Clifford and spinor fields of all kinds.

We emphasize that the *DECℓ^l* and the *DHE*, although related, are of different mathematical natures. This issue has been particularly scrutinized in sections 4 and 5. We studied also the local Lorentz invariance and the electromagnetic gauge invariance and showed that only for the *DHE* such transformations are of the same mathematical nature, something that suggests by itself a possible link between them.

A Covariant Derivatives of Clifford and Spinor Fields

A.1 Covariant Derivative of Clifford Fields

In this appendix, $(M, g, \nabla, \tau_g, \uparrow)$ denotes a general *Riemann-Cartan* spacetime (see definition 3). Since $\mathcal{C}\ell(M, g) = \tau M/J(M, g)$, it is clear that any metric compatible ($\nabla g = 0$) connection defined in τM passes to the quotient $\tau M/J(M, g)$, and thus define an algebra bundle connection [10]. In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M, g)$ is completely determined.

We will find formulas for the covariant derivative of Clifford fields and of *DHSF* using the general theory of connections in principal bundles and covariant derivatives in associate vector bundles, as described in many excellent textbooks, e.g., ([8],[15],[29],[30]).

Let then (E, M, π_1, G, F) denoted $E = P \times_{\rho} F$ be a vector bundle associated to a PFB (P, M, π, G) by the linear representation ρ of G in $F = \mathbf{V}$.

Definition 51 Let $\sigma : \mathbb{R} \supset I \rightarrow M, t \mapsto \sigma(t)$ be a curve in M with $x_0 = \sigma(0) \in M$, and let $p_0 \in \pi^{-1}(x_0)$. The parallel transport of p_0 along σ is given by the curve $\hat{\sigma} : \mathbb{R} \ni I \rightarrow P, t \mapsto \hat{\sigma}(t)$ defined by

$$\frac{d}{dt} \hat{\sigma}(t) = \Gamma_p \left(\frac{d}{dt} \sigma(t) \right), \quad (63)$$

with $p_0 = \hat{\sigma}(0)$ and $\pi(\hat{\sigma}(t)) = \sigma(t)$. We also denote $p|_t = \hat{\sigma}(t)$.

In Eq. (63), $\Gamma_p : T_x M \rightarrow T_p P$ is a connection²¹ on (P, M, π, G) . Consider the trivializations of P

$$\Phi_i : \pi^{-1}(U_i) \rightarrow U_i \times G, \quad \Phi_i(p) = (\pi(p), \phi_{i,x}(p))$$

²¹See, e.g., definition (a) on page 358 of [8].

and E

$$\Xi_i : \pi_1^{-1}(U_i) \rightarrow U_i \times F, \quad \Xi_i(q) = (\pi_1(q), \chi_i(q)) = (x, \chi_i(q))$$

Then, we have the

Definition 52 *The parallel transport of $\Psi_0 \in E$, $\pi_1(\Psi_0) = x_0$, along the curve $\sigma : \mathbb{R} \supset I \rightarrow M$, $t \mapsto \sigma(t)$ from $x_0 = \sigma(0) \in M$ to $x = \sigma(t)$ is the element $\Psi_{\parallel t} \in E$ such that:*

- (i) $\pi_1(\Psi_{\parallel t}) = x$,
- (ii) $\chi_i(\Psi_{\parallel t}) = \rho(\phi_i(p_{\parallel t}) \circ \phi_i(p_0)^{-1})\chi_i(\Psi_0)$.
- (iii) $p_{\parallel t} \in P$ is the parallel transport of $p_0 \in P$ along σ from x_0 to x .

Definition 53 *Let v be a vector at x_0 tangent to the curve σ (as defined above). The covariant derivative of $\Psi \in \sec E$ along v is denoted $(D_v^E \Psi)_{x_0} \in \sec E$ and*

$$(D_v^E \Psi)(x_0) \equiv (D_v^E \Psi)_{x_0} = \lim_{t \rightarrow 0} \frac{1}{t} (\Psi_{\parallel t}^0 - \Psi_0), \quad (64)$$

where $\Psi_{\parallel t}^0$ is the parallel transport of the vector $\Psi_t \equiv \Psi(\sigma(t))$ of the given section $\Psi \in \sec E$ along σ from $\sigma(t)$ to x_0 . The only requirements on σ are that $\sigma(0) = x_0$ and

$$\left. \frac{d}{dt} \sigma(t) \right|_{t=0} = v. \quad (65)$$

Proposition 54 *Let $V \in \sec TM$. The covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M, g)$ is given by*

$$\nabla_V A = V(A) + \frac{1}{2} [\omega_V, A], \quad (66)$$

where $V(A) := V(A^I)E_I$ and ω_V is the connection 1-form $V \mapsto \omega_V = -\frac{1}{2} V^a \Gamma_{abc} e^b \wedge e^c$, written in the basis $\{e_a\}$, with Γ_{abc} given by $\nabla_{e_a} e_b = \Gamma_{ab}^c e_c$.

Proof Writing $A(t) = A(\sigma(t))$ in terms of the multivector basis $\{e_I\}$ of sections associated to a given spin frame, as in section 5.2, we have $A(t) = A^I(t)E_I(t) = A^I(t)[(\Xi(t), E_I)] = [(\Xi(t), A^I(t)E_I)] = [(\Xi(t), a(t))]$, with $a(t) := A^I(t)E_I \in \mathbb{R}_{1,3}$. It follows from item (ii) of definition 52 that

$$A_{\parallel t}^0 = [(\Xi(0), g(t)a(t)g(t)^{-1})] \quad (67)$$

for some $g(t) \in \text{Spin}_{1,3}^e$, with $g(0) = 1$. Thus

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} (g(t)a(t)g(t)^{-1} - a(0)) &= \left[\frac{dg}{dt} a g^{-1} + g \frac{da}{dt} g^{-1} + g a \frac{dg_t^{-1}}{dt} \right]_{t=0} = \\ &= \dot{a}(0) + \dot{g}(0)a(0) - a(0)\dot{g}(0) = \\ &= V(A^I)E_I + [\dot{g}(0), a(0)], \end{aligned}$$

where $\dot{g}(0) \in Lie(\text{Spin}_{1,3}^e) = \Lambda^2(\mathbb{R}^{1,3})$. Therefore

$$\nabla_V A = V(A^I)e_I + \frac{1}{2}[\omega_V, A],$$

for some $\omega_V \in \sec \bigwedge^2(M)$. In particular, calculating the covariant derivative of the basis 1-vector fields e_b yields $V^a \Gamma_{ab}{}^c e_c = \nabla_V e_b = \frac{1}{2}[\omega_V, e_b]$, so that $\omega_V = -\frac{1}{2}V^a \Gamma_{abc} e^b \wedge e^c$. ■

Remark 55 *Eq.(66) shows that the covariant derivative preserves the degree of an homogeneous Clifford field, as can be easily verified.*

The general formula Eq. (66) and the associative law in the Clifford algebra immediately yields the

Corollary 56 *The covariant derivative ∇_V on $\mathcal{C}\ell(M, g)$ acts as a derivation on the algebra of sections, i.e., for $A, B \in \sec \mathcal{C}\ell(M, g)$ and $V \in \sec TM$, it holds*

$$\nabla_V(AB) = (\nabla_V A)B + A(\nabla_V B) \quad (68)$$

Under a change of gauge (local Lorentz transformation) $e^a \mapsto e'^a = Ue^a U^{-1}$, with $U \in \sec \mathcal{C}\ell(M, g)$, $UU = \tilde{U}U = 1$, the corresponding transformation law for ω_V is as follows.

Corollary 57 *Under a change of gauge (local Lorentz transformation), ω_V transforms as*

$$\frac{1}{2}\omega_V \mapsto U \frac{1}{2}\omega_V U^{-1} + (\nabla_V U)U^{-1}, \quad (69)$$

Proof. It is a simple calculation using Eq.(66). ■

A.2 Covariant Derivatives of Spinor Fields

The spinor bundles introduced in section 2, like $I(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{\ell} I$, $I = \mathbb{R}_{1,3} \frac{1}{2}(1 + E_0)$, and $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$, $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$ (and subbundles) are examples of vector bundles. Thus, the general theory of covariant derivative operators on associate vector bundles can be used (as in the previous section) to obtain formulas for the covariant derivatives of sections of these bundles. Given $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\Phi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$, we denote the corresponding covariant derivatives by $\nabla_V^s \Psi$ and $\nabla_V^s \Phi$ ²².

Proposition 58 *Given $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\Phi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$,*

$$\nabla_V^s \Psi = V(\Psi) + \frac{1}{2}\omega_V \Psi, \quad (70)$$

$$\nabla_V^s \Phi = V(\Psi) - \frac{1}{2}\Psi \omega_V. \quad (71)$$

²²Recall that $I^l(M) \hookrightarrow \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $I^r(M) \hookrightarrow \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$.

Proof. It is analogous to that of proposition 54, with the difference that Eq. (67) should be substituted by $\Psi_{||t}^0 = [(\Xi(0), g(t)a(t))]$ and $\Phi_{||t}^0 = [(\Xi(0), a(t)g(t)^{-1})]$. ■

Proposition 59 *Let ∇ be the connection on $\mathcal{C}\ell(M, g)$ to which ∇^s is related. Then, for any $V \in \sec TM$, $A \in \sec \mathcal{C}\ell(M, g)$, $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ and $\Phi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$,*

$$\nabla_V^s(A\Psi) = A\nabla_V^s\Psi + (\nabla_V A)\Psi, \quad (72)$$

$$\nabla_V^s(\Phi A) = \Phi\nabla_V A + (\nabla_V^s\Phi)A. \quad (73)$$

Proof. Recalling that $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ ($\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M)$) is a module over $\mathcal{C}\ell(M, g)$, the result follows from a simple computation. ■

Finally, let $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M)$ be such that $\Psi\mathbf{e} = \Psi$ where $\mathbf{e}^2 = \mathbf{e} \in \mathbb{R}_{1,3}$ is a primitive idempotent. Then, since $\Psi\mathbf{e} = \Psi$,

$$\begin{aligned} \nabla_V^s\Psi &= \nabla_V^s(\Psi\mathbf{e}) = V(\Psi\mathbf{e}) + \frac{1}{2}\omega_V\Psi\mathbf{e} \\ &= [V(\Psi) + \frac{1}{2}\omega_V\Psi]\mathbf{e} = (\nabla_V^s\Psi)\mathbf{e}, \end{aligned} \quad (74)$$

from where we verify that the covariant derivative of a *LIASF* is indeed a *LIASF*.

Acknowledgments: The authors are grateful to Doctors V. V. Fernández, A. Jadczyk, P. Lounesto, D. Miralles, A. M. Moya, I. Porteous and J. Vaz, Jr. for discussions and to F. G. Rodrigues for help with latex. They are also grateful to an anonymous referee for important observations. W.A.R. is also grateful to CNPq for a research grant (contract 201560/82-8) and to the Department of Mathematical Sciences of the University of Liverpool for providing a Visiting Professor position and an enjoyable atmosphere for research during the academic year 2001/2002. R.A.M. is grateful to FAPESP (process number 98/16486-8). This paper is dedicated to the late P. Lounesto.

References

- [1] Avis, S. J. and Isham, C. J., Generalized Spin Structures and Four Dimensional Space-Times, *Commun. Math. Phys.* **72**, 103-118 (1980).
- [2] Chevalley, C., *The Algebraic Theory of Spinors and Clifford Algebras. Collect Works vol. 2*, Springer-Verlag, Berlin, 1997.
- [3] Been, I. M. and Tucker, R. W., *An Introduction to Spinors and Geometries with Applications in Physics*, Adam Hilger, Bristol and New York, 1987.
- [4] Been, I. M. and Tucker, R. W., Representing Spinors with Differential Forms, in Trautman, A. and Furlan, G. (eds.), *Spinors in Physics and Geometry*, World Scientific, Singapore, 1988.

- [5] Blaine Lawson, H. Jr., and Michelshon, M. L., *Spin Geometry*, Princeton University Press, Princeton, 1989.
- [6] Bleecker, D., *Gauge Theory and Variational Principles*, Addison-Wesley Publ. Co., Inc., Reading, MA, 1981.
- [7] Choquet-Bruhat, Y., *Géométrie Différentielle et Systèmes Extérieurs*, Dunod, Paris, 1968.
- [8] Choquet-Bruhat, Y., DeWitt-Morette, C, and Dillard-Bleick, M., *Analysis, Manifolds and Physics*, revised edition, North Holland Publ. Co., Amsterdam, 1982.
- [9] Choquet-Bruhat, Y., DeWitt-Morette, C, and Dillard-Bleick, M., *Analysis, Manifolds and Physics*, Part II: 92 Applications, North Holland Publ. Co., Amsterdam, 1989.
- [10] Crummeyrolle, A., *Orthogonal and Symplectic Clifford Algebras*, Kluwer Acad. Publ., Dordrecht, 1991.
- [11] Dirac, P. A. M., The Quantum Theory of the Electron, *Proc. Roy. Soc. A* **117**, 610-624 (1928).
- [12] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Euclidean Clifford Algebra Space, *Adv. Appl. Clifford Algebras* **11** (S3), 1-21 (2001).
- [13] Fernández, V. V., Moya, A. M. and Rodrigues, W. A. Jr., Extensors, *Adv. Appl. Clifford Algebras* **11**(S3), 23-43 (2001).
- [14] Fernández, V. V. , Moya, A. M. and Rodrigues, W. A. Jr., Metric Tensor Vs. Metric Extensor, *Adv. Appl. Clifford Algebras* **11**(S3), 43-51 (2001).
- [15] Frankel, T., *The Geometry of Physics*, Cambridge University Press, Cambridge, 1997.
- [16] Geroch, R. Spinor Structure of Space-Times in General Relativity. I, *J. Math. Phys.* **9**, 1739-1744 (1968).
- [17] Geroch, R. Spinor Structure of Space-Times in General Relativity. II, *J. Math. Phys.* **11**, 343-348 (1970).
- [18] Graf, W., Differential Forms as Spinors, *Ann. Inst. Henri Poincaré*, **XXIV**, 85-109 (1978).
- [19] Hestenes, D., *Space-Time Algebra*, Gordon and Breach, New York, 1966.
- [20] Kähler, E., Der Innere Differentialkalkül, *Rediconti di Matematica e delle sue Applicazioni* **21**, 425-523 (1962).
- [21] Lounesto, P., *Clifford Algebras and Spinors*, second edition, Cambridge University Press, Cambridge, 2001.

- [22] Milnor, J., Spin Structures on Manifolds, *L'Enseignement Mathématique* **9**, 198-203 (1963).
- [23] Mosna, R. A., Miralles, D., and Vaz, J., Jr., Multivector Dirac Equations and \mathbb{Z}_2 -Gradings of Clifford Algebras, *Int. J. Theor. Phys.* **41**, 1651- 1671 (2002).
- [24] Mosna, R. A., Miralles, D., and Vaz, J., Jr., \mathbb{Z}_2 -Gradings of Clifford Algebras and Multivector Structures, *J. Phys. A: Math. Gen.* **36**, 4395-4405 (2003), math-ph/0212020.
- [25] Moya, A. M., Fernández, V. V. and Rodrigues, W. A. Jr., Metric Clifford Algebra , *Adv. Appl. Clifford Algebras* **11**(S3), 53-73 (2001).
- [26] Moya, A. M., Fernández, V. V. and Rodrigues, W. A. Jr., Multivector Functions of a Real Variable, *Adv. Appl. Clifford Algebras* **11**(S3), 75-83 (2001).
- [27] Moya, A. M., Fernández, V. V. and Rodrigues, W. A. Jr., Multivector Functions of Mutivector Variable, *Adv. Appl. Clifford Algebras* **11**(S3), 85-98 (2001).
- [28] Moya, A. M., Fernández, V. V. and Rodrigues, W. A. Jr., Multivector Functionals, *Adv. Appl. Clifford Algebras* **11**(S3), 99-109 (2001).
- [29] Naber, G. L., Topology, *Geometry and Gauge Fields. Interactions*, Appl. Math. Sci. 141, Springer-Verlag, New York, 2000.
- [30] Nakahara M., Geometry, *Topology and Physics*, Institute of Physics Publ., Bristol and Philadelphia, 1990.
- [31] Nicolescu, L. I., *Notes on Seiberg-Witten Theory*, Graduate Studies in Mathematics 28, Am. Math. Soc., Providence, Rhode Island, 2000.
- [32] Osborn, H., *Vector Bundles*, vol. I, Acad. Press, New York, 1982.
- [33] Penrose, R. and Rindler W., *Spinors and Spacetime*, vol. 1, Cambridge University Press, Cambridge, 1986.
- [34] Rodrigues, W. A. Jr., *Algebraic and Dirac-Hestenes Spinors and Spinor Fields*, math-ph/0212030, subm. for publication.
- [35] Sachs, R. K., and Wu, H., *General Relativity for Mathematicians*, Springer-Verlag, New York, 1997.