Geometric aspects of stochastic delay differential equations on manifolds

Pedro J. Catuogno
Departamento de Matemáticas (FCEYN), Universidad Nacional de Mar del Plata, Funes 3350 - Mar del Plata - BA, Argentina.
e-mail: pedrojc@mdp.edu.ar

and

Paulo R. C. Ruffino
Departamento de Matemática, Universidade Estadual de Campinas, 13.081-970 - Campinas - SP, Brazil. e-mail: ruffino@ime.unicamp.br

Abstract

We show some analytical properties of SDDE including a closed formula for a strong solution of $\dot{x}(t) = x(t-1) \circ dB_t$ with an initial adapted process φ_t in the delay interval $[-1, 0]$. SDDE's on a manifold M depend intrinsically on a connection ∇. The main geometric result in this article concerns the horizontal lift of solutions of SDDE on a manifold M to an SDDE in the frame bundle BM, hence the lifted equation should come together with the prolonged horizontal connection ∇^H on BM.

Key words: stochastic delay equations, frame bundles, horizontal lift.

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1 Introduction

For many different intrinsic reasons, most of systems, not to say all of them, react retarded with respect to an input. In this context we mean delays of nanoseconds in electronic systems, delays of few minutes in biochemical processes and even delays of years in gravitational systems. Hence, this fact alone is enough to show the importance of delay differential equations as an appropriate mathematical model for these systems. Many relevant mathematical contributions appeared in the last couple of decades, among many others, we remark the book of J. Hale [5] and the work of S. Mohammed and contributors [8], [9], [7] (and references therein), whose stochastic approach is closer to our interest in this paper.

To illustrate with a simple example in a space with elementary geometry, consider the following exponential delay equation in the real line:

$$\dot{x}(t) = x(t-1), \tag{1}$$

with initial condition $u : [-1,0] \to \mathbb{R}$ in the delayed interval $[-1,0]$, where $u \in L^1([-1,0])$. The solution is not expected to be C^1: it would require that $u(-1) = \dot{u}(0)$. Note that, in fact, even the continuity of the solution in the integers (multiple of the delay $r = 1$ in this case) is not a necessary condition for a certain function $x(t)$ be a solution of equation (1) with initial function u. In this sense, the set of (initial) conditions which determines uniquely a certain solution must include conditions for the sequence $x(n)$, $n = 0, 1, 2, \ldots$ of values in the integers. In this paper, most of time, we shall look for continuous solutions in the integers.

To find solutions of equation (1), instead of considering the more intricate and difficult approach of integrating in time:

$$x(t) = x(0) + \int_0^{t-1} x(s)\ ds,$$

one can divide the equation into intervals of the same length of the delay $r = 1$. Assume for instance that the initial function in the delayed interval is constant $u(t) \equiv 1$, for $t \in [-1,0]$. To obtain a continuous solution, one immediately finds that if $t \in [0,1]$ then $x(t) = 1 + t$, for t in $[1,2]$ one computes $x(t) = t^2/2 - t + 3/2$, and so on. By induction, one sees that if $t \in [n-1, n]$ then $x(t)$ is a polynomial on t of degree n. A nice and nested formula for these polynomials reads:

$$x(t) = \sum_{k=0}^{n} \frac{(t-k+1)^k}{k!} \qquad \text{for } t \in (n-1, n]. \tag{2}$$

See e.g. Driver and Driver [3].

Delay equations in differentiable manifolds involve a parallel transport in order to map vectors from a tangent spaces to another (in the above example in $\mathbb{R}$ one uses the parallel transport induced by the canonical flat connection). Hence, in this richer geometrical context, delay equations depends strongly on a chosen connection. Let M be a differentiable manifold, X a vector field of M, ∇ a connection on M and $\alpha : [-1,0] \to M$ an initial continuous trajectory. The solution of a delay equation on M (with retard

$r = 1$, say), when it exists, is a curve $\gamma(t)$ such that the derivative $\dot\gamma(t)$ equals the parallel transport of $X(\gamma(t-1))$ along γ from $T_{\gamma(t-1)}M$ to $T_{\gamma(t)}M$, for $t \geq 0$. In, symbols:

$$\begin{aligned} \frac{d\gamma}{dt}(t) &= P^{\nabla}_{t,t-1}(\gamma)(X(\gamma(t-1))) \\ \gamma(t) &= \alpha(t) \quad \text{for all } t \in [-1,0] \end{aligned}$$

where $P^{\nabla}_{t,s}(\gamma) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ induced by ∇.

Horizontal processes in the frame bundle BM of a manifold M is a basic concept in stochastic geometry and stochastic dynamical systems, e.g. parallel transport, development, anti-development, horizontal Brownian motion and others, they are all constructed based on horizontal processes. The main question we address in this article is the following: is the horizontal lift of a stochastic differential delay equation (SDDE) a SDDE in BM as well? As we said before, once a SDDE depends on the connection in the manifold, this question carries intrinsically another one: the lifted SDDE in BM (if exists at all!) is taken with respect to which (prolonged) connection in BM?

The article is organized as follows: in the next section we deal with some extensions of well known results for deterministic systems to stochastic systems in the real line. In section 3 we present the main geometric results of this paper, in particular we prove that the horizontal lift of a solution of the delay equations in M with connection ∇ corresponds to a solution of a delay equation on BM where the parallel transport in BM is done with respect to the horizontal prolonged connection ∇^H. Moreover we show that, surprisingly, this delay equation in BM can be substituted to another stochastic equation (without delay) but the integration is performed with respect to a different semimartingale. Finally, in the appendix we explore further the formulas presented in section 2 to get formulas for left invariant delay equations in Lie groups with flat connection.

2 Stochastic delay equation in the real line

Consider the following stochastic delay exponential equation:

$$du(t) = u(t-1) \ \circ dB_t \tag{3}$$

It is well known that in this case, the Stratonovich and Itô equations coincide, see e.g. S. Mohammed [9].

Assume that the initial condition $u(t) = 1$ for $t \in [-1, 0]$. Like in the deterministic case, one can divide the equation into intervals of the same length of the delay $r = 1$. We shall denote by $u(t)$ the corresponding continuous solution. Next proposition presents a formula for this unique (strong) solution in terms of polynomials of Brownian motions.

Proposition 2.1 *The unique strong continuous solution $u(t)$ of the stochastic delay exponential equation (3) is given by:*

$$u(t) = \sum_{k=0}^{n} \frac{(B_{(t-k+1)})^k}{k!} \qquad \text{for } t \in (n-1, n].$$

Proof:

Continuity is obvious. By Itô formula we have that

$$B^k_{(t-k+1)} = k \int_0^{(t-k+1)} B_s^{k-1} \ \circ dB_s.$$

So, by definition:

$$u(t) = 1 + \sum_{k=1}^{n} \frac{1}{(k-1)!} \int_0^{(t-k+1)} B_s^{k-1} \ \circ dB_s.$$

Hence:

$$\begin{aligned} du(t) &= \left(\sum_{k=1}^{n} \frac{B^{k-1}_{(t-k+1)}}{(k-1)!} \right) \circ dB_t. \\ &= u(t-1) \ \circ dB_t \end{aligned}$$

Uniqueness follows by construction and the requirement of continuity at the integers.

□

Like in the deterministic case (Driver and Driver [3]), the study of the constant initial condition $u(t) = 1$ in $t \in [-1, 0]$ is not only interesting by itself, but, above all, it helps on finding solutions for general initial process $(\varphi)_{t\in[-1,0]}$ by variation of parameters. We shall denote by $u^{(j)}(t)$ the translations by j, $j = 1, 2, \ldots$ of the stochastic exponential solution $u(t)$ $(= u^{(0)}(t))$, precisely, $u^{(j)}(t)$ is a solution of equation (3) with

$$u^{(j)}(t) = \begin{cases} 0 & \text{for} \quad t \in [-1, j-1) \\ 1 & \text{for} \quad t \in [j-1, j] \\ \displaystyle\sum_{k=0}^{n-j} \frac{(B_{(t-k+1)} - B_j)^k}{k!} & \text{for} \quad t \geq j, \text{ with } t \in (n-1, n]; \end{cases}$$

Note that the unique point of discontinuity of $u^{(j)}(t)$ is at $t = j-1$. Next proposition provides a variation of parameter formula for the continuous solution $x(t)$ given an initial process $(\varphi)_{t\in[-1,0]}$.

Proposition 2.2 *The unique continuous (strong) solution $x(t)$ of equation (3) given an initial process $(\varphi)_{t\in[-1,0]}$ which is $\mathcal{F}_{t+1}$–adapted is*

$$\begin{aligned} x(t) &= u^{(1)}(t)\varphi(0) \\ &\quad + \sum_{k=1}^{n-1} u^{(k+1)}(t) \int_{k-1}^{k}\int_{k-2}^{s_k-1}\cdots\int_{1}^{s_3-1}\int_{0}^{s_2-1} \varphi_{(s_1-1)} \circ dB_{s_1}\ldots \circ dB_{s_k} \\ &\quad + \int_{n-1}^{t}\int_{n-2}^{s_n-1}\cdots\int_{0}^{s_2-1} \varphi_{(s_1-1)} \circ dB_{s_1}\ldots \circ dB_{s_n}, \\ &\quad \textit{for } t\in(n-1,n]. \end{aligned} \tag{4}$$

Proof:

The continuity follows easily from the fact the all the summands are continuous in the interval $(n-1,n]$. Moreover, since each $u^{(j)}(t)$, $j = 1,2,\ldots,n$ are solutions of the delay equation (3) and $u^{(n)}(t)$ is constant in the interval $(n-1,n]$ then the Stratonovich differential

$$\begin{aligned} dx(t) &= u^{(1)}(t-1)\varphi_0 \ \circ dB_t \\ &\quad + \sum_{k=1}^{n-2} u^{(k+1)}(t-1)\left(\int_{k-1}^{k}\cdots\int_{1}^{s_3-1}\int_{0}^{s_2-1} \varphi_{(s_1-1)} \circ dB_{s_1}\ldots \circ dB_{s_k}\right) \circ dB_t \\ &\quad + \int_{n-2}^{t-1}\cdots\int_{0}^{s_2-1} \varphi_{(s_1-1)} \circ dB_{s_1}\ldots \circ dB_t. \\ &= x(t-1) \ \circ dB_t \end{aligned}$$

□

The following is a well expected result:

Corollary 2.3 *If the initial condition φ is deterministic, then the solution $x(t)\in\mathcal{H}_n$ for $t\in(n-1,n]$ where $\mathcal{H}_n$is the n–th Wiener chaos.*

Proof: The first summand in formula (4) is a product of an $(n-k)$–degree polynomial of Brownian motion times

$$\int_{[0,k]^k} f(s_1,\ldots,s_k) \circ dB_{s_1}\ldots \circ dB_{s_k}$$

where $f : [0,k]^k \to \mathbb{R}$ is given by

$$f(s_1, \ldots, s_k) =$$
$$\varphi(s_1 - 1) \cdot 1_{[0,s_2-1]}(s_1) \cdot 1_{[1,s_3-1]}(s_2) \ldots 1_{[k-2,s_k-1]}(s_k - 1) \cdot 1_{[k-1,k]}(s_k).$$

If $\varphi \neq 0$, the above integral is in $\mathcal{H}_k$ (see, e.g. Nualart [11]), hence, the product is in $\mathcal{H}_n$. For the second summand the argument is analogous.

□

Corollary 2.4 *The continuous solution $x(t)$ is linear with respect to the initial function φ in the delayed interval* $[-1, 0]$.

Proof:
It follows immediately from formula (4).

□

We remark that the linearity above refers a.s. to fixed $f, g \in C([-1, 0])$ and $\lambda \in \mathbb{R}$. For almost all ω the corresponding linear operator is not continuous, cf. Mohammed [9, Cor. III.3.1].

2.1 A remark on a weak solution

A weak solution of the SDDE (3) can be nicely written in terms of a non-delay stochastic integral when one observes the following trivial change of variables:

Lemma 2.1 *Let $x(t)$ be an $\mathcal{F}_t$ adapted process. Then for $a, b \geq 1$*

$$\int_a^b x(t-1)\ dB_t = \int_{a-1}^{b-1} x(t)\ d\widetilde{B}_t$$

where $\widetilde{B}_t$ is the Brownian motion $B_t - B_1$.

Proof: The result follows trivially when one uses the limit in probability of the Riemann sum in the integrals.

□

Using the above formula, a weak solution can be written, for $t \geq 1$:

$$x(t) = x_1 + \int_0^{t-1} x(t) \circ d\widetilde{B}_t.$$

3 Delay Equations in Manifolds

We start this section defining formally a SDDE on a differentiable manifold M endowed with a connection ∇. Let $A_1, \dots, A_m$ be vector fields in M and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions. Let $(M_t)_{t\geq 0}$ be a $\mathbb{R}^m$-semimartingale adapted to $(\mathcal{F}_t)_{t\geq 0}$, we suppose that $M_0 = 0$. Finally, let (α_t) be a deterministic trajectory in M. We consider a Stratonovich SDDE on the manifold M:

$$\begin{aligned} dx_t &= \sum_{k=1}^{m} P^{\nabla}_{t,t-1}(x)(A_k(x_{t-1})) \circ dM^k_t, \\ x_t &= \alpha_t \quad \text{for} \quad t \in [-1, 0]. \end{aligned} \tag{5}$$

The theory of SDDE is a particular case of the theory of stochastic functional differential equations (see [7], [8], [9]).

A stochastic process ϕ_t on M adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a local solution of the (5) if for all $t \in [-1, 0]$, $\phi_t = \alpha_t$, there exists a stopping time $T > 0$ such that for all $t \leq T$ and for any $F \in \mathcal{C}^\infty(M)$:

$$F(\phi_t) = F(\phi_0) + \sum_{k=1}^{m} \int_0^t P^{\nabla}_{r,r-1}(\phi)(A_k(\phi_{r-1}))F(\phi_r) \circ dM^k_r$$

In this section we prove that the horizontal lift of a solution of equation (5) is a solution of the following SDDE on BM:

$$\begin{aligned} dx_t &= \sum_{k=1}^{m} P^{\nabla^H}_{t,t-1}(x)(A^H_k(x_{t-1})) \circ dM^k_t, \\ x_t &= \alpha^H_{p\ t} \quad \text{for } t \in [-1, 0], \end{aligned} \tag{6}$$

where $p \in \pi^{-1}(\alpha(0))$, α^H_p is the horizontal lift of α such that $\alpha^H_{p\ 0} = p$, A^H_k is the horizontal lift of A_k for $k = 0, 1, \dots, m$ and the connection ∇^H is the horizontal lift of ∇.

We begin by recalling some fundamental facts on differential geometry, we indicate e.g. Bishop and Crittenden [1], Cordero et al. [2] or Kobayashi and Nomizu [6]. Let M be a differentiable manifold, BM the frame bundle of M consists of all linear isomorphism $p : \mathbb{R}^n \to T_xM$ for some $x \in M$, with projection $\pi(p) = x$. The fibre bundle BM is a principal bundle over M with structure group $GL(n, \mathbb{R})$ and Lie algebra denoted by $\mathcal{G}l(n, \mathbb{R})$.

Let $\alpha : I \to M$ be a curve in M. The horizontal lift of α to BM, can be written as the composition

$$\alpha^H_p := P^{\nabla}_{t,0}(\alpha) \circ p \tag{7}$$

where $P_{t,s}^{\nabla}(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$ is the parallel transport along the curve α. A connection ∇ on M determines a decomposition of each tangent space T_pBM into the direct sum of the vertical subspace $V_pBM = Ker(\pi_*(p))$ and the horizontal subspace H_pBM of the tangent at p of horizontal lifts of curves in M. This decomposition naturally defines the horizontal lift of $v \in T_xM$ at $p \in BM$ ($\pi(p) = x$) as the unique tangent vector $v^H \in H_pBM$ such that $\pi_*(p)v^H = v$. Let $\{e_1, \dots, e_d\}$ be the standard basis of $\mathbb{R}^n$, the standard vector fields $\{E[e_1], \dots, E[e_n]\}$ in BM are the unique horizontal fields such that $\pi_*(p)E[e_i](p) = p(e_i)$ for every $p \in BM$. The distribution $\{H_p : p \in BM\}$ is the span of the standard vector fields $E[e_i]$. Let $A \in \mathcal{G}l(n, \mathbb{R})$, $A^*(p) = p_*(Id)A$ where p is considered as the application $p : GL(n, \mathbb{R}) \to BM$, $p(g) = p \circ g$. Obviously, $A^*(p)$ is a vertical vector. Let $\{E_{i,j} : 1 \le i, j \le n\}$ be the standard basis of $\mathcal{G}l(n, \mathbb{R})$, the distribution $\{V_pBM : p \in BM\}$ is the span of the vertical vector fields $E^*_{i,j}$. We observed that $\{E[e_i], E^*_{i,j} : 1 \le i, j \le n\}$ parallelizes BM.

There are many ways of extending the connection ∇ of M to BM. We are interested in the horizontal lift ∇^H, see e.g. Cordero et al. [2, Chap. 6]. In order to simplify the exposition, from here on, we assumed that the connections are torsion free. The horizontal lift ∇^H is defined as the unique connection on BM which satisfies:

$$\begin{aligned}
\nabla^H_{A^*} B^* &= (AB)^* \\
\nabla^H_{A^*} X^H &= 0 \\
\nabla^H_{X^H} A^* &= 0 \\
\nabla^H_{X^H} Y^H &= (\nabla_X Y)^H
\end{aligned}$$

We have the following commutative property of the parallel transport in BM with respect to the parallel transport in M:

Proposition 3.1 *Let ∇ be a connection on M, ∇^H its horizontal lift to BM and α a curve in BM. Then, for any $v \in T_{\pi\circ\alpha(0)}M$ we have that*

$$P_{0,t}^{\nabla^H}(\alpha)(v^H) = (P_{0,t}^{\nabla}(\pi \circ \alpha)(v))^H$$

Proof:

See Cordero et al. [2, Prop. 6.2.21].

□

Now, we present a fundamental lemma for the next results of this section.

Lemma 3.1 *Let γ be a solution of the deterministic delay differential equation*

$$\begin{aligned}\frac{dx}{dt}(t) &= P^{\nabla}_{t,t-1}(x)(X(x(t-1)))\\ x(t) &= \alpha(t) \quad \textit{for } t \in [-1,0].\end{aligned}$$

where ∇ is a connection on M, X a vector field in M and $\alpha : [-1,0] \to M$ a differentiable curve. Then the horizontal lift γ_p^H is a solution of

$$\begin{aligned}\frac{dx}{dt}(t) &= P^{\nabla^H}_{t,t-1}(x)(X^H(x(t-1)))\\ x(t) &= \alpha_p^H(t) \quad \textit{for} \quad t \in [-1,0].\end{aligned}$$

Proof:

We apply Proposition 3.1 to γ_p^H and $X(\gamma(t-1))$, thus:

$$\begin{aligned}\frac{d\gamma_p^H}{dt}(t) &= (\frac{d\gamma}{dt}(t))^H\\ &= (P^{\nabla}_{t,t-1}(\gamma)(X(\gamma(t-1)))^H\\ &= P^{\nabla^H}_{t,t-1}(\gamma_p^H)(X^H(\gamma_p^H(t-1)))\end{aligned}$$

and obviously $\gamma_p^H(t) = \alpha_p^H(t)$ for $t \in [-1,0]$.

□

The deterministic result of last lemma extends to stochastic systems as well:

Proposition 3.2 *Let γ be a solution of the SDDE (5) on M with connection ∇. Then γ_p^H is solution of the SDDE (6) on BM with connection ∇^H.*

Proof:

Apply the above lemma and the transfer principle (see e.g. Emery [4]).

□

Our second main result in this section shows that, surprisingly, the horizontal lift of γ which was written as a solution of a delay equation, can also be written as a solution of a stochastic equation without delay, but the integration here is taken with respect to another semimartingale. This SDE (without delay) is going to be written in the following canonical form:

$$dx_t = \sum_{i=1}^{d} E[e_i](x_t) \circ dN_t^i, \tag{8}$$

where N_t is the $\mathbb{R}^d$-semimartingale $\int_0^t \theta \circ dx_p^H{}_t$ and θ is the canonical 1-form on BM defined by $\theta(p) = p^{-1}\pi_*(p)$ (see Shigekawa [12]). Note that the global fields of frames $E[v]$ (where $v \in \mathbb{R}^n$ and $E[v] = v_1 E[e_1] + \ldots + v_n E[e_n]$), are parallel for ∇^H along horizontal curves.

Proposition 3.3 *Let γ be a curve in M, and $v \in \mathbb{R}^n$. Then*

$$P_{t,0}^{\nabla^H}(\gamma_p^H)(E[v](\gamma_p^H(0))) = E[v](\gamma_p^H(t))$$

Proof:

Again, by Proposition 3.1, formula (7) and definition:

$$\begin{aligned}
P_{t,0}^{\nabla^H}(\gamma_p^H)(E[v](\gamma_p^H(0))) &= (P_{t,0}^{\nabla}(\gamma)(\pi_*(p)(E[v](\gamma(0)))))^H \\
&= (P_{t,0}^{\nabla}(\gamma)(p(v)))^H \\
&= (\gamma_p^H(t)(v))^H \\
&= E[v](\gamma_p^H(t)).
\end{aligned}$$

□

Corollary 3.4 *Let γ be a solution of the SDDE (5) on M with connection ∇. Then γ_p^H is solution of the SDE (8) on BM.*

Proof:

It follows immediately form the fact that the canonical vector fields $E[e_i]$ are invariant for parallel transport along γ_p^H.

□

Appendix: Delay equations in Lie groups

In this appendix we explore further applications of the general form of the (strong) solutions given in Section 2 to get explicit formulas for solutions of delay equations in Lie groups. Consider for example the following SDDE in $GL(n,\mathbb{R})$:

$$dg(t) = A\,g(t-1)\,dt + g(t-1)\,B \circ dW_t \tag{9}$$

with initial condition, say, $g(t) = g_0$, for $t \in [-1, 0]$, where A and B are $n \times n$-matrices, This equation corresponds to a delay equation in the Lie group $GL(n,\mathbb{R})$ which involves vectors fields which are right invariant ($A(\cdot)$, in the deterministic part) and left invariant ($(\cdot)B$, in the diffusion part), endowed with the flat connection. By representation, a delay equation in any

Lie group involving right and/or left invariant vector fields can be written in this matrix form. We remark that applications of this kind of linear delay stochastic equations are particularly interesting in mathematical biology once most of biological transition of phases depends naturally on a certain delay, see e.g. Murray [10]. Next proposition presents some formulas for (strong) solutions of equations of this kind.

Proposition 3.5 *Given the initial condition $g(t) = g_0$ for $t \in [-1, 0]$, the (strong) solutions of the SDDE's:*

1. $dg(t) = A\, g(t)\, dt + g(t-1)\, B \circ dW_t$;
2. $dg(t) = A\, g(t-1)\, dt + g(t-1)\, B \circ dW_t$;
3. $dg(t) = A\, g(t)\, dt + B\, g(t-1) \circ dW_t$;
4. $dg(t) = A\, g(t-1)\, dt + B\, g(t-1) \circ dW_t$,

are given, respectively, by:

1. $g(t) = e^{At}\, g_0 \left(\sum_{k=0}^{n} \frac{(BW_{(t-k+1)})^k}{k!} \right)$;
2. $g(t) = \left(\sum_{k=0}^{n} \frac{A^k\,(t-k+1)^k}{k!} \right) \cdot g_0 \cdot \left(\sum_{k=0}^{n} \frac{B^k (W_{(t-k+1)})^k}{k!} \right)$;
3. $g(t) = e^{At}\, g_0 \left(\sum_{k=0}^{n} \frac{(BW_{(t-k+1)})^k}{k!} \right)$, *whenever A commutes with B;*
4. $g(t) = \left(\sum_{k=0}^{n} \frac{AB(t-k+1)W_{(t-k+1)}^k}{k!} \right) \cdot g_0$, *whenever A commutes with B.*

Proof:
Straightforward from Itô's formula.

□

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