Gradings on the algebra of upper triangular matrices and their graded identities

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Abstract

Let K be an infinite field and denote $UT_n(K)$ the algebra of $n \times n$ upper triangular matrices over K. We describe all elementary gradings on this algebra. Further we produce linear bases of the respective relatively free graded algebras, and prove that one can distinguish the elementary gradings by their graded identities. We describe bases of the graded polynomial identities in several "typical" cases. Although we consider elementary gradings by the cyclic group of order two the same methods serve for elementary gradings by any finite group.

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Introduction

Graded polynomial identities play an important role in the structure theory of PI algebras. Very many properties of the ideals of identities are described in the language of graded identities and graded algebras, see for example [6, 17]. That was why the graded identities became an object of independent interest. It turned out that graded identities can be successfully applied to various difficult problems in the theory of PI algebras. Let us recall some of the keystone results concerning graded polynomial identities. Let G be a finite abelian group and let A be a G-graded algebra. Then A is PI algebra if and only if its 0-component is PI, see [2, 11]. It was soon discovered that one may consider the graded identities satisfied by an algebra as an "approximation" of the ordinary ones. It is well known that the multilinear identities satisfied by an algebra determine its identities when the algebra is considered over a field of characteristic 0. So it makes sense to study graded multilinear identities, their cocharacters, codimensions, and so on. These have been extensively studied, see for example [2, 7, 8, 9, 10, 14, 15], and [3] and its bibliography.

An important task in PI theory is describing the graded identities satisfied by a given algebra. Since matrix algebras are very important their graded identities are of significant interest. In [13] the $\mathbb{Z}_2$-graded identities of the matrix algebra of order two, $M_2(K)$ over a field of characteristic 0, were described. Furthermore a finite basis of these identities was explicitly given, and the cocharacter of the corresponding graded T-ideal was computed. Let E denote the infinite dimensional Grassmann (or exterior) algebra over K and $M_{1,1}(E)$ the algebra of 2×2 matrices over E whose entries on the main diagonal are even (i.e. central) elements of E, while the entries on the second diagonal are odd elements of E.

Recall that in [13] also bases of the graded identities satisfied by the algebras $M_{1,1}(E)$ and $E \otimes E$ were obtained, and as a consequence, it was shown that in characteristic 0, these two algebras are PI-equivalent (that is they satisfy the same polynomial identities). This result is well known and it is part of the classification of the T-prime algebras given by Kemer, see [17]. The original proof uses heavily structure theory of PI algebras; later an alternative proof was given by Regev [23]. The proof of the PI equivalence of $M_{1,1}(E)$ and $E \otimes E$ in [13] is much more elementary than the other two. Further in this direction, in [19] the results of Di Vincenzo [13] were extended to algebras over infinite fields of characteristic $\neq 2$, and a third, elementary proof of the PI equivalence of $M_{1,1}(E)$ and $E \otimes E$ in characteristic 0, was given. We only note that in positive characteristic these two algebras are not PI equivalent.

We recall that graded identities were one of the most important tools in Kemer's positive solution of the famous Specht problem (see [17]). Graded identities, together with other kinds of "weaker" identities are indispensable means in the study of polynomial identities. These may include weak identities, see [22], and [18] for more recent bibliography, and/or trace identities, see for example [21, 22], identities with involution, and so on.

An interesting and quite important problem is describing the graded identities satisfied by a given algebra. Surely matrix algebras and related algebras are of great interest in PI theory, that is why a lot of work has been done in investigating graded identities in such algebras. Thus for example, in [25, 26], Vasilovsky described the $\mathbb{Z}_n$ and the $\mathbb{Z}$-graded identities of the matrix algebra $M_n(K)$ of order n over a field of characteristic 0. Later the result of Vasilovsky about the $\mathbb{Z}_n$-graded identities was established over infinite fields, see [1]. In [20] the $\mathbb{Z}_n$-graded identities of the algebra $UT_n(K)$ of $n \times n$ upper triangular matrices were described over any infinite field. In all of the above cases, the respective gradings are the natural ones.

Now let G be a finite abelian group. For the algebra $M_n(K)$ of $n \times n$ matrices there are two important classes of G-gradings: the elementary gradings and the fine gradings. In fact in [5] it was proved that if K is an algebraically closed field, every G-grading on $M_n(K)$ is a tensor product of an elementary and a fine grading.

For the algebra $UT_n(K)$, it was proved in [27] that if one further assumes that $\operatorname{char} K = 0$ then every G-grading is elementary.

Motivated by this result, in this paper we study the elementary gradings on the algebra $UT_n(K)$ of $n \times n$ upper triangular matrices over an infinite field. We describe these gradings by means of the graded identities that they satisfy. We start with the $\mathbb{Z}_2$-gradings in order to outline the main ideas of the proofs, and then we proceed briefly with the G-graded identities. Note that in the latter case only the notation is much more complicated compared to the former case.

1 Preliminaries

We consider associative algebras only. All algebras, vector spaces etc., are over a fixed infinite field K. Denote by $UT_n = UT_n(K)$ the algebra of $n \times n$ upper triangular matrices over K. If G is a multiplicative group then a G-grading on an associative algebra is a decomposition of A as a direct sum of vector subspaces $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{gh}$ for every g, $h \in G$. The vector space A_g is the g-th homogeneous component in the grading, A_1 is the neutral (or identity) component. One defines analogously graded vector spaces (no multiplication, only direct sum decomposition). A possible $\mathbb{Z}_n$-grading of the algebra $UT_n = UT_n(K)$ is the following. Let $\mathbb{Z}_n = \langle g \rangle$, $g^n = 1$, so that $\mathbb{Z}_n = \{1, g, g^2, \ldots, g^{n-1}\}$ and let $(UT_n)_{g^k}$ be the span of all matrix units e_{ij} with $j - i = k$. Here e_{ij} is the matrix whose (i, j)-th entry equals 1 and all other entries equal 0. Recall that the graded identities for this specific grading on UT_n were described in [20]. When $n = 2$ and K is of characteristic 0, the above is the only $\mathbb{Z}_2$-grading allowed. A detailed description of the $\mathbb{Z}_2$-graded identities of $UT_2(K)$ and their numerical invariants were given in [24].

Let V be an n-dimensional vector space with a basis $v_1, v_2, \ldots, v_n$, and let $\tilde{g} = (g_1, g_2, \ldots, g_n) \in G^n$ be an n-tuple of elements of G. Then V is G-graded if we set v_i to be of homogeneous degree g_i, $\deg v_i = g_i$, $i = 1, 2, \ldots, n$. This grading induces one on the algebra of linear transformations on V. So the matrix algebra $M_n(K)$ is G-graded and furthermore the matrix units are homogeneous of degrees $\deg(e_{ij}) = g_i^{-1} g_j$ for all i and j. The induced grading on $M_n(K)$ is called *elementary*. One can give an alternative definition of the elementary gradings on $M_n(K)$.

Theorem 1 ([12]) *Let G be a group and let $M_n(K)$ be G-graded. The grading is elementary if and only if all matrix units e_{ij} are homogeneous.* $\diamondsuit$

Now we recall the notion of G-graded polynomial identity. Let $X = \cup X_g$ be a union of the disjoint countable sets $X_g = \{x_{g1}, x_{g2}, \ldots\}$. The free associative algebra $K(X)$ freely generated over K by X is equipped in a natural way with a structure of G-graded algebra. Namely the homogeneous degree $\deg(x_{gi}) = g$ for every $x_{gi} \in X_g$, and then one extends this grading to the monomials on X. Hence $\deg(x_{g_1 i_1} x_{g_2 i_2} \ldots x_{g_r i_r}) = g_1 g_2 \ldots g_r$. So $K(X)$ is the free G-graded algebra freely generated by X. In some instances we shall use symbols like x_g to indicate a variable in $K(X)$ of homogeneous degree

g, or still other letters like y and z for $\mathbb{Z}_2$-gradings. In such occasions y with or without lower index stands for 1-variable and z for -1-variable. Let $f(x_{g_1 i_1}, x_{g_2 i_2}, \ldots, x_{g_r i_r}) \in K(X)$ be a polynomial. If $A = \oplus A_g$ is a G-graded algebra then f is a G-graded polynomial identity (or simply a G-graded identity) for A if $f(a_{g_1}, a_{g_2}, \ldots, a_{g_r}) = 0$ in A for every homogeneous substitution $a_{g_t} \in A_g$. The ideal $T^G(A) = Id^G(A)$ of all graded identities in $K(X)$ is closed under all G-graded (or homogeneous) endomorphisms of $K(X)$; such ideals are called G-graded T-ideals. When the grading is explicitly given we shall simply write $Id(A)$ for the ideal of the G-graded identities of A. One defines the notions of variety of G-graded algebras, relatively free G-graded algebra, and so on, in analogy with the case of ordinary polynomial identities. Moreover as in the case of ordinary polynomial identities, it can be shown that over an infinite field, every G-graded T-ideal is generated as such by its multihomogeneous polynomials.

The following elementary fact will be used frequently (and without mentioning). It seems to us it is some kind of *folklore*, that is why we do not give credit for it.

Lemma 2 *Let the G-grading on $UT_n(K)$ be elementary. Then all idempotent matrix units e_{ii} belong to the homogeneous 1-component. The same is true for the matrix algebra $M_n(K)$ as well. Hence the identity matrix I belongs to the 1-component.*

Proof. It suffices to observe that $e_{ii}e_{ij} = e_{ij}$ for every i and j. If g and $h \in G$ are the respective homogeneous components of the grading on UT_n then $gh = h$ and $g = 1$. $\diamondsuit$

We recall that this need not be the case if the grading on $M_n(K)$ is not elementary, see for examples [12].

Let A be a unitary algebra over an infinite field K. It is well known that the polynomial identities of A are determined by its proper (or commutator) identities. Let $L(X)$ be the free Lie algebra freely generated by the set X. One may assume that $L(X) \subseteq K(X)$ as vector spaces, and that $L(X)$ is the Lie subalgebra of $K(X)$ that is generated by X under commuting instead of the usual multiplication in $K(X)$. Let $L'(X) = L(X)^2 = [L(X), L(X)]$ be the (Lie) ideal in $L(X) \subset K(X)$. Then the above fact can be restated in the following way. If $B(X)$ is the associative subalgebra of $K(X)$ generated by 1 and by $L'(X)$ then every T-ideal T in $K(X)$ is generated as T-ideal by $T \cap B(X)$. When one considers graded polynomial identities, a modification of this fact holds. Namely we have the following proposition.

Proposition 3 *Let K be an infinite field and let T be the ideal of graded identities for the G-graded algebra A. Suppose that $1 \in A$ belongs to the neutral homogeneous component in the grading of A. Then T is generated as an ideal of graded identities by its elements such that every neutral variable participates in commutators only.*

Proof. See [16], pp. 42–44 for the proof of the ungraded version of the proposition. $\diamondsuit$

We shall write $[a, b] = ab - ba$ for the commutator of a and b, and we assume that the commutators are left-normed. This means that $[a, b, c] = [[a, b], c]$ and so on.

2 G-graded identities for $UT_n(K)$

Throughout this section we shall assume that G is an arbitrary group and K is an infinite field. We shall also assume that UT_n is equipped with an elementary G-grading induced by the n-tuple $\tilde{g} = (g_1, g_2, \ldots, g_n) \in G^n$ of (not necessarily distinct) elements of G. Notice that we may always assume that $g_1 = 1$. In fact, it is easy to check that the two n-tuples $(g_1, g_2, \ldots, g_n)$ and $(1, g_1^{-1}g_2, \ldots, g_1^{-1}g_n)$ give rise to the same grading on UT_n. Hence we shall always assume, as we may, that in the n-tuple inducing the grading $g_1 = 1$.

We start with the following general fact that holds for any group G and for any field K.

Lemma 4 *Suppose that $\tilde{g} = (g_1, g_2, \ldots, g_n) \in G^n$ and let h_1, h_2, $\ldots$, h_m be the distinct elements of G that appear in $\tilde{g}$. Suppose that for $i = 1$, 2, $\ldots$, m, h_i appears a_i times. Then there is an isomorphism of K-algebras $(UT_n)_1 \cong UT_{a_1} \oplus UT_{a_2} \oplus \cdots \oplus UT_{a_m}$.*

Proof. Let $1 < k_1 < k_2 < \cdots < k_{a_i}$ be the positions in the sequence of g that are equal to h_i: $g_{k_1} = g_{k_2} = \ldots = g_{k_{a_i}}$. Then $g_{k_p}^{-1} g_{k_q} = h_i^{-1} h_i = 1 \in G$ hence the matrix units $e_{k_p k_q} \in (UT_n)_1$ are homogeneous for all $1 \leq p \leq q \leq a_i$. Set $A_1^{h_i}$ to be the K-span of the matrix units $e_{k_p k_q}$ for $1 \leq p \leq q \leq a_i$. Then obviously $A_1^{h_i} \cong UT_{a_i}$ as algebras and therefore $(UT_n)_1 \cong A_1^{h_1} \oplus A_1^{h_2} \oplus \cdots \oplus A_1^{h_m}$. Thus we have the algebra isomorphism $(UT_n)_1 \cong UT_{a_1} \oplus UT_{a_2} \oplus \cdots \oplus UT_{a_m}$. Observe that the sum is direct since the h_i are distinct elements of G. $\diamondsuit$

Let us establish another result that holds in the generic case i.e., for any group and any field.

Proposition 5 *Every elementary G-grading on UT_n is uniquely determined by the homogeneous degrees of the elements in the first row of a matrix of UT_n.*

Proof. Let $UT_n = \oplus_{g\in G} A_g$, and let $e_{1r} \in A_{g_r}$. We have that all e_{ii} belong to A_1, the neutral component. Suppose that $e_{ij} \in A_g$ for some $g \in G$ and $i < j$. Then since $e_{1i}e_{ij} = e_{1j}$ we get that $g_i g = g_j$ and hence $g = g_i^{-1} g_j$ is uniquely determined. $\diamondsuit$

Later on we shall establish a convenient algorithm for recovering the grading given only the grading on the first row, in the case when G is cyclic.

Lemma 6 *Suppose that the elementary grading on UT_n is induced by $\tilde{g} = (g_1, g_2, \ldots, g_n) \in G^n$ where the elements $g_1, g_2, \ldots, g_n$ are pairwise distinct. If $\dim_K(UT_n)_{g_i} = n-1$ for some i, $2 \le i \le n$, then the grading is induced by $(1, g_2, g_2^2, \ldots, g_2^{n-1})$.*

Proof. Fix some p, $1 \le p \le n$. Then among the matrix units e_{pp}, $e_{p,p+1}$, $\ldots$, e_{pn} at most one is of homogeneous degree g_i. The same holds if one considers the matrix units e_{1p}, e_{2p}, $\ldots$, e_{pp}. Set $UT_n = A$. Since $\dim A_{g_i} = n-1$ it follows that A_{g_i} is the span of the matrix units e_{12}, e_{23}, $\ldots$, $e_{n-1,n}$. Therefore $g_2 = g_2^{-1}g_3 = g_3^{-1}g_4 = \ldots = g_{n-1}^{-1}g_n$ and hence $g_3 = g_2^2$, $g_4 = g_3 g_2^{-1} g_3 = g_2^3$, $\ldots$, $g_n = g_{n-1}g_{n-2}^{-1}g_{n-1} = g_2^{n-1}$ by an obvious induction. $\diamondsuit$

Corollary 7 *If the elementary grading on UT_n is not induced by an n-tuple $(1, h, h^2, \ldots, h^{n-1})$, $h \in G$, then $\dim(UT_n)_{g_i} \le n-2$ for all $i \ge 2$.* $\diamondsuit$

3 Gradings on UT_n by cyclic groups

In this section we study in detail the graded identities satisfied by UT_n in case G is a cyclic group.

Lemma 8 *Let $G = \langle g \rangle$ be the cyclic group of order n and suppose that $\tilde{g} = (g^{i_1}, g^{i_2}, \ldots, g^{i_n})$ induces an elementary G-grading on UT_n, where the elements $g^{i_1}, g^{i_2}, \ldots, g^{i_n}$ are all distinct. If for some k, $\dim_K(UT_n)_{g^{i_k}} = 1$ then the grading is induced by $(1, g^{i_2}, g^{2i_2}, \ldots, g^{(n-1)i_2})$.*

Proof. The grading is elementary one, hence e_{1r} belongs to the homogeneous component indexed by $g^{-i_1}g^{i_r}$. This says that the matrix units e_{1j}, $j = 1, 2, \ldots, n$, belong to (pairwise) different homogeneous components; the same is true for the matrix units e_{in}, $i = 1, 2, \ldots, n$. It follows that $i_k = i_n$. The homogeneous degrees of the matrix units $e_{23}, e_{24}, \ldots, e_{2n}$ are respectively $g^{i_3-i_2}, g^{i_4-i_2}, \ldots, g^{i_n-i_2}$, and these must be different from g^{i_n}. Therefore $i_r - i_2 \not\equiv i_n \pmod{n}$ for all $r \geq 3$. Now observe that $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$ and $i_2 + i_n \not\equiv i_2 \pmod{n}$ since $i_n \not\equiv 0 \pmod{n}$. In this way we conclude that $i_2 + i_n \equiv i_1 \equiv 0 \pmod{n}$ and $i_n \equiv -i_2 \pmod{n}$.

Repeating the same argument for the matrix units e_{3r}, we obtain that $i_r - i_3 \not\equiv i_n \equiv -i_2 \pmod{n}$, $r \geq 4$. It follows that $i_3 - i_2$ is not congruent to either i_r, $r \geq 4$. So it remains that $i_3 - i_2 \equiv 0, i_2, i_3 \pmod{n}$. Since neither of i_2 and i_3 is congruent to $0 \pmod{n}$ we have that $i_3 \equiv 2i_2 \pmod{n}$. In a similar manner one goes on by induction and shows that $i_r \equiv (r-1)i_2 \pmod{n}$ for all r. $\diamondsuit$

Now we turn our attention to the case when G is cyclic of order two. We write $G = \{1, -1\}$, and the corresponding gradings are called 2-gradings. In this case we shall use the letters y_i for the variables in $K(X)$ whose homogeneous degree is 1, and z_i for the variables of homogeneous degree -1.

We start with an algorithm that describes the grading given the one on the first row.

Lemma 9 *Let $a \in UT_n$. If the elements of the i-th row are graded by $\varepsilon_{ii} = 1, \varepsilon_{i,i+1}, \ldots, \varepsilon_{in}$ respectively, then the elements of the $i+1$-st row are graded by*

$$\varepsilon_{i+1,i+1} = 1, \varepsilon_{i,i+1}\varepsilon_{i+1,i+2}, \varepsilon_{i,i+1}\varepsilon_{i+1,i+3}, \ldots, \varepsilon_{i,i+1}\varepsilon_{i+1,n},$$

respectively. In other words, the grading on the $i+1$-st row is exactly the same as that on the corresponding entries in the i-th row when $e_{i,i+1} \in (UT_n)_1$, and it is the opposite when $e_{i,i+1} \in (UT_n)_{-1}$. Here opposite means that we multiply by -1 the grading on the i-th row.

Proof. The proof is straightforward consequence of Proposition 5. $\diamondsuit$

Lemma 10 *Let $\tilde{g} = (g_1, \ldots, g_{k_1}, g_{k_1+1}, \ldots, g_{k_2}, \ldots, g_{k_t+1}, \ldots, g_{k_{t+1}}) \in G^n$ where $g_1 = \ldots = g_{k_1} = 1$, $g_{k_1+1} = \ldots = g_{k_2} = -1$, $g_{k_2+1} = \ldots = g_{k_3} = 1$, etc. Then, in the corresponding grading of UT_n, $z_1z_2\ldots z_{t+1} = 0$ is a 2-graded identity but $z_1z_2\ldots z_t = 0$ is not.*

Proof. We have that $g_{k_{i-1}+1}^{-1} g_{k_i+1} = -1$ for all $i = 1, 2, \ldots, t$ (we put $k_0 = 0$). Hence the matrix units $e_{k_{i-1}+1,k_i+1}$ are of homogeneous degree -1 for every i, and their product

$$e_{1,k_1+1} e_{k_1+1,k_2+1} \ldots e_{k_{t-1}+1,k_t+1} \neq 0$$

in UT_n. This shows that $z_1 z_2 \ldots z_t = 0$ is not a graded identity for UT_n with the respective 2-grading.

In order to prove that $z_1 z_2 \ldots z_{t+1} = 0$ is a 2-graded identity we observe that $e_{ij} \in A_{-1}$ if and only if $g_i^{-1} g_j = -1$ i.e. $g_i g_j = -1$. Consider a product of matrix units $e_{i_1,i_2} e_{i_2,i_3} \ldots e_{i_t,i_{t+1}} e_{i_{t+1},i_{t+2}}$ all belonging to A_{-1}. Then $i_1 < i_2 < \ldots < i_{t+1} < i_{t+2}$, and $g_{i_r} g_{i_{r+1}} = -1$ for all r. So g_{i_r} belongs to some of the groups of consecutive 1's or -1's, $g_{i_{r+1}}$ belongs to another group of -1's, respectively 1's. But we have $t+1$ such groups. Therefore such product should be identically 0, and this proves the statement of the lemma. $\diamondsuit$

Lemma 11 *Let $\tilde{g} = (1, 1, \ldots, 1, -1, -1, \ldots, -1)$ where the first k entries are 1's and the last $n-k$ are -1's. Consider the elementary grading on UT_n induced by $\tilde{g}$. If $k > n-k$ then*

1. The ideal $Id(UT_n, \tilde{g})$ of 2-graded identities for UT_n is generated by

$$z_1 z_2, \quad [y_1 y_2] \ldots [y_{2k-1}, y_{2k}], \quad z[y_1, y_2] \ldots [y_{2(n-k)-1}, y_{2(n-k)}].$$

2. A linear basis for the proper polynomials in the relatively free graded algebra $K(X)/Id(UT_n, g)$ consists of the polynomials

$$[y_{i_1,1}, y_{i_2,1}, \ldots, y_{i_p,1}] \ldots [y_{i_1,r}, \ldots, y_{i_{p_r},r}] z [y_{j_1,1}, \ldots, y_{j_p,1}] \ldots [y_{j_1,s}, \ldots, y_{j_{p_s},s}]$$

where $0 \leq r \leq k-1$, $0 \leq s \leq n-k-1$, and

$$[y_{i_1,1}, y_{i_2,1}, \ldots, y_{i_{t_1},1}] \ldots [y_{i_1,r}, \ldots, y_{i_{t_r},r}]$$

for $0 \leq r \leq k-1$. For both types of polynomials we require that the commutators $[y_{p_1}, y_{p_2}, \ldots, y_{p_q}]$ satisfy the inequalities $p_1 > p_2 \leq p_3 \leq \ldots \leq p_q$.

Proof. Set I the graded ideal generated by the three graded identities of item (1). Then any nonzero homogeneous element of the quotient $K(X)/I$ is of one of the following two types:

$$y_{i_1} \ldots y_{i_t} z y_{j_1} \ldots y_{j_v} + I, \qquad y_{i_1} \ldots y_{i_t} + I.$$

Since $y_{i_1}\dots y_{i_t}zy_{j_1}\dots y_{j_v}\notin I$ it follows that $y_{i_1}\dots y_{i_t}$ does not belong to the graded ideal generated by $[y_1,y_2]\dots[y_{2k-1},y_{2k}]$. Therefore by $y_{i_1}\dots y_{i_t}$ can be written as a linear combination of products of at most $k-1$ commutators in $K(X)/I$ (see for example [16], Chapter 5.2. Analogously $zy_{j_1}\dots y_{j_v}$ is not in the graded T-ideal generated by $z[y_1,y_2]\dots[y_{2(n-k)-1},y_{2(n-k)}]$ and it must be a linear combination of products of at most $n-k-1$ commutators multiplied on the left hand side by z. Therefore the given monomials do span the relatively free graded algebra. They are linearly independent modulo $Id(UT_n,\tilde{g})$ due to the same reasoning as in [16], pp. 52–55. Thus the proof of the lemma is completed. $\diamondsuit$

The proofs of the following three lemmas are quite similar to the one above and that is why we omit them.

Lemma 12 *Let $\tilde{g}=(1,1,\dots,1,-1,-1,\dots,-1)$ and let UT_n be graded by the elementary grading induced by g. Suppose that in $\tilde{g}$ there are k entries 1 and $n-k$ entries -1, and that $k<n-k$. Then*

1. The ideal $Id(UT_n,\tilde{g})$ is generated by the polynomials

$$z_1z_2,\quad [y_1,y_2]\dots[y_{2(n-k)-1},y_{2(n-k)}],\quad [y_{2(n-k)-1},y_{2(n-k)}]z.$$

2. A basis for the vector space of the proper polynomials in the relatively free graded algebra $K(X)/Id(UT_n,\tilde{g})$ consists of the polynomials

$$[y_{i_1,1},y_{i_2,1}\dots,y_{i_{t_1},1}]\dots[y_{i_1,r},y_{i_2,r}\dots,y_{i_{t_r},r}],\qquad 1\le r\le n-k-1,$$

and the polynomials

$$\begin{aligned}&[y_{i_1,1},y_{i_2,1}\dots,y_{i_{p_1},1}]\dots[y_{i_1,r},y_{i_2,r}\dots,y_{i_{p_r},r}]z\times\\ &\qquad\times[y_{j_1,1},y_{j_2,1}\dots,y_{j_{q_1},1}]\dots\quad[y_{j_1,s},y_{j_2,s}\dots,y_{j_{q_s},s}]\end{aligned}$$

for $0\le r\le n-k-1$, $0\le s\le k-1$, with the same restrictions on the variables y in the commutators as in the preceding lemma. $\diamondsuit$

Lemma 13 *Let $\tilde{g}=(1,1,\dots,1,-1,-1,\dots,-1)$ with k entries 1 and k entries -1, $2k=n$. Then:*

1. The ideal $Id(UT_n,\tilde{g})$ is generated by

$$z_1z_2,\quad [y_1,y_2]\dots[y_{2k-1},y_{2k}].$$

2. A basis for the vector space of the proper polynomials in the relatively free graded algebra consists of the following polynomials:

$$[y_{i_1,1}, y_{i_2,1} \dots, y_{i_{t_1},1}] \dots [y_{i_1,r}, y_{i_2,r} \dots, y_{i_{t_r},r}], \qquad 1 \le r \le k-1,$$

and also the polynomials

$$[y_{i_1,1}, y_{i_2,1} \dots, y_{i_{p_1},1}] \dots [y_{i_1,r}, y_{i_2,r} \dots, y_{i_{p_r},r}] z \times$$
$$\times [y_{j_1,1}, y_{j_2,1} \dots, y_{j_{q_1},1}] \dots \quad [y_{j_1,s}, y_{j_2,s} \dots, y_{j_{q_s},s}]$$

for $0 \le r \le k-1$, $0 \le s \le k-1$, with the same restrictions on the variables y in the commutators as in the preceding two lemmas. $\diamondsuit$

Lemma 14 *Let $\tilde{g} = (1, -1, 1, -1, 1, -1 \dots)$. Then the ideal $Id(UT_n, \tilde{g})$ is generated by*

$$[y_1, y_2] \dots [y_{t-1}, y_t], \quad f_j = w_0^j z_1 w_1^j z_2 \dots z_t w_{t+1}^j$$

where $t = [(n+1)/2]$ (the integer part), and w_i^j are monomials of the type $[y_1, y_2] \dots [y_{2i-1}, y_{2i}]$; the monomials w_0^j and w_{t+1}^j may be empty, and the total degree of every f_j equals n. $\diamondsuit$

Observe that in the last lemma one has $[A_1 A_1] = A_{-1} A_{-1}$, hence the above identities can be deduced from $z_1 z_2 \dots z_n = 0$. In general one has only the inclusion $A_{-1} A_1 \subseteq A_1 A_1$.

Now we deal with the generic case. Suppose that the elementary 2-grading on UT_n is induced by $\tilde{g} = \varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ where $\varepsilon_i = \pm 1 \in \mathbb{Z}_2$ and $\varepsilon_1 = 1$. Then the matrix unit e_{ij} is of homogeneous degree $\varepsilon_i \varepsilon_j$, for every $1 \le i \le j \le n$. Consider the set of all finite non decreasing sequences whose terms are positive integers in $[1, n]$. If $a = (a_1, a_2, \dots, a_t, a_{t+1})$ is such a sequence then one can form the *staircase* of the following matrix units: $s(a) = (e_{a_1,a_2}, e_{a_2,a_3}, \dots, e_{a_{t-1},a_t}, e_{a_t,a_{t+1}})$. Since their product is $e_{a_1,a_{t+1}}$ it is nonzero. Let us consider the sequence $\varepsilon(a) = (\varepsilon_{a_1}, \varepsilon_{a_2}, \dots, \varepsilon_{a_{t+1}})$. Then $e_{a_k,a_{k+1}}$ is of homogeneous degree $\varepsilon_{a_k} \varepsilon_{a_{k+1}}$. This observation justifies our next definition.

We call a sequence $(\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_{t+1}})$, $t \ge 1$, *admissible* if it can be obtained as $\varepsilon(a)$ for some sequence a as above. Let $m = x_{i_1} x_{i_2} \dots x_{i_t}$ be a monomial in the free 2-graded algebra $K(X)$, $x_{i_j} = y_{i_j}$ or z_{i_j}. We call it admissible if the homogeneous degree of the variable x_{i_j} is $\varepsilon_{i_j} \varepsilon_{i_{j+1}}$ for all $j = 1$, 2, ..., t. Therefore the signs $+$ and $-$ alternate in the monomial m exactly in the same way as in the sequence $(\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_3, \ldots, \varepsilon_{t-2}\varepsilon_{t-1}, \varepsilon_t\varepsilon_{t+1})$. Let us denote the monomial m by $m(\varepsilon)$. If $m = m(\varepsilon)$ is an admissible monomial, $(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{t+1}})$, then an *admissible* substitution in $m(\varepsilon)$ is the substitution where the first variable is replaced by $e_{i_1 i_2}$, the second by $e_{i_2 i_3}$ and so on.

Proposition 15 *The multilinear monomial $m(\varepsilon)$ is not a graded identity for UT_n if and only if the sequence ε is admissible.*

Proof. Suppose first that $m(\varepsilon)$ is not a graded identity for UT_n. There exist homogeneous elements $a_1, a_2, \ldots, a_t \in UT_n$ such that $m(a_1, a_2, \ldots, a_t) \neq 0$ in UT_n. Write every a_i as a linear combination of the matrix units (which are homogeneous), and then expand all products. At least one of the summands will be nonzero. Hence we can choose a_i among the matrix units. If $a_k = e_{i_k, j_k}$ then $i_k \leq j_k$ and $j_k = i_{k+1}$ for all k. Therefore we can choose the matrix units e_{i_1,i_2}, e_{i_2,i_3}, ..., e_{i_{t-1},i_t}, $e_{i_t,i_{t+1}}$ such that the evaluation of m on them is nonzero. Now taking into account that the homogeneous degree of e_{ij} equals that of $e_{1i}e_{1j}$ we get that the sequence $(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_t}, \varepsilon_{i_{t+1}})$ is admissible since it is a subsequence of the sequence $(\varepsilon_1 = 1, \varepsilon_2, \ldots, \varepsilon_n)$.

Now let the sequence ε be admissible. Then as above we form the respective admissible monomial that evaluates on some matrix unit hence is nonzero on UT_n. $\diamondsuit$

Theorem 16 *There are 2^{n-1} different elementary 2-gradings on the algebra UT_n. Two different gradings satisfy different graded polynomial identities.*

Proof. The first statement is straightforward since there are 2^{n-1} different sequences $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of ± 1 and with $\varepsilon_1 = 1$. In order to prove the second statement we observe that for any grading $\tilde{g}$ the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \tilde{g}$ is admissible. It follows that the corresponding admissible monomial $m_{\tilde{g}} = x_1 x_2 \ldots x_{n-1}$ is a product of $n-1$ homogeneous variables x_i, ($x_i = y_i$ or z_i), with homogeneous degree $\varepsilon_i\varepsilon_{i+1}$, $i = 1, 2, \ldots, n-1$. Hence $x_i = y_i$ or z_i according as $\varepsilon_i\varepsilon_{i+1} = 1$ or -1, respectively. Let $\overline{m_{\tilde{g}}}$ the polynomial obtained from $m_{\tilde{g}}$ by substituting any variables of homogeneous degree 1 with a commutator of two variables of homogeneous degree 1. So, if $\varepsilon_i\varepsilon_{i+1} = 1$ then we substitute $x_i = y_i$ with $\overline{y}_i = [y_i, y_{n-1+i}]$. It follows that $\overline{m_{\tilde{g}}}$ is a polynomial in the variables $\overline{y}_i$ and z_j of degree $n-1$. Moreover these variables can be evaluated on matrix units $e_{p,q}$ with $p < q$. Since the elements of the staircase $e_{12}, e_{23}, \ldots, e_{n-1,n}$ have the same homogeneous degree as the variables $x_1, x_2, \ldots, x_{n-1}$, respectively, it follows that $\overline{m_{\tilde{g}}}$ is not an identity for the grading g.

If $\tilde{g}'$ is a different grading then the admissible monomials $m_{\tilde{g}}$ and $m_{\tilde{g}'}$ corresponding to the sequences $\tilde{g}, \tilde{g}'$ will be different. Then $\overline{m_{\tilde{g}}}$ is a graded identity for the grading g' and $\overline{m_{\tilde{g}'}}$ is a graded identity for the grading g. $\diamondsuit$

Corollary 17 *There are 2^{n-1} nonisomorphic elementary 2-gradings on UT_n.*

Proof. Isomorphic gradings satisfy the same graded identities. $\diamondsuit$

Instead of describing bases of the graded identities for the different gradings we describe the corresponding relatively free algebras (assuming that the field K is infinite).

Theorem 18 *Let $\tilde{g}$ be an elementary 2-grading on UT_n. Then the admissible monomials form a basis of the vector space of all multilinear polynomials in the relatively free graded algebra.*

Proof. The admissible monomials span the multilinear component of the relatively free algebra. We show that the admissible monomials are linearly independent. Suppose on the contrary that $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_k m_k = 0$ is a graded identity for UT_n with the elementary grading induced by some g. Here all $\alpha_i \in K$ are nonzero scalars, and m_i are different admissible monomials of the same multidegree. Let $m_1 = x_{i_1} x_{i_2} \ldots x_{i_t}$ where $x_{i_j} = y_{i_j}$ or z_{i_j}. Then one makes an admissible substitution (with respect to m_1) in order to obtain nonzero element in UT_n. But the same substitution will vanish the rest of the monomials since they differ from m_1. Hence $\alpha_1 = 0$, a contradiction.

Remark 19 *The last theorem describes completely the relatively free graded algebra in the case when $\operatorname{char} K = 0$.*

Remark 20 *In order to describe the graded identities satisfied by given elementary grading on UT_n one considers the* nonadmissible *monomials. It is rather straightforward to show that one may choose a basis of the identities among such monomials. Furthermore one may consider multilinear monomials only. The last observation is justified by the fact that no variable z (i.e. odd variable) can participate twice in a monomial, and if some variable y participates k times in a monomial m then $m = m_1 y^k m_2$ for some monomials m_1 and m_2.*

Remark 21 *The last several assertions dealt with 2-gradings on UT_n. But their proofs do not depend on the specific group $\mathbb{Z}_2$. Therefore one modifies the assertions accordingly and gets the same results for elementary gradings by* any *group G. Thus if G is of order m then there will be m^{n-1} different (nonisomorphic) elementary G-gradings on UT_n. Two different such gradings will satisfy different graded identities, and so on. In the same manner as above one shows that the relatively free graded algebras admit linear bases consisting of the admissible monomials.*

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