ON BEZOUT AND DISTRIBUTIVE SUBRINGS OF LOCAL RINGS

MIGUEL FERRERO* AND RYSZARD MAZUREK†

Abstract

Let T be a right chain ring with nonzero maximal ideal J. In this paper we study rings R such that $J \subseteq R \subseteq T$ and determine conditions for R to be right distributive (right Bezout).

1. Introduction

Let R be a ring and M be a right R-module. Recall that M is said to be a distributive module if the lattice of submodules of M is distributive, that is if the distributive law $A \cap (B + C) = (A \cap B) + (A \cap C)$ holds for all submodules A, B, C of M. Recall also that M is said to be a Bezout module if every finitely generated submodule of M is cyclic.

A ring R is called a right distributive ring if R is a distributive right module over itself. Similarly, R is called a right Bezout ring if R is a Bezout right module over itself. The classes of right distributive rings and right Bezout rings are incomparable. Recall that a ring R is said to be a right chain ring if the set of right ideals of R is linearly ordered by inclusion. Right chain rings form an important class of rings contained in both these classes.

In [FP], the authors studied a right distributive ring R obtained as pullback of a right chain ring T with maximal ideal J and a right distributive domain $D \subseteq T/J$, such that the left skew field of fractions of D does exist and is equal T/J. As a consequence, the ring R is a subring of T and contains the maximal ideal J of T, which is a completely prime ideal and a left waist of R. The converse is also proved under the additional assumption that R is a prime right distributive ring. Thus it is natural to study this situation in general.

Let T be a right chain ring with maximal ideal J and R a ring such that $J \subseteq R \subseteq T$. We determine conditions under which the ring R is right distributive (right Bezout). We will see that in this way we are studying right distributive (Bezout) rings with left waists contained in local rings. The main result of the paper shows that several conditions are equivalent, for example: the ring R is a right distributive (right Bezout)

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ring containig a nonzero completely prime ideal which is a left waists if and only if R is a right distributive (right Bezout) ring and there exists a local ring T with maximal ideal J such that $0 \neq J \subseteq R \subseteq T$ if and only if there exists a right chain ring T with maximal ideal J such that $0 \neq J \subseteq R \subseteq T$ and T/J is a distributive (Bezout) right R/J-module. Thus we obtain an extension of the main results in [FP].

Finally, we also prove some results on waists, extending known results for right distributive rings and which are new for right Bezout rings.

Throughout the paper all rings have an identity, and subrings have the same identity. If R is a ring and L is a subset of R, then the right annihilator of L in R is denoted by $r_R(L)$. The right annihilator of an element $a \in R$ is denoted simply by $r_R(a)$. To denote that L is a left ideal of R we write $L <_l R$. The Jacobson radical of R is denoted by $J(R)$.

2. Waists

Let R be a ring. Recall that a left waist of R is a proper left ideal L of R such that L is comparable, with respect to inclusion, with all left ideals of R (cf [AGR]), i.e., $L \neq R$ and for every $L_1 <_l R$ either $L_1 \subseteq L$ or $L \subseteq L_1$. Right waists of R are defined similarly. Obviously, a proper left ideal L of R is a left waist if and only if $L \subseteq Ra$ for every $a \in R \setminus L$. Moreover, every left waist L of R is contained in any maximal left ideal of R, and thus $L \subseteq J(R)$.

It is clear that the sum of a family of left waists of a ring R is a left waist. Thus for every ring R there exists the largest left waist of R.

In the paper we concentrate mainly on such left waists of a ring R that are completely prime ideals of R. Recall that a left ideal L of R is completely prime if for every $a, b \in R$, $ab \in L$ implies either $a \in L$ or $b \in L$. Let us note that if L is a completely prime left ideal of R which is a left waist, then $L = La$ for every $a \in R \setminus L$ (see [MP, p. 467]).

Let R be a ring and L a left ideal of R. Let $P_l(L) = \{a \in R \mid as \in L \text{ for some } s \in R \setminus L\}$. Moreover, for $s \in R$ we put

$$(L : s) = \{a \in R \mid as \in L\}.$$

It easy to see that $(L : s)$ is a left ideal of R and for every $a, b \in R$, $ab \in P_l(L)$ implies either $a \in P_l(L)$ or $b \in P_l(L)$.

Lemma 1. *Let R be a ring and L a left waist of R such that $r_R(L) \subseteq L$. Then*
(i) $L + (L : s)$ is a left waist of R for every $s \in R \setminus L$.
(ii) $P_l(L)$ is a completely prime ideal and a left waist of R containing L.

Proof. (i) Let $s \in R \setminus L$. If $(L : s) \subseteq L$, then $L + (L : s) = L$ is a left waist of R. Thus we assume $L \subseteq (L : s)$, and we have to show that $L + (L : s) = (L : s)$ is a left waist of R. Since $s \notin L$, it follows that $1 \notin (L : s)$ and thus $(L : s)$ is a proper left ideal of R. To show that $(L : s)$ is comparable with every left ideal of R assume that $a \in (L : s)$ and $b \notin (L : s)$. Then $as \in L$ and $bs \notin L$, and thus $as \in L \subseteq Rbs$. Hence

$as = rbs$ for some $r \in R$, and so $(a - rb)s = 0$. If $L \subseteq R(a - rb)$, then $Ls = 0$ and we get $s \in r_R(L) \subseteq L$, a contradiction. Thus $a - rb \in L$, which gives $a \in L + Rb$. Since by assumption $L \subsetneq (L : s)$, it follows that $b \notin L$. Hence $L \subseteq Rb$ and consequently $a \in L + Rb \subseteq Rb$, which shows that $(L : s)$ is a left waist of R.

(ii) Obviously, $P_l(L) = \bigcup_{s \in R \setminus L} (L : s) = \bigcup_{s \in R \setminus L} (L + (L : s))$, since $L = (L : 1)$ is a left waist of R. Now (i) implies that $P_l(L)$ is a left waist of R. Moreover, from the definition of $P_l(L)$ it follows that $P_l(L)$ is a completely prime left ideal of R. Since $P_l(L)$ is a left waist of R, we get $P_l(L) \subseteq J(R)$ and thus [FT, Lemma 2.5] implies that $P_l(L)$ is a completely prime ideal of R. □

In [TZ, Example 4.5] a distributive ring R was constructed with no completely prime ideal inside the Jacobson radical $J(R)$ and with a waist L which is a nonzero prime ideal of R. This example shows that in Proposition 1 the condition $r_R(L) \subseteq L$ cannot be omitted.

Theorem 2. *For any ring R, the following conditions are equivalent:*
(i) R contains a left waist L such that $r_R(L) \subseteq L$.
(ii) The largest left waist of R is a nonzero completely prime ideal of R.
(iii) R contains a nonzero left waist which is a completely prime ideal of R.

Proof. (i) $\Rightarrow$ (ii) Assume that R contains a left waist L such that $r_R(L) \subseteq L$ and let Q be the largest left waist of R. Then $L \subseteq Q$ and thus $r_R(Q) \subseteq r_R(L) \subseteq L \subseteq Q$. Now Lemma 1(ii) implies that $Q = P_l(Q)$ is a completely prime ideal of R. Since $r_R(Q) \subseteq Q \neq R$, it follows that Q is a nonzero ideal of R.

(ii) $\Rightarrow$ (iii) This implication is obvious.

(iii) $\Rightarrow$ (i) Let Q be a nonzero left waist of R which is a completely prime ideal of R. Suppose that $r_R(Q) \not\subseteq Q$. Then there exists $a \in r_R(Q) \setminus Q$. Since $a \notin Q$, it follows that $Q = Qa = 0$, a contradiction. Thus Q is a left waist of R such that $r_R(Q) \subseteq Q$. □

3. Distributive and Bezout subrings of local rings

In this section we study right distributive rings and right Bezout rings which are subrings of a local ring T and contain the Jacobson radical of T. At the end of the section we present examples (Example 10 and Example 11) showing that these classes of rings are incomparable.

Recall that a ring T is local if T has a unique maximal left ideal, or, equivalently, if T has a unique maximal right ideal. In this case the maximal right (left) ideal of T coincides with the Jacobson radical $J(T)$ of T. In the rest of this paper we put $J = J(T)$. If T is local, then $T \setminus J$ coincides with the set of invertible elements of T.

Proposition 3. *Let T be a local ring. If R is a subring of T such that $J \subseteq R$, then J is a completely prime ideal of R which is a left and right waist of R.*

Proof. Obviously J is a completely prime ideal of R. Now let $a \in R \setminus J$, so an invertible element of T and $J = Ja = aJ$ follows. Since $J \subseteq R$, we get $J \subseteq Ra$ and $J \subseteq aR$. Hence J is a left and right waist of R. □

In the proof of the next lemma we will need the following well known characterizations of distributive modules and Bezout modules (see [T], p. 33, 34).

Proposition 4. *Let R be a ring and M a right R-module. Then*

(i) M is distributive if and only if for every $m, n \in M$ there exist $x, y, a, b \in R$ such that $x + y = 1, mx = na, ny = mb$.

(ii) M is Bezout if and only if for every $m, n \in M$ there exist $c, d, e, f \in R$ such that $m(1 - ce) = nde, n(1 - df) = mcf$. □

Let R be a ring and Q be a completely prime ideal of R. A ring T is called a right localization of R at Q if R is a subring of T, every element of the set $S = R \setminus Q$ is invertible in T and every element of T has the form rs^{-1}, where $r \in R$ and $s \in S$. It is well known that if $r_R(s) = 0$ for every $s \in S$, then a right localization of R at Q exists if and only if S is a right Ore subset of R, that is if for every $r \in R$ and $s \in S$, $rS \cap sR \neq \emptyset$. If R is a domain and $S = R \setminus 0$ is a right Ore subset of R, then we say that R is a right Ore domain. In this case the right localization T of R at 0 is a division ring; we call it the right skew field of fractions of R. Obviously, all the above notions have their left-side counterparts.

Lemma 5. *Let T be a local ring with $J \neq 0$ and R a subring of T such that $J \subseteq R$. If R is either right distributive or right Bezout, then T is a right localization of R at J and T is a right chain ring.*

Proof. Using Proposition 4 it is easy to verify that a right localization of a right distributive (right Bezout) ring is a right distributive (right Bezout) ring. Moreover, it is well known that local right distributive rings as well as local right Bezout rings are right chain rings (see [BBT, Proposition 1.3]). Thus to end the proof it is enough to show that T is a right localization of R at J.

Let t be an arbitrary element of T. Obviously, to prove that T is a right localization of R at J it is enough to show that there exists $s \in R \setminus J$ such that $ts \in R$.

We begin with the case when R is a right distributive ring. By assumption there exists an element $j \in J \setminus 0$. Since $j, jt \in J \subseteq R$, Proposition 4(i) implies that there exist $x, y, a, b \in R$ with $x + y = 1, jx = jta, jty = jb$. Since $j(x - ta) = j(ty - b) = 0$ and $j \neq 0$, it follows that $x - ta, ty - b \in J \subseteq R$. In particular, $ta \in R$ and $ty \in R$. If $a \in J$ and $y \in J$, then $ta \in J$ and $x = 1 - y \notin J$, and so $x - ta \notin J$, a contradiction. Thus either $a \notin J$ or $y \notin J$, which shows that $ts \in R$ for some $s \in R \setminus J$.

Suppose now that R is a right Bezout ring. As above, for $j \in J \setminus 0$ we have $j, jt \in J \subseteq R$, and Proposition 4(ii) implies that there exist $c, d, e, f \in R$ with $j(1-ce) = jtde$ and $jt(1 - df) = jcf$. Since $j(1 - ce - tde) = j(t(1 - df) - cf) = 0$ and $j \neq 0$, we get

$$(1) \quad 1 - ce - tde \in J \quad \text{and} \quad t(1 - df) - cf \in J,$$

and thus $tde \in R$ and $t(1 - df) \in R$. If $e \in J$, then (1) implies that $1 \in J$, a contradiction. Hence $e \in R \setminus J$. If $d \in R \setminus J$, then $de \in R \setminus J$ and $tde \in R$. If $d \in J$, then $1 - df \in R \setminus J$ and $t(1 - df) \in R$. From the above it follows that there exists $s \in R \setminus J$ such that $ts \in R$. □

Lemma 6. *Let T be a right chain ring and R a subring of T such that $J \subseteq R$. If I_1, I_2 are incomparable right ideals of R (i.e., $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$), then there exists $a \in R \setminus 0$ such that $aJ \subseteq I_1 \subseteq aT$ and $aJ \subseteq I_2 \subseteq aT$.*

Proof. Since I_1, I_2 are incomparable right ideals of R, there exist $a \in I_1 \setminus I_2$ and $b \in I_2 \setminus I_1$. Since

$$(2) \quad bJ \subseteq bR \subseteq I_2 \text{ and } aJ \subseteq aR \subseteq I_1,$$

we get that $a \notin bJ$ and $b \notin aJ$. Since T is a right chain ring, it follows that $bJ \subseteq aJ$ and $aJ \subseteq bJ$. Hence $aJ = bJ$ and consequently $aT = bT$. Now (2) implies that $aJ \subseteq I_1$ and $aJ \subseteq I_2$. Suppose that $I_1 \not\subseteq aT$. Then $I_1T \not\subseteq aT$ and so $b \in bT = aT \subseteq I_1J \subseteq I_1R \subseteq I_1$, a contradiction. Thus $I_1 \subseteq aT$. Similar arguments show that $I_2 \subseteq aT$. □

Proposition 7. *Let T be a right chain ring with $J \neq 0$ and R a subring of T such that $J \subseteq R$. Then*

(i) R is a right distributive ring if and only if T/J is a distributive right R/J-module.

(ii) R is a right Bezout ring if and only if T/J is a Bezout right R/J-module.

Proof. In the proof of both parts of the proposition we will apply the observation that for every nonzero $a \in T$ the map $\varphi : T/J \to aT/aJ$, $\varphi(t + J) = at + aJ$, is an isomorphism of right R/J-modules. To verify this is enough to show that $ker\varphi = 0$. But this is obvious, since $at \in aJ$ and $t \notin J$ gives $a \in aJ$, and consequently $a = 0$.

(i) Suppose that R is a right distributive ring. By assumption there exists an element $j \in J \setminus 0$. Since jT is a right ideal of R, jT is a distributive right R-module and jJ is a submodule of jT. Thus jT/jJ is a distributive right R/J-module. As we have shown above, the module jT/jJ is isomorphic to T/J. Hence T/J is a distributive right R/J-module.

Suppose now that T/J is a distributive right R/J-module. Let I_1, I_2, I_3 be right ideals of R. If one of them is contained in other, then obviously I_1, I_2, I_3 satisfy the distributive law $I_1 \cap (I_2 + I_3) = (I_1 \cap I_2) + (I_1 \cap I_3)$. Thus we assume that I_1, I_2, I_3 are incomparable right ideals of R. By Lemma 6 there exist $a, b \in R \setminus 0$ such that $aJ \subseteq I_1, I_2 \subseteq aT$ and $bJ \subseteq I_1, I_3 \subseteq bT$. Suppose that $aT \neq bT$. Since T is a right chain ring it follows that either $aT \subseteq bJ$ or $bT \subseteq aJ$. Therefore we get $I_1 \subseteq I_3$ or $I_1 \subseteq I_2$, a contradiction. Thus $aT = bT$ and consequently $aJ \subseteq I_1, I_2, I_3 \subseteq aT$. Hence $I_1/aJ, I_2/aJ, I_3/aJ$ are submodules of the right R/J-module aT/aJ. But the module $aT/aJ \simeq T/J$ is distributive and so I_1, I_2, I_3 satisfy the distributive law.

(ii) Suppose that R is a right Bezout ring. Then analogously as in the first part of the proof of (i) it can be shown that T/J is a Bezout right R/J-module.

Suppose now that T/J is a Bezout right R/J-module. Let $r_1, r_2 \in R$. We will show that $r_1R + r_2R$ is a principal right ideal of R. If $r_1R \subseteq r_2R$ or $r_2R \subseteq r_1R$, we are done. Thus we assume that r_1R, r_2R are incomparable right ideals of R. By Lemma 6 there exists $a \in R \setminus 0$ such that $aJ \subseteq r_1R, r_2R \subseteq aT$. As we have noted earlier, the right R/J-module aT/aJ is isomorphic to the Bezout right R/J-module T/J. Thus the submodule $r_1R/aJ + r_2R/aJ = (r_1R + r_2R)/aJ$ of aT/aJ is generated by an element

$b + aJ$. Now it is easy to verify that $r_1R + r_2R = bR$, which ends the proof. □

Lemma 8. *Let R be a ring and T a left localization of R at a completely prime ideal Q of R. Then*

(i) ([S, Proposition 3.3]) *If R is a right distributive ring, then T is a distributive right R-module.*

(ii) If R is a right Bezout ring, then T is a Bezout right R-module.

Proof. We need only prove (ii). Let $t_1 = s^{-1}r_1, t_2 = s^{-1}r_2$ be arbitrary elements of T. There exist $c, d, e, f \in R$ such that $r_1(1 - ce) = r_2de, r_2(1 - df) = r_1cf$. Hence $t_1(1 - ce) = t_2de$ and $t_2(1 - df) = t_1cf$, and thus T is a Bezout right R-module. □

Immediately from Proposition 7 and Lemma 8 we get the following

Corollary 9. *Let T be a right chain ring. Assume that R is a subring of T such that $J \subseteq R$, R/J is a left Ore domain and T/J is the left skew field of fractions of R/J. Then*

(i) ([FP, Theorem 2.1]) *R is a right distributive ring if and only if R/J is a right distributive ring.*

(ii) R is a right Bezout ring if and only if R/J is a right Bezout ring. □

The following example shows that a right Bezout ring R contained in a right chain ring T and containing J need not be right distributive.

Example 10. Let $B = \mathbb{H}[x]$ be the polynomial ring in one variable over the ring of Hamilton's real quaternions $\mathbb{H}$. Since $\mathbb{H}$ is a division ring, every one-sided ideal of B is principal and therefore B is a left and right Bezout domain. Moreover, since $\mathbb{H}$ is not a commutative ring, B is not a right distributive ring (see [T, p. 218]).

Let C be the left skew field of fractions of B and let $T = C[[t]]$, the ring of power series over C. Then T is a right chain ring and $J = J(T) = tT$. Moreover, $R = B + tT$ is a subring of T and $J \subseteq R$. Since $R/J \cong B$ is a left Ore domain and $T/J \cong C$ is a left skew field of fractions of B, we can apply Corollary 9. It follows that the ring R is right Bezout but not right distributive. □

The next example shows that a right distributive ring R contained in a right chain ring T and containing J need not be right Bezout, even in the commutative case.

Example 11. Let $D = \mathbb{Z}[\sqrt{-5}\,]$. Since D is the ring of algebraic integers of the field $\mathbb{Q}(\sqrt{-5}\,)$, D is a Dedekind domain and consequently D is a distributive domain. Since the ideal of D generated by 2 and $1 + \sqrt{-5}$ is not principal (see [M, p. 132]), D is not a Bezout ring.

Let $C = \mathbb{Q}(\sqrt{-5}\,)$ and $T = C[[t]]$. Then T is a commutative chain ring and $R = D + tT$ is a subring of T such that $J \subseteq R$. As in Example 10 one can show that the ring R is distributive but R is not a Bezout ring. □

4. Distributive and Bezout rings with waists

Lemma 12. *Suppose that R is either a right distributive ring or R is a right Bezout ring. Let Q be a completely prime ideal of R such that $Q \subseteq J(R)$. Then Q is a right waist of R. If furthermore Q is a left waist of R and R is semiprime, then there exists the right localization of R at Q.*

Proof. Q is a right waist of R by [S, Proposition 2.1] if R is right distributive. The same was proved in [MT, Proof of Corollary 9] if R is right Bezout. We include here a proof of this fact: Let $a \in Q$ and $b \notin Q$. Then $aR + bR = cR$, for some $c \in R$. Hence there exist $a_1, b_1, x, y \in R$ such that $a = ca_1$, $b = cb_1$ and $c = ax + by$. Thus $c \notin Q$ since $b \notin Q$. Also $c(1 - a_1x - b_1y) = 0 \in Q$ and $ca_1 = a \in Q$, consequently $1 - a_1x - b_1y \in Q \subseteq J(R)$ and $a_1 \in Q \subseteq J(R)$. Therefore $1 - b_1y \in J(R)$ and it follows that b_1y is invertible in R. Then $a = ca_1 = cb_1y(b_1y)^{-1}a_1 = by(b_1y)^{-1}a_1 \in bR$, which shows that Q is a right waist or R.

Using Proposition 4 it can be proved analogously as in [T, p. 67] that $S = R \setminus Q$ is a right Ore subset of R.

Suppose that Q is also a left waist of R. To end the proof it is enough to show that for every $s \in R \setminus Q$, $r_R(s) = 0$. Let $a \in r_R(s)$, i.e., $sa = 0$. Then $sa \in Q$ and Q is a completely prime ideal of R, consequently $a \in Q$. Since Q is a left waist of R and $s \notin Q$, we get $Q = Qs$. Hence $(RaR)^2 = RaRaR \subseteq QaR = QsaR = 0$. Since R is semiprime, it follows that $a = 0$ and thus $r_R(s) = 0$. □

Theorem 13. *For any semiprime ring R, the following conditions are equivalent:*

(i) R is a right distributive (right Bezout) ring with a nonzero left waist L such that $r_R(L) \subseteq L$.

(ii) R is a right distributive (right Bezout) ring and the largest left waist of R is a nonzero completely prime ideal of R.

(iii) R is a right distributive (right Bezout) ring with a nonzero left waist which is a completely prime ideal of R.

(iv) R is a right distributive (right Bezout) subring of a local ring T such that $0 \neq J \subseteq R$.

(v) R is a right distributive (right Bezout) subring of a right chain ring T such that $0 \neq J \subseteq R$.

(vi) R is a subring of a right chain ring T such that $0 \neq J \subseteq R$ and T/J is a distributive (Bezout) right R/J-module.

Proof. The conditions (i), (ii) and (iii) are equivalent by Theorem 2.

(iii) ⇒ (iv) Let R be a semiprime right distributive (right Bezout) ring and Q be a nonzero completely prime ideal of R which is a left waist. Then $Q \subseteq J(R)$ and consequently, by Lemma 12, there exists a right localization T of R at Q. Obviously, T is a local ring and $J = J(T) = \{qs^{-1} | q \in Q, s \in R \setminus Q\}$. To end the proof of the implication it is enough to show that $J \subseteq R$. Let $j = qs^{-1} \in J$. Since Q is a left waist of R and $s \in R \setminus Q$, it follows that $q \in Q \subseteq Rs$ and so $j \in R$.

(iv) ⇒ (v) This implication is a consequence of Lemma 5.

(v) ⇔ (vi) These conditions are equivalent by Proposition 7.

(v) ⇒ (iii) This implication is a consequence of Proposition 3. □

If R is a ring and I is a right ideal of R, then analogously as in Section 2 we put $P_r(I) = \{a \in R \mid sa \in I \,\text{for some}\, s \in R \setminus I\}$. Moreover, we denote the set $P_r(0)$ by $N_r(R)$. Obviously $N_r(R)$ is the set of right zero-divisors of R.

Lemma 14. *Let R be a ring which is either right distributive or right Bezout. If I is a proper right ideal of R such that $P_r(I) \subseteq J(R)$, then I is a right waist of R.*

Proof. The result is already known if R is a right distributive ring ([FT, Corollary 3.8]). Thus we assume that R is right Bezout and $P_r(I) \subseteq J(R)$. Take $a \notin I$ and $b \in I$. By Proposition 4(ii) there exist $c, d, e, f \in R$ such that $a(1 - ce) = bde$ and $b(1 - df) = acf$. If $df \notin J(R)$, then $f \notin J(R)$ and from $acf \in I$ we get $ac \in I$. Thus $c \in P_r(I) \subseteq J(R)$ and so $1 - ce$ is invertible in R. Hence $a \in bR \subseteq I$, a contradiction. Therefore we have $df \in J(R)$ and consequently $1 - df$ is invertible in R. This gives $b \in aR$ and the proof is complete. □

In the next proposition we point out a condition under which the inclusion $P_r(I) \subseteq J(R)$ holds for every right waist I of R. The proposition is a generalization of [FS, Proposition 4.4].

Proposition 15. *Let R be a ring such that $N_r(R) \subseteq J(R)$. If I is a right waist of R, then $P_r(I) \subseteq J(R)$.*

Proof. Assume that I is a right waists and take $a \in P_r(I) \setminus J(R)$. Then there exists $s \notin I$ with $sa \in I$. Hence $I \subseteq sR$ and we have $I = sB$, where $B = \{x \in R \mid sx \in I\}$. Since $a \notin J(R)$ and B is a right ideal of R, there exists a maximal right ideal M of R such that $B \not\subseteq M$. Thus either $I \subset sM$ or $sM \subseteq I$. The last possibility implies that $M \subset B$, thus $B = R$ and we get the contradiction $s \in I$. So we must have $sB = I \subseteq sM$. Take any $b \in B$. There exists $c \in M$ with $sb = sc$. Thus $s(b - c) = 0$ and so $b - c \in J(R)$. It follows that $b \in M$, i.e, $B \subseteq M$, again a contradiction. □

Corollary 16. *Suppose that T is a local ring and R is a subring of T containing J which is either right distributive or right Bezout. Then a proper right ideal I of R is a right waist if and only if $P_r(I) \subseteq J(R)$.*

Proof. Obviously $N_r(R) \subseteq J$. Moreover, by Proposition 3 we have $J \subseteq J(R)$. Hence $N_r(R) \subseteq J(R)$ and we can apply Proposition 15. The rest is a consequence of Lemma 14. □

Let I be a right ideal of R and S a multiplicative subset of R. The S-saturation of I in R is defined as

$$IS^{-1} = \{x \in R \mid xs \in I \,\text{for some}\, s \in S\}.$$

The right ideal I is said to be S-saturated if $IS^{-1} = I$ (cf [TZ]).

We conclude this paper with the following more precise result in case the Jacobson radical of R is equal to J.

Proposition 17. *Suppose that T is a local ring and R is a subring of T containing*

J and such that $J(R) = J$. If R is either right distributive or right Bezout, then for every proper right ideal I of R the following conditions are equivalent:

(i) $P_r(I) \subseteq J(R)$.

(ii) I is a right waist of R.

(iii) I is S-saturated, where $S = R \setminus J(R)$.

(iv) I is a right ideal of T.

Proof. The conditions (i) and (ii) are equivalent by Corollary 16.

(i) $\Rightarrow$ (iii) Let $a \in IS^{-1}$. Then $as \in I$ for some $s \in R \setminus J(R)$. By (i) we have $s \notin P_r(I)$ and thus $a \in I$. Hence $IS^{-1} \subseteq I$. Since obviously $I \subseteq IS^{-1}$, we get (iii).

(iii) $\Rightarrow$ (iv) We can assume $I \neq 0$. Since $I \neq R$, (iii) implies $I \subseteq J(R)$, and in particular $J \neq 0$. Let $a \in I$ and $t \in T$. From Lemma 5 we get $t = rs^{-1}$ for some $r \in R, s \in R \setminus J(R)$. Since $s \notin J(R)$, Proposition 3 implies that $J(R) \subseteq Rs$ and thus $ar = bs$ for some $b \in R$. Since $ar \in I$, $b \in IS^{-1}$ and so $b \in I$ by (iii). Hence $at = ars^{-1} = b \in I$.

(iv) $\Rightarrow$ (i) Assume that $s \in P_r(I)$. Then $as \in I$ for some $a \in R \setminus I$. If $s \notin J(R)$, then $s^{-1} \in T$ and hence $a \in I$, a contradiction. □

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Instituto de Matemática, Universidade Federal do Rio Grande do Sul, 91509-900, Porto Alegre, Brasil

ferrero@mat.ufrgs.br

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland
mazurek@cksr.ac.bialystok.pl