FULLY AND STRONGLY ALMOST SUMMING MULTILINEAR MAPPINGS

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Abstract. In this paper the classes of strongly almost summing multilinear mappings and fully almost summing multilinear mappings are introduced. We investigate the connections of these classes and other classes of absolutely summing mappings. Besides, we prove some structural properties such as a Dvoretzky-Rogers Theorem, some coincidence results and generalize a theorem of S. Kwapien which asserts that T is absolutely $(1;1)$-summing whenever T^* is absolutely $(q;q)$-summing for some $q \geq 1$.

1. Introduction

The success of the theory of absolutely summing linear operators has motivated the investigation of new classes of multilinear mappings and polynomials between Banach spaces. The first possible directions of a multilinear theory of absolutely summing multilinear mappings were outlined by A. Pietsch [15] and several related concepts have been exhaustively studied by several authors (Botelho [3], Matos [9], Meléndez-Tonge [12] among many others). Recently a question of Pietsch about Hilbert-Schmidt multilinear mappings was answered by Matos in [10] and this work motivated the study of a new space of continuous multilinear mappings, called the space of fully absolutely summing multilinear mappings (see Souza [16] and other results will appear in Matos [11]).

The linear concept of almost summing operators was first considered for the multilinear and polynomial cases by Botelho [3] and Botelho-Braunss-Junek [4]. In [13] and [14] it is shown that whenever $n \geq 2$ and $E_1, ..., E_n$ are $\mathcal{L}_\infty$-spaces, every continuous n-linear mapping from $E_1 \times ... \times E_n$ into any Banach space F is almost 2-summing. Other natural directions for extending the concepts of almost summing linear operators to polynomial and multilinear mappings are considered in this article. Our first definition lead us to the space of strongly almost summing mappings which is strictly contained in the space of almost summing mappings. Among other results, we will show that every continuous scalar valued bilinear mapping defined on $\mathcal{L}_\infty$-spaces is strongly almost 2-summing, generalizing a result of Botelho [3] about almost summing bilinear mappings. The second definition we will work with, inspired in [11], creates the space of fully almost summing mappings and furnishes some new other interesting results and generalizations of already known theorems.

Throughout, $E, E_1, ..., E_n, F$ will stand for Banach spaces. If $2 \leq q \leq \infty$ and $(r_j)_{j=1}^{\infty}$ are the Rademacher functions, we say that E has cotype q if there exists $C_q(E) \geq 0$ such that for any $k \in \mathbb{N}$ and $x_1, ..., x_k \in E$,

$$(\sum_{j=1}^{k} \|x_j\|^q)^{\frac{1}{q}} \leq C_q(E)(\int_0^1 \|\sum_{j=1}^{k} r_j(t)x_j\|^2 dt)^{\frac{1}{2}}.$$

To cover the case $q = \infty$ we replace $(\sum_{j=1}^{k} \|x_j\|^q)^{\frac{1}{q}}$ by $max_{j \leq n}\|x_j\|$.We will denote $\cot E = \inf\{q; E$ has cotype $q\}$. If $1 \leq q \leq 2$, we say that E has type q if there exists $T_q(E) \geq 0$ such that for any

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$k \in \mathbb{N}$ and $x_1, ..., x_k \in E$,

$$(\int_0^1 \| \sum_{j=1}^k r_j(t) x_j \|^2 dt)^{\frac{1}{2}} \leq T_q(E)(\sum_{j=1}^k \|x_j\|^q)^{\frac{1}{q}}$$

The next concept of absolutely summing multilinear mappings is, perhaps, the most natural generalization of the linear definition of absolutely summing operators and has been explored by several authors (see [2], [5], [13], [9], [16]).

Definition 1. *(Alencar-Matos* [1] *and Matos* [9]*) A continuous multilinear mapping*

$$T : E_1 \times ... \times E_n \to F$$

is absolutely $(p; q_1, ..., q_n)$*-summing (or* $(p; q_1, ..., q_n)$*-summing) if*

$$(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^{\infty} \in l_p(F)$$

for all $(x_j^{(s)})_{j=1}^{\infty} \in l_{q_s}^w(E_s)$*,* $s = 1, ..., n$*. Equivalently,* T *is absolutely* $(p; q_1, ..., q_n)$*-summing if there exists* $C > 0$ *such that*

$$(1.1) \qquad (\sum_{j=1}^{\infty} \|T(x_j^{(1)}, ..., x_j^{(n)})\|^p)^{\frac{1}{p}} \leq C \|(x_j^{(1)})_{j=1}^{\infty}\|_{w,q_1} ... \|(x_j^{(n)})_{j=1}^{\infty}\|_{w,q_n} \ \forall (x_j^{(k)})_{j=1}^{\infty} \in l_{q_k}^w(E_k).$$

In order to avoid trivialities we assume that $\frac{1}{p} \leq \frac{1}{q_1} + ... + \frac{1}{q_n}$. Henceforth we will denote the space of all absolutely $(p; q_1, ..., q_n)$-summing n-linear mappings from $E_1 \times ... \times E_n$ into F by $\mathcal{L}_{as(p;q_1,...,q_n)}(E_1, ..., E_n; F)$. When $q_1 = ... = q_n = q$, we write $\mathcal{L}_{as(p;q)}(E_1, ..., E_n; F)$.

The infimum of the $C > 0$ for which inequality (1.1) always holds defines a norm for the space of all absolutely $(p; q_1, ..., q_n)$-summing multilinear mappings. This norm is denoted by $\|.\|_{as(p;q_1,...,q_n)}$ and $(\mathcal{L}_{as(p;q_1,...,q_n)}(E_1, ..., E_n; F), \|.\|_{as(p;q_1,...,q_n)})$ is a Banach space.

The $(\frac{p}{n}; p)$-summing n-linear mappings will be called p-dominated n-linear mappings and constitutes an important particular case due the strong analogy with the linear case (see [12], [9]).

2. Almost and strongly almost summing multilinear mappings

The first attempts to a concept of almost summability for polynomials and multilinear mappings are due to Botelho [3] and Botelho-Braunss-Junek [4].

Definition 2. *(Botelho-Braunss-Junek* [4]*) If* $p_1, ..., p_n \geq 1$*, a continuous* n*-linear mapping* $T : E_1 \times ... \times E_n \to F$ *is said to be almost* $(p_1, ..., p_n)$*-summing if there exists* $C \geq 0$ *such that*

$$\left(\int_0^1 \| \sum_{j=1}^k T(x_j^{(1)}, ..., x_j^{(n)}) r_j(t) \|^2 dt \right)^{\frac{1}{2}} \leq C \|(x_j^{(n)})_{j=1}^k\|_{w,p_1} ... \|(x_j^{(n)})_{j=1}^k\|_{w,p_n}$$

for every k *and any* $x_j^{(l)}$ *in* $E_l, l = 1, ..., n$ *and* $j = 1, ..., k$*.*

The space of all almost $(p_1, ..., p_n)$-summing multilinear mappings will be denoted by

$$\mathcal{L}_{al(p_1,...,p_n)}(E_1, ..., E_n; F).$$

When $p_1 = ... = p_n$ we write $\mathcal{L}_{al,p}(E_1, ..., E_n; F)$.

The infimum of the $C > 0$ for which last inequality holds defines a norm and turns the space of all almost $(p_1, ..., p_n)$-summing multilinear mappings a Banach space.

The first nontrivial coincidence result for almost summing mappings is due to Botelho.

Theorem 1. *(Botelho* [3]*) Every scalar valued bilinear mapping defined on* $\mathcal{L}_{\infty}$*-spaces is almost* 2*-summing.*

Further recent work of the first named author [13] showed other important coincidence situations.

Theorem 2. *(Pellegrino [13], [14]) If $n \geq 2$ and E is an $\mathcal{L}_\infty$-space, then*

$$\mathcal{L}(^nE;F) = \mathcal{L}_{al,2}(^nE;F),$$

regardless of the Banach space F.

As we have mentioned, motivated by these several coincidence theorems, we will give a more restrictive concept, related to the definition of almost summing mappings and next we will show that we still have nontrivial coincidence results in this new situation.

Definition 3. *A continuous n-linear mapping is strongly almost $(q_1,...,q_n)$-summing if there exists $C > 0$ such that*

$$\left(\int_0^1 \left\| \sum_{j_1,...j_n=1}^{k} T(x_{j_1}^{(1)},...,x_{j_n}^{(n)}) r_{\pi(j_1,...j_n)}(t) \right\|^2 dt\right)^{\frac{1}{2}} \leq C \left\| (x_j^{(1)})_{j=1}^k \right\|_{w,q_1} \cdots \left\| (x_j^{(n)})_{j=1}^k \right\|_{w,q_n}$$

for every k, where π is any permutation from $\mathbb{N}$ into $\mathbb{N} \times ... \times \mathbb{N}$.

It is important to observe that the particular choice of π is irrelevant. The linear space composed by the n-linear strongly almost $(q_1,...,q_n)$-summing mappings from $E_1 \times ... \times E_n$ into F will be denoted by $\mathcal{L}_{sal(q_1,...,q_n)}(E_1,...,E_n;F)$. When $q_1 = ... = q_n = q$ we denote by $\mathcal{L}_{sal,q}(E_1,...,E_n;F)$.

The next Proposition shows some analogy with the linear definition of almost summing operators since, as in the linear case, if $p > 2$ the only n-linear mapping which is almost p-summing is the trivial mapping.

Proposition 1. *If $p > 2$, the unique multilinear mapping which is strongly almost $(p,...,p)$-summing is the null mapping.*

Proof. If $T \in \mathcal{L}_{sal,p}(^mE;F)$, then

$$\left(\int_0^1 \left\| \sum_{j_1\cdots j_m=1}^{n} T(x,...,x) r_{\pi(j_1,...,j_m)}(t) \right\|^2 dt\right)^{\frac{1}{2}}$$

$$= \|T(x,...,x)\| \left(\int_0^1 \left| \sum_{j_1\cdots j_m=1}^{n} r_{\pi(j_1,...,j_m)}(t) \right|^2 dt\right)^{\frac{1}{2}} = \|T(x,...,x)\| \left(\int_0^1 \left| \sum_{j=1}^{n^m} r_j(t) \right|^2 dt\right)^{\frac{1}{2}}$$

$$\geq C_2(\mathbb{K})^{-1} \|T(x,...,x)\| \left(\left| \sum_{j=1}^{n^m} 1^2 \right|\right)^{\frac{1}{2}} = C_2(\mathbb{K})^{-1} \|T(x,...,x)\| \, n^{\frac{m}{2}}.$$

Thus, since T is strongly almost $(p,...,p)$-summing, we will be able to find $C > 0$ such that

$$n^{\frac{m}{2}} \|T(x,...,x)\| \leq C \left\| (x)_{j=1}^n \right\|_{w,p}^m = C \|x\|^m n^{\frac{m}{p}}.$$

Therefore

$$\|T\| \leq C n^{\frac{m}{p} - \frac{m}{2}} \ \forall n \in \mathbb{N}.$$

Making $n \to \infty$, we have $\|T\| = 0$, whenever $p > 2$. Q.E.D.

The next Proposition is the first indication that one can expect a Dvoretzky-Roges Theorem for strongly almost summing mappings.

Proposition 2. *If $\dim E < \infty$ and $p \leq 2$, $\mathcal{L}_{sal,p}(^nE;E) = \mathcal{L}(^nE;E)$.*

Proof. It suffices to prove for $p=2$. Since every finite dimensional Banach space has type 2, there exists a positive constant C such that

$$\left(\int_0^1\left\|\sum_{j_1,\dots j_n=1}^{k}T(x_{j_1}^{(1)},...,x_{j_n}^{(n)})r_{\pi(j_1,\dots j_n)}(t)\right\|^2 dt\right)^{\frac{1}{2}}\leq C\left(\sum_{j_1,\dots j_n=1}^{k}\left\|T(x_{j_1}^{(1)},...,x_{j_n}^{(n)})\right\|^2\right)^{\frac{1}{2}}$$

$$\leq C\left\|T\right\|\left(\sum_{j=1}^{k}\left\|x_j^{(1)}\right\|^2\right)^{\frac{1}{2}}...\left(\sum_{j=1}^{k}\left\|x_j^{(n)}\right\|^2\right)^{\frac{1}{2}}$$

But, since $\dim E<\infty$, we have $l_2^w(E)=l_2(E)$ and thus

$$\left(\int_0^1\left\|\sum_{j_1,\dots j_n=1}^{k}T(x_{j_1}^{(1)},...,x_{j_n}^{(n)})r_{\pi(j_1,\dots j_n)}(t)\right\|^2 dt\right)^{\frac{1}{2}}\leq C\left\|T\right\|\|(x_j^{(1)})_{j=1}^k\|_{w,2}...\left\|(x_j)_{j=1}^k\right\|_{w,2}.\text{ Q.E.D}$$

The following result, known as Contraction Principle, clarify the behavior of random variables and will be fundamental to justify the terminology "strongly".

Theorem 3. *(Contraction Principle [6]) Let E be a Banach space and $1\leq p<\infty$. Consider a sequence of independent symmetric real valued random variables $(\varkappa_k)_{k=1}^{\infty}$ on a probability space (Ω,Σ,P). Then, regardless of the choice of real numbers $a_1,...,a_n$ and $x_1,...,x_n$ in E,*

$$\left(\int_\Omega\left\|\sum_{k\leq n}a_k\varkappa_k(w)x_k\right\|^p dP(w)\right)^{\frac{1}{p}}\leq\max|a_k|\left(\int_\Omega\left\|\sum_{k\leq n}\varkappa_k(w)x_k\right\|^p dP(w)\right)^{\frac{1}{p}}.$$

In particular, if A and B are subsets of $\{1,...,n\}$ such that $A\subset B$, then

$$\left(\int_\Omega\left\|\sum_{k\in A}\varkappa_k(w)x_k\right\|^p dP(w)\right)^{\frac{1}{p}}\leq\left(\int_\Omega\left\|\sum_{k\in B}\varkappa_k(w)x_k\right\|^p dP(w)\right)^{\frac{1}{p}}.$$

The next result is an immediate outcome of the Contraction Principle, and justify the denomination "strongly" in our definition.

Proposition 3. *Every strongly almost $(q_1,...,q_n)$-summing mapping is almost $(q_1,...,q_n)$-summing.*

Since Definition 3 preserves independent, symmetric random variables we can invoke the concepts of type and cotype and obtain some natural connections. Firstly, we need some definitions.

Definition 4. *(Matos [11]) A continuous n-linear mapping $T:E_1\times...\times E_n\to F$ is said to be fully $(p;q_1,...,q_n)$-summing if there exists $C\geq 0$ such that*

$$\left(\sum_{j_1,...,j_n=1}^{\infty}\left\|T(x_{j_1}^{(1)},...,x_{j_n}^{(n)})\right\|^p\right)^{\frac{1}{p}}\leq C\|(x_j^{(1)})_{j=1}^{\infty}\|_{w,q_1}...\left\|(x_j)_{j=1}^{\infty}\right\|_{w,q_n}$$

whenever $(x_k^{(l)})_{k=1}^{\infty}\in l_{q_l}^w(E_l), l=1,...,n$. In this case we will write

$$T\in\mathcal{L}_{fas(p;q_1,...,q_n)}(E_1,...,E_n;F).$$

Following the same line of thought, we say that an n-linear mapping $T:E_1\times...\times E_n\to F$ is weakly fully $(p;q_1,...,q_n)$-summing if

$$\sup_{\varphi\in B_{E'}}\left(\sum_{j_1,...,j_n=1}^{\infty}\left|<\varphi,T(x_{j_1}^{(1)},...,x_{j_n}^{(n)})>\right|^p\right)^{\frac{1}{p}}\leq C\left\|(x_j^{(1)})_{j=1}^{\infty}\right\|_{w,q_1}...\left\|(x_j^{(n)})_{j=1}^{\infty}\right\|_{w,q_n}$$

whenever $(x_k^{(l)})_{k=1}^{\infty} \in l_{q_l}^w(E_l), l = 1, ..., n$. *In this case we will write*

$$T \in \mathcal{L}_{wfas(p;q_1,...,q_n)}(E_1, ..., E_n; F).$$

Several results about fully summing mappings can be found in [11] and [16].

Now the same standard reasoning used for almost summing mappings (see [3]) can be analogously used in order to obtain the following Proposition:

Proposition 4. *If F has finite cotype q, then every strongly almost $(p_1, ..., p_n)$-summing multilinear mapping is fully $(q; p_1, ..., p_n)$-summing. On the other hand, if F has type q, then every fully $(q; p_1, ..., p_n)$-summing multilinear mapping is strongly almost $(p_1, ..., p_n)$-summing. In particular, if F is a Hilbert space, then*

$$\mathcal{L}_{fas(2;2,...,2)}(^nE; F) = \mathcal{L}_{sal,2}(^nE; F). \tag{2.1}$$

Next corollary is a generalization of Theorem 7.1 of [3].

Corollary 1. *If E is an $\mathcal{L}_\infty$-space then $\mathcal{L}_{sal,2}(^2E; \mathbb{K}) = \mathcal{L}(^2E; \mathbb{K})$.*

Proof. Since every scalar valued continuous bilinear mapping defined on $\mathcal{L}_\infty$-spaces is 2-dominated, and since

$$\mathcal{L}_{as(1;2,2)}(^2E; \mathbb{K}) \subset \mathcal{L}_{fas(2;2,2)}(^2E; \mathbb{K}) \text{ ([13])}$$

then

$$\mathcal{L}_{fas(2;2,2)}(^2E; \mathbb{K}) = \mathcal{L}(^2E; \mathbb{K})$$

and (2.1) yields

$$\mathcal{L}_{sal,2}(^2E; \mathbb{K}) = \mathcal{L}(^2E; \mathbb{K}).$$

We also have some structural properties, such as:

Theorem 4. *If every continuous multilinear $T : E_1 \times ... \times E_n \to F$ is strongly almost $(q_1, ..., q_n)$-summing, then every continuous multilinear $T : E_{j_1} \times ... \times E_{j_r} \to F$, with $1 \le r \le n$, $j_1, ..., j_r \in \{1, ..., n\}$ mutually distincts, is strongly almost $(q_{j_1}, ..., q_{j_n})$-summing.*

If $p > 1$, we know that $\mathcal{L}_{al,p}(E; E) \neq \mathcal{L}(E; E)$ [4, Example 4]. As a corollary of this result and Theorem 4, we have a Dvoretzky-Rogers Theorem for strongly almost summing mappings.

Corollary 2. *If $1 < p \le 2$ we have*

$$\mathcal{L}_{sal,p}(^nE; E) \neq \mathcal{L}(^nE; E) \Leftrightarrow \dim E < \infty.$$

3. Examples of strongly almost summing mappings

We will show that despite the definition of strongly almost summing mappings is restrictive, we do not have to look further to give examples of such mappings. A simple computation give us the example below, which can inspire many others.

Example 1. *If $u : E \to F$ is an almost p-summing linear mapping and φ is a continuous linear functional, then*

$$T : E \times E \to F : T(x, y) = u(x)\varphi(y)$$

is strongly almost $(p, 2)$-summing.

We can also construct more general examples. Our first statement in an inclusion result.

Proposition 5. *Every weakly fully $(1; 1, ..., 1)$-summing mapping is strongly almost $(1, ..., 1)$-summing.*

Proof. We just need to observe that

$$\left(\int_0^1 \| \sum_{j_1,...,j_n=1}^{n} (T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) r_{\pi(j_1,...,j_n)}(t) \|^2 dt\right)^{\frac{1}{2}} \le$$

$$\leq \sup_{t\in[0,1]} \| \sum_{j,k=1}^{n} (T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) r_{\pi(j_1,...,j_n)}(t) \|$$

$$= \sup_{t\in[0,1]} \sup_{\varphi\in B_{X'}} |< \varphi, \sum_{j,k=1}^{n} (T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) r_{\pi(j_1,...,j_n)}(t) >|$$

$$\leq \sup_{\varphi\in B_{X'}} \sum_{j,k=1}^{n} |< \varphi, T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) >|$$

$$\leq \|(T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}))_{j_1,...,j_n=1}^{n}\|_{w,1}. \text{ Q.E.D.}$$

The following Lemma which proof is a simple exercise will help us to construct other examples.

Lemma 1. *If*

$$\mathcal{L}(E_1, ..., E_n; \mathbb{K}) = \mathcal{L}_{fas(q;r_1,...,r_n)}(E_1, ..., E_n; \mathbb{K}),$$

then

$$\mathcal{L}(E_1, ..., E_n; F) = \mathcal{L}_{wfas(q;r_1,...,r_n)}(E_1, ..., E_n; F).$$

Now, the following Proposition is a consequence of Proposition 5 and Lemma 1.

Proposition 6. *If*

$$\mathcal{L}(E_1, ..., E_n; \mathbb{K}) = \mathcal{L}_{fas(1;1,...,1)}(E_1, ..., E_n; \mathbb{K}), \tag{3.1}$$

then for every Banach space F we have

$$\mathcal{L}(E_1, ..., E_n; F) = \mathcal{L}_{sal(1,...,1)}(E_1, ..., E_n; F).$$

Results in which we have a coincidence result as in (3.1) will appear in [11].

4. Fully almost summing mappings

The next concept, suggested by M. C. Matos, is also natural and furnishes various interesting consequences.

Definition 5. *If $p, p_1, ..., p_n \geq 1$, a continuous n-linear mapping is fully almost $(p; p_1, ..., p_n)$-summing if there exists $C > 0$ such that*

$$\left(\int_{[0,1]^n} \left\| \sum_{j_1,...j_n=1}^{k} T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)}) r_{j_1}(t_1)...r_{j_n}(t_n) \right\|^p d\lambda \right)^{\frac{1}{p}} \leq C \left\| (x_j^{(1)})_{j=1}^k \right\|_{w,p_1} ... \left\| (x_j^{(n)})_{j=1}^k \right\|_{w,p_n}$$

for every natural k, where λ denotes the Lebesgue measure over the Borelians of $[0,1]^n$.

The linear space of all fully almost $(p; p_1, ..., p_n)$-summing n-linear mappings from $E_1 \times ... \times E_n$ into F will be denoted by $\mathcal{L}_{fal(p;p_1,...,p_n)}(E_1, ..., E_n; F)$.

The infimum of the constants C defines a norm $\|.\|_{fal(p;p_1,...,p_n)}$ and

$$(\mathcal{L}_{fal(p;p_1,...,p_n)}(E_1, ..., E_n; F), \|.\|_{fal(p;p_1,...,p_n)})$$

is a Banach space.

In the case $p_1 = ... = p_n = q$ we write $\mathcal{L}_{fal(p;q)}(E_1, ..., E_n; F)$ and $\|T\|_{fal(p;q)}$.

It must be mentioned that we are no longer able to explore type and cotype as we did in last section, since we do not have independent random variables anymore. In fact it is not hard to see that the random variables

$$r_{jk} : [0,1]^2 \to [0,1] : r_{jk}(t,s) = r_j(t) r_k(s)$$

are not independent since

$$\lambda(r_{11}^{-1}(1) \cap r_{12}^{-1}(1) \cap r_{21}^{-1}(1) \cap r_{22}^{-1}(-1)) = 0,$$

whereas

$$\lambda(r_{11}^{-1}(1)).\lambda(r_{12}^{-1}(1)).\lambda(r_{21}^{-1}(1)).\lambda(r_{22}^{-1}(-1)) = \frac{1}{8},$$

where λ denotes the Lebesgue measure over the Borel sets of $[0,1]^2$.

On the other hand, this new definition will allow us to explore deeply the Rademacher functions.

5. Examples and results of fully almost summing mappings

The next result which proof is analogous to the proof of Proposition 1 shows that in order to obtain non trivial examples of n-linear fully almost $(p;q)$-summing mappings we must have $q \leq 2$.

Proposition 7. *If $q > 2$ and $T \in \mathcal{L}_{fal(p;q)}(^nE;F)$ for some $n \in \mathbb{N}$ then $T = 0$.*

The following property shows more similarity with the definition of strongly almost summing mappings.

Proposition 8. *If $\mathcal{L}_{fal(p;p_1,...,p_n)}\left(E_1,...,E_n;F\right) = \mathcal{L}\left(E_1,...,E_n;F\right)$, then*

$$\mathcal{L}_{fal\left(p;p_{k_1},...,p_{k_j}\right)}\left(E_{k_1},...,E_{k_j};F\right) = \mathcal{L}\left(E_{k_1},...,E_{k_j};F\right)$$

whenever $k_j \in \{1,...,n\}$, with $1 \leq j < n$, mutually distincts.

Proof. Take $T \in \mathcal{L}\left(E_1;F\right)$. We are going to show that $T \in \mathcal{L}_{fal(p;p_1)}\left(E_1;F\right)$.

Let $\varphi \in E'$ and $a \in E$ such that $\varphi\left(a\right) = 1$. Define

$$R : E_1 \times E_2 \longrightarrow F$$
$$(x,y) \longrightarrow R\left(x,y\right) = T\left(x\right)\varphi\left(y\right)$$

Once we know that $R \in \mathcal{L}\left(E_1,E_2;F\right)$, then by hypothesis $R \in \mathcal{L}_{fal(p;p_1,p_2)}\left(E_1,E_2;F\right)$ and making $y_1 = a$, $y_2 = y_3 = ... = 0$, we get

$$\begin{aligned}
&\left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)T(x_j)\right\|^p dt\right) \\
&= \left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)R(x_j,a)\right\|^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \left\|\sum_{j,k=1}^m r_j(t)R(x_j,y_k)\right\|^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^1\int_0^1 \left\|\sum_{j,k=1}^m r_j(t)R(x_j,y_k)\right\|^p dtd\theta\right)^{\frac{1}{p}} \\
&= \left(\int_0^{\frac{1}{2}}\int_0^1 \left\|\sum_{j,k=1}^m r_j(t)r_k\left(\theta\right)R(x_j,y_k)\right\|^p dtd\theta + \int_{\frac{1}{2}}^1\int_0^1 \left\|\sum_{j,k=1}^m r_j(t)r_k\left(\theta\right)R(x_j,y_k)\right\|^p dtd\theta\right)^{\frac{1}{p}} \\
&= \left(\int_0^1\int_0^1 \left\|\sum_{j,k=1}^m r_j(t)r_k\left(\theta\right)R(x_j,y_k)\right\|^p dtd\theta\right)^{\frac{1}{p}} \\
&\leq \|R\|_{fal(p;p_1,p_2)} \left\|(x_j)_{j=1}^m\right\|_{w,p_1} \|(y_k)_{k=1}^m\|_{w,p_2} \\
&\leq \|R\|_{fal(p;p_1,p_2)} \left\|(x_j)_{j=1}^m\right\|_{w,p_1}.
\end{aligned}$$

This shows that $T \in \mathcal{L}_{fal(p;p_1)}(E_1;F)$. The same reasoning furnishes

$$\mathcal{L}_{fal(p;p_2)}(E_2;F) = \mathcal{L}(E_2;F). \text{ Q.E.D.}$$

It also can be checked that every finite type multilinear mapping is fully almost $(p;2)$-summing and an adequate use of the Rademacher functions furnishes

$$\mathcal{L}(E_1,...,E_n;F) = \mathcal{L}_{fal(p;1,...,1)}(E_1,...,E_n;F),$$

for every $0 < p < \infty$.

The next theorem asserts that we have an inclusion theorem concerning r-dominated mappings and fully $(r;2)$-summing mappings.

Theorem 5. *If E_1,...,E_n and F are Banach spaces, we have* $\mathcal{L}_{as(\frac{r}{n};r)}(E_1,...,E_n;F) \subset \mathcal{L}_{fal(r;2)}(E_1,...,E_n;F)$, *for every $r \in (0,\infty)$.*

Proof. Take $T \in \mathcal{L}_{as(\frac{r}{n};r)}(E_1,...,E_n;F)$. Using the Grothendieck-Pietsch Domination Theorem for multilinear mappings, Fubini's Theorem and Khinchin's Inequality, we obtain

$$\begin{aligned}
&\left(\int_0^1 ... \int_0^1 \left\| \sum_{j_1,...,j_n=1}^m r_{j_1}(t_1)...r_{j_n}(t_n)T(x_{j_1}^1,...,x_{j_n}^n)\right\|^r dt_1...dt_n\right)^{\frac{1}{r}} \\
&= \left(\int_0^1 ... \int_0^1 \left\| T\left(\sum_{j_1=1}^m r_{j_1}(t_1)x_{j_1}^1,...,\sum_{j_n=1}^m r_{j_n}(t_n)x_{j_n}^n\right)\right\|^r dt_1...dt_n\right)^{\frac{1}{r}} \\
&\leq \left\{\int_0^1 ... \int_0^1 \|T\|_{as(\frac{r}{n};r)}^r \prod_{k=1}^n \left[\int_{B_{E_k'}} \left|\varphi_k\left(\sum_{j_k=1}^m r_{j_k}(t_k)x_{j_k}^k\right)\right|^r d\mu_k(\varphi_k)\right]^{\frac{1}{r}.r} dt_1...dt_n\right\}^{\frac{1}{r}} \\
&= C\left\{\int_0^1 ... \int_0^1 \int_{B_{E_1'}} \left|\sum_{j_1=1}^m r_{j_1}(t_1)\varphi_1(x_{j_1}^1)\right|^r d\mu_1 .... \int_{B_{E_n'}} \left|\sum_{j_n=1}^m r_{j_n}(t_n)\varphi_n(x_{j_n}^n)\right|^r d\mu_n dt_1...dt_n\right\}^{\frac{1}{r}} \\
&= C\left\{\int_{B_{E_1'}} ... \int_{B_{E_n'}} \int_0^1 \left|\sum_{j_1=1}^m r_{j_1}(t_1)\varphi_1(x_{j_1}^1)\right|^r dt_1 .... \int_0^1 \left|\sum_{j_n=1}^m r_{j_n}(t_n)\varphi_n(x_{j_n}^n)\right|^r dt_n d\mu_n...d\mu_1\right\}^{\frac{1}{r}} \\
&\leq C\left\{\int_{B_{E_1'}} ... \int_{B_{E_n'}} (B_r)^r \left(\sum_{j_1=1}^m \left|\varphi_1(x_{j_1}^1)\right|^2\right)^{\frac{r}{2}} ....(B_r)^r \left(\sum_{j_n=1}^m \left|\varphi_n(x_{j_n}^n)\right|^2\right)^{\frac{r}{2}} d\mu_n...d\mu_1\right\}^{\frac{1}{r}} \\
&\leq C_1\left\{\int_{B_{E_1'}} ... \int_{B_{E_n'}} \left[\prod_{k=1}^n \left[\sup_{x_k' \in B_{E_k'}} \left(\sum_{j_k=1}^m \left|\langle x_k', x_{j_k}^k\rangle\right|^2\right)^{\frac{1}{2}}\right]^r\right] d\mu_n...d\mu_1\right\}^{\frac{1}{r}} \\
&= C_1\left\{\int_{B_{E_1'}} ... \int_{B_{E_n'}} \|(x_j^1)_{j=1}^m\|_{w,2}^r ... \|(x_j^n)_{j=1}^m\|_{w,2}^r d\mu_n...d\mu_1\right\}^{\frac{1}{r}} \\
&= C_1 \prod_{k=1}^n \|(x_j^k)_{j=1}^m\|_{w,2},
\end{aligned}$$

where $C_1 = \|T\|_{as(\frac{r}{n};r)} B_r^n$ and B_r is the constant of Khinchin's inequality.

Therefore, $T \in \mathcal{L}_{fal(r;2)}(E_1,...,E_n;F)$ and $\|T\|_{fal(r;2)} \leq \|T\|_{as(\frac{r}{n};r)} B_r^1...B_r^n$. Q.E.D.

6. Composition theorems

Both definitions are well behaved for composition. In the following we prove some results for fully almost summing mappings, but one can easily check that the same properties are also true for strongly almost summing mappings.

Proposition 9. *(Ideal Property) If $R : F \longrightarrow G$ is a bounded linear operator and*

$$T \in \mathcal{L}_{fal(p;p_1,...,p_n)}(E_1,...,E_n;F),$$

then $RT \in \mathcal{L}_{fal(p;p_1,...,p_n)}(E_1, ..., E_n; F)$.

Proof. Observe that

$$\left(\int_0^1 ... \int_0^1 \left\| \sum_{j_1,...,j_n=1}^{m} r_{j_1}(t_1)...r_{j_n}(t_n) RT(x_{j_1}^1, ..., x_{j_n}^n) \right\|^p dt_1...dt_n \right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 ... \int_0^1 \left\| R\left(\sum_{j_1,...,j_n=1}^{m} r_{j_1}(t_1)...r_{j_n}(t_n) T(x_{j_1}^1, ..., x_{j_n}^n) \right) \right\|^p dt_1...dt_n \right)^{\frac{1}{p}}$$

$$\leq \|R\| \left(\int_0^1 ... \int_0^1 \left\| \sum_{j_1,...,j_n=1}^{m} r_{j_1}(t_1)...r_{j_n}(t_n) T(x_{j_1}^1, ..., x_{j_n}^n) \right\|^p dt_1...dt_n \right)^{\frac{1}{p}}$$

$$\leq \|R\| \, \|T\|_{fal(p;p_1,...,p_n)} \prod_{k=1}^{n} \|(x_j^k)_{j=1}^m\|_{w,p_k}. \text{ Q.E.D.}$$

Theorem 6. *If* $T = (T_1, ..., T_n), T_k \in \mathcal{L}(E_k; F_k)$, *for every* $k = 1, ..., n$. *and*

$$R \in \mathcal{L}_{fal(p;p_1,...,p_n)}(F_1, ..., F_n; G),$$

then $RT \in \mathcal{L}_{fal(p;p_1,...,p_n)}(E_1, ..., E_n; F)$.

Proof. For each $k = 1, ..., n$, chose $\left(x_j^k\right)_{j=1}^m \in l_{p_k}^w(E_k)$. Since $T_k \in \mathcal{L}(E_k; F_k)$, we have $\left(T_k\left(x_j^k\right)\right)_{j=1}^m \in l_{p_k}^w(F_k)$ and

$$\|(T_k\left(x_j^k\right))_{j=1}^m\|_{w,p_k} \leq \|T_k\| \, \|(x_j^k)_{j=1}^m\|_{w,p_k}.$$

Hence

$$\left[\int_0^1 ... \int_0^1 \left\| \sum_{j_1,...,j_n=1}^{m} r_{j_1}(t_1)...r_{j_n}(t_n) RT(x_{j_1}^1, ..., x_{j_n}^n) \right\|^p dt_1...dt_n \right]^{\frac{1}{p}}$$

$$= \left[\int_0^1 ... \int_0^1 \left\| \sum_{j_1,...,j_n=1}^{m} r_{j_1}(t_1)...r_{j_n}(t_n) R(T_1\left(x_{j_1}^1\right), ..., T_n\left(x_{j_n}^n\right)) \right\|^p dt_1...dt_n \right]^{\frac{1}{p}}$$

$$\leq \|R\|_{fal(p;p_1,...,p_n)} \prod_{k=1}^{n} \|(T_k\left(x_j^k\right))_{j=1}^m\|_{w,p_k}$$

$$\leq \|R\|_{fal(p;p_1,...,p_n)} \prod_{k=1}^{n} \|T_k\| \, \|(x_j^k)_{j=1}^m\|_{w,p_k}. \text{ Q.E.D.}$$

7. A multilinear version for a Theorem of S.Kwapien

Our last result is an interesting generalization of the following result, due to S. Kwapien.

Theorem 7. *(Kwapien [7]) Let X be a Banach space and H a Hilbert space. If $u \in \mathcal{L}(X; H)$ is such that u^* is q-summing for some $1 \leq q < \infty$, then u is 1-summing and $\|u\|_{as,1} \leq A_1^{-1} B_q \|u^*\|_{as,q}$, where A_1 and B_q are the constants of Khinchin's Inequality.*

Before we state the main result, we shall soon introduce an appropriate definition for the adjoint of a multilinear operator.

Definition 6. *Let $E_1, ..., E_n$ and F be Banach spaces. If $T \in \mathcal{L}(E_1, ..., E_n; F)$, we define the adjoint of T by*

$$T^* : F^* \longrightarrow \mathcal{L}(E_1, ..., E_n; \mathbb{K})$$
$$\varphi \longrightarrow T^*\varphi : E_1 \times ... \times E_n \longrightarrow \mathbb{K}$$

with $(T^*\varphi)(x_1, ..., x_n) = \varphi(T(x_1, ..., x_n))$

One can see that for every $S \in \mathcal{L}(F; G)$, we have $(S \circ T)^* = T^* \circ S^*$.

Theorem 8. *If E_1,..., E_N are Banach spaces, H is a Hilbert space and*

$$T \in \mathcal{L}(E_1, ..., E_N; H)$$

is such that T^* is almost 2-summing, then T is absolutely $(1; 1, ..., 1)$-summing and

$$\|T\|_{as(1;1,...,1)} \leq A_1^{-1} \|T^*\|_{al,2}.$$

Proof. We first consider the case of an operator $T : E_1 \times ... \times E_N \longrightarrow l_2^n$ $(n \in \mathbb{N})$.

Consider $x^{(k,1)}, ..., x^{(k,m)} \in E_k, 1 \leq k \leq N$. Invoking Khinchin's Inequality (see [6, Theorem 1.10]), we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\|_{l_2^n} \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle T(x^{(1,j)}, ..., x^{(N,j)}), e_k \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \left| \left\langle (x^{(1,j)}, ..., x^{(N,j)}), T^* e_k \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{j=1}^{m} \left[A_1^{-1} \left(\int_0^1 \left| \sum_{k=1}^{n} \left\langle (x^{(1,j)}, ..., x^{(N,j)}), T^* e_k \right\rangle r_k(t) \right| dt \right) \right] \\
&= A_1^{-1} \int_0^1 \sum_{j=1}^{m} \left| \left\langle (x^{(1,j)}, ..., x^{(N,j)}), \sum_{k=1}^{n} r_k(t) T^* e_k \right\rangle \right| dt \\
&\leq A_1^{-1} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|_{as(1;1,...,1)} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^m \right\|_{w,1} dt
\end{aligned}$$

Thus, since $\mathcal{L}(E_1, ..., E_N; \mathbb{K}) = \mathcal{L}_{as(1;1,...,1)}(E_1, ..., E_N; \mathbb{K})$ holds isometrically, we have

$$\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\|_{l_2^n} \leq A_1^{-1} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^m \right\|_{w,1} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\| dt. \tag{7.1}$$

On the other hand, since T^* is almost summing we have

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) T^* e_k \right\|^2 dt \right)^{\frac{1}{2}} \leq \|T^*\|_{al,2} \|(e_k)_{k=1}^n\|_{w,2} = \|T^*\|_{al,2}. \tag{7.2}$$

The proof can now be completed if we finally consider any operator $T \in \mathcal{L}(E_1, ..., E_N; H)$ which adjoint $T^* : H \longrightarrow \mathcal{L}(E_1, ..., E_N; \mathbb{K})$ is almost summing.

Now, fix $x^{(k,1)}, ..., x^{(k,m)} \in E_k$, with $1 \leq k \leq N$. Identify the span of the $T(x^{(1,j)}, ..., x^{(N,j)})'s, j = 1, ..., m$, with l_2^n for an appropriate n and define by Ψ such identification. This is possible, since such span is a finite dimensional Hilbert space. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto this span. We have $P^* = P$ and by (7.1) and (7.2), we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| T(x^{(1,j)}, ..., x^{(N,j)}) \right\|_{l_2^n} \\
&= \sum_{j=1}^{m} \left\| \Psi \circ P \circ T(x^{(1,j)}, ..., x^{(N,j)}) \right\|_{l_2^n} \\
&\leq A_1^{-1} \left\| (\Psi \circ P \circ T)^* \right\|_{al,2} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} \\
&= A_1^{-1} \left\| T^* \circ P^* \circ \Psi^* \right\|_{al,2} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} \\
&\leq A_1^{-1} \left\| T^* \right\|_{al,2} \left\| P^* \right\| \left\| \Psi^* \right\| \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} \\
&\leq A_1^{-1} \left\| T^* \right\|_{al,2} \left\| P \right\| \left\| \Psi \right\| \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1} \\
&= A_1^{-1} \left\| T^* \right\|_{al,2} \prod_{i=1}^{N} \left\| (x^{(i,j)})_{j=1}^{m} \right\|_{w,1}.
\end{aligned}$$

Therefore, T is absolutely $(1; 1, ..., 1)$-summing and $\|T\|_{as(1;1,...,1)} \leq A_1^{-1} \|T^*\|_{al,2}$.

Remark 1. *The analogous of Theorem 8 for fully absolutely summing operators does not hold. For a counterexample it suffices to take $H = \mathbb{K}$, $N = 2$, $E_1 = E_2 = c_0$ and $T \in \mathcal{L}(c_0, c_0; \mathbb{K})$ such that $T \notin \mathcal{L}_{fas(1;1)}(c_0, c_0; \mathbb{K})$. There exists such T since a well known result due to Littlewood (see* [8]*) asserts that there exists $T \in \mathcal{L}(c_0, c_0; \mathbb{K})$ such that*

$$\sum_{j,k=1}^{\infty} \|T(e_j, e_k)\| = \infty.$$

On the other hand it is obvious that the adjoint of T is almost summing.

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