A class of real integrals by means of complex variables

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Abstract

Using the residue theorem with an appropriate contour of integration, we discuss a class of real improper integrals. By means of a change of variables we can simplify calculations by converting an integral which depends on a parameter in a complex plane to another that does not depend on the parameter. We then have a branching point in addition to a simple pole in place of one single (a unique) pole.

1 Introduction

Earlier or later, a student of physics will encounter the improper integrals in a variety of applications in a problem, for example, where the Fourier transform procedure is the tool to be used. We remember that the Fourier transform methodology takes the problem in the configuration spaces to the momentum spaces. We solve the so-called transformed problem in the momentum spaces, which are supposedly simpler, and get the solution of the problem in the configuration spaces by means of the inverse Fourier transform.

On the other hand, a student of engineering use the methodology of Laplace transform to reduce the initial (or boundary) problem, as in an electric circuit or a mass-spring problem, for example, to another apparently more simple one. In the same way as with the Fourier transform, the student takes the solution back to the initial (first) parameter problem by means of the inverse Laplace transform.

Both, the Fourier and Laplace transforms[1] are defined by improper integrals. In general, the inverse transform are made by using complex variables, particularly the residue theorem with an appropriate contour of integration.

In this study we are interested in calculating a certain type of real improper integral by means of complex variables, i.e., we use the residue theorem with an appropriate contour of integration to calculate the real improper integral. We emphasize the importance of firstly introducing an appropriate change of the variable before the calculation involving a complex variable.

This paper is organized as follows: in section two, we introduce the main problem, associated with the respective improper integrals in a simple way, without use the complex variable (this can be seen in a first calculus course). In section three we introduce the complex variable to simplify the initial problem and we then obtain the solution of the initial problem. In section four we introduce an appropriate change of the variable before performing the integral and finally, in section five we present our comments.

2 A class of improper integrals

Our aim in this paper is to calculate the following real integral which appear, for example, where sometimes is necessary to consider an area of a region which extends indefinitely to the right (or to the left) along the axis x:

$$\Omega(k) = \int_0^\infty \frac{dx}{ax^{2k} + 2bx^k + c} \tag{1}$$

with a,b, $c \in \mathbf{R}$, $a \neq 0$ and $k = 1, 2, \ldots$ We emphasize that the special case $k = 1$ can be performed in terms of the hypergeometric function.[2]

Before approaching the general problem it is appropriate to consider a special case, where the denominator has no zeros in the positive real axis. Take, for example, the case where $k = 1 = a = c$ and $b = 0$, i.e., the integral

$$\Omega(1) = \int_0^\infty \frac{dx}{1 + x^2}. \tag{2}$$

Probably this integral was and still is, maybe, the first improper integral calculated explicitly, in one of the first courses of calculus. As it is known,

[1] For the other types of integral transforms, we refer the reader to the reference [1, 2].

[2] $\Omega(1) = \displaystyle\int_0^\infty \frac{dx}{ax^2 + 2bx + c} = \frac{1}{\sqrt{ac}}\, {}_2F_1(1/2, 1/2; 3/2; 1-b^2/ac)$ with $a > 0$ and $b^2 < ac$.

this integral can be calculated without having real zeros in the denominator by means of a trigonometric change of variable[3] $x = \tan\theta$ and the result is $\Omega(1) = \pi/2$. We note that the integral in eq.(2) is also a special case ($k = 2$ and $\mu = 1$) of the integral

$$\Omega_\mu(k) = \int_0^\infty \frac{dx}{\mu + x^k} \tag{3}$$

with $\Re(\mu) > 0$, that, after changing the variable $x = \mu^\xi y$, can be written as the following

$$\Omega_\mu(k) = \mu^{\xi-1} \int_0^\infty \frac{dy}{1 + y^k} \tag{4}$$

with $k = 2, 3, \ldots$ $\Re(\mu) > 0$ and $\xi = 1/k$.

Using eq.(4) one can ask, for example, if the calculus of $\Omega_\mu(3)$ is analogous. In this case we have a real zero in the denominator and two others complex conjugate roots. And in the case of $k = 4$? Here we don't have any real roots. In general, if k is an odd number, $x = -1$ is always a zero in the denominator but if k is an even number we don't have real roots in the denominator. In addition, we can consider the special cases $k = 2, 3, 4$ and 5 as partial fractions, but for $n > 5$ the job is extremely laborous.

Going back to our original problem, i.e., the integral in eq.(1), we introduce the following change of a variable

$$x^k = t$$

and we can write[4]

$$\Omega(k) = \frac{1}{k} \int_0^\infty \frac{t^\nu}{at^2 + 2bt + c} dt$$

where $a \neq 0$ and we have defined the parameter $\nu = (1 - k)/k$.

Using partial fraction we can write for the above expression

$$\Omega(k) = \frac{1/k}{t_2 - t_1} \int_0^\infty \left(\frac{t^\nu}{t - t_2} - \frac{t^\nu}{t - t_1} \right) dt$$

[3]This integral can also be calculated by means of partial fractions but, in this case, we must extend this method to the complex variable because the denominator has no real zeros.

[4]For $a > 0$, $b^2 < ac$ we can express the results in terms of the hypergeometric function, i.e., $\Omega(k) = \frac{1}{k} a^{-(\nu+1)/2} c^{(\nu-1)/2} B(\nu+1, 1-\nu)\, {}_2F_1[(\nu+1)/2, (1-\nu)/2; 3/2; 1-b^2/ac]$ where $B(p, q)$ is the beta function.

where $at_{1,2} = -b \pm \sqrt{\Delta}$ with $\Delta = b^2 - ac$ which we admit is different from zero.[5]

Turning back to the initial variable, x, we get

$$\Omega(k) = \frac{1}{t_2 - t_1} \int_0^\infty \left(\frac{1}{x^k - t_2} - \frac{1}{x^k - t_1} \right) dx.$$

Admitting that the roots t_1 and t_2 are such that $-t_1 = \mu_1$ and $-t_2 = \mu_2$ we obtain[6]

$$\Omega(k) = \frac{1}{t_2 - t_1} \int_0^\infty \left(\frac{1}{x^k + \mu_2} - \frac{1}{x^k + \mu_1} \right) dx$$

which can be written as follows

$$\Omega(k) = \frac{(-t_2)^\mu - (-t_1)^\nu}{t_2 - t_1} \int_0^\infty \frac{dx}{1 + x^k} = \beta(t_1, t_2, \nu)\, \Omega_\mu(k) \qquad (5)$$

where we introduce the parameter

$$\beta(t_1, t_2, \nu) = \frac{(-t_2)^\nu - (-t_1)^\nu}{t_2 - t_1}.$$

Eq.(5) shows that, with the restriction quoted above, eqs.(1) and (4) are proportional, and thus we need to consider only one of them. For simplicity we'll discuss the eq.(4) and furthermore, without loss of generality, we'll take $\mu = 1$.

3 Using complex variables

For reasons presented in section 2, we believe that it is clear that the integral in eq.(1) or eq.(5) for $k > 5$ is difficult to calculate by using partial fractions because the calculus is extremely laborous. Thus, at this point we must introduce the complex variables to simplify our initial problem.

As we know, the integral transforms methodology changes the original problem into another problem that is simpler them the first one. Here, we

[5] The case $\Delta = 0$, where $t_1 = t_2$ we then have a pole of the second order.

[6] We exclude only the case where the roots t_1 and t_2 are real and positives. In this case we have the result of the integration given by the expression $\int_0^\infty \frac{dx}{1 - x^q} = \frac{\pi}{q} \cot\left(\frac{\pi}{q}\right)$, where $q > 1$.

are looking for another more appropriate integral to calculate our original integral, eq.(4). We introduce the complex variables and use the residue theorem with an appropriate contour of integration. As we have seen in section 2, first we need to consider a special case, $k = 2$. In this case, it is probably also the first case studied when the calculation of the real integrals by means of complex variables is discussed.

We considere the following complex integral

$$\oint_C \frac{dz}{1+z^2} \tag{6}$$

where the integral is taken into the complex plane (for this change we call it the transformed problem) counterclockwise and the contour C is composed of a straight right line (real axis) from $-R$ up to $+R$, with $R > 0$ and closed by a semicircle, in the upper half-plane, centered at the origin and radius R (the same result can be obtained if we close the contour in the lower half-plane), as in Figure 1

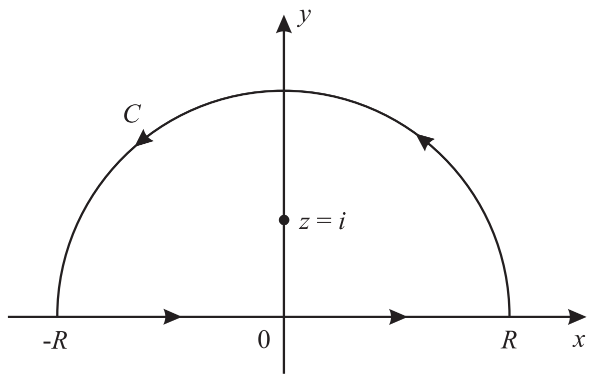

Figure 1: Contour for the integral in eq.(6).

Using the residue theorem we can write

$$\oint_C \frac{dz}{1+z^2} = 2\pi i \,\mathrm{Res}\,(z=i)$$

because the unique point (a simple pole) which is inside the contour of integration is $z = i$.

Rounding the contour of integration in the positive sense we get

$$\int_{-R}^{R}\frac{dx}{1+x^2}+\int_{C_R}\frac{dz}{1+z^2}=2\pi i\lim_{z\to i}\left[(z-i)\frac{1}{1+z^2}\right]$$

where C_R is a semicircle centered at the origin and radius R. Taking the limit $R\to\infty$ the integral on C_R goes to zero, by means of the Jordan Lemma[3] and if we know that the function on the integral is an even function, we obtain

$$\int_0^\infty\frac{dx}{1+x^2}=\frac{\pi}{2}=\Omega(1)$$

which is exactly the same result obtained in section 2, without using the complex variables.

Here we also ask if is that easy to calculate the integral for other values of the parameter? The answer is no. For example, we quote that the case $k=3$, where we have the complex integral

$$\oint_C\frac{dz}{1+z^3} \tag{7}$$

but the contour, closed and oriented in the positive sense is composed of two straight right line, one of them from the origin up to R ($R>0$) and another from $R\exp(2\pi i/3)$ up to the origin, joined by an arc of a circle centered at the origin and radius R. In this contour only a pole (a simple pole) is inside the contour, i.e., in this case $z=$exp$(\pi i/3)$ is the unique pole which is inside the contour, as in Figure 2.

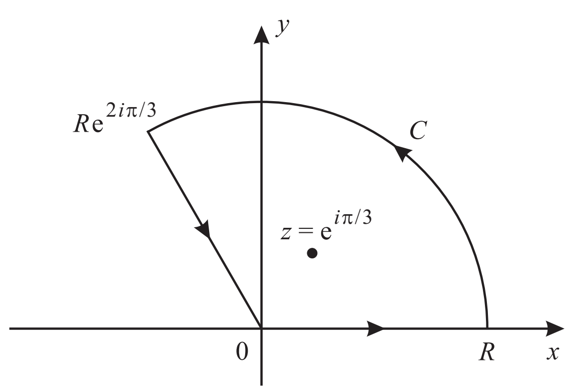

Figure 2: Contour for the integral in eq.(7).

Again, using the residue theorem we can show that the improper integral, for $k = 3$, is $2\pi/3\sqrt{3}$.

This reasoning could be extended to the other values of k as follows: we consider the same complex integral

$$\oint_C \frac{1}{1+z^k}$$

where C is now constructed by two straight right line, one of them from the origin up to R and another of $R\exp(2\pi i/k)$ up to the origin and an arc of the circle centered at the origin and radius R. For this contour only one of the k roots of -1 is inside the contour. If we proceed as in the case $k = 3$ we conclude that

$$\int_0^\infty \frac{dx}{1+x^k} = \frac{\pi/k}{\operatorname{sen}(\pi/k)} \equiv \frac{1}{k} B\left(\frac{1}{k}, \frac{k-1}{k}\right) \tag{8}$$

where $k = 2, 3, \ldots$ and $B(p, q)$ is the beta function.[7]

4 An appropriate change of variables

As we have already mentioned, our aim is to calculate real integrals as in eq.(1) which can be transferred to equations like the eq.(4) type as we have seen in section 2. In section 3, we have used the residue theorem with an appropriate contour of integration. We note that despite the similarity, the contour depends on the value of the parameter k, i.e., it is always composed of two straight right line and an arc of a circle in which the length depends on the parameter k.

Here, we consider another type of contour that is not dependent on the parameter k. We change the contour of the integration described above, which was chosen in such a way that we have only a simple pole inside the contour, with another contour that contains a branching point at $z = 0$ and a one unique simple pole at $z = -1$. It is appropriate because we take the contour of integration that lets the branching point outside the contour

[7] If we had the general integral $\displaystyle\int_0^\infty \frac{x^{\mu-1}}{1+x^\nu} dx$ with $\Re(\nu) \geq \Re(\mu) > 0$ we could make the same change of variable and apply this integral to our integral with the appropriate identification of the parameters with the result as $\displaystyle\frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right)$.

and our problem is reduced to calculating the residue in point $z = -1$, independent of the parameter k.

To this end we must firstly introduce a change of variable $x^k = t$ in eq.(4) and without loss of generality we use $\mu = 1$ and get

$$\Omega_1(k) = \frac{1}{k}\int_0^\infty \frac{t^{\frac{1-k}{k}}}{1+t}dt$$

where $k = 2, 3, \ldots$ To calculate this integral we use the complex variables, i.e., we consider the following complex integral

$$\oint_\Gamma \frac{z^{\xi-1}}{z+1}dz \tag{9}$$

with $\xi = 1/k$ and Γ is an appropriate contour of integration. We note that the integral presents a function having a simple pole at $z = -1$ and a branching point at $z = 0$ because the exponent of z is always in the interval $0 < \xi < 1$. We choose the contour Γ, oriented in the positive sense, composed by two straight right line L_1 and L_2 and two concentric circles C_1 and C_2 centered at the origin with radii ϵ $(0 < \epsilon < 1)$ and R $(R > 1)$, respectively, as in Figure 3.

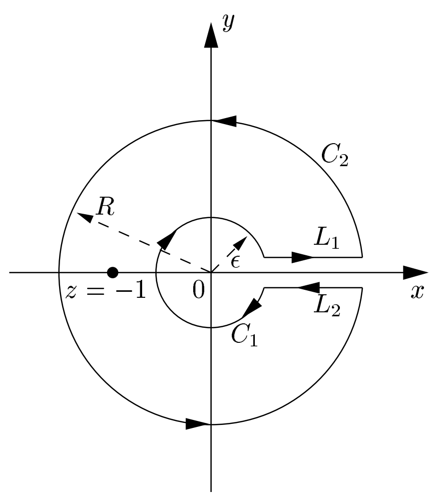

Figure 3: Contour for the integral in eq.(9)

Now, running the contour of integration and using the residue theorem we can write

$$\begin{aligned}\oint_{\Gamma} \frac{z^{\xi-1}}{1+z} dz &= \int_{\epsilon}^{R} \frac{x^{\xi-1}}{1+x} + \oint_{\Gamma_R} \frac{z^{\xi-1}}{1+z} dz + \int_{R}^{\epsilon} \frac{(x\,\mathrm{e}^{2\pi i})^{\xi-1}}{1+x} dx + \int_{\Gamma_\epsilon} \frac{z^{\xi-1}}{1+z} dz \\ &= 2\pi i\,\mathrm{Res}(z=-1).\end{aligned}$$

Using Jordan lemma[3] we can show that the integral on Γ_R goes to zero, for $R \to \infty$. For the integral on Γ_ϵ, we parameterize the circle $z = \epsilon \exp(i\theta)$ with $0 < \theta < 2\pi$ and taking the limit $\epsilon \to 0$, we can also show that the integral on Γ_ϵ goes to zero. We note that with both limits, $\epsilon \to 0$ and $R \to \infty$, we recover the extremes of integration and then we get

$$\left\{1 - \mathrm{e}^{2\pi i(\xi-1)}\right\} \int_0^{\infty} \frac{x^{\xi-1}}{1+x} dx = 2\pi i \lim_{z \to -1} \left\{(z+1)\left(\frac{x^{\xi-1}}{z+1}\right)\right\} = 2\pi i\,\mathrm{e}^{\pi i(\xi-1)}.$$

On the other hand, we note as we have said in Section 2, that this contour is not dependent on the parameter k. Then, multiplying the expression above for $\exp[-\pi i(\xi-1)]$ we can write

$$\left\{\mathrm{e}^{\pi i(\xi-1)} - \mathrm{e}^{-\pi i(\xi-1)}\right\} \int_0^{\infty} \frac{x^{\xi-1}}{1+x} dx = 2\pi i.$$

Using an Euler relation for the function $\sin z$ we obtain

$$\int_0^{\infty} \frac{x^{\xi-1}}{1+x} dx = -\frac{\pi}{\sin[\pi(\xi-1)]}$$

which can be write as

$$\int_0^{\infty} \frac{x^{\xi-1}}{1+x} dx = \frac{\pi}{\sin(\pi\xi)}.$$

Finally, turning back in the initial variable, $x = t^k$ we get

$$\int_0^{\infty} \frac{dt}{1+t^k} = \frac{\pi/k}{\sin(\pi/k)}$$

which is exactly the same result obtained in eq.(8) by means of another contour of integration depending on the parameter k.

5 Comments

In this paper we have discussed, by means of an appropriate change of variable, the integration of a class of real functions using complex variables. Another type of contour with the same aim can be seen in Capelas de Oliveira[4].

It is important to note that with a simple change of a variable, we reduce the problem depending on a parameter k, to another one not dependent on k. With this change of a variable, our integral can be calculated, independent of k, by means of a unique contour of integration, which exclude the branching point and which has only a simple pole inside the contour.

References

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