Poincaré-Hopf and Morse Inequalities for Lyapunov Graphs*

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Abstract

Lyapunov graphs carry dynamical information of gradient-like flows as well as topological information of its phase space which is taken to be a closed orientable n-manifold. In this article we will show that an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ in dimension n greater than two, with cycle number κ, satisfies the Poincaré-Hopf inequalities if and only if it satisfies the Morse inequalities and the first Betti number $\gamma_1 \geq \kappa$. We also show a continuation theorem for abstract Lyapunov graphs with the presence of cycles. Finally, a family of Lyapunov graphs $\mathcal{L}(h_0, \ldots, h_n, \kappa)$ with fixed pre-assigned data $(h_0, \ldots, h_n, \kappa)$ is associated with the Morse polytope, $\mathcal{P}_\kappa(h_0, \ldots, h_n)$, determined by the Morse inequalities for the given data.

1 Introduction

Lyapunov graphs were initially introduced by Franks in [10]. We use these graphs as bookkeeping devices that retain at the same time local and global homological information of the flow and its phase space, a closed orientable n-manifold M.

Given a continuous flow $\phi_t : M \to M$, on a closed n-manifold M, results of Conley [3] imply the existence of a continuous Lyapunov function $f : M \to \mathbb{R}$ associated with the flow with the property

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that it strictly decreases along the orbits outside the chain recurrent set, that is, if $x \notin R$ then $f(\phi_t(x)) < f(\phi_s(x))$ for $t > s$ and is constant on the chain recurrent components of R. We assume that R is a finite component chain recurrent set where each component R_k is an isolated invariant set.

Define the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. We call $M/\sim_f$ a *Lyapunov graph*. Each vertex v_k represents components R_k of the chain recurrent set R and hence can be labelled with dynamical invariants. Each edge represents a level set times an interval and hence can be labelled with topological invariants of the level set.

One can also define an *abstract Lyapunov graph in dimension* n as a finite, connected, oriented graph, that has no oriented cycles. Also, each vertex is labelled with a chain recurrent flow on a compact n-dimensional space and each edge is labelled with topological invariants of a closed $(n-1)$-dimensional manifold. This definition is far too general for our purposes. We will label the vertex v_k of an abstract Lyapunov graph with the dimensions of the Conley homology indices, $\dim CH_j(R_k) = h_j(v_k)$, with $j = 0, \ldots n$. Hence, each vertex is labelled with a list of nonnegative integers $(h_0(v_k), \ldots, h_n(v_k), \kappa(v_k))$, where $\kappa(v_k)$ is the *cycle number of the vertex* v_k, which is a nonnegative integer weight on v_k. An alternative notation is to label the vertex with $h_j(v_k) = n_j$ whenever $n_j \neq 0$. And $\kappa(v_j) = k_j$ whenever $k_j \neq 0$. The latter notation is convenient whenever $(h_0(v_k), \ldots, h_n(v_k), \kappa(v_k))$ has many zero entries. Also, for simplicity we will omit reference to the vertex v_k whenever possible. We choose to label the edges with the Betti numbers of a closed $(n-1)$-dimensional manifold, a Betti number vector. A *Betti number vector* is a list of nonnegative integers $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$, where $\gamma_{n-k} = \gamma_k$, $\gamma_0 = \gamma_n = 1$ and $\gamma_{n/2}$ is even if n is even[1]. An abstract Lyapunov graph of Morse type will be defined subsequently, but basically it is an abstract Lyapunov graph with each vertex v labelled with a non-degenerate singularity of Morse index j, i.e., $h_j(v) = 1$, and with all cycle number of vertices equal to zero.

Given an abstract Lyapunov graph L with vertex set V and cycle rank[2] κ_L, we will denote it by $L(h_0, \ldots, h_n, \kappa)$, where $h_j = \sum_{v_k \in V} h_j(v_k)$, and $\kappa_V = \sum_{v_k \in V} \kappa(v_k)$ and $\kappa = \kappa_L + \kappa_V$. We will refer to κ as the *cycle number of the graph.* This definition is easily extended to Lyapunov semi-graphs. It is easy to see that the cycle number of an abstract Lyapunov graph of Morse type is equal to its cycle rank.

[1]We relax the condition of γ_i being even in the case $2i = 0 \mod 4$ when considering Betti number vectors in the Morse inequalities.

[2]The *cycle rank* of a graph is the maximum number of edges that can be removed without disconnecting the graph.

In [1] the Poincaré-Hopf inequalities (2)–(4) with $\kappa = 0$ are introduced for flows on isolating blocks B and their Lyapunov semi-graph[3] L_B, in order to ensure the continuation of L_B to a Morse type Lyapunov semi-graph. These inequalities involve the Betti numbers of the exiting and entering boundaries of B. However, in [1] we considered only abstract Lyapunov semi-graphs in dimensions greater than two with no non-oriented cycles, that is, $\kappa = 0$. The dimension two case is much easier and was completely characterized in [1] with $\kappa \geq 0$.

In this article we will treat Lyapunov graphs in dimensions greater than two in full generality, that is, we will consider graphs with cycle number $\kappa \geq 0$. In Theorem 1.1 and Theorem 1.2, we prove more general continuation theorems than the ones in [1].

We say that a Lyapunov graph (resp., a Lyapunov semi-graph) $L(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities if the data $(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities (5)–(7) (resp., (2)–(4)).

Theorem 1.1 *Consider an abstract Lyapunov semi-graph L_B where the vertex v associated to B is labelled with $(h_0(v), \ldots, h_n(v), \kappa_v)$. It admits continuations to Morse type Lyapunov semi-graphs with cycle rank greater or equal to κ_v if and only if it satisfies the Poincaré-Hopf inequalities* (2)–(4), *where $\kappa_v \leq \min\{h_1 - (e^- + h_0 - 1), \quad h_{n-1} - (e^+ + h_n - 1)\}$. Moreover, the number of possible continuations is obtained.*

The inequalities (2)–(4) involve the number of exiting, e^-, and entering, e^+, boundaries of B as well as their Betti numbers. Theorem 1.1 implies the following theorem.

Theorem 1.2 *Consider an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$. It admits continuations to abstract Lyapunov graphs of Morse type with cycle rank greater or equal to κ if and only if it satisfies the Poincaré-Hopf inequalities* (2)–(4) *at each vertex, where $\kappa \leq \min\{h_1 - (h_0 - 1), h_{n-1} - (h_n - 1)\}$. Moreover, the number of possible continuations is obtained.*

Certain topological invariants of the manifold, as well as invariants of the flow, impose restrictions on the Lyapunov graph. For instance, in [6] it was shown that the cycle rank κ of a Lyapunov graph is a lower bound to the Cornea genus[4] of the manifold $g(M)$, which in turn is always less than or equal to the first Betti number of M, $\kappa \leq g(M) \leq \gamma_1(M)$. This generalizes a theorem

[3]This terminology is used for the first time in this article and formalizes the distinction between a graph and a dangling graph. See Section 2

[4]This genus is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect the smooth closed manifold M. See [4].

of Franks [10] which asserts that if M is simply connected then $\kappa = 0$. Other characterization theorems of flows on manifolds are made possible by using Lyapunov graphs, see [5], [7], [8], [9]. A more difficult question which has been answered in some cases, is that of the realization of an abstract Lyapunov graph as a flow on some manifold. If the graph carries enough dynamical and topological invariants this is possible.

One of the objectives of this paper is a small step in this direction. It is a well known result of Morse that if an abstract dynamical data list does not satisfy the Morse inequalities for a closed manifold M then there is no flow on M with this data. Note that an abstract Lyapunov graph carries dynamical data but carries no information of the manifold it is realizable on. This means that the Morse inequalities cannot be used to verify the realizability or not of this dynamical data. However, note that the Poincaré-Hopf inequalities (5)–(7) are verifiable for the dynamical data of an abstract Lyapunov graph. We show in Theorem 1.3 that for an abstract Lyapunov graph with $\kappa \geq 0$ the Poincaré-Hopf inequalities (5)–(7) are necessary and sufficient conditions for the generalized Morse inequalities (1) to hold for some Betti number vector. As a consequence of this fact, if the graph does not satisfy the Poincaré-Hopf inequalities, it will not satisfy the Morse inequalities (1) for any choice of Betti number vector and hence we screen out Lyapunov graphs which cannot be realized as a continuous flow on any manifold. An abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ satisfies the Morse inequalities and the *κ-connectivity inequality* $\gamma_1 \geq \kappa$ if there exists a Betti number vector $(\gamma_0, \ldots, \gamma_n)$ satisfying (1) and $\gamma_1 \geq \kappa$.

Theorem 1.3 *Given an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$, it satisfies the Poincaré-Hopf inequalities (5)–(7) if and only if it satisfies the Morse inequalities (1) and the inequality $\gamma_1 \geq \kappa$ for some Betti number vector $(\gamma_0, \ldots, \gamma_n)$.*

Conley in [3] proves that the following generalized Morse inequalities are valid, where γ_i is the i-th Betti number of M and h_i is the dimension of the i-th Conley homology index as defined previously.

$$
\begin{array}{rcll}
\gamma_n - \gamma_{n-1} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & = & h_n - h_{n-1} + - \ldots \pm h_2 \pm h_1 \pm h_0 & \text{(n)} \\
\gamma_{n-1} - \gamma_{n-2} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_{n-1} - h_{n-2} + - \ldots \pm h_2 \pm h_1 \pm h_0 & \text{(n} - 1) \\
\vdots & & \vdots & \\
\gamma_j - \gamma_{j-1} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_j - h_{j-1} + - \ldots \pm h_2 \pm h_1 \pm h_0 & \text{(j)} \\
\gamma_{j-1} - \gamma_{j-2} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_{j-1} - h_{j-2} + - \ldots \pm h_2 \pm h_1 \pm h_0 & \text{(j} - 1) \\
\vdots & & \vdots & \\
\gamma_2 - \gamma_1 + \gamma_0 & \leq & h_2 - h_1 + h_0 & (2) \\
\gamma_1 - \gamma_0 & \leq & h_1 - h_0 & (1) \\
\gamma_0 & \leq & h_0 & (0)
\end{array}
\qquad (1)
$$

Once an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities, Theorem 1.3 guarantees the existence of a Betti number vector which satisfies the Morse inequalities (1) and $\gamma_1 \geq \kappa$ and vice-versa. A natural problem is to determine all possible Betti number vectors which satisfy the Morse inequalities and $\gamma_1 \geq \kappa$ for the dynamical data $(h_0, \ldots, h_n, \kappa)$ on L.

Results from network-flow theory used in the proof of the above theorem lead to a method to construct all Betti number vectors satisfying (1) and $\gamma_1 \geq \kappa$. We also establish that the inequalities (1), $\gamma_1 \geq \kappa$ and the constraints a Betti number vector must satisfy, namely $\gamma_0 = \gamma_n = 1$, $\gamma_j = \gamma_{n-j}$, for $j = 1, \ldots, n-1$, $\gamma \geq 0$, define an integral polytope, i.e., a bounded polyhedron with extreme points. Furthermore, this polytope contains a (componentwise) maximum element and its extreme points are Betti number vectors. We refer to this polytope as the *Morse polytope*. We will see that this is the same as saying that the Morse polytope coincides with the convex hull of the Betti number vectors. The mentioned construction method establishes a relationship between graphs in the family of Lyapunov graphs $\mathcal{L}(h_0, \ldots, h_n, \kappa)$ with cycle number κ, with integral elements in the Morse polytope. These results generalize analogous ones obtained in [2] for the case $\kappa = 0$. Finally, for the family of Lyapunov graphs $\mathcal{L}(h_0, \ldots, h_n, \kappa)$, the associated polytope $\mathcal{P}_\kappa(h_0, \ldots, h_n)$ is contained in $\mathcal{P}(h_0, \ldots, h_n)$, the polytope corresponding to the case $\kappa = 0$. Note that if $\kappa = 0$, $\mathcal{P}_\kappa(h_0, \ldots, h_n) = \mathcal{P}(h_0, \ldots, h_n)$.

This article is divided in the following sections. Section 2 will briefly introduce explosions and implosions of abstract Lyapunov graphs in order to define the continuation of a graph. Section 3 will introduce more general Poincaré-Hopf inequalities with parameter κ (2)–(4) for isolating blocks than the ones obtained in [1] in the case $\kappa = 0$ and the more general Poincaré-Hopf inequalities with parameter κ (5)–(7) for closed manifolds than the ones obtained in [2] in the case $\kappa = 0$.

Section 4 presents the explosion algorithm with cycles which produces a linear h_κ^{cd}-system whose solution guarantees the continuation results in the presence of cycles. In Section 5, we show that this h_κ^{cd}-system has a solution if and only if the Poincaré-Hopf inequalities with parameter κ for isolating blocks are satisfied. Hence, we prove a continuation theorem for abstract Lyapunov graphs in the presence of cycles. In Section 6 we present the reduced h_κ^{cd}-system which has a solution if and only if the Poincaré-Hopf inequalities with parameter κ for closed manifolds are satisfied. We use the solutions of the reduced h_κ^{cd}-system to establish the sufficiency of the Poincaré-Hopf inequalities with parameter κ in order for the Morse inequalities and $\gamma_1 \geq \kappa$ to hold and lastly, we prove the necessity. In Section 7 we describe the Morse polytope $\mathcal{P}_\kappa(h_0, \ldots, h_n)$. Also, additional geometric properties of the polytope are presented.

2 Lyapunov Graphs, Semi-Graphs and Isolating Blocks

There is a natural correspondence between a closed manifold and a Lyapunov graph. In order to establish a correspondence with the isolating block we need to define a Lyapunov semi-graph. For this purpose, we need to extend the notion of a directed graph to allow for a distinguished vertex, which we will denote by ∞.

Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or dangling edges as in [7]). Note that whenever G' does not contain semi-edges G' is a graph in the usual sense. The graphical representation of the graph will have the semi-edges cut short.

An isolating block N of an isolated invariant set Λ with entering set for the flow N^+ and exiting set for the flow N^-, can be associated to a Lyapunov semi-graph L_N, consisting of one vertex labelled with the dimensions of the Conley homology indices of Λ and entering and exiting labelled semi-edges. The number of incoming (outgoing) edges e^+ (e^-) correspond to the number of connected components of N^+ (N^-). The labels on the edges correspond to the Betti numbers of the closed codimension one submanifolds N^+ and N^-.

The following definition is crucial to what follows in this paper. It will classify singularities with $h_\ell = 1$ (for Morse flows these correspond to the non-degenerate singularities of Morse index ℓ) by distinguishing the effect it causes on the level sets N^- and N^+.

A singularity, respectively a vertex, labelled with $h_\ell = 1$ is ℓ-d if it has the algebraic effect of increasing the ℓ-th Betti number of N^+ or respectively, the corresponding β_ℓ label on the incoming edge. A singularity, respectively a vertex labelled with $h_\ell = 1$ is $(\ell-1)$-c if it has the algebraic effect of decreasing the $(\ell-1)$-th Betti number of N^+ or respectively, the corresponding $\beta_{\ell-1}$ label on the incoming edge. In the case $n = 2i = 0 \mod 4$, a singularity, respectively a vertex labelled with $h_i = 1$ is β-i, if all Betti numbers are kept constant. See the corresponding semi-graphs in Figure 1.

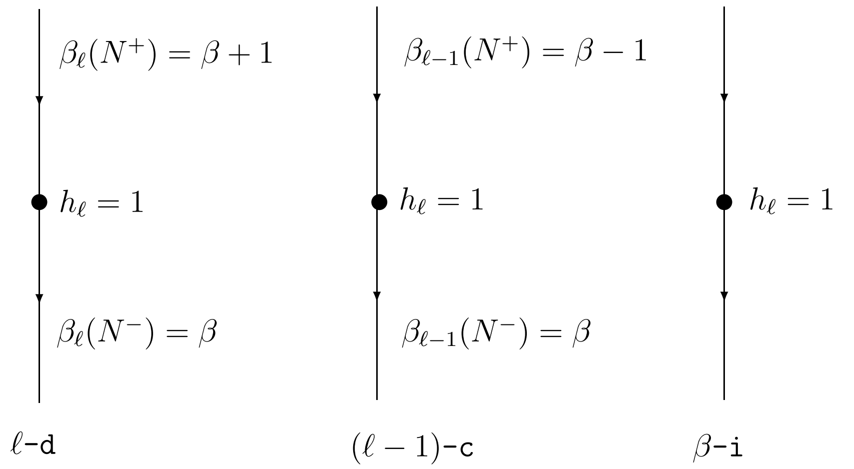


Figure 1: The three possible algebraic effects.

An *abstract Lyapunov graph of Morse type* is an abstract Lyapunov graph that satisfies the following:

1. every vertex is labelled with $h_j = 1$ for some $j = 0, \ldots, n$ and the cycle number of each vertex equal to zero.

2. the number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:

   (a) if $h_j = 1$ for $j \neq 0, 1, n-1, n$ then $e^+ = 1$ and $e^- = 1$;

   (b) if $h_1 = 1$ then $e^+ = 1$ and $0 < e^- \leq 2$; if $h_{n-1} = 1$ then $e^- = 1$ and $0 < e^+ \leq 2$;

   (c) if $h_0 = 1$ then $e^- = 0$ and $e^+ = 1$; if $h_n = 1$ then $e^+ = 0$ and $e^- = 1$.

3. every vertex labelled with $h_\ell = 1$ must be of type ℓ-d or $(\ell-1)$-c. Furthermore if $n = 2i = 0 \mod 4$ and $h_i = 1$ then v may be labelled with β-i.

Inspired by Conley's idea of continuation of an isolated invariant set to a simpler one, Reineck in [12] proved a continuation result showing that any isolated invariant set on a manifold can be continued to an isolated invariant set of a gradient flow. How exactly can this be interpreted in the Lyapunov graph setting? This was answered in [1] with a continuation theorem for all Lyapunov graphs in dimension 2 and in dimension $n > 2$ with $\kappa = 0$. A graph which does not admit a continuation to a Morse type Lyapunov graph cannot be realized as a flow on any manifold.

The notion of *vertex explosion* is used to define continuation of abstract Lyapunov graphs. Let v be a vertex on an abstract Lyapunov graph labelled with $(h_0(v), h_1(v) \ldots, h_n(v), \kappa_v)$. A vertex v can be exploded if v can be removed and replaced by an abstract Lyapunov graph I of Morse type with cycle rank greater or equal to κ_v. The graph I must respect the orientations and labels of the incoming and outgoing edges of v. In other words, the new graph obtained must be oriented and with cycle number greater or equal to κ_v. The incoming (outgoing) edges of v, must be incoming (outgoing) edges on vertices of I and all labels on the edges must respect the restrictions of the Morse type vertices. Moreover,

$$h_\lambda(v) = \sum_{j=1}^{k} h_\lambda(v_j), \quad \text{for } \lambda = 1, \ldots, n-1, \text{ where } v_j \in I.$$

An abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ admits a *continuation* to an abstract Lyapunov graph of Morse type L_M if each vertex can be exploded such that L_M has cycle rank greater or equal to κ.

Given an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ with cycle number equal to κ, one can define a *graph implosion of* L as an abstract Lyapunov graph L_C with:

1. one saddle type vertex ν labelled with $(h_1(\nu), \ldots, h_{n-1}(\nu), \kappa)$ where
$$\sum_{v \in V} h_\lambda(v) = h_\lambda(\nu),$$
where V is the set of vertices of L;

2. the vertex ν will have $\sum_{v \in V} h_n(v) = e^+$, incoming edges and will have $\sum_{v \in V} h_0(v) = e^-$ outgoing edges;

3. the incoming edges of ν are outgoing edges of e^+ vertices labelled with $h_n = 1$ and the outgoing edges of ν are incoming edges of e^- vertices labelled with $h_0 = 1$;

4. the labels of all the edges satisfy $B_j^+ = B_j^- = 0$, for all $j \neq 0, n-1$, and $B_j^+ = B_j^- = 1$, for $j = 0, n-1$.

In Section 6 examples are presented.

3 Poincaré-Hopf Inequalities with Connectivity Parameter κ

3.1 Poincaré-Hopf Inequalities with κ for Isolating Blocks

In [1] we consider the Poincaré-Hopf inequalities (2)–(4), in the case $\kappa = 0$ for an isolated invariant set Λ with isolating block N, with entering set for the flow N^+ and exiting set for the flow N^-, under the hypothesis that the flow satisfies the Conley index duality condition on components of the chain recurrent set. These inequalities were obtained by analysis of long exact sequences of the index pairs (N, N^-) and (N, N^+), for Λ and for the isolated invariant set of the reverse flow, Λ', where $\operatorname{rank} H_i(N, N^-) = h_i$, $\operatorname{rank} H_i(N, N^+) = h_{n-i}$, $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$, $\operatorname{rank} H_0(N) = 1$ and $\operatorname{rank}(H_i(N^{\pm})) = B_i^{\pm}$.

We will now consider these inequalities in the presence of a parameter κ for an isolating block N. Thus, the Poincaré-Hopf inequalities for isolating blocks with this parameter will be the collection of the following constraints (2)–(4).

$$
\left\{
\begin{array}{l}
n = 2i+1 \left\{
\begin{array}{l}
h_i \geq -(B_{i-1}^{+} - B_{i-1}^{-}) + (B_{i-2}^{+} - B_{i-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \\
\qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \\
\qquad \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^{+} - e^{-})] \\
\\
h_{i+1} \geq -\left[-(B_{i-1}^{+} - B_{i-1}^{-}) + (B_{i-2}^{+} - B_{i-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \right. \\
\qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \\
\qquad \left. \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^{+} - e^{-})] \right]
\end{array}
\right. \\
\\
n = 2i \left\{
\begin{array}{l}
h_i \geq -(B_{i-1}^{+} - B_{i-1}^{-}) + (B_{i-2}^{+} - B_{i-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \\
\qquad \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^{+} - e^{-})] \\
\\
h_i \geq -\left[-(B_{i-1}^{+} - B_{i-1}^{-}) + (B_{i-2}^{+} - B_{i-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \right. \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \\
\qquad \left. \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^{+} - e^{-})] \right]
\end{array}
\right. \\
\\
\qquad\qquad \vdots \\
\\
\left\{
\begin{array}{l}
h_j \geq -(B_{j-1}^{+} - B_{j-1}^{-}) + (B_{j-2}^{+} - B_{j-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^{+} - e^{-})] \\
\\
h_{n-j} \geq -\left[-(B_{j-1}^{+} - B_{j-1}^{-}) + (B_{j-2}^{+} - B_{j-2}^{-}) + - \ldots \pm (B_2^{+} - B_2^{-}) \pm (B_1^{+} - B_1^{-}) \right. \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\qquad \left. \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^{+} - e^{-})] \right]
\end{array}
\right. \\
\qquad\qquad \vdots \\
\left\{
\begin{array}{l}
h_2 \geq -(B_1^{+} - B_1^{-}) - (h_{n-1} - h_1) + (h_n - h_0) + (e^{+} - e^{-}) \\
h_{n-2} \geq -\left[-(B_1^{+} - B_1^{-}) - (h_{n-1} - h_1) + (h_n - h_0) + (e^{+} - e^{-}) \right]
\end{array}
\right. \\
\\
\left\{
\begin{array}{l}
h_1 \geq h_0 - 1 + e^{-} + \kappa \\
h_{n-1} \geq h_n - 1 + e^{+} + \kappa
\end{array}
\right.
\end{array}
\right.
\tag{2}
$$

Furthermore, the Poincaré-Hopf equality must be considered in the odd-dimensional case $n = 2i + 1$:

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1} (-1)^j h_j \tag{3}$$

where

$$\mathcal{B}^+ = \frac{(-1)^i}{2} B_i^+ \pm B_{i-1}^+ \pm \ldots - B_1^+$$
$$\mathcal{B}^- = \frac{(-1)^i}{2} B_i^- \pm B_{i-1}^- \pm \ldots - B_1^-$$

Moreover, in the even dimensional case $n = 2 \mod 4$, the condition that

$$h_i - \sum_{j=1}^{i-1} (-1)^{j+1} (B_j^+ - B_j^-) - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^- - e^+) \text{ be even} \tag{4}$$

must be imposed.

3.2 Poincaré-Hopf Inequalities for Closed Manifolds

In [2] we consider a particular case of the Poincaré-Hopf inequalities for isolating blocks (2)–(4), which are the Poincaré-Hopf inequalities for closed manifolds (5)–(7). Given a Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ with cycle number equal to κ, the implosion L_C of $L(h_0, \ldots, h_n, \kappa)$ is a graph with only one saddle vertex ν labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$ and has the following properties: $e^+ = h_n$, $e^- = h_0$, $B_j^- = B_j^+ = 0$. By substituting the information of the vertex ν of L_C in the Poincaré-Hopf inequalities for isolating blocks (2)–(4), the Poincaré-Hopf inequalities for closed manifolds (5)–(7) are obtained.

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
n = 2i+1 \left\{ \; -h_i \le (h_{i+2} - h_{i-1}) - (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \le h_{i+1} \right. \\[2em]
n = 2i \left\{ \; -h_i \le (h_{i+1} - h_{i-1}) - (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i} - h_0) \le h_i \right.
\end{array}
\right. \qquad \text{(i)} \\[4em]
\left\{
\begin{array}{l}
-h_j \le (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \le h_{n-j} \;\text{(j)} \\
\qquad\qquad \vdots \\
-h_2 \le (h_{n-1} - h_1) - (h_n - h_0) \le h_{n-2} \;\text{(2)} \\[2em]
\left\{
\begin{array}{l}
h_1 \ge h_0 - 1 + \kappa \\
h_{n-1} \ge h_n - 1 + \kappa
\end{array}
\right. \quad \text{(1)}
\end{array}
\right.
\end{array}
\right.
\tag{5}
$$

In the case $n = 2i+1$ we have

$$\sum_{j=0}^{2i+1} (-1)^j h_j = 0 \tag{6}$$

and in the case $n = 2i = 2 \mod 4$ we have the additional contraint that

$$h_i - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) \;\text{ be even.} \tag{7}$$

4 Explosion Algorithm in the Presence of Cycles

In this section we will generalize the continuation theorem for abstract Lyapunov graphs in [1] by inserting a new step in the explosion algorithm permitting the presence of cycles in the graph. This expanded algorithm will be expressed in terms of a linear system h_κ^{cd} as was done in [1] where $\kappa = 0$.

4.1 STEP 1 - Explosion to a saddle type vertex with parameter κ

A vertex on an abstract Lyapunov graph labelled with $(h_0, h_1, \ldots, h_n, \kappa)$ is a *repeller vertex* if it has indegree zero and $h_n > 0$ (respectively, is an *attractor vertex* if it has outdegree zero and $h_0 > 0$). Otherwise, a vertex with positive indegree and outdegree is a *generalized saddle vertex.* A particular

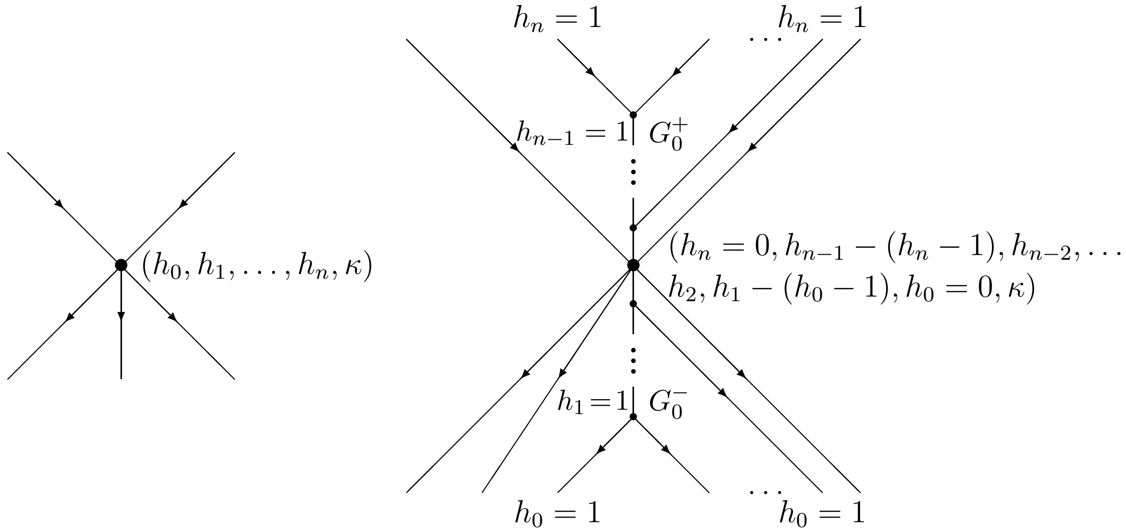


Figure 2: Generalized saddle vertex and its partial explosion.

case of this occurs when the vertex is labelled with $(h_0 = 0, h_1, \ldots, h_n = 0, \kappa)$ and will be called a *saddle vertex*. Our explosion algorithm consists of initially performing a partial explosion on a vertex which reduces it to a saddle type vertex.

Given a generalized saddle vertex or a repeller (respectively attractor) vertex labelled with $(h_0, h_1, \ldots, h_n, \kappa)$, a partial explosion will be done in order to obtain a saddle vertex labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$.

1. G_0^+ is formed with h_n vertices labelled with $h_n = 1$ and $h_n - 1$ vertices labelled with $h_{n-1} = 1$ of the type $(n-1)$-d and has one outgoing edge which connects to v.

2. G_0^- is formed with h_0 vertices labelled with $h_0 = 1$ and $h_0 - 1$ vertices labelled with $h_1 = 1$ of the type 0-c and has one incoming edge which comes from v.

This explosion is possible if the last two inequalities in (2) are satisfied. There is no need to be concerned with the labels on the edges since these type of vertices only alter β_0 and dually β_{n-1} which remains equal to one on every edge.

4.2 STEP 2 - Explosion of a saddle type vertex in the presence of cycles

In STEP 1, a partial explosion of v to a saddle type vertex was done. In this step we continue to denote the partially exploded vertex by v.

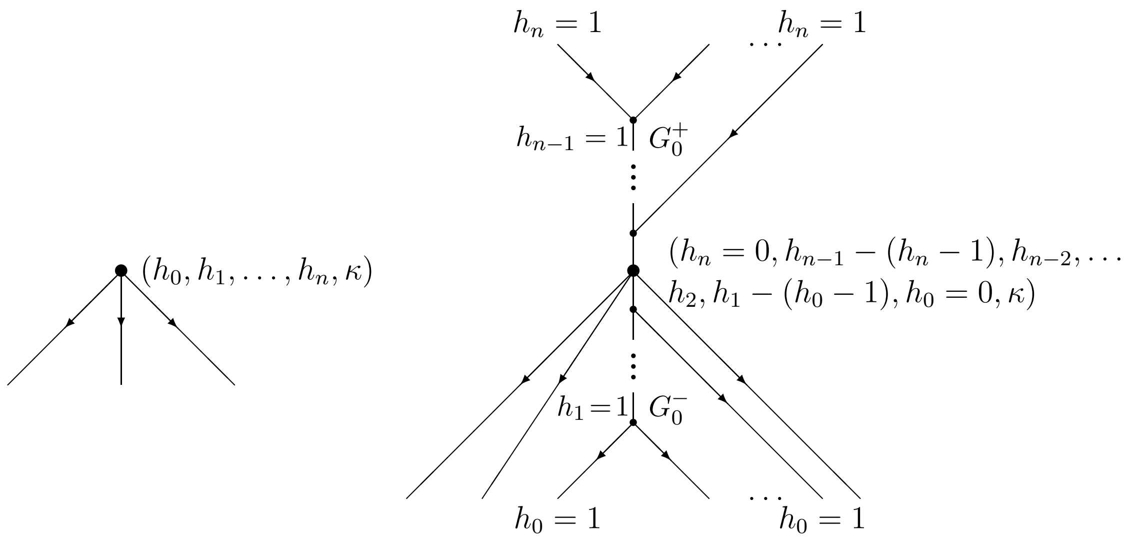


Figure 3: Repeller vertex and its partial explosion.

Now v is a saddle type vertex labelled vertex with $(0, h_1, h_2, \ldots, h_{n-1}, 0, \kappa)$ and incoming edges labelled with $((\beta_0^+, \ldots, \beta_{n-1}^+)_i)_{i=1}^{e^+}$ and outgoing edges labelled with $((\beta_0^-, \ldots, \beta_{n-1}^-)_i)_{i=1}^{e^-}$, where i denotes the edge. Let $B_j^+ = \sum_{i=1}^{e^+} (\beta_j^+)_i$ and $B_j^- = \sum_{i=1}^{e^-} (\beta_j^-)_i$. See Figure 4. Observe that $B_0^- = e^-$ e $B_0^+ = e^+$.

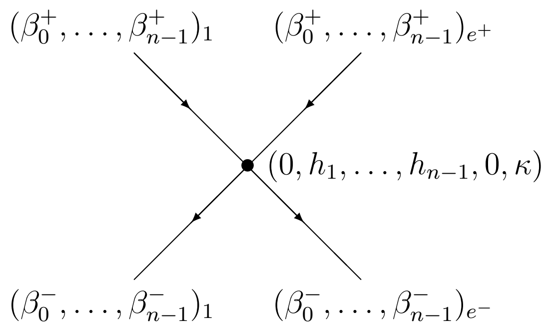


Figure 4: Vertex to be exploded.

Schematically the algorithm which explodes a saddle vertex has four basic parts:

1. adjusting the incident edges $\longrightarrow$ defines G^+ and G^-;

2. inserting cycles;

3. the linear explosion without middle dimensions $\longrightarrow$ defines $G^+ \cup \bigcup_{j=1}^{\ell} L_j^+$ and $G^- \cup \bigcup_{j=1}^{\ell} L_j^-$ where $\ell <$ mid-dimension;

4. middle dimension explosion $\longrightarrow$ consider n odd, $n = 0 \mod 4$, $n = 2 \mod 4$.

4.2.1 Adjusting the incident edges

In this step we wish to define G^+ and G^-.

Choose $e^- - 1$ vertices labelled with $h_1 = 1$ of type 0-c. This is possible by the last inequality in (2). By choosing this number of vertices labelled with 1-singularities, G^- is formed with e^- outgoing edges and one incoming edge. Singularities of type 0-c do not alter the β_i with $0 < i < n-1$. This type of singularity decreases β_0 and by duality β_{n-1}. Hence, the incoming edge of G^- has $B_0^- = B_{n-1}^- = 1$ and $B_j^- = \sum_{i=1}^{e^-} (\beta_j^-)_i$ with $j = \{1, \dots, n-2\}$. See Figure 6.

Similarly, the graph G^+ is formed by choosing $e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $n-1$-d.

4.2.2 The insertion of cycles

An *elementary cycle* is a pair of (h_1^c, h_{n-1}^d) with one edge labelled with $(1, 0, \dots, 0, 1)$ and the other edge labelled with $(1, \beta_1, \dots, \beta_{n-2}, 1)$. See Figure 5.

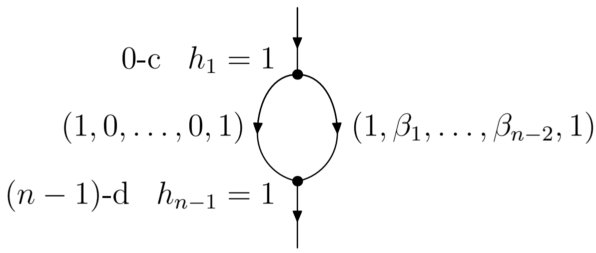


Figure 5: (h_1^c, h_{n-1}^d) pair

Without loss of generality, attach to G^-, κ elementary cycles where $(1, \beta_1, \dots, \beta_{n-2}, 1) = (1, B_1^-, \dots, B_{n-2}^-, 1)$. Of course this attachment can also be done to G^+. It is clear that once κ cycles are inserted the number of vertices labelled with $h_1 = 1$ of type 0-c is greater or equal to κ. Similarly, the number of vertices labelled with $h_{n-1} = 1$ of type $n-1$-d is greater or equal to κ.

Hence all together we have inserted $h_1^c = \kappa + e^- - 1$ vertices labelled with $h_1 = 1$ of type 0-c. Similarly, we have inserted $h_{n-1}^d = \kappa + e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $(n-1)$-d. This is possible due to the last two inequalities in (2), which asserts that for this saddle vertex v

$(h_0 = h_n = 0)$:

$$\begin{cases} h_1 \geq -1 + e^- + \kappa \\ h_{n-1} \geq -1 + e^+ + \kappa \end{cases} \tag{8}$$

However more cycles can appear in the explosion.

The last cycle inserted has incoming edge labelled with $(1, B_1^-, B_2^-, \ldots, B_{mid}^-, \ldots, B_{n-2}^-, 1)$.

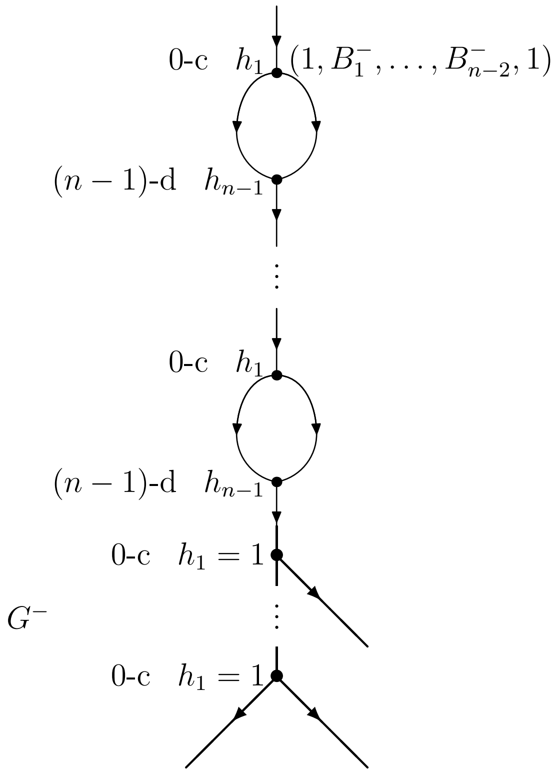


Figure 6: Outgoing edges exploded and cycles inserted.

4.2.3 The linear explosion without middle dimensions (Step 1, ..., Step ℓ)

This is done by an induction argument. For more details see [1].

Assume that the adjustments of B_j for $j < \ell$ and by duality B_{n-j-1} for $j < \ell$ have been made in increasing order for j. Hence, several linear graphs have been added to G^+ forming at this point a graph $G^+ \cup \bigcup_{i=1}^{\ell-1} L_i^+$ whose outgoing edge is labelled with

$$(1, B_1, \ldots, B_{\ell-1}, B_\ell^+, \ldots, B_{mid}^+, \ldots, B_{n-\ell-1}^+, B_{n-\ell}, \ldots, B_{n-2}, 1).$$

Similarly, several linear graphs have been added to G^- forming at this point a linear graph $G^- \cup \bigcup_{i=1}^{\ell-1} L_i^-$ whose incoming edge is labelled with

$$(1, B_1, \ldots, B_{\ell-1}, B_\ell^-, \ldots, B_{mid}^-, \ldots, B_{n-\ell-1}^-, B_{n-\ell}, \ldots, B_{n-2}, 1).$$

See Figure 7 for the case $\ell = 1$.

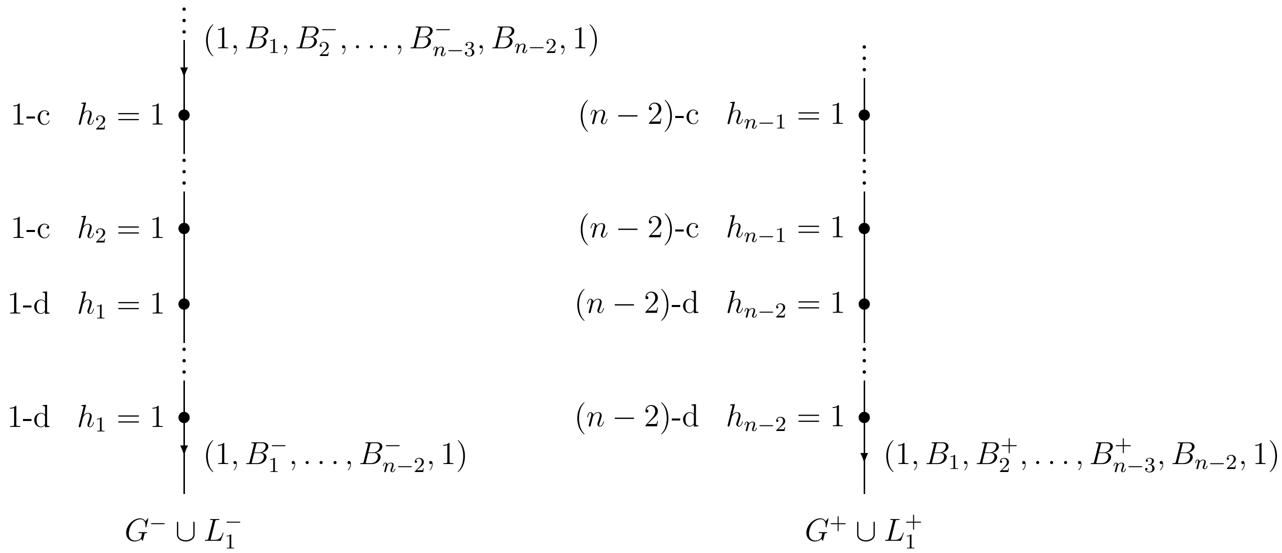


Figure 7: Linear explosion: L^- and L^+.

In order to adjust B_ℓ and by duality $B_{n-\ell-1}$ add to the graph above L_ℓ^- formed with h_ℓ^d vertices $h_\ell = 1$ of type ℓ-d, and $h_{\ell+1}^c$ vertices $h_{\ell+1} = 1$ of type ℓ-c forming $G^- \cup \bigcup_{i=1}^{\ell} L_i^-$. Similarly, the insertion of $h_{n-\ell}^c$ vertices $h_{n-\ell} = 1$ of type $(n-\ell-1)$-c and the insertion of $h_{n-\ell-1}^d$ vertices $h_{n-\ell-1} = 1$ of type $(n-\ell-1)$-d will form $G^+ \cup \bigcup_{i=1}^{\ell} L_i^+$.

Since the insertion of any other type of vertex will not alter the ℓ-th and the $(n-\ell-1)$-th Betti number it is necessary that

$$B_\ell = B_\ell^- + h_\ell^d - h_{\ell+1}^c = B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c. \tag{9}$$

The labels of B_ℓ and $B_{n-\ell-1}$ for $0 < \ell < mid$ have all been adjusted. It remains to adjust the middle dimensional labels. This is done in the next step.

4.2.4 Middle dimensional explosion

At this point the adjustments of the labels at the middle dimensions must be made. It is necessary to consider the case when $n-1$ is odd and there are two middle dimensional labels. If $n = 2i$ the vertices inserted, alter the labels as in (9) with $\ell = i$. In the case $n = 0 \mod 4$ the presence of the β^i vertices does not alter this equation.

If $n-1$ is even, there is only one middle dimensional label $B_{\frac{n-1}{2}}$ and in this case the insertion of the vertices alters the labels as in,

$$B_i = B_i^- + 2h_i^d = B_i^+ + 2h_{i+1}^c. \tag{10}$$

4.2.5 h_κ^{cd}-Systems

Hence, all these adjustments are recorded in the following h_κ^{cd}-systems which describes the explosion of a saddle vertex labelled with $(h_0 = 0, h_1, \ldots, h_{n-1}, h_n = 0, \kappa)$. Hence, these linear systems of equations must be solved for $(h_1^c, h_1^d, \ldots, \beta_i, \ldots, h_{2i}^c, h_{2i-1}^d, \kappa)$ in order for the saddle explosion algorithm to work.

$$n = 2i+1 \begin{cases} e^- - 1 - h_1^c + \kappa = 0 \\ \{h_j = h_j^c + h_j^d, \;\; j = 1, \ldots, 2i \\ e^+ - 1 - h_{2i}^d + \kappa = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ \vdots \\ \frac{-(B_i^+ - B_i^-)}{2} + h_i^d - h_{i+1}^c = 0 \end{cases} \end{cases} \tag{11}$$

or

$$n = 2i \begin{cases} e^- - 1 - h_1^c + \kappa = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \;\; j = 1, \ldots, 2i-1, \beta^i = 0 \text{ if } j \neq i \text{ and } 2i \neq 0 \mod 4 \\ e^+ - 1 - h_{2i-1}^d + \kappa = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{cases} \end{cases} \tag{12}$$

5 Continuation Results in the Presence of Cycles

The explosion algorithm will provide an intermediate step for the proof of Theorem 1.1. Proposition 5.1 establishes that the existence of a continuation is equivalent to the existence of a nonnegative integral solution of the h^{cd}_{κ}-system. Then in Proposition 5.2 it is shown that the h^{cd}_{κ}-system admits a nonnegative integral solution if and only if so do the Poincaré-Hopf inequalities.

Proposition 5.1 *A saddle vertex v labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$ can be exploded to a Lyapunov semi-graph of Morse type with cycle rank greater than or equal to κ, and $\kappa \leq \min\{h_1 - (e^- - 1), h_{n-1} - (e^+ - 1)\}$, if and only if the appropriate h^{cd}_{κ}-system,* (11) *or* (12)*, has a nonnegative integral solution $(h^c_1, h^d_1, \ldots, h^c_{n-1}, h^d_{n-1})$.*

Proof: This follows directly from the fact that the steps in the saddle explosion algorithm are described by the h^{cd}_{κ}-systems. ■

In order to use results already established in [1], we reproduce in (13), (14), the generic forms of the linear system considered therein:

$$n = 2i+1 \left\{ \begin{array}{l} h^c_1 = b_0 \\ \{h^c_j + h^d_j = b_j, \quad j = 1, \ldots, 2i \\ h^d_{2i} = b_{2i+1} \\ \left\{ \begin{array}{l} h^d_1 - h^c_2 - h^c_{2i} + h^d_{2i-1} = \delta_1 \\ h^d_2 - h^c_3 - h^c_{2i-1} + h^d_{2i-2} = \delta_2 \\ \vdots \\ h^d_i - h^c_{i+1} = \delta_i \end{array} \right. \end{array} \right. \tag{13}$$

or

$$n = 2i \left\{ \begin{array}{l} h^c_1 = b_0 \\ \{h^c_j + h^d_j + \beta^i = b_j, \quad j = 1, \ldots, 2i-1, \;\; \beta^i = 0 \text{ if } j \neq i \text{ and } 2i \neq 0 \mod 4 \\ h^d_{2i-1} = b_{2i} \\ \left\{ \begin{array}{l} h^d_1 - h^c_2 - h^c_{2i-1} + h^d_{2i-2} = \delta_1 \\ h^d_2 - h^c_3 - h^c_{2i-2} + h^d_{2i-3} = \delta_2 \\ \vdots \\ h^d_{i-1} - h^c_i - h^c_{i+1} + h^d_i = \delta_{i-1} \end{array} \right. \end{array} \right. \tag{14}$$

Systems (11) and (12) are special cases of (13) and (14), respectively, which may be obtained with the following substitutions:

$$n = 2i+1 \quad \begin{cases} b_0 &= -1 + e^- + \kappa \\ b_j &= h_j, \quad \text{for } j = 1, \ldots, 2i \\ b_{2i+1} &= -1 + e^+ + \kappa \\ \delta_j &= B_j^+ - B_j^-, \quad \text{for } j = 1, \ldots, i-1 \\ \delta_i &= (B_i^+ - B_i^-)/2 \end{cases} \tag{15}$$

$$n = 2i \quad \begin{cases} b_0 &= -1 + e^- + \kappa \\ b_j &= h_j, \quad \text{for } j = 1, \ldots, 2i-1 \\ b_{2i} &= -1 + e^+ + \kappa \\ \delta_j &= B_j^+ - B_j^-, \quad \text{for } j = 1, \ldots, i-1. \end{cases} \tag{16}$$

It was shown in [1] that system (13) may be recast as a network-flow problem by a suitable change of sign of half of the equations. The h^{cd} variables are interpreted as flows on the arcs of the network and each equation may be read as "flow in − flow out = node constant". The networks corresponding to the case $n = 2i+1$ are depicted in Figures 8 and 9 , with the node constant shown inside the node. In the planar embedding adopted in this picture, the zig-zag shape of the digraph component of the network resembles the lateral structure of a clotheshorse. Arcs corresponding to flow variables $(h_1^c, h_1^d, h_2^c, h_2^d, \ldots, h_{2i}^c, h_{2i}^d)$, in this order, form an Eulerian nonoriented path covering the whole digraph.

When $n = 2i+1$ and i is even, the network is slightly different, see the rightmost cycle in the network of Figure 9. Both networks contain a chain of $i-1$ cycles of length four and the arc sequence associated with $(h_1^c, h_1^d, h_{2i}^c, h_{2i}^d)$ forms a nonoriented path that is adjacent to the first cycle. The arcs in the j-th cycle are associated with variables h_{j+1}^d, h_{2i-j}^c, h_{2i-j}^d and h_{j+1}^c, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle.

The set of equations (14) may be recast as a network-flow problem with additional restrictions. Figure 10 gives the "network" for the even case $n = 0 \mod 4$. Quotes around the word network for the even cases are needed on account of the flows corresponding to the downward dangling arcs on the right, which must be equal. If we fix the value of h_i^d we obtain a regular network-flow.

Figure 11 depicts the "network" corresponding to the case $n = 2 \mod 4$. Again the problem is not a pure network-flow problem, although it is close enough to a network-flow problem to facilitate the necessary adaptations of the general theory and methods to this special case. Therefore, to simplify the discussion, we will refer to all problems as network-flow problems, calling attention to

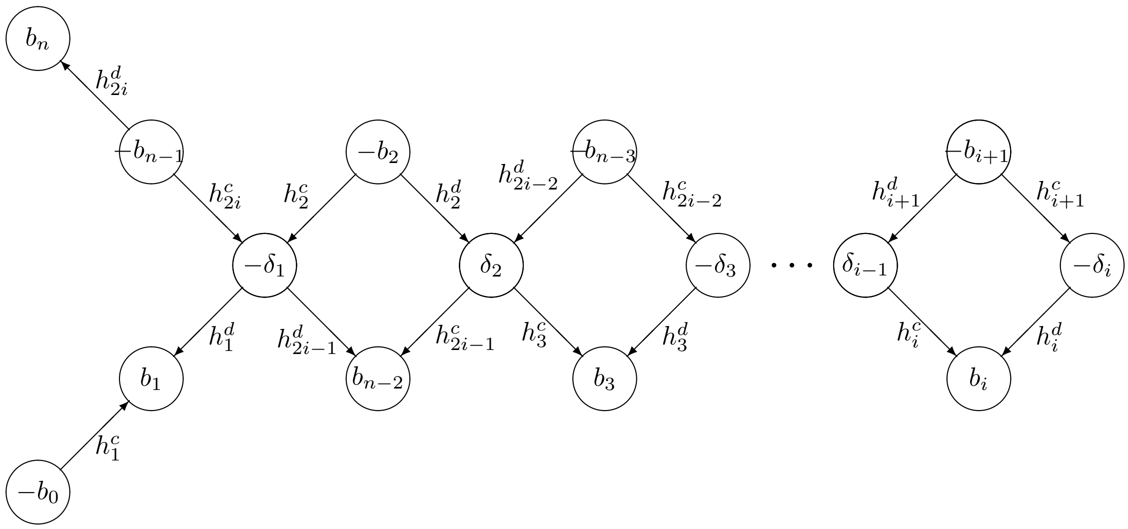


Figure 8: Network for the case $n = 2i + 1$, i odd.

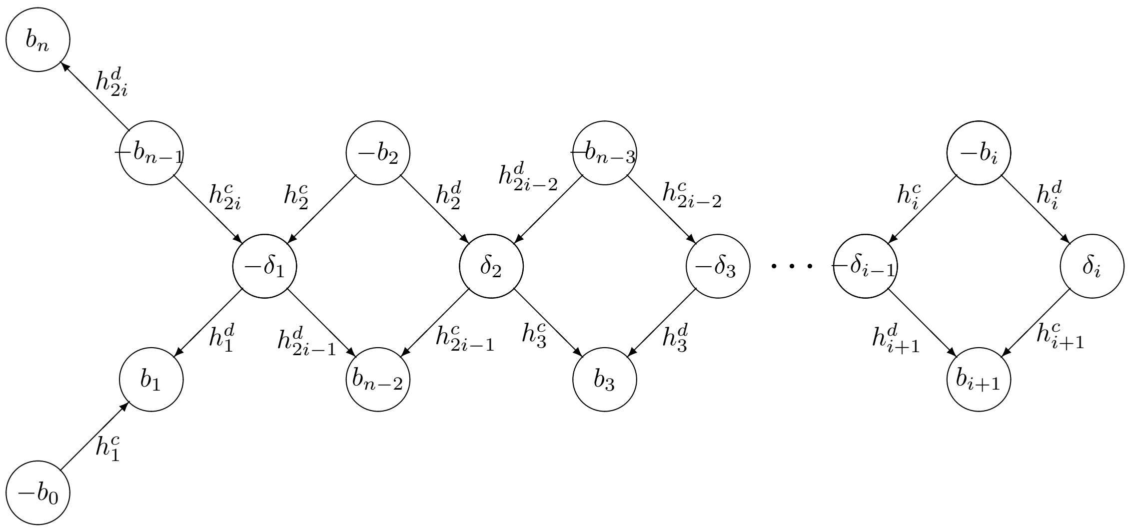


Figure 9: Network for $n = 2i + 1$, i even.

the special considerations the even cases require when need arises.

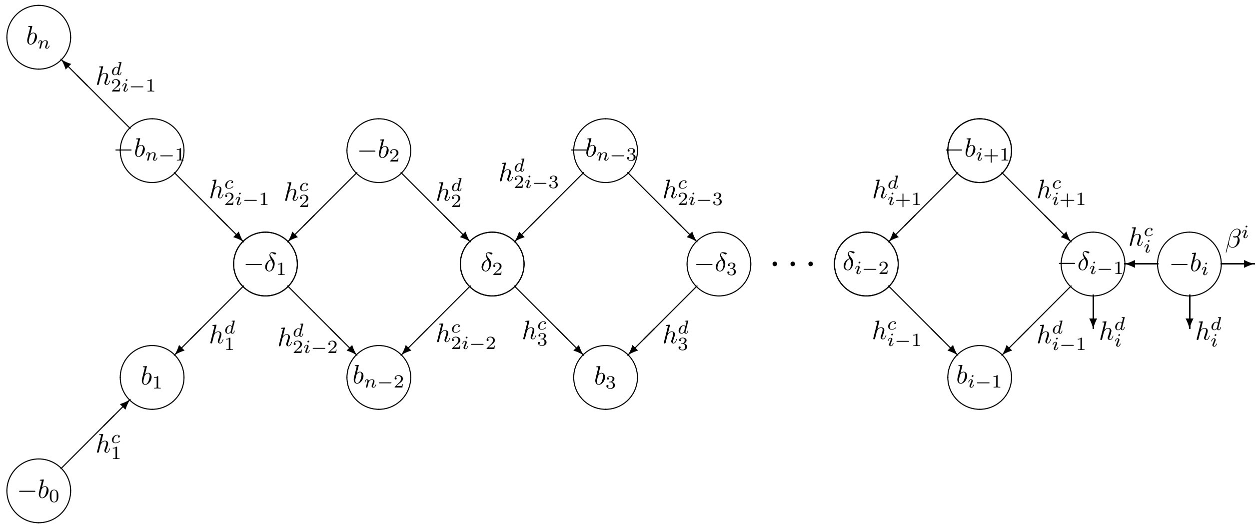


Figure 10: Network for the case $n = 2i = 0 \mod 4$.

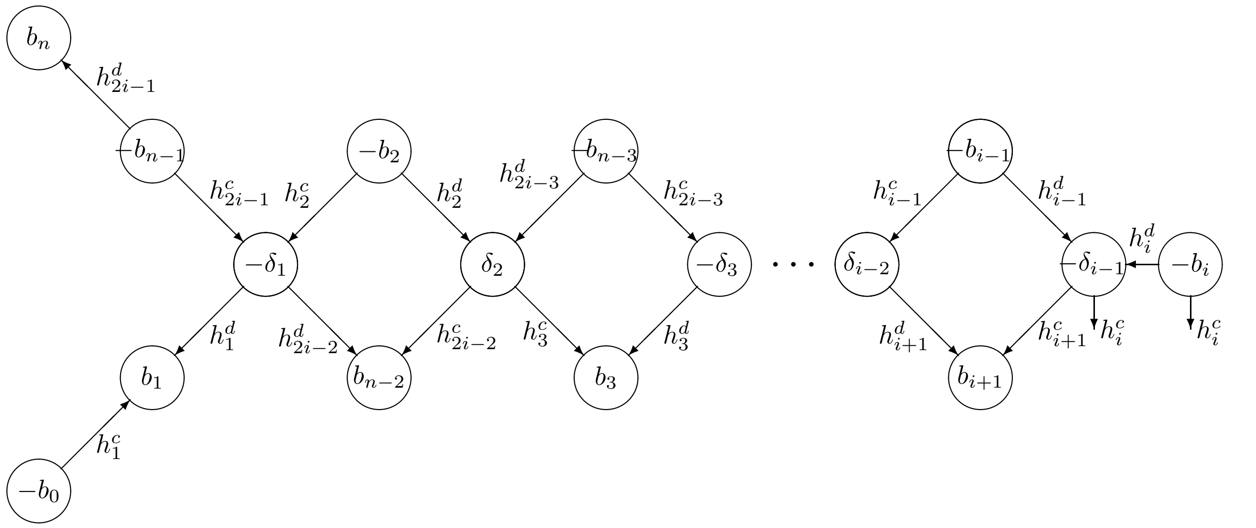


Figure 11: Network for the case $n = 2i = 2 \mod 4$.

The general solution of system (13) (or (14)) is the sum of a particular solution and a solution of the homogeneous version of the system, that is, a solution that satisfies the condition "flow in = flow out" at every node. The former one is called a *flow* and, the latter one, a *circulation.* Numerical examples are given in Subsection 7.1, see Figure 19. Thus the set of circulations is a linear subspace, the null space of the coefficient matrix of the linear system. Rockafellar [13] showed that a vector of a linear space is the sum of elementary vectors of this space. An *elementary vector* of a subspace is a vector of this subspace with minimal support. It is well known that the elementary circulations are those whose supports correspond to simple cycles of the network.

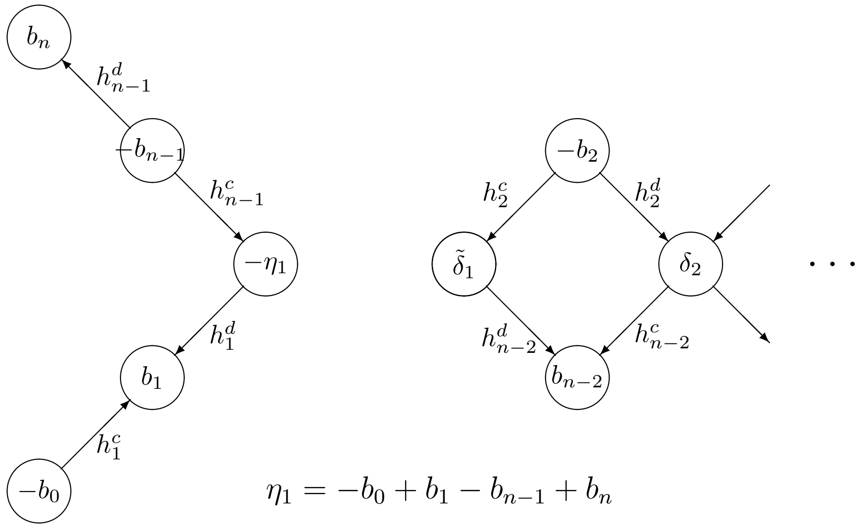
Therefore the elementary solutions of (13) and (14) are easy enough to determine. In the odd and even cases there is an elementary circulation associated with each cycle. In the even cases there is an additional elementary circulation, associated with the rightmost flows of the networks depicted in Figures 10 and 11. In the case $n = 2i = 0 \mod 4$ (resp., $n = 2i = 2 \mod 4$), this additional circulation has $(h_i^c, \beta^i, h_i^d) = (1, -2, 1)$ (resp., $(h_i^c, h_i^d) = (1, 1)$) and remaining components equal to zero. Finally, if we remove one arc from each cycle (and set h_i^d (resp., h_i^c) to zero in the $n = 2i = 0 \mod 4$ (resp., $n = 2i = 2 \mod 4$) case), the remaining arcs form a tree. The columns of the coefficient matrix of the h_κ^{cd}-system associated with the remaining arcs (unknowns) are linearly independent and thus this subsystem has a unique solution, if it has a solution (this submatrix has more rows than columns). A solution of a linear system whose support correspond to linearly independent columns of the coefficient matrix of the system is called a *basic solution*. Geometrically, the nonnegative basic solutions correspond to extreme points, or vertices, of the polyhedron defined by the flow balance equations and nonnegativity constraints. Basic (tree) solutions of system (13) (or (14)) are easy to calculate: start at the leaves of the tree (nodes with only one incident arc) and work your way in. In algebraic parlance, this is equivalent to permutating the rows and columns of the associated submatrix to make it lower triangular and then solve the corresponding system by back substitution.

Proposition 5.2 *The h_κ^{cd}-systems* (11) *and* (12) *have nonnegative integral solutions if and only if the Poincaré-Hopf inequalities* (2), (3) *and* (4), *for isolating blocks are satisfied. Moreover, the set of all solutions to the h_κ^{cd}-system may be obtained from a single basic solution and the elementary circulations of the network.*

Proof: The trick to show the equivalence between the network-flow problems and the Poincaré-Hop inequalities in [1] was to split the network-flow problem into a set of i, in the odd n case, or $i+1$, in the other case, independent smaller network-flow problems. The first step taken was to split the problem in two. This was done by eliminating from the system the variables h_1^c, h_1^d, h_{n-1}^c and h_{n-1}^d, whose values are easily determined from the system. In network terms, this is equivalent to splitting the network in two, as depicted in Figure 12.

Four variables in (13), or (14), are uniquely determined:

$$\left\{\begin{array}{rcl} -h_1^c & = & -b_0 \\ h_1^c + h_1^d & = & b_1 \\ -h_{n-1}^c - h_{n-1}^d & = & -b_{n-1} \\ h_{n-1}^d & = & b_n \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{rcl} h_1^c & = & b_0 \\ h_1^d & = & b_1 - b_0 \\ h_{n-1}^d & = & b_n \\ h_{n-1}^c & = & b_{n-1} - b_n \end{array}\right. \tag{17}$$



$$\eta_1 = -b_0 + b_1 - b_{n-1} + b_n$$
$$\tilde{\delta}_1 = -\delta_1 + \eta_1$$

Figure 12: Splitting the problem in two.

The network on the left of Figure 12 contains the four equations from (13), or (14), shown above, plus the redundant (by definition of η_1) equation $h^c_{n-1} - h^d_1 = -\eta_1$. Substituting the values determined above in the only equation of (13), or (14), that contains some of the four variables h^c_1, h^d_1, h^c_{n-1} and h^d_{n-1}, we obtain

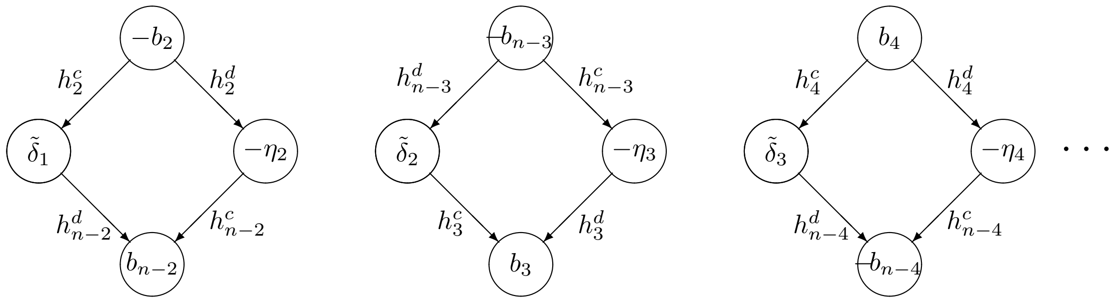
$$\begin{aligned} h^d_1 - h^c_2 - h^c_{n-1} + h^d_{n-2} &= \delta_1 \\ &\Downarrow \\ h^c_2 - h^d_{n-2} &= -\delta_1 + \eta_1, \end{aligned}$$

which is precisely the equation associated with the leftmost node in the network on the right in Figure 12. Since the rest of the network is a copy of the original network, all the remaining equations are dutifully represented therein. Therefore the original network-flow problem was successfully split into two independent network-flow problems.

The same splitting can be done at each of the nodes in the intermediate row of the network, the nodes with degree 4. Figure 13 gives the next three splittings. Notice that $\tilde{\delta}_j$ gives the cumulative sum of node constants for node with constant $\pm\delta_j$ in the original network and all nodes on its left. That is,

$$\tilde{\delta}_j = \sum_{k=0}^{j}((-1)^{k+1}b_k + (-1)^k b_{n-k}) + \sum_{k=1}^{j}(-1)^k\delta_k, \quad \text{for } j = 1, \dots, i-1. \tag{18}$$

Thus all node constants of the independent networks produced with the splittings may be expressed in terms of the originals constants.



$\eta_1 = -b_0 + b_1 - b_{n-1} + b_n$

$\tilde{\delta}_1 = -\delta_1 + \eta_1$ $\qquad$ $\tilde{\delta}_2 = \delta_2 + \eta_2$ $\qquad$ $\tilde{\delta}_3 = -\delta_3 + \eta_3$

$\eta_2 = \tilde{\delta}_1 - b_2 + b_{n-2}$ $\qquad$ $\eta_3 = \tilde{\delta}_2 + b_3 - b_{n-2}$ $\qquad$ $\eta_4 = \tilde{\delta}_3 + b_4 - b_{n-4}$

Figure 13: Further splitting.

Figure 14 gives the last splitting for all cases. Notice that the splitting of networks corresponding to the case $n = 2i+1$ produce only two types of networks: the path-network on the left of Figure 12 and the cycle-network, a typical instance of which is depicted in Figure 15, whose digraph component is a 4-length nonoriented cycle. On the other hand, the splitting of networks corresponding to the case $n = 2i$ produce three types of networks, two of them coincide with the ones obtained in the former case, but the last is not altogether a network since it contains the troublesome downward dangling arcs, see Figure 14.

The problem of determining whether (13) (or (14)) has a nonnegative integral solution reduces to asserting whether each of the independent linear systems produced with the splitting has one. This task is now straightforward, given the simple structure of the independent linear systems.

The first linear system, corresponding to the path-network, is the easiest, since it has a unique solution. Taking into account that b and δ are nonnegative integral vectors, the unique solution, given in (17), is trivially integral. Thus the solution will be nonnegative and integral if and only if the following inequalities are satisfied:

$$\begin{aligned} b_0 &\geq 0 \\ b_1 &\geq b_0 \\ b_n &\geq 0 \\ b_{n-1} &\geq b_n. \end{aligned} \tag{19}$$

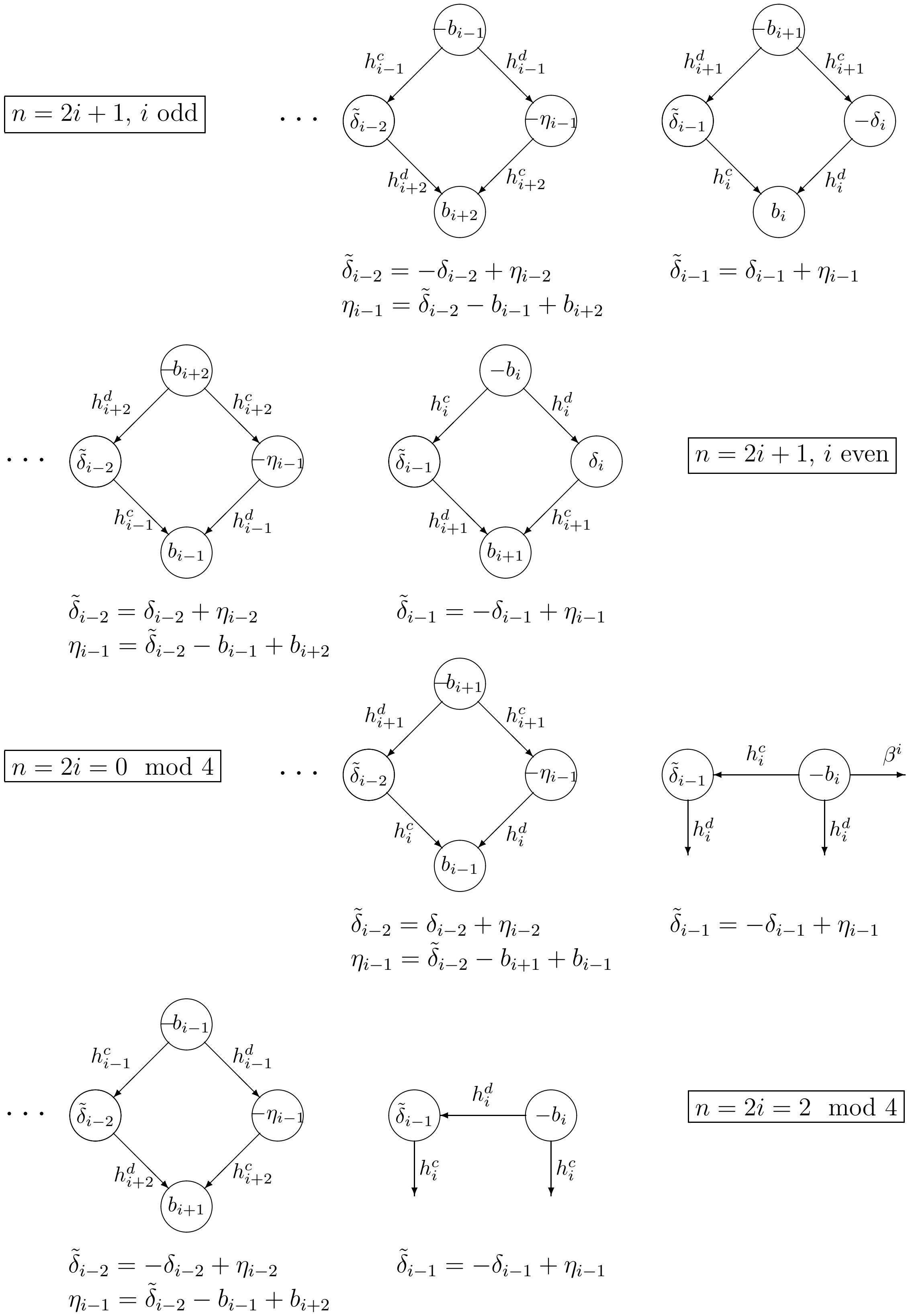


Figure 14: Last independent problems.

It was shown in [1] that the linear system corresponding to the cycle-network of Figure 15 has a nonnegative integral solution if and only if the following conditions hold:

$$-\theta + \alpha + d + \xi = 0 \tag{20}$$

$$\begin{aligned} d &\geq -\alpha \\ \theta &\geq \alpha \end{aligned} \tag{21}$$

Furthermore, the general solution (particular solution plus multiple of circulation) to this linear system is $(x, y, z, w) = (\alpha, \theta - \alpha, d, 0) + m(1, -1, -1, 1)$. Notice that (20) is satisfied by construction for all networks produced by the splitting, except the last one obtained in the decomposition of the original network for $n = 2i + 1$. Thus for $n = 2i + 1$, one condition in the set of necessary and sufficient conditions for the existence of nonnegative integral solutions of (13) is

$$\begin{aligned} 0 &= \tilde{\delta}_{i-1} + (-1)^{i+1} b_i + (-1)^i b_{i+1} + (-1)^i \delta_i \\ &= \sum_{k=0}^{i-1} ((-1)^{k+1} b_k + (-1)^k b_{n-k}) + \sum_{k=1}^{i-1} (-1)^k \delta_k + (-1)^{i+1} b_i + (-1)^i b_{i+1} + (-1)^i \delta_i \\ &= \sum_{k=0}^{i} ((-1)^{k+1} b_k + (-1)^k b_{n-k}) + \sum_{k=1}^{i} (-1)^k \delta_k. \end{aligned} \tag{22}$$

Substituting the values of b and δ given by (15) in equation (22) we obtain the Poincaré equality (3), which must be satisfied in the $n = 2i + 1$ case.

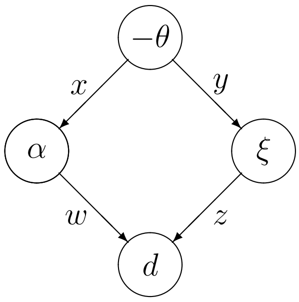


Figure 15: Generic instance of cycle-network.

If $n = 2i = 0 \mod 4$, then it was shown in [1] that the last linear system, which involves h_i^c, h_i^d and β^i (see Figure 14), as a nonnegative integral solution if and only if the following inequalities hold:

$$\begin{aligned} b_i &\geq -\tilde{\delta}_{i-1} \\ b_i &\geq \tilde{\delta}_{i-1} \end{aligned} \tag{23}$$

The general solution to this system is $(h_i^c, \beta^i, h_i^d) = (\tilde{\delta}_{i-1}, b_i - \tilde{\delta}_{i-1}, 0) + m(1, -2, 1)$.

In the case $n = 2i = 2 \mod 4$, the last system has the unique solution $(h_i^c, h_i^d) = ((b_i - \tilde{\delta}_{i-1})/2, (b_i + \tilde{\delta}_{i-1})/2)$, which is nonnegative and integral if and only if

$$b_i - \tilde{\delta}_{i-1} \quad \text{is} \quad \text{even} \tag{24}$$

$$\begin{aligned} b_i &\geq -\tilde{\delta}_{i-1} \\ b_i &\geq \tilde{\delta}_{i-1}. \end{aligned} \tag{25}$$

Notice that we have the means to construct solutions to all subsystems, and hence, to the original system. We simply need to collect all particular solutions in one vector and add multiples of the various elementary circulations, one for each cycle.

Finally, substituting (15) (or (16) in the appropriate conditions, from the ones obtained, namely (19), (19), (21), (20), (23), (24) and (25), we conclude that the h_κ^{cd}-system has a solution if and only if the Poincaré-Hopf inequalities (2)–(4) hold. ■

Proof of Theorem 1.1.

Given a vertex v labelled with $(h_0, h_1, \ldots, h_n, \kappa)$ we can perform a partial explosion to a vertex v of saddle type labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$ since the last two inequalities in (2) are satisfied. It is easy to see that the vertex v continues to satisfy the Poincaré-Hopf inequalities. By Proposition 5.1 v can be exploded to semi-graphs of Morse type, if and only if the h_κ^{cd}-system has nonnegative integer solution. By Proposition 5.2 the h_κ^{cd}-system has nonnegative integer solution if and only if the Poincaré-Hopf inequalities are satisfied. Hence, L_B admits continuations to abstracts Lyapunov semi-graphs of Morse type if and only if the Poincaré-Hopf inequalities are satisfied.

Also, note that the Poincaré-Hopf inequalities assert that

$$\kappa \leq h_1 - h_0 + 1 - e^-$$

$$\kappa \leq h_{n-1} - h_n + 1 - e^+$$

Since $\kappa = \kappa_v + \kappa_L$ and in this case $\kappa = \kappa_v$ and $\kappa_L = 0$ we have that

$$\kappa_v \leq \min\{h_1 - (e^- + h_0 - 1), \quad h_{n-1} - (e^+ + h_n - 1)\}.$$

The number of continuations, i.e., the number of nonnegative integral flows, has been calculated in [1] for $\kappa = 0$. From Proposition 5.2, this number is the number of admissible multiples of elementary circulations of the network. Since the values of h_1^c, h_1^d, h_{2i}^c and h_{2i}^d are uniquely determined, this is the number of nonnegative integral flows of the smaller network obtained after the elimination of these four variables from the system of equations (13) or (14). It is straightforward to verify that this elimination will result in the same subnetwork, regardless of the value of κ. Thus the total number of continuations is just the number of possible values of κ $(1+\min\{h_1-(e^-+h_0-1),\ \ h_{n-1}-(e^++h_n-1)\})$ times the number of continuations for $\kappa=0$.■

Proof of Theorem 1.2:

By the definition of continuation we have that $L(h_0,\ldots,h_n,\kappa)$ can be continued to an abstract Lyapunov graph of Morse type if each vertex can be exploded. Since, by hypothesis $L(h_0,\ldots,h_n,\kappa)$ satisfies the Poincaré-Hopf inequalities at each vertex then by Theorem 1.1 each vertex admits continuations to abstract Lyapunov semi-graphs of Morse type. Hence, $L(h_0,\ldots,h_n,\kappa)$ can be continued to an abstract Lyapunov graph of Morse type.

Moreover, Theorem 1.1 asserts that $\kappa_v\le\min\{h_1-(e^-+h_0-1),\ \ h_{n-1}-(e^++h_n-1)\}$.

Let V be the vertex set of L and $\mathcal{V}$ its cardinality. Similarly, let E be the edge set of L and $\mathcal{E}$ its cardinality. By definition $\kappa=\kappa_V+\kappa_L$ where $\kappa_V=\sum_{v_k\in V}\kappa(v_k)$ and κ_L is the cycle rank of L. Recall that $\kappa_L-1=\mathcal{E}-\mathcal{V}$. Hence,

$$\begin{aligned}\kappa_V &\le \textstyle\sum_{v_k\in V}\min\{h_1(v_k)-(e^-(v_k)+h_0(v_k)-1),\ \ h_{n-1}(v_k)-(e^+(v_k)+h_n(v_k)-1)\}\\ &\le \min\{h_1-h_0-\mathcal{E}+\mathcal{V},h_{n-1}-h_n-\mathcal{E}+\mathcal{V}\}\\ &= \min\{h_1-h_0-\kappa_L+1,h_{n-1}-h_n-\kappa_L+1\}\\ &= \min\{h_1-h_0+1,h_{n-1}-h_n+1\}-\kappa_L\end{aligned}$$

where $h_j=\sum_{v_k\in V}h_j(v_k)$ and $\sum_{v_k\in V}(1-e^-(v_k))=\mathcal{V}-\mathcal{E}$.

Therefore, $\kappa=\kappa_V+\kappa_L\le\min\{h_1-h_0+1,h_{n-1}-h_n+1\}$.

The number of continuations is merely the product of the number of continuations at each vertex as in Theorem 1.1. ■

Given $L(h_0,\ldots,h_n,\kappa)$ satisfying the Poincaré-Hopf inequalities at each vertex (2)–(4), the set of all possible continuations determines the family $\mathcal{L}(h_0,\ldots,h_n,\kappa)=\cup_{j\ge\kappa}L(h_0,\ldots,h_n,j)$ of Lyapunov graphs $L(h_0,\ldots,h_n,j)$, $\kappa\le j\le\min\{h_1-(h_0-1),\,h_{n-1}-(h_n-1)\}$.

6 Morse Inequality Results

In this section we will prove Theorem 1.3.

As was shown in the previous section every abstract Lyapunov graph $L(h_0,\ldots,h_n,\kappa)$ that satisfies the Poincaré-Hopf inequalities at each vertex (2)–(4) can be continued to some graph in the family $\mathcal{L}(h_0,\ldots,h_n,\kappa)$ of Lyapunov graphs $L(h_0,\ldots,h_n,j)$, $\kappa \leq j \leq \min\{h_1-(h_0-1),$ $h_{n-1}-(h_n-1)\}$.

Also, by setting $B_j^- = B_j^+$, $e^- = h_0$ and $e^+ = h_n$ we obtain the particularized versions (26) and (27) of systems (11) and (12). Hence, Corollary 6.1 follows directly from Proposition 5.2.

Corollary 6.1 *The systems below have a nonnegative integral solution $(h_1^c\, h_1^d,\ldots,h_{n-1}^c\, h_{n-1}^d)$ if and only if the Poincaré-Hopf inequalities for closed manifolds* (5)–(7) *are satisfied.*

$$n=2i+1\left\{\begin{array}{l} h_0-1-h_1^c+\kappa=0\\ \{h_j=h_j^c+h_j^d,\ \ j=1,\ldots,2i\\ h_n-1-h_{2i}^d+\kappa=0\\ \left\{\begin{array}{l} h_1^d-h_2^c-h_{2i}^c+h_{2i-1}^d=0\\ h_2^d-h_3^c-h_{2i-1}^c+h_{2i-2}^d=0\\ \vdots\\ h_i^d-h_{i+1}^c=0\end{array}\right.\end{array}\right. \tag{26}$$

$$n=2i\left\{\begin{array}{l} h_0-1-h_1^c+\kappa=0\\ \{h_j=h_j^c+h_j^d+\beta^i,\ \ j=1,\ldots,2i-1,\ \beta^i=0 \textit{ if } j\neq i \textit{ and } 2i\neq 0 \mod 4\\ h_n-1-h_{2i-1}^d+\kappa=0\\ \left\{\begin{array}{l} h_1^d-h_2^c-h_{2i-1}^c+h_{2i-2}^d=0\\ h_2^d-h_3^c-h_{2i-2}^c+h_{2i-3}^d=0\\ \vdots\\ h_{i-1}^d-h_i^c-h_{i+1}^c+h_i^d=0\end{array}\right.\end{array}\right. \tag{27}$$

Hence, it follows that:

Corollary 6.2 *Given an abstract Lyapunov graph $L(h_0,\ldots,h_n,\kappa)$, it admits continuations to abstract Lyapunov graphs of Morse type with cycle rank greater or equal to κ if and only if it satisfies the Poincaré-Hopf inequalities* (5)–(7)*, where $\kappa \leq \min\{h_1-(h_0-1), h_{n-1}-(h_n-1)\}$.*

We will refer systems (26), (27) above as reduced h_κ^{cd}-systems.

It is worthwhile to stress that the constructive proof of the equivalence between the existence of nonnegative integral solutions to the reduced h_κ^{cd}-systems and the feasibility of the Poincaré-Hopf inequalities (5)–(7) also provided the means to construct all solutions of the reduced h_κ^{cd}-systems, a fact which will be explored in Section 7.

The proof that the Poincaré-Hopf inequalities imply the Morse inequalities is also constructive. We will provide formulas to produce a Betti number vector $(\gamma_0, \ldots, \gamma_n)$ that satisfies the Morse inequalities from a certain basic solution of the (appropriate) reduced h^{cd}-system.

First we consider the case $n = 2i + 1$. Let h^{cd} be the complementary solution of the reduced h^{cd}_κ-system (26)

$$h^c_j h^d_{2i+1-j} = 0, \quad \text{for } j = 2, \ldots, i. \tag{28}$$

This follows from the fact that if $(\hat{x}, \hat{y}, \hat{z}, \hat{w})$ is a nonnegative integral solution of the cycle network-flow problem depicted in Figure 15, then $(x', y', z', w') = (\hat{x}, \hat{y}, \hat{z}, \hat{w}) - \min\{x, w\}(-1, 1, 1, -1)$ is yet another nonnegative integral solution, satisfying $x' \ w' = 0$. Thus, given an arbitrary nonnegative integral solution h^{cd} we may transform it into a nonnegative integral solution satisfying (28) by summing appropriate multiples of elementary circulations. Observe that this new solution is a basic solution.

Now suppose $n = 2i = 0 \mod 4$. Without loss of generality we may consider a nonnegative integral solution of the reduced h^{cd}_κ-system (27) that satisfies (28). The argument for $2 \le j \le i - 1$ is the same as before. Now if $(\hat{h}^c_i, \hat{\beta}^i, \hat{h}^d_i)$ is a nonnegative integral solution of the last independent problem depicted in Figure 14, then $(\tilde{h}^c_i, \tilde{\beta}^i, \tilde{h}^d_i) = (\hat{h}^c_i, \hat{\beta}^i, \hat{h}^d_i) + \min\{\hat{h}^c_i, \hat{h}^d_i\}(-1, 2, -1)$ is another solution of this subproblem that satisfies (28).

In order to tackle the case $n = 2i = 2 \mod 4$, we introduce the auxiliary system below.

$$n = 2i = 2 \mod 4 \begin{cases} h_0 - 1 - h^c_1 + \kappa = 0 \\ \{h_j = h^c_j + h^d_j + \delta, \ \ j = 1, \ldots, 2i-1, \ \delta = 0 \text{ if } j \neq i \\ h_n - 1 - h^d_{2i-1} + \kappa = 0 \\ \begin{cases} h^d_1 - h^c_2 - h^c_{2i-1} + h^d_{2i-2} = 0 \\ h^d_2 - h^c_3 - h^c_{2i-2} + h^d_{2i-3} = 0 \\ \vdots \\ h^d_{i-1} - h^c_i - h^c_{i+1} + h^d_i = 0 \end{cases} \end{cases} \tag{29}$$

If (27) has a nonnegative integral solution $(h^c_1, h^d_1, \ldots, h^c_i, h^d_i, \ldots, h^c_{2i}, h^d_{2i})$, then (29) has the solution $(h^c_1, h^d_1, \ldots, h^c_i, 0, h^d_i, \ldots, h^c_{2i}, h^d_{2i})$ (simply let $\delta = 0$). The advantage of introducing (29) is that now we have room to transform a given nonnegative integral solution $(h^c_1, h^d_1, \ldots, h^c_i, 0, h^d_i, \ldots, h^c_{2i}, h^d_{2i})$ of (29) into one that satisfies (28) much in the same way as we did in the case $n = 2i = 0 \mod 4$.

Given these basic solutions it can be shown that the formulas below define Betti number vectors

$(\gamma_0, \ldots, \gamma_n)$ that satisfy the Morse inequalities.

$$\boldsymbol{n = 2i+1} \qquad \begin{aligned} &\gamma_0 = \gamma_{2i+1} = 1 \\ &\gamma_j = \begin{cases} h_1^d - h_2^c + \kappa, & \text{if } j = 1 \\ h_j^d - h_{j+1}^c, & \text{if } 2 \le j < i \\ h_i^d, & \text{if } j = i \\ h_{i+1}^c, & \text{if } j = i+1 \\ -h_{j-1}^d + h_j^c, & \text{if } i+2 \le j \le 2i-1 \\ -h_{2i-1}^d + h_{2i}^c + \kappa, & \text{if } j = 2i \end{cases} \end{aligned} \tag{30}$$

$$\boldsymbol{n = 2i} \qquad \begin{aligned} &\gamma_0 = \gamma_{2i} = 1 \\ &\gamma_j = \begin{cases} h_1^d - h_2^c + \kappa, & \text{if } j = 1 \\ h_j^d - h_{j+1}^c, & \text{if } 2 \le j \le i-1 \\ \beta^i, & \text{if } j = i \text{ and } 2i = 0 \mod 4 \\ \delta, & \text{if } j = i \text{ and } 2i = 2 \mod 4 \\ -h_{j-1}^d + h_j^c, & \text{if } i+1 \le j \le 2i-2 \\ -h_{2i-2}^d + h_{2i-1}^c + \kappa, & \text{if } j = 2i-1 \end{cases} \end{aligned} \tag{31}$$

The only role of the complementarity condition (28) is to ensure nonnegativity of the γ defined above. Other solutions of the reduced h_κ^{cd}-systems lead to Betti number vectors that satisfy the Morse inequalities and duality conditions but may have negative components. These Betti number vectors will not be considered in this context.

Given Corollary 6.1, the sufficiency part of Theorem 1.3 is established by the following proposition, which was proven in [2] for $\kappa = 0$.

Proposition 6.1 *If the reduced h_κ^{cd}-system has a nonnegative integral solution, then there exists a nonnegative integral Betti number vector $(\gamma_0, \ldots, \gamma_n)$ that satisfies the Morse inequalities* (1) *and the inequality $\gamma_1 \ge \kappa$.*

Proof:

Case $\boldsymbol{n = 2i+1}$.

By definition (30), γ is clearly integral. Equations $h_j^d - h_{j+1}^c - h_{2i+1-j}^c + h_{2i-j}^d = 0$ in (26), for $j = 1, \ldots, i-1$, imply $\gamma_j = \gamma_{2i+1-j}$, for $j = 1 \ldots, i-1, i+2, \ldots, 2i$. Likewise, equation $h_i^d - h_{i+1}^c = 0$ in (26) implies $\gamma_i = \gamma_{i+1}$. Therefore the vector γ given by (30) satisfies conditions $\gamma_j = \gamma_{2i+1-j}$, for $j = 0, \ldots, 2i+1$.

In order to show that $\gamma \geq 0$ it suffices to show that γ_j or γ_{2i+1-j}, for $1 \leq j \leq i-1$, is nonnegative, since $\gamma_j = \gamma_{2i+1-j}$, for $j = 0, \ldots, 2i+1$, $\gamma_i = \gamma_{i+1} = h^d_i \geq 0$ and $\gamma_0 = \gamma_{2i+1} = 1$. The definition of γ and (28) imply this, since

$$h^c_{j+1} h^d_{2i-j} = 0 \quad \Rightarrow \quad \begin{cases} h^c_{j+1} = 0 \Rightarrow \gamma_{2i+1-j} = \gamma_j = h^d_j - h^c_{j+1} = h^d_j \geq 0 \\ \text{or} \\ h^d_{2i-j} = 0 \Rightarrow \gamma_j = \gamma_{2i+1-j} = -h^d_{2i-j} + h^c_{2i+1-j} = h^c_{2i+1-j} \geq 0 \end{cases} \qquad \text{for } 2 \leq j \leq i-1$$

and

$$h^c_2 h^d_{2i-1} = 0 \quad \Rightarrow \quad \begin{cases} h^c_2 = 0 \Rightarrow \gamma_{2i} = \gamma_1 = h^d_1 - h^c_2 + \kappa = h^d_1 + \kappa \geq \kappa \geq 0 \\ \text{or} \\ h^d_{2i-1} = 0 \Rightarrow \gamma_1 = \gamma_{2i} = -h^d_{2i-1} + h^c_{2i} + \kappa = h^c_{2i} + \kappa \geq \kappa \geq 0 \end{cases} \qquad \text{for } 2 \leq j \leq i-1.$$

Given that $h_j = h^c_j + h^d_j$, $j = 1, \ldots, 2i$, we have that:

$$\begin{aligned}
\sum_{j=0}^{2i+1} (-1)^{j+1} h_j &= -h_0 + h^c_1 + \sum_{j=1}^{2i-1} (-1)^{j+1} (h^d_j - h^c_{j+1}) - h^d_{2i} + h_{2i+1} \\
&= -1 + \kappa + \sum_{j=1}^{2i-1} (-1)^{j+1} (h^d_j - h^c_{j+1}) - \kappa + 1 \\
&= -\gamma_0 + h^d_1 - h^c_2 + \kappa + \sum_{j=2}^{i-1} (-1)^{j+1} (h^d_j - h^c_{j+1}) \\
&\quad + (-1)^{i+1} h^d_i - (-1)^{i+1} h^c_{i+1} \\
&\quad + \sum_{j=i+1}^{2i-2} (-1)^{j+1} (h^d_j - h^c_{j+1}) + h^d_{2i-1} - h^c_{2i} - \kappa + \gamma_{2i+1} \\
&= -\gamma_0 + \gamma_1 + \sum_{j=2}^{i-1} (-1)^{j+1} \gamma_j + (-1)^{i+1} \gamma_i + (-1)^{i+2} \gamma_{i+1} \\
&\quad + \sum_{j=i+1}^{2i-2} (-1)^{j+1} (-\gamma_{j+1}) + (-1)^{2i+1} \gamma_{2i} + \gamma_{2i+1} \\
&= \sum_{j=0}^{2i+1} (-1)^{j+1} \gamma_j.
\end{aligned}$$

Hence we have proved the top Morse equality.

Similarly, it can be shown that

$$\begin{aligned}\sum_{j=0}^{\ell}(-1)^{j+1}h_j &= -h_0+h_1^c+\sum_{j=1}^{\ell-1}(-1)^{j+1}(h_j^d-h_{j+1}^c)+(-1)^{\ell+1}h_\ell^d\\ &= -h_0+h_1^c+\sum_{j=1}^{\ell}(-1)^{j+1}(h_j^d-h_{j+1}^c)+(-1)^{\ell+1}h_{\ell+1}^c.\end{aligned}$$

The above equations imply that

$$\sum_{j=0}^{\ell}(-1)^{j+1}h_j=\begin{cases}\sum_{j=0}^{\ell}(-1)^{j+1}\gamma_j+(-1)^{\ell+1}h_\ell^d, & \text{if } i+1\le\ell\le 2i\\ \sum_{j=0}^{i}(-1)^{j+1}\gamma_j, & \text{if } \ell=i\\ \sum_{j=0}^{\ell}(-1)^{j+1}\gamma_j+(-1)^{\ell+1}h_{\ell+1}^c, & \text{if } 1\le\ell\le i-1.\end{cases} \tag{32}$$

Thus, invoking the nonnegativity of $(h_1^c,h_1^d,\ldots,h_{2i}^c,h_{2i}^d)$, we obtain all but the last of the remaining Morse inequalities from (32).

By the reduced $h_{(\kappa)}^{cd}$-system we have that $h_0=h_1^c-(\kappa-1)$. Since $\gamma_0=1$, this implies that $h_0=h_1^c-\kappa+\gamma_0$. Since $h_1^c\ge\kappa$ it follows that $h_0\ge\gamma_0$, thus establishing the last Morse inequality.

Case $n=2i$.

This case is similar to the odd-dimensional case. Integrality and nonnegativity follow from the integrality and nonnegativity of the h^{cd} solution and the complementarity condition. Regarding the satisfaction of the Morse inequalities, details will be given for the case $2i=0 \mod 4$. The arguments may be easily adapted for the case $2i=2 \mod 4$ simply be replacing β^i with δ.

Summing the first $2i+1$ equations of (27) we have

$$\begin{aligned}
\sum_{j=0}^{2i}(-1)^{j+1}h_j &= -h_0 + h_1^c + \sum_{j=1}^{2i-2}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}\beta^i + h_{2i-1}^d - h_{2i} \\
&= -1 + \kappa + \sum_{j=1}^{2i-2}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}\beta^i + \kappa - 1 \\
&= -\gamma_0 + h_1^d - h_2^c + \kappa + \sum_{j=2}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}\beta^i \\
&\quad + \sum_{j=i}^{2i-3}(-1)^{j+1}(h_j^d - h_{j+1}^c) - h_{2i-2}^d + h_{2i-1}^c + \kappa - \gamma_{2i} \\
&= -\gamma_0 + \gamma_1 + \sum_{j=2}^{i-1}(-1)^{j+1}\gamma_j + (-1)^{i+1}\gamma_i \\
&\quad + \sum_{j=i}^{2i-2}(-1)^{j+1}(-\gamma_{j+1}) + \gamma_{2i-1} - \gamma_{2i} \\
&= \sum_{j=0}^{2i}(-1)^{j+1}\gamma_j.
\end{aligned}$$

Hence we have proved the top Morse equality.

The next equations are obtained analogously:

$$\begin{aligned}
\sum_{j=0}^{\ell}(-1)^{j+1}h_j &= \begin{cases} -h_0 + h_1^c + \sum_{j=1}^{\ell-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}\beta^i + (-1)^{\ell+1}h_\ell^d, & \text{if } i \le \ell \le 2i-1 \\ -h_0 + h_1^c + \sum_{j=1}^{\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{\ell+1}h_{\ell+1}^c, & \text{if } 1 \le \ell < i \end{cases} \\
&= \begin{cases} \sum_{j=0}^{\ell}(-1)^{j+1}\gamma_j + (-1)^{\ell+1}h_\ell^d, & \text{if } i \le \ell \le 2i-1 \\ \sum_{j=0}^{\ell}(-1)^{j+1}\gamma_j + (-1)^{\ell+1}h_{\ell+1}^c, & \text{if } 1 \le \ell \le i-1. \end{cases}
\end{aligned} \tag{33}$$

The nonnegativity of $(h_1^c, h_1^d, \ldots, h_{2i}^c, h_{2i}^d)$ implies all but the last of the remaining Morse inequalities. The last inequality is established exactly as in the $n = 2i+1$ case. ■

We now prove the necessity of the Poincaré-Hopf inequalities in Theorem 1.3 in order for the Morse inequalities to hold.

Conclusion of the proof of Theorem 1.3:

By inequality (1) of (1) we have that

$$h_1 \geq \gamma_1 - \gamma_0 + h_0.$$

Using the fact that $\gamma_1 \geq \kappa$ we have

$$h_1 \geq \kappa - 1 + h_0.$$

By Conley duality on the indices we have that and using the fact that $\gamma_{n-1} \geq \kappa$ we have

$$h_{n-1} \geq \kappa - 1 + h_n,$$

and therefore we obtain inequalities (1) of (5).

For $j = 2, ..., i$, follow the procedure described bellow. From inequality (j) of (1) we have that

$$h_j \geq \gamma_j - \gamma_{j-1} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 + h_{j-1} - h_{j-2} + - \ldots \pm h_2 \pm h_1 \pm h_0$$

$$\geq -\gamma_{j-1} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 + h_{j-1} - h_{j-2} + - \ldots \pm h_2 \pm h_1 \pm h_0.$$

Hence,

$$h_j + \gamma_{j-1} - \gamma_{j-2} - + \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 \geq h_{j-1} - h_{j-2} + - \ldots \pm h_2 \pm h_1 \pm h_0 \tag{34}$$

Also it follows from the dual Morse inequalities that

$$\gamma_{j-1} - \gamma_{j-2} + - \ldots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 \leq h_{n-(j-1)} - h_{n-(j-2)} + - \ldots \pm h_{n-2} \pm h_{n-1} \pm h_n \tag{35}$$

Substituting (35) in (34) we have that

$$h_j + h_{n-(j-1)} - h_{n-(j-2)} + - \ldots \pm h_{n-2} \pm h_{n-1} \pm h_n \geq h_{j-1} - h_{j-2} + - \ldots \pm h_2 \pm h_1 \pm h_0$$

$$\Rightarrow h_j \geq -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) - + \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0)$$

Multiplying by (-1) we have that

$$-h_j \leq (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \tag{36}$$

Using Conley duality on the indices we have that

$$-\overline{h}_j \leq (\overline{h}_{n-(j-1)} - \overline{h}_{j-1}) - (\overline{h}_{n-(j-2)} - \overline{h}_{j-2}) + - \ldots \pm (\overline{h}_{n-1} - \overline{h}_1) \pm (\overline{h}_n - \overline{h}_0) \tag{37}$$

Finally multiplying by (-1) and using the duality of the Conley index, $\overline{h}_j = h_{n-j}$, we have that

$$h_{n-j} \geq (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \qquad (38)$$

Hence we obtain the inequalities (j) of (5) with j = 2, ..., i − 1.

In order to obtain the inequalities in mid-dimensions the procedure is analogous. However, one must observe that if $n = 2i$ we have that $n - i = i$ and hence h_i is dual to itself. If $n = 2i + 1$ the dual of h_i is h_{i+1}. ■

We now present an example in Figure 16 of an abstract Lyapunov graph in dimension 7, that may possibly represent a flow on a 7-dimensional manifold M. If this is the case, recall that each edge of the graph represents a closed connected 6-dimensional manifold N cross an interval and hence it need only be labelled with the Betti numbers $(\beta_1, \beta_2, \beta_3)$ of the manifold N. In Figure 16 the abstract Lyapunov graph $L(h_0, h_1, \ldots, h_7, \kappa) = L(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$ is presented. In theory, each vertex v should be labelled with $(h_0(v), h_1(v), \ldots, h_7(v), \kappa_v)$. However, if this vector contains many zeros, e.g., $(0, 1, 0, 0, 0, 0, 0, 0, \kappa = 0)$ we adote the alternative notation $h_1 = 1$ as was mentioned in the introduction. Whenever the vertex is labelled with $h_j = 1$, $j \neq 0, 7$, we also specify if the vertex is j-disconnecting, j-d, or $(j - 1)$-connecting, $(j - 1)$-c.

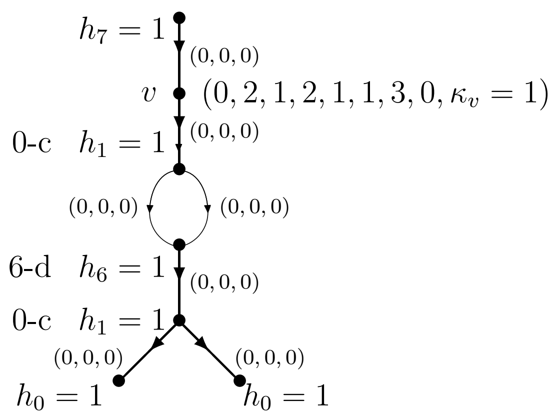


Figure 16: Abstract Lyapunov graph $L(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$

In Figure 17 we present a couple of possible continuations of $L(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$ which belongs to the family $\mathcal{L}(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$. Note that any graph in $\mathcal{L}(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$ must have cycle number $2 \leq\leq \min\{h_1 - (h_0 - 1), h_6 - (h_7 - 1)\} = 3$. Recall that κ is related to the first Betti number γ_1 of the manifold M by the inequality $\gamma_1 \geq \kappa$.

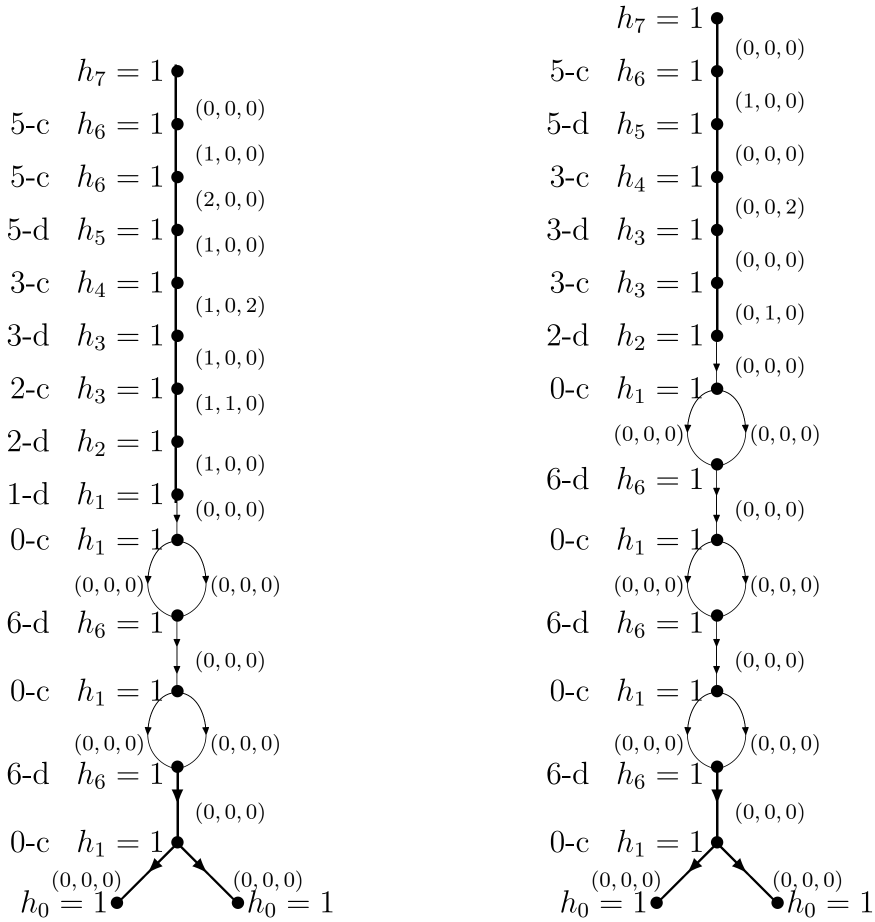


Figure 17: Graphs in the family $\mathcal{L}(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$.

Finally, in Figure 18 we present the implosion of the leftmost graph in Figure 17 $L(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$.

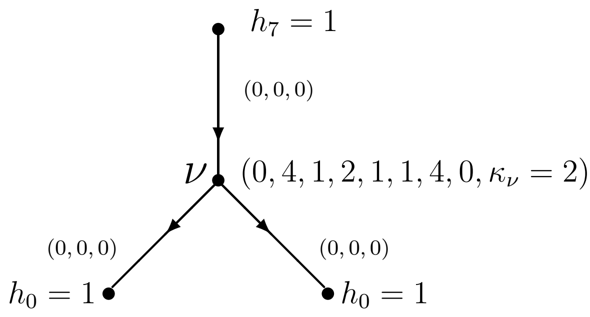


Figure 18: Implosion of $L(2, 4, 1, 2, 1, 1, 4, 1, \kappa = 2)$.

7 Morse Polytope

Consider a fixed pre-assigned index data set $(h_0, h_1, \ldots, h_n)$ and let κ be an integer in the interval $[0, \min\{h_1-(h_0-1), h_{n-1}-(h_n-1)\}]$. The allowance of cycles adds the extra inequality $\gamma_1 \geq \kappa$ to the set of Morse inequalities. This larger set of inequalities, plus the boundary constraints $\gamma_0 = \gamma_n = 1$, the duality constraints $\gamma_j = \gamma_{n-j}$, for $j = 0, \ldots, n$, and the nonnegative constraints $\gamma \geq 0$, define a polyhedron $\mathcal{P}_\kappa(h_0, \ldots, h_n)$, which for simplicity we refer to as $\mathcal{P}_\kappa$, in $\mathbb{R}^{n+1}$, which will be the subject of this section. This polyhedron is in fact a polytope, that is, a bounded polyhedron, since $0 \leq \gamma \leq h_j$ (upper bound is implied by inequalities (j) and $(j-1)$ of (1)), for $1 \leq j \leq n-1$. Much will be inherited from the study of the Morse polytope done in [2], since the addition of the inequality $\gamma \geq \kappa$ will lead to minor modifications of the polytope.

The sets of nonnegative solutions to the reduced h^{cd}_κ-system (special cases of (11) or (12)) also constitutes a polytope. It is remarkable that both polytopes, $\mathcal{P}_\kappa$ and the one determined by the h^{cd}-system, have integral vertices, and that there is a relationship (though not 1-to-1) between the integral elements in each polytope.

7.1 Case n odd

The study of $\mathcal{P}_\kappa$ may be significantly simplified by considering the reduced polytope $\mathcal{P}_\kappa^r$, obtained by eliminating γ_0, γ_{i+1}, $\ldots$, γ_{2i+1}, using the boundary and duality equations. These conditions imply that

$$\sum_{j=0}^{2i+1-k}(-1)^{j+1}\gamma_j = -1 + \sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j.$$

Therefore, $\mathcal{P}_\kappa^r$ is the set of $\gamma^r \in \mathbb{R}^i$ that satisfies

$$\begin{aligned}
0 = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j, \quad 0 \le h_0 - 1, \quad 0 \le h_{2i+1} - 1 \\
\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j \le 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, \text{ for } 2 \le k \le 2i, \; k \text{ even} \\
\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j \ge 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, \text{ for } 2 \le k \le 2i, \; k \text{ odd} \\
\gamma^r \ge 0 \\
\gamma_1 \ge \kappa.
\end{aligned} \tag{39}$$

Clearly, there is a 1-to-1 relationship between $\gamma^r = (\gamma_1, \gamma_2, \ldots, \gamma_i)$ in $\mathcal{P}_\kappa^r$ and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i+1})$ in $\mathcal{P}_\kappa$.

The following proposition extends the result obtained in [2] for $\kappa = 0$.

Proposition 7.1 *The polytope $\mathcal{P}_\kappa^r$ given by* (39) *satisfies the following properties:*

1. *The vertices of $\mathcal{P}_\kappa^r$ are integral.*

2. *Each vertex of $\mathcal{P}_\kappa^r$ belongs to one of the faces: $\mathcal{F}_t = \{\gamma \in \mathcal{P}_\kappa^r \mid \sum_{j=1}^{i}(-1)^{j+1}\gamma_j = 1 + \sum_{j=1}^{i}(-1)^{j+1}h_j\}$ or $\mathcal{F}_0 = \{\gamma \in \mathcal{P}_\kappa^r \mid \gamma_i = 0\}$.*

3. *If $\tilde{\gamma}^r \in \mathcal{F}_t$, then $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{i-1}, 0) \in \mathcal{F}_0$, that is, $\mathcal{F}_0$ is the projection of $\mathcal{F}_t$ on the plane $\gamma_i = 0$.*

4. *Each (integral) γ^r in $\mathcal{F}_t$ corresponds to an (integral) nonnegative h^{cd} satisfying* (26).

5. *If $\kappa \ge \kappa'$ then $\mathcal{P}_\kappa \subseteq \mathcal{P}_{\kappa'}$.*

Proof:

1. In [2] it was shown that the matrix of coefficients corresponding to all, except the last, inequalities in (39) is totally unimodular. Now the row of coefficients corresponding to the last inequality is $(1, 0, \ldots, 0)$. Appending such a row to a totally unimodular matrix preserves this property. Since the right-hand-side is obviously integral, the vertices of $\mathcal{P}_\kappa^r$ are integral.

2. The proof for the corresponding proposition in [2] may be applied verbatim here.

3. Only two inequalities in (39) contain γ_i: $\gamma_i \geq 0$ and $(-1)^{i+1}\sum_{j=1}^{i}(-1)^{j+1}\gamma_j \leq (-1)^{i+1}(1 + \sum_{j=0}^{i}(-1)^{j+1}h_j)$ (the inequality obtained when $k = i+1$). The latter one is equivalent to

$$(-1)^{i+1}(-1)^{i+1}\gamma_i = \gamma_i \leq (-1)^{i+1}\left(1 - h_0 + h_1 \sum_{j=2}^{i}(-1)^{j+1}(h_j - \gamma_j)\right).$$

   For $\tilde{\gamma}^r \in \mathcal{F}_t$ the above inequality is tight, that is,

$$\tilde{\gamma}_i = \tilde{u} = (-1)^{i+1}\left(1 - h_0 + h_1 \sum_{j=2}^{i}(-1)^{j+1}(h_j - \tilde{\gamma}_j)\right).$$

   Therefore $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{i-1}, \xi) \in \mathcal{P}_\kappa^r$ for $0 \leq \xi \leq \tilde{u}$.

4. Given an (integral) γ^r in $\mathcal{F}_t$, consider its natural extension γ in $\mathcal{P}_\kappa$, that is, let $\gamma_0 = \gamma_{2i+1} = 1$ and $\gamma_{2i+1-\ell} = \gamma_\ell$, for $1 \leq \ell \leq i$. The proof of Proposition 3.2 in [2] may be easily adapted to show that the h^{cd} vector defined below is a nonnegative (integral) solution to (26):

$$\begin{aligned}
h^d_{2i} &= -\sum_{j=0}^{2i}(-1)^{j+1}(h_j - \gamma_j) + \kappa, \\
h^d_{2i+1-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \text{ for } 2 \leq \ell \leq i \\
h^c_{2i+2-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \text{ for } i+2 \leq \ell \leq 2i \\
h^c_1 &= h_0 - \gamma_0 + \kappa, \\
h^d_1 &= \gamma_1 + h^c_2 - \kappa, \\
h^d_\ell &= \gamma_\ell + h^c_{\ell+1}, && \text{for } 2 \leq \ell \leq i-1 \\
h^c_\ell &= \gamma_\ell + h^d_{\ell-1}, && \text{for } i+2 \leq \ell \leq 2i-1 \\
h^c_{2i} &= \gamma_{2i} + h^d_{2i-1} - \kappa, \\
h^d_i &= \gamma_i \\
h^c_{i+1} &= \gamma_{i+1}.
\end{aligned}$$

5. This fact is trivial, since the inequality $\gamma_1 \geq \kappa$ implies $\gamma_1 \geq \kappa'$ when $\kappa \geq \kappa'$. ∎

Proposition 7.1 thus implies that $\mathcal{P}_\kappa$ is the convex hull of nonnegative Betti number vectors that satisfy the Morse inequalities and the inequality $\gamma_1 \geq \kappa$. Suppose there exist nonnegative integral solutions of (26). Let $\bar{h}^{cd}$ be the nonnegative integral solution of the reduced h_0^{cd}-system that satisfies the complementarity conditions (28). Let $\bar{\gamma}$ be the Betti number vector in $\mathcal{P}_\kappa$ constructed from $\bar{h}^{cd}$ using (30).

Proposition 7.2 *Suppose the reduced h_κ^{cd}-system* (26) *admits nonnegative solutions. The $\mathcal{P}_\kappa^r$ may be rewritten as*

$$\mathcal{P}_\kappa^r = \{0 \leq \gamma^r \in \mathbb{R}^i \mid \gamma_1 \geq \kappa, \text{ and } (-1)^{k+1}\sum_{j=0}^{k}(-1)^{j+1}\gamma_j \leq (-1)^{k+1}\sum_{j=0}^{k}(-1)^{j+1}\bar{\gamma}_j, \text{ for } 1 \leq k \leq i\}.$$

Furthermore, $\bar{\gamma}$ is the maximum vector of $\mathcal{P}_\kappa^r$, componentwise.

Proof: This proposition was proved in [2] for $\kappa = 0$, so the first assertion is trivially true, since $\mathcal{P}_\kappa^r = \mathcal{P}_0^r \cap \{0 \leq \gamma^r \in \mathbb{R}^i \mid \gamma_1 \geq \kappa\}$. Now notice that if $\bar{h}^{cd}$ is the solution of the reduced h_0^{cd}-system, then $\bar{h}^{cd}(\kappa)$ given by

$$\begin{array}{rcll} \bar{h}_1^c(\kappa) &=& \bar{h}_1^c + \kappa, & \\ \bar{h}_1^d(\kappa) &=& \bar{h}_1^d - \kappa = h_1 - (h_0 - 1) - \kappa, & \\ \bar{h}_j^c(\kappa) &=& \bar{h}_j^c, & \text{for } 2 \leq j \leq 2i-1 \\ \bar{h}_j^d(\kappa) &=& \bar{h}_j^d, & \text{for } 2 \leq j \leq 2i-1 \\ \bar{h}_{2i}^c(\kappa) &=& \bar{h}_{2i}^c - \kappa = h_{2i} - (h_{2i+1} - 1) - \kappa, & \\ \bar{h}_{2i}^d(\kappa) &=& \bar{h}_{2i}^d + \kappa. & \end{array}$$

is a nonnegative integral solution of the reduced h_κ^{cd}-system, for integral κ in $[0, \min\{h_1 - (h_0 - 1), h_{2i} - (h_{2i+1} - 1)\}]$. Furthermore, its corresponding Betti number vector given by (30) is the same $\bar{\gamma}$. Therefore $\bar{\gamma}^r \in \mathcal{P}_\kappa^r$ for κ in $[0, \min\{h_1 - (h_0 - 1), h_{2i} - (h_{2i+1} - 1)\}]$. Since $\mathcal{P}_{\kappa_1}^r \supset \mathcal{P}_{\kappa_2}^r$ if $\kappa_1 \leq \kappa_2$, and $\bar{\gamma}^r$ belongs to $\mathcal{P}_\kappa^r$, for all κ in $[0, \min\{h_1 - (h_0 - 1), h_{2i} - (h_{2i+1} - 1)\}]$, we conclude that $\bar{\gamma}^r$ is the maximum vector of $\mathcal{P}_\kappa^r$, for all κ in $[0, \min\{h_1 - (h_0 - 1), h_{2i} - (h_{2i+1} - 1)\}]$. ∎

The various facts established so far are illustrated in the next example. Notice that $\mathcal{P}_\kappa^r$ is also the convex hull of $\mathcal{F}_t \cup \mathcal{F}_0$. Additionally, since $\mathcal{F}_0$ is the projection of $\mathcal{F}_t$ and all elements in $\mathcal{F}_t$ may be obtained from solutions of the reduced h_κ^{cd}-system, we have means, albeit indirect, of obtaining all elements in $\mathcal{P}_\kappa^r$ from the h^{cd} vectors.

Example Let $n = 2i + 1 = 7$ and $(h_0, \ldots, h_7) = (1, 5, 11, 10, 5, 3, 4, 3)$. The solution of the reduced h_0^{cd}-system that satisfies (28) is $\bar{h}^{cd} = (0, 5, 3, 8, 5, 5, 5, 0, 3, 0, 2, 2)$, and the elementary circulations are $\text{circ}^1 = (0, 0, 1, -1, 0, 0, 0, 0, -1, 1, 0, 0)$ and $\text{circ}^2 = (0, 0, 0, 0, 1, -1, -1, 1, 0, 0, 0, 0)$, corresponding to cycles 1 and 2 depicted in Figure 19. The maximum element of $\mathcal{P}_0$ is $\bar{\gamma} = (1, 2, 3, 5, 5, 3, 2, 1)$. Thus, Proposition 7.2 implies the polytope $\mathcal{P}_\kappa^r$ is given by the inequalities

$$\begin{array}{ccccccc} \gamma_1 & & & & & \leq & 2 \\ \gamma_1 & - & \gamma_2 & & & \geq & -1 \\ \gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 4 \\ \gamma_1 & & & & & \geq & \kappa \\ & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0 \end{array}$$

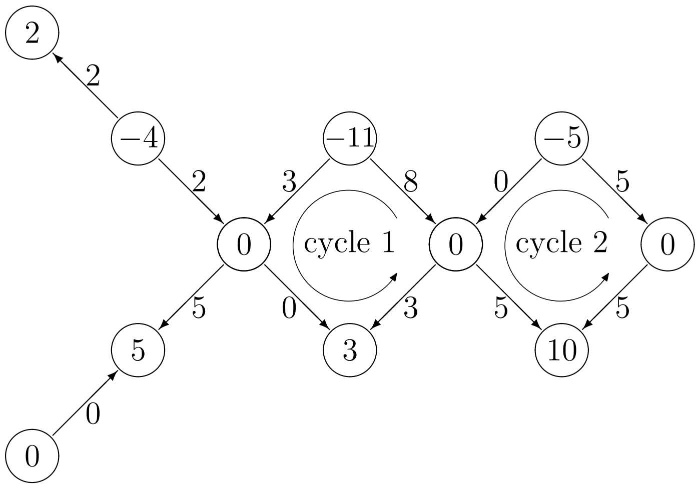
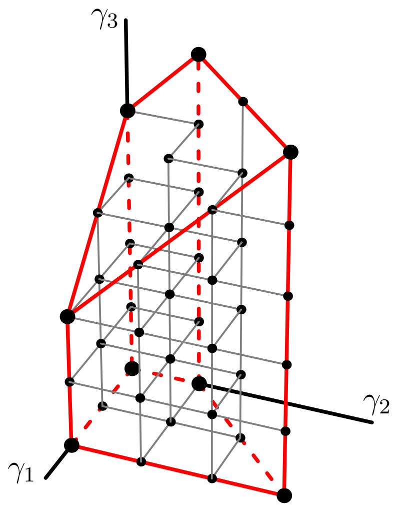


Figure 19: Solution $\tilde{h}^{cd}$ of example.

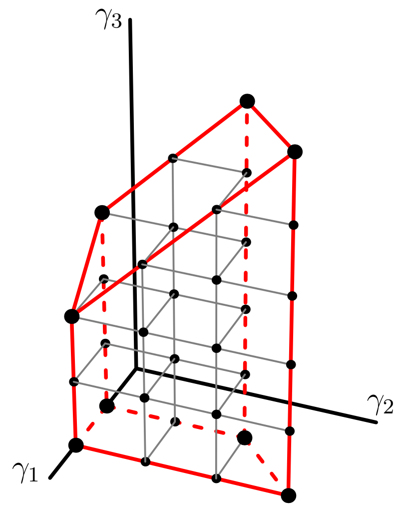
In this case κ may assume the values 0, 1 and 2. Figure 20 depicts the three polytopes, delineating their edges and emphasizing their integral elements. The relationship between two γ^r's in $\mathcal{F}_t$ and their corresponding h^{cd}'s may be obtained by the basic ones given in the following table.

$\hat{\gamma}^r - \tilde{\gamma}^r$	$\hat{h}^{cd} - \tilde{h}^{cd}$
$(-1, -1, 0)$	circ^1
$(0, -1, -1)$	circ^2

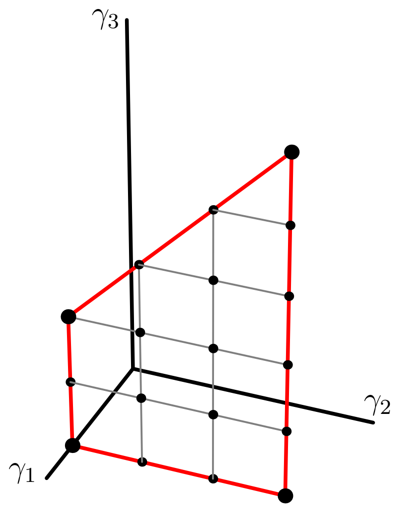
Thus the reduced h_0^{cd} system solution correponding to $\hat{\gamma}^r = (1, 2, 5) = \bar{\gamma}^r + (-1, -1, 0)$ is $h^{cd} = \bar{h}^{cd} + \text{circ}^1 = (0, 5, 4, 7, 5, 5, 5, 0, 2, 1, 2, 2)$. Likewise, if $\hat{\gamma}^r = (1, 1, 4) = \bar{\gamma}^r + (-1, -2, -1) = \bar{\gamma}^r + (-1, -1, 0) + (0, -1, -1)$, then $h^{cd} = \bar{h}^{cd} + \text{circ}^1 + \text{circ}^2 = (0, 5, 4, 7, 6, 4, 4, 1, 2, 1, 2, 2)$.



(a) $\mathcal{P}_0^r$ and its integral elements.



(b) $\mathcal{P}_1^r$ and its integral elements.



(c) $\mathcal{P}_2^r$ and its integral elements.

Figure 20: Polytopes $\mathcal{P}_\kappa^r$.

Out of curiosity we depict in Figure 21 all the integral γ^r's that are generated from integral solutions not necessarily nonnegative to the reduced h_κ^{cd}-system. Since the flow in cycle 1 may vary from 0 to 3 and in cycle 2 from 0 to 5, there are 24 nonnegative integral flows that solve the said system. As expected, these γ^r's satisfy the Morse inequalities (in fact, they belong to $\{\gamma^r \in \mathbb{R}^3 \mid \gamma_1 - \gamma_2 + \gamma_3 = 4\}$), but are not necessarily nonnegative.

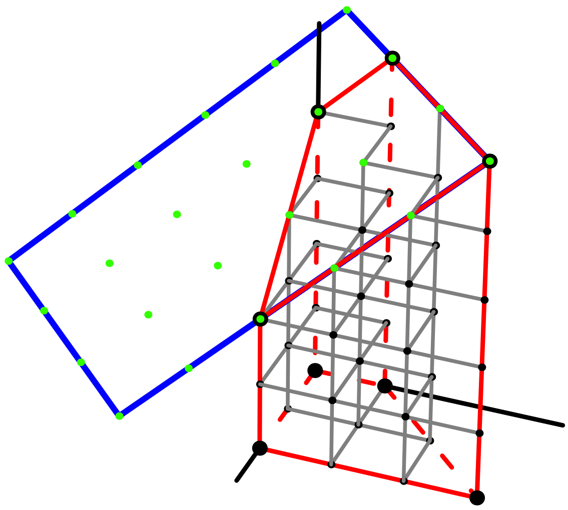


Figure 21: Extended face $\mathcal{F}_t$ produced by circulation.

7.2 Case n even

Initially, let $n = 2i$, where $i \geq 2$ is even, and suppose the pre-assigned index data set $(h_0, \ldots, h_{2i})$ is such that $\sum_{j=0}^{2i}(-1)^j h_j$ is even. In this case we the duality conditions to eliminate $\gamma_{i+1}, \ldots, \gamma_{2i-1}$, the boundary conditions to eliminate γ_0 and γ_{2i}, and the first equation in (1) to eliminate γ_i from the system of inequalities that defines $\mathcal{P}_\kappa$. As before, the resulting system of inequalities (40) in $(\gamma_1, \ldots, \gamma_{i-1})$ defines a polytope $\mathcal{P}_\kappa^r$ whose elements are in an 1-to-1 relationship with the elements of $\mathcal{P}_\kappa$. Details of the deduction of (40) are provided in [2].

$$
\begin{array}{rcll}
0 & \geq & 1 - h_{2i} & \\
\sum_{j=1}^{k}(-1)^{j+1}\gamma_j \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{c} \leq \\ \geq \end{array} & \begin{array}{l} 1 + \sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \\ 1 + \sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \end{array} & \begin{array}{l} \text{if } 1 \leq k \leq i-1,\ k \text{ odd} \\ \text{if } 1 \leq k \leq i-1,\ k \text{ even} \end{array} \\
(-1)^i \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j & \geq & (-1)^i \left(1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right) & \\
\sum_{j=1}^{2i-k}(-1)^{j+1}\gamma_j \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{c} \leq \\ \geq \end{array} & \begin{array}{l} 1 + \sum_{j=0}^{2i-k}(-1)^{j+1}h_j, \\ 1 + \sum_{j=0}^{2i-k}(-1)^{j+1}h_j, \end{array} & \begin{array}{l} \text{if } i+1 \leq k \leq 2i-1,\ k \text{ odd} \\ \text{if } i+1 \leq k \leq 2i-1,\ k \text{ even} \end{array} \\
0 & \geq & 1 - h_0 & \\
\gamma_j & \geq & 0, & \text{for } 1 \leq j \leq i-1 \\
\gamma_1 & \geq & \kappa. &
\end{array}
\tag{40}
$$

The following proposition was shown in [2] for the case $\kappa = 0$.

Proposition 7.3 *The polytope $\mathcal{P}^r_\kappa$ defined by* (40) *has integral vertices and each (integral) γ^r in the polytope corresponds to an (integral) nonnegative h^{cd} satisfying* (27)*. Each vertex of $\mathcal{P}^r_\kappa$ belongs to one of three faces:*

$$
\begin{aligned}
\mathcal{F}_t &= \left\{\gamma \in \mathcal{P}^r_\kappa \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \min\left\{\sum_{j=i+1}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{i-1}(-1)^{j+1}h_j\right\}\right\} \\
\mathcal{F}_b &= \left\{\gamma \in \mathcal{P}^r_\kappa \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right\} \\
\mathcal{F}_0 &= \{\gamma^r \in \mathcal{P}^r_\kappa \mid \gamma_{i-1} = 0\}.
\end{aligned}
$$

Proof: The proof in [2] may be easily adpated to encompass the $\kappa \neq 0$ case, as done for Proposition 7.1. ∎

The inequalities defining $\mathcal{P}^r_\kappa$ may be greatly simplified if we resort to the use of $\bar{\gamma}$, the Betti number vector in $\mathcal{P}_\kappa$ corresponding to the solution $\bar{h}^{cd}$ of the reduced h^{cd}_κ-system that satisfies the complementarity conditions (28). The next proposition is thus the analogous to Proposition 7.2 for the $n = 0 \mod 4$ case.

Proposition 7.4 *Suppose* (27) *has a nonnegative solution. The polytope* $\mathcal{P}^r_\kappa$ *may be recast as*

$$\mathcal{P}^r_\kappa = \left\{ 0 \leq \gamma^r \;\middle|\; \begin{array}{rcl} (-1)^{k+1}\displaystyle\sum_{j=0}^{k}(-1)^{j+1}\gamma_j & \leq & (-1)^{k+1}\displaystyle\sum_{j=0}^{k}(-1)^{j+1}\tilde{\gamma}_j, \quad \textit{for } 1 \leq k \leq i-1 \\ \displaystyle\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j & \geq & \dfrac{1}{2}\displaystyle\sum_{j=0}^{2i}(-1)^{j+1}h_j \\ \gamma_1 & \geq & \kappa \end{array} \right\} \tag{41}$$

Furthermore, $\bar{\gamma}^r$*) is the maximum vector in* $\mathcal{P}^r_\kappa$*, componentwise.*

Proof: The adpatation of the proof of the corresponding proposition in [2] may be adapted as done in the proof of Proposition 7.2. ■

Analogous developments, i.e., reduction in the number of variables and versions of Propositions 7.3 and 7.4 may be obtained for the case $n = 2 \mod 4$ in a straightforward way.

We conclude this section with the remark that the family $\mathcal{L}(h_0, \ldots, h_n, \kappa)$ is associated to $\mathcal{P}^r_\kappa$. This association is established by the fact that any graph in $\mathcal{L}(h_0, \ldots, h_n, \kappa)$ has a fixed data set which determines $\mathcal{P}^r_\kappa$. Although every graph in $\mathcal{L}(h_0, \ldots, h_n, \kappa)$ satisfies the Morse inequalities for any Betti number vector in $\mathcal{P}^r_\kappa$ it is an open problem if these graphs can be realized as flows on manifolds with these Betti numbers.

What we do know is that for a given continuation, $L(h_0, h^c_1, h^d_1, \ldots, h^c_{n-1}, h^d_{n-1}, h_n, \kappa)$, of the graph $L(h_0, h_1, \ldots, h_{n-1}, h_n, \kappa)$ there is a correspondence with an integral element v on the extended face $\mathcal{F}_t$ of $\mathcal{P}(h_0, \ldots, h_n)$ as in Figure 21. In fact, it is the data in $L(h_0, h^c_1, h^d_1, \ldots, h^c_{n-1}, h^d_{n-1}, h_n, \kappa)$ which defines the coordinates of the Betti number vector which determines v. Also, there is no topological interpretation for those graphs $L(h_0, h^c_1, h^d_1, \ldots, h^c_{n-1}, h^d_{n-1}, h_n, \kappa)$ which determine Betti number vectors not in the nonnegative orthant.

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