# Sharp Orders of $n$-Widths of Sobolev's Classes on Compact Globally Symmetric Spaces of Rank 1

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**Abstract**

Sharp orders of Kolmogorov's $n$-widths $d_n(W_p^r(M^d),\ L_q(M^d))$ of Sobolev's classes $W_p^r(M^d)$ on compact globally symmetric spaces of rank 1 are found for different $p$ and $q$. In particular, we are considering the cases $1 \le p \le 2 \le q < \infty$, $2 \le p \le q < \infty$ and $1 < q \le p \le 2$.

## 1 Introduction

Let $X$ be a Banach space, and $A$ be a (convex, compact, centrally symmetric) subset of $X$. The Kolmogorov's $n$–width of $A$ in $X$ is defined by

$$d_n(A, X) := \inf_{X_n} \sup_{f \in A} \inf_{g \in X_n} \|f - g\|,$$

where $X_n$ runs over all subspaces of $X$ of dimension $n$ or less. The Gel'fand's $n$-width of $A$ in $X$ is defined by

$$d^n(A, X) := \inf_{L^n} \sup_{x \in L^n \cap A} \|x\|,$$

where $L^n$ runs over all subspaces of codimension at most $n$. The Bernstein's $n$-width of $A$ in $X$ is defined by

$$b_n(A, X) := \sup_{X_{n+1}} \sup\{\epsilon > 0 : (\epsilon B \cap X_{n+1}) \subset A\},$$

where $X_{n+1}$ is any $(n+1)$-dimensional subspace of $X$, and $B = \{x \in X : \|x\| \leq 1\}$ is the unit ball of $X$.

In the present paper we investigate the asymptotic behavior of the $n$-widths of Sobolev's classes $W_p^r(M^d)$ in $L_q(M^d)$ on a compact globally symmetric space of rank 1 or two point homogeneous space $M^d$. Each of such manifolds $M^d$ can be considered as the orbit space of some compact subgroup $\mathcal{H}$ of the orthogonal group $\mathcal{G}$, that is $M^d = \mathcal{G}/\mathcal{H}$. On any such manifold there is an invariant Riemannian metric $\rho(\cdot,\cdot)$, an invariant Haar measure $d\nu$ and a Laplace-Beltrami operator $\Delta$. A function $Z(\cdot) : M^d \to R$ is called zonal if $Z(h^{-1}\cdot) = Z(\cdot)$ for any $h \in \mathcal{H}$. Two-point homogeneous spaces admit essentially only one invariant differential operator, the Laplace-Beltrami operator. A complete classification of the two point homogeneous spaces was given by Wang [15]. They are the spheres $S^d$, d=1,2,3,...; the real projective spaces $P^d(\mathbb{R})$, d=2,3,4...; the complex projective spaces $P^d(\mathbb{C})$, d=4,6,8,...; the quaternionic projective spaces $P^d(\mathbb{H})$, d=8,12,... and the Cayley elliptic plane $P^{16}(Cay)$.

Suppose that $\varphi$ is a function on $M^d$ with finite $L_p(M^d) = L_p$ norm given by $\|\varphi\|_p = (\int_{M^d} |\varphi|^p d\nu)^{1/p}$ if $1 \leq p < \infty$, by $\|\varphi\|_\infty = \operatorname{ess\,sup} |\varphi|$ if $p = \infty$ and let $U_p := \{\varphi | \; \|\varphi\|_p \leq 1\}$. The real Hilbert space $L_2(M^d)$ with usual scalar product $\langle\langle f, g\rangle\rangle = \int_{M^d} f(x)g(x)d\nu(x)$ has the decomposition $L_2(M^d) = \oplus_{l=0}^{\infty} H_l$. There is a unique, real zonal element $Z^{(l)} \in H_l$, which is a kernel of an integral operator for an orthogonal projection onto $H_l$. Let $\{Y_k^l\}_{k=1}^{d_l}$ be an orthonormal basis of $H_l$, $d_l = \dim H_l$. It is known that $\dim H_l \asymp l^{d-1}$ (see [1]). Let $Z$ be a zonal integrable function on $M^d$. For any integrable function $g$ we can define the convolution $h$ as the following $h(\cdot) = (Z * g)(\cdot) = \int_{M^d} Z(\cos(\rho(\cdot, x))g(x)d\nu(x)$. For the convolution on $M^d$ we have the Young's inequality $\|Z*g\|_q \leq \|Z\|_r \|g\|_p$, where $1/q = 1/p + 1/r - 1$ and $1 \leq p, q, r \leq \infty$. It is possible to show that for any $r > 0$, the function

$$g_r(\cdot) \sim \sum_{k=1}^{\infty} (k(k+\alpha+\beta+1))^{-r/2} Z^{(k)}(\cdot)$$

is integrable on $M^d$. The Sobolev's class $W_p^r(M^d)$, $r > 0$, can be defined as the set of all functions $f$ on $M^d$ given by $f = g_r * \phi + c$ where $\|\phi\|_{L_p(M^d)} \leq 1$, $c \in \mathbb{R}$ and $\phi \perp 1$. References to the previously-mentioned results from harmonic analysis can be found in [1, 3, 5].

In this paper there are several universal constants which enter into the estimates. These positive constants are mostly denoted by the letters $C, C_1, C_2, \ldots$

We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper. For easy of notation we will write $a_n \gg b_n$ for two sequences, if $a_n \geq Cb_n$ for $n \in \mathbb{N}$ and $a_n \asymp b_n$ if $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$ and some constants $C, C_1$ and $C_2$. Also, we shall put

$$(a)_+ := \begin{cases} a, & a > 0, \\ 0, & a \leq 0. \end{cases}$$

Through the text $[a]$ means entire part of $a \in R$.

The main result establishes sharp in power scale asymptotic behavior of Kolmogorov's $n$-widths.

**Theorem 1.1.** *For Sobolev's classes on $M^d$ we have the following estimates as $n \to \infty$. If $2 \leq p \leq q < \infty$, $r > d/2$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \asymp n^{-r/d}.$$

*If $2 \leq p \leq q = \infty$, $r > d/2$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d} (\log n)^{1/2}.$$

*If $1 \leq p \leq 2 \leq q < \infty$, $r > d/p$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \asymp n^{-r/d+1/p-1/2}.$$

*If $1 \leq p \leq 2 \leq q = \infty$, $r > d/p$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d+1/p-1/2} (\log n)^{1/2}.$$

**Theorem 1.2.** *For Sobolev's classes on $M^d$ we have the following estimates as $n \to \infty$. If $1 < q \leq p \leq 2$, $r > 0$, then*

$$d_n(W_p^r,\ L_q(M^d)) \asymp b_n(W_p^r,\ L_q(M^d)) \asymp n^{-r/d}.$$

*If $1 = q \leq p \leq 2$, $r > 0$, then*

$$n^{-r/d} (\log n)^{-1/2} \ll b_n(W_p^r,\ L_q(M^d)) \leq d_n(W_p^r,\ L_q(M^d)) \ll n^{-r/d}.$$

**Remark 1.1.** *For Sobolev's classes on $M^d$ we have the following estimates as $n \to \infty$ (see [11]). If $2 \le p \le q < \infty$, $r > d/2$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d}(\log n)^{1/2}.$$

*If $2 \le p \le q = \infty$, $r > d/2$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d}\log n.$$

*If $1 \le p \le 2 \le q < \infty$, $r > d/p$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d+1/p-1/2}(\log n)^{1/2}.$$

*If $1 \le p \le 2 \le q = \infty$, $r > d/p$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \ll n^{-r/d+1/p-1/2}\log n.$$

**Remark 1.2.** *For Sobolev's classes on $M^d$ we have the following estimates as $n \to \infty$ (see [1]):*

$$d_n(W_p^r(M^d), L_q(M^d)) \gg \begin{cases} n^{-r/d}, & 2 \le p, q < \infty, r > 0, \\ n^{-r/d+1/p-1/2}, & 1 \le p \le 2 \le q \le \infty, r > d/p. \end{cases}$$

*If $1 \le p = q \le \infty, r > 0$ or $2 \le q \le p < \infty, r > 0$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \asymp n^{-r/d}.$$

*If $1 \le p \le q \le 2, r > d(1/p - 1/q)$, then*

$$d_n(W_p^r(M^d), L_q(M^d)) \asymp n^{-r/d+1/p-1/q}.$$

# 2 Estimates of Levy Means

Let $E = (\mathbb{R}^n, \|\cdot\|)$ be an $n$-dimensional Banach space. Furthermore, let $|||x|||=(\sum_{k=1}^n |x_k|^2)^{1/2}$ be the Euclidean norm on $\mathbb{R}^n$ and let

$$S^{n-1} = \{x \in \mathbb{R}^n \,:\, |||x||| = 1\}$$

be the Euclidean unit sphere in $\mathbb{R}^n$. The Levy mean is defined by

$$M = M(\mathbb{R}^n, \|\cdot\|) = \left(\int_{S^{n-1}} \|x\|^2 \, d\mu\right)^{1/2},$$

where $\mu$ denotes the normalized rotation invariant measure on $S^{n-1}$. In order to specify the norm $\|\cdot\|$ whose Levy mean we want to estimate, we consider an arbitrary system of harmonics $\{\xi_k(\tau)\}_{k=1}^n \subset \oplus_{l=0}^N H_l$ orthonormal in $L_2(M^d, d\nu(\tau))$. Set $\Xi_n = \text{span}\,\{\xi_1, \ldots, \xi_{\text{n}}\}$ and let $J : \mathbb{R}^n \to \Xi_n$ be

the coordinate isomorphism that assigns to $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ the function $\xi^\alpha = \sum_{k=1}^n \alpha_k \, \xi_k \in \Xi_n$. The definition

$$\|\alpha\|_{(p)} := \|J\alpha\|_{L_p(M^d)}$$

induces a norm on $\mathbb{R}^n$. The following result gives estimates for the Levy mean $M(\|\cdot\|_{(p)})$.

**Theorem 2.1.** *Let $\{\xi_k\}_{k=1}^n$ be an arbitrary system of orthonormal harmonics in $\oplus_{l=0}^N H_l$, $n = \dim \oplus_{l=0}^N H_l$. Then the following inequalities hold:*
*1) if $2 \le p < \infty$, then $1 \le M(\|\cdot\|_{(p)}) \le Cp^{1/2}$;*
*2) if $p = \infty$, then $1 \le M(\|\cdot\|_{(\infty)}) \le C(\ln n)^{1/2}$;*
*3) if $1 \le p \le 2$, then $0 < C \le M(\|\cdot\|_{(p)}) \le 1$;*

*4) if $p = 2$, then $M(\|\cdot\|_{(2)}) = 1$.*

**Proof.** It is easy to check that 4) follows from the definition. Since the Levy mean $M(\|\cdot\|_{(p)})$ is a monotone increasing function in $p$ for $1 \le p \le \infty$, the lower bounds in 1), 2) and the upper bound in 3) follow from 4).

We now turn to the upper estimate in 1) and the lower estimate in 3). Let $d\gamma(x) = e^{-\pi |||x|||^2} dx$ be the Gaussian measure on $\mathbb{R}^n$. For an arbitrary function $f \in C(S^{n-1})$ we define an extension $\tilde{f}$ to $\mathbb{R}^n \setminus \{0\}$ by $\tilde{f}(x) = |||x||| \cdot f(x/|||x|||)$. In [2, p. 71] it is shown that

$$\int_{S^{n-1}} f(x) d\mu(x) = \frac{(2\pi)^{1/2}}{n^{1/2}} \int_{\mathbb{R}^n} \tilde{f}(x) \, d\gamma(x). \tag{1}$$

Let $(r_k)$ denotes the sequence of Rademacher functions $r_k(\theta) = \text{sign} \sin(2^{\text{k}}\pi\theta)$, for $\theta \in [0, 1]$ and $k = 1, 2, \ldots$, and for $m = 1, 2, \ldots$; $i = 1, 2, \ldots, n$ let

$$\delta_i^m(\theta) = m^{-1/2} \left( r_{(i-1)m+k}(\theta) + \cdots + r_{im}(\theta) \right).$$

It follows by Lemma 2.1 in [12, p. 585] that if $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying

$$h(x_1, \ldots, x_n) e^{-\sum_{i=1}^n |x_i|} \to 0 \text{ uniformly when } \sum_{i=1}^n |x_i| \to \infty,$$

then

$$\int_{\mathbb{R}^n} h(x) d\gamma(x) = \lim_{m\to\infty} \int_0^1 h\left((2\pi)^{-1/2}(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))\right) d\theta. \qquad (2)$$

Applying (1) and (2) for the functions $f(x) = \|x\|_{(p)}$, $x \in S^{n-1}$ and $h(x) = \tilde{f}(x) = \|x\|_{(p)}$, $x \in \mathbb{R}^n$, we have

$$\int_{S^{n-1}} \|x\|_{(p)} d\mu(x) = \frac{(2\pi)^{1/2}}{n^{1/2}} \lim_{m\to\infty} \int_0^1 \|(2\pi)^{-1/2}(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))\|_{(p)} d\theta$$

$$= n^{-1/2} \lim_{m\to\infty} \int_0^1 \left( \int_{M^d} |\sum_{i=1}^n \delta_i^m(\theta) \xi_i(\tau)|^p d\nu(\tau) \right)^{1/p} d\theta.$$

Setting $\tilde{\xi}_{(i-1)m+k}(\tau) = m^{-1/2} \xi_i(\tau)$, $\tau \in M^d$, for $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$ and $m = 1, 2, \ldots$, we have

$$\sum_{i=1}^n \delta_i^m(\theta) \xi_i(\tau) = \sum_{j=1}^{mn} r_j(\theta) \tilde{\xi}_j(\tau)$$

and therefore

$$\int_{S^{n-1}} \|x\|_{(p)} d\mu(x) = n^{-1/2} \lim_{m\to\infty} \int_0^1 \left( \int_{M^d} |\sum_{j=1}^{mn} r_j(\theta) \tilde{\xi}_j(\tau)|^p d\nu(\tau) \right)^{1/p} d\theta. \qquad (3)$$

The Khintchine inequality (see [14, p. 41], [4]) states that

$$b(p) \left( \sum_{s=1}^u |c_s|^2 \right)^{1/2} \le \left( \int_0^1 |\sum_{s=1}^u r_s(\theta)\, c_s|^p \, d\theta \right)^{1/p} \le c(p) \left( \sum_{s=1}^u |c_s|^2 \right)^{1/2},$$

where $b(1) \ge 1/2$ and

$$c(p) = 2^{1/2} \left( \frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{1/p} \asymp p^{1/2}, \quad p \to \infty.$$

From the addition formula (see e.g. [5]) and from the definition of $\tilde{\xi}_j(\tau)$, it follows that

$$\sum_{j=1}^{mn} |\tilde{\xi}_j(\tau)|^2 = \sum_{i=1}^{n} |\xi_i(\tau)|^2 = n.$$

Therefore from the Khintchine inequality and (3) we have

$$\int_{S^{n-1}} \|x\|_{(p)} d\mu(x) \leq n^{-1/2} \lim_{m\to\infty} \left( \int_{M^d} \int_0^1 |\sum_{j=1}^{mn} r_j(\theta) \tilde{\xi}_j(\tau)|^p d\theta d\nu(\tau) \right)^{1/p}$$

$$\leq c(p) n^{-1/2} \lim_{m\to\infty} \left( \int_{M^d} \left( \sum_{j=1}^{mn} |\tilde{\xi}_j(\tau)|^2 \right)^{p/2} d\nu(\tau) \right)^{1/p}$$

$$= c(p) n^{-1/2} \lim_{m\to\infty} \left( \int_{M^d} n^{p/2} d\nu(\tau) \right)^{1/p} = c(p) \qquad (4)$$

and

$$\int_{S^{n-1}} \|x\|_{(p)} d\mu(x) \geq n^{-1/2} \lim_{m\to\infty} \int_{M^d} \int_0^1 |\sum_{j=1}^{mn} r_j(\theta) \tilde{\xi}_j(\tau)| d\theta d\nu(\tau)$$

$$\geq b(1) n^{-1/2} \lim_{m\to\infty} \int_{M^d} \left( \sum_{j=1}^{mn} |\tilde{\xi}_j(\tau)|^2 \right)^{1/2} d\nu(\tau) \geq \frac{1}{2} \qquad (5)$$

Any polynomial $t_N \in \oplus_{k=0}^N H_k$ can be presented in the form $t_N = D_N * t_N$, where $D_N = \sum_{k=0}^N Z^{(k)}$. Using Hölder inequality we can show that

$$\|t_N\|_{L_\infty(M^d)} \leq \|D_N\|_{L_\infty(M^d)} \|t_N\|_{L_1(M^d)}$$

and since $D_N = D_N * D_N$, then

$$\|D_N\|_{L_\infty(M^d)} \leq \|D_N\|^2_{L_2(M^d)} = \sum_{k=0}^N \|Z^{(k)}\|^2_{L_2(M^d)} = n.$$

It means that

$$\|I\|_{L_1(M^d)\cap \mathcal{T}_N \to L_\infty(M^d)\cap \mathcal{T}_N} \leq n,$$

where $I$ is the embedding operator and $\mathcal{T}_N := \oplus_{k=0}^N H_k$. It is easy to see that $\|I\|_{L_\infty(M^d)\to L_\infty(M^d)} = 1$. Application of Riesz-Thorin Interpolation Theorem gives

$$\|t_N\|_{L_\infty(M^d)} \leq n^{1/p} \|t_N\|_{L_p(M^d)}, \quad 1 \leq p \leq \infty. \qquad (6)$$

Hence,

$$n^{-1/2}\|x\|_{(p)} \le |||x||| \le \|x\|_{(p)}, \quad x \in \mathbb{R}^n,\ 2 \le p \le \infty. \tag{7}$$

From Young's inequality we have

$$\|t_N\|_{L_2(M^d)} \le \|D_N\|_{L_2(M^d)}\|t_N\|_{L_1(M^d)} \le n^{1/2}\|t_N\|_{L_1(M^d)}$$

and consequently,

$$\|x\|_{(p)} \le |||x||| \le n^{1/2}\|x\|_{(p)}, \quad x \in \mathbb{R}^n,\ 1 \le p \le 2. \tag{8}$$

The following result is in [2, p. 60]:

$$(\gamma(\rho))^{-1} \le \left(\int_{S^{n-1}} \|x\|^\rho d\mu(x)\right)\left(\int_{S^{n-1}} \|x\| d\mu(x)\right)^{-\rho} \le \gamma(\rho),$$

where $\gamma(\rho)$ depends just on $\rho > 0$ and $\|\cdot\|$ is an arbitrary norm on $R^n$ such that $a|||\cdot||| \le \|\cdot\| \le b|||\cdot|||$, $b/a \le n^{1/2}$. Since we have (7) and (8), then this result is true for the norms $\|\cdot\| = \|\cdot\|_{(p)}$, $1 \le p \le \infty$. Thus from (4) and (5)

$$\gamma(2)^{-1/2}/2 \le M(\|\cdot\|_{(p)}) \le \gamma(2)^{1/2}c(p), \quad 1 \le p < \infty.$$

Finally to get the upper bound in 2) we apply (6) for $p = \log n$ and thus

$$M(\|\cdot\|_{(\infty)}) = \left(\int_{S^{n-1}} \|x\|^2_{(\infty)} d\mu(x)\right)^{1/2} = \left(\int_{S^{n-1}} n^{2/p}\|x\|^2_{(p)} d\mu(x)\right)^{1/2}$$

$$\le \gamma(2)^{1/2} n^{1/p} c(p) \ll (\log n)^{1/2}.$$

■

We remark that some estimates of Levy means in functional spaces have been found in [6, 7, 9, 10, 11, 8].

# 3 Estimates of $n$-Widths

Let us fix a norm $\|\cdot\|$ on $\mathbb{R}^n$ and denote by $E$ the Banach space $(\mathbb{R}^n, \|\cdot\|)$ with unit ball $B_E$. The dual space $E^o = (\mathbb{R}^n, \|\cdot\|_o)$ is endowed with the norm

$$\|x\|_o = \sup\{|\langle x, y\rangle| : y \in B_E\},$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^n$.

**Theorem 3.1.** *(see [13]) Let $E^o = (\mathbb{R}^n, \|\cdot\|_o)$. For every $0 < \lambda < 1$, there exists a subspace $F_k \subset \mathbb{R}^n$, with $\dim F_k = k > \lambda n$, such that*

$$|||\alpha||| \le C \cdot M_o (1-\lambda)^{-1/2} \|\alpha\|, \ \forall \alpha \in F_k,$$

*where $C$ is an universal constant and*

$$M_o = \left( \int_{S^{n-1}} \|\alpha\|_0^2 d\mu \right)^{1/2}.$$

**Proof of the Theorem 1.1.** Let us put $\mathcal{T}_N := \oplus_{l=0}^N H_l$, $n := \dim \mathcal{T}_N$, $B_p^n := U_p \cap \mathcal{T}_N$ and $B_{(p)}^n := J^{-1} B_p^n$. It is easy to see that $n \asymp N^d$.

Now, let $0 < \lambda < 1$ and $1 \le q \le 2$. By Theorem 3.1 there exists a subspace $F_k$ of $\mathbb{R}^n$, with $\dim F_k = k > \lambda n$, such that

$$\|x\|_{(2)} \le C \cdot M \left( \|\cdot\|_{(q')} \right) (1-\lambda)^{-1/2} \|x\|_{(q)}, \ x \in F_k.$$

If $m = n - k$, then $(1-\lambda)^{-1/2} \le (n/m)^{1/2}$. From Theorem 2.1, for all $x \in F_k$, we have that

$$\|x\|_{(2)} \le C \|x\|_{(q)} \left( \frac{n}{m} \right)^{1/2} \cdot \begin{cases} (q')^{1/2}, & 1 < q \le 2, \\ (\log n)^{1/2}, & q = 1. \end{cases}$$

Therefore

$$d^m(B_q^n, \ L_2 \cap \mathcal{T}_N) \le \sup_{x \in F_k \cap B_{(q)}^n} \|x\|_{(2)}$$

$$\le \sup_{x \in F_k \cap B_{(q)}^n} \|x\|_{(q)} C \cdot \left( \frac{n}{m} \right)^{1/2} \cdot \begin{cases} (q')^{1/2}, & 1 < q \le 2, \\ (\log n)^{1/2}, & q = 1. \end{cases}$$

$$\le C \cdot \left( \frac{n}{m} \right)^{1/2} \cdot \begin{cases} (q')^{1/2}, & 1 < q \le 2, \\ (\log n)^{1/2}, & q = 1. \end{cases} \tag{9}$$

It was shown in [1] that there is a sequence of polynomial operators $T_N : W_p^r(M^d) \to L_q(M^d) \cap \mathcal{T}_N$, $N \in \mathbb{N}$, such that

$$\sup_{f \in W_p^r(M^d)} \|f - T_N(f)\|_q \ll N^{-r + d(1/p - 1/q)_+}, \quad 1 \le p, q \le \infty, r > d(1/p - 1/q)_+. \tag{10}$$

Using this fact and a discretization technique, we can show that for some absolute constant $C$ and all $1 \le p, q \le \infty$ with $r > d(1/p - 1/q)_+$,

$$W_p^r(M^d) \subset C \left[ \left( \bigoplus_{k=0}^{[\epsilon^{-1} \log n]} n_k^{-r/d} B_p^{n_{k+1}} \right) \bigoplus \left( \bigoplus_{k=[\epsilon^{-1}\log n]+1}^{\infty} n_k^{-r/d + (1/p - 1/q)_+} B_q^{n_{k+1}} \right) \right],$$

where $\epsilon > 0$ is a fixed parameter, $N_k = [2^{k/dr} n^{1/d}]$ and $n_k = \dim \mathcal{T}_{N_k} \asymp 2^{k/r} n$. Now consider the sequence $(M_k)_{k \in \mathbb{N}}$ given by

$$M_k = \begin{cases} n, & k = 0, \\ [2^{-\epsilon k} n + 1], & 1 \leq k \leq \epsilon^{-1} \log n, \\ 0, & k > \epsilon^{-1} \log n. \end{cases}$$

It is easy to check that $\sum_{k=0}^{\infty} M_k \leq C(\epsilon) n$, where $C(\epsilon)$ depends just on $\epsilon$. Using definition and properties of Gel'fand's $n$-widths (see [15, p. 238]), we have

$$d^{[C(\epsilon)n+1]}(W_q^r(M^d),\ L_2(M^d)) \leq C \sum_{k=0}^{[\epsilon^{-1} \log n]} n_k^{-r/d} d^{M_k}(B_q^{n_{k+1}}, L_2(M^d) \cap \mathcal{T}_{N_{k+1}})$$

$$+ C \text{diam} \left( \bigoplus_{k=[\epsilon^{-1} \log n]+1}^{\infty} n_k^{-r/d + (1/q - 1/2)_+} B_2^{n_{k+1}}, \quad L_2(M^d) \right). \qquad (11)$$

Combining (9) and (11) for $r > d/2$, $0 < \epsilon < 2/d - 1/r$ and $1 \leq q \leq 2$, we get estimates for Gel'fand's $n$-widths

$$d^n(W_q^r(M^d),\ L_2(M^d)) \ll \begin{cases} n^{-r/d}, & 1 < q \leq 2, \\ n^{-r/d} (\log n)^{1/2}, & q = 1. \end{cases}$$

Now, let $1 \leq p, q \leq \infty$; $\delta, r > 0$; $s = r - d(1/p - 1/q)_+ - \delta$. Then by (10) we have $W_p^{r-s}(M^d) \subset C \cdot U_q$ and hence by the definition of Sobolev's classes

$$W_p^r(M^d) \subset C \cdot W_q^{r - d(1/p - 1/q)_+ - \delta}(M^d). \qquad (12)$$

Finally, we just need to apply the well known duality result between Kolmogorov's and Gel'fand's $n$-widths

$$d^n(W_q^r(M^d),\ L_2(M^d)) = d_n(W_2^r(M^d),\ L_{q'}(M^d)),$$

where $1/q + 1/q' = 1$, the inclusion (12) for $1 \leq p \leq 2$, $q = 2$ and standard embedding arguments to get the upper bounds in the Theorem 1.1. The lower bounds follow from Remark 1.2. ■

**Proof of the Theorem 1.2.** Let $x \in J^{-1} \mathcal{T}_N = \mathbb{R}^n$. The Hölder's inequality implies that

$$\|x\|_{(q)}^o = \sup\{|\langle x, y \rangle| \ : \ y \in B_{(q)}^n\} = \sup_{y \in B_{(q)}^n} \left| \int_{M^d} Jx \cdot Jy d\nu \right| \leq \|Jx\|_{q'} = \|x\|_{(q')},$$

where $1/q + 1/q' = 1$. Hence fixing $0 < \lambda < 1$, Theorem 3.1 yield the existence of a subspace $F_k$, $\dim F_k = k > \lambda n$, such that

$$\|x\|_{(2)} = \|Jx\|_2 \leq CM(\|\cdot\|^o_{(q)})(1-\lambda)^{-1/2}\|Jx\|_q$$

$$\leq CM(\|\cdot\|_{(q')})(1-\lambda)^{-1/2}\|Jx\|_q$$

for all $x \in F_k$. The last estimate tells us that for

$$\epsilon = C^{-1}(1-\lambda)^{1/2}\left(M(\|\cdot\|_{(q')})\right)^{-1}$$

we have

$$\epsilon B^n_{(q)} \bigcap F_k \subset B^n_{(2)},$$

so that, since $\dim F_k > \lambda n$, we can get

$$b_{\lambda n-1}(B^n_2,\ L_q(M^d)) \geq \epsilon.$$

It is easy to check that

$$n^{-r/d}B^n_2 \subset W^r_2(M^d).$$

Therefore setting $\lambda = 1/2$ and applying Theorem 2.1 we obtain that

$$\begin{aligned} b_{\frac{n}{2}-1}(W^r_2(M^d), L_q(M^d)) &\geq n^{-r/d} b_{\frac{n}{2}-1}(B^n_2, L_q(M^d)) \\ &\geq C\begin{cases} n^{-r/d}, & 1 < q \leq 2, \\ n^{-r/d}(\log n)^{-1/2}, & q = 1. \end{cases} \end{aligned}$$

This gives the proof of the Theorem 1.2 in the part of lower bounds for all $1 \leq q \leq p \leq 2$, by embedding. The upper bounds

$$d_n(W^r_p(M^d),\ L_q(M^d)) \ll n^{-r/d}, \quad 1 \leq q \leq p \leq 2,$$

follow from the estimate (10). ■

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