Geometric proprieties of invariant connections on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$

Marco A. N. Fernandes*
Instituto de Matemática
Universidade Federal da Bahia
Av. Ademar de Barros s/n
40.170-110 Salvador – BA, Brasil

Luiz A. B. San Martin†
Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065
13.081-970 Campinas – SP, Brasil.

Abstract

This article describes some geometric aspects of a class of affine connections in homogeneous spaces, that emerged in an earlier paper by the authors, related to the geometry of statistical models. We describe the geodesics as well some properties of the curvature of these connections.

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*e-mail: marcoanf@ufba.br
†e-mail: smartin@ime.unicamp.br. Research partially supported by CNPq Grant n° 301060/94-0

1 Introduction

The α-connections where introduced in the statistical literature in the eighties (see Amari [1] and Barndorff-Nielsen [2]) as a differential geometric tool for studying parametric models. The idea was to refine the classical concept of Fisher information, which is a Riemannian metric attached to a statistical model. Both these geometric objects are defined by means of integrals over some measure space, making them very hard to analyze by the standard differential geometric methods.

In [4] the authors consider a set up, based on Lie group theory, in which it is possible to take advantage of the symmetries and describe the possible affine connections in homegenous spaces arising as α-connections of the so-called transformational statistical models. In particular, we were laid to consider invariant affine connections on symmetric spaces. It was proved in [4] that only those symmetric spaces whose restricted root system are of type A_l admit such connections which are different from the canonical Riemannian one. In particular, the non-compact symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ has a 1-parameter family of invariant connections, which at the origin $o = \mathrm{SO}\,(n)$ is given by

$$\overset{(\alpha)}{\nabla}_A B = \alpha\left(\frac{AB+BA}{2} - \frac{\mathrm{tr}(AB)}{n} I_n\right)$$

Here $A, B \in \mathfrak{s}$, the subspace of the symmetric matrices with zero trace, which we identify to the tangent space of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ at o and I_n is the identity $n \times n$ matrix. In [4] some properties of these connections were already discussed. However, many questions related to their geometry remained unsolved. The purpose of this article is to develop these properties further.

When $\alpha = 1$ we denote the connection simply by $\nabla_A B$. This is the only case to be considered, since for general $\alpha \neq 0$, the computations are similar. For this connection we describe its geodesics in Section 2. Afterwards, in Section 3, we prove that the curvature tensor $R(A,B,C)$ and all its covariant derivatives belong to the subespace spanned by A and B. We apply this fact to prove the following properties of ∇: (i) the Ricci tensor is zero and (ii) ∇ is not compatibile with any Riemannian metric.

Before starting, it is convenient to fix some notations, remind a few of the geometry of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, more details on the subject can be seen in [5]. Write $M = G/K = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, the symmetric space of the positive definite matrices and $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}\,(n) \oplus \mathfrak{s}$, for the Cartan decomposition of the Lie algebra of G given by splitting the tangent space at the origin of M.

The group G acts transitively in M by $g(hK) = (gh)K$, and for each $g \in G$, the map $g : M \to M$ defined for $g(\xi) = g\xi$ is a diffeomorfism that satisfies:

$$(dg)_\xi \left(\tilde{A}(\xi)\right) = (\mathrm{Ad}(g)(A))^\sim (g\xi) \tag{1}$$

where

$$\tilde{A}(\xi) = \frac{d}{dt} (\exp(tA)(\xi)) |_{t=0}$$

and $\mathrm{Ad}(g) : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$, $g \in G$, is the adjoint map. Still in relation to the Lie algebra of G, we know that the roots of $\mathfrak{sl}(n, \mathbb{R})$ relative at $\mathfrak{h}$, Cartan subalgebra formed for the diagonal matrices of trace zero, are given by

$$(\lambda_i - \lambda_k)(H) = (H, E_{ii} - E_{kk}) = \mathrm{tr}(H(E_{ii} - E_{kk})),$$

for each $i \neq k$, $i, k = 1, \ldots, n$ where E_{ik} is the basic $n \times n$ matrix whose ik entry is 1 and all the others are zero.

2 Geodesics

Using a geometric caracterization of the α-connections, made in [4] we shall obtain here a description of the geodesics for the α-connections in the symmetric space $M = G/K$ of positive definite matrices.

The α-connections for a models considered here have ageometric interpretation which were described in [4]. Let $\mathcal{S}$ be the vector space of all symmetric $n \times n$ matrices. We have that $\mathfrak{s} = \{A \in \mathcal{S} : \mathrm{tr}A = 0\}$ is a subspace of codimension one of $\mathcal{S}$, complemented bythe line spanned by the identity 1. The trace form $\mathrm{tr}(AB)$ an inner product on $\mathcal{S}$. With respect to this inner product the line of scalar matrices is orthogonal to $\mathfrak{s}$. We denote by $\mathcal{S}^+$ the cone of the positive semi-definite matrices in $\mathcal{S}$.

There is a natural action of $\mathrm{Sl}(n, \mathbb{R})$ on $\mathcal{S}$ given by the law

$$(g, A) \longmapsto g \cdot s = gAg^*$$

where g^* means transposition of matrix. The induced infinitesimal action of $\mathfrak{sl}(n, \mathbb{R})$ on $\mathcal{S}$ is given by the derivative

$$\frac{d}{dt}\left(e^{tX} s e^{tX^*}\right)_{|t=0} = Xs + sX.$$

Each $X \in \mathfrak{sl}(n, \mathbb{R})$ induces the linear vector field

$$\tilde{X}(s) = Xs + sX.$$

If $a \cdot 1$, $a \neq 0$ is a scalar matrix in $\mathcal{S}$ then its orbit $\mathcal{O}(a)$ under $\mathrm{Sl}(n, \mathbb{R})$ is the subset of matrices with determinant a^n which are positive definite if $a > 0$ or negative definite if $a < 0$. Since $g(a \cdot 1) g^* = a \cdot 1$ if and only if g is an orthogonal matrix, it follows that $\mathcal{O}(a)$, $a \neq 0$, identifies with the homogeneous space $\mathrm{Sl}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$. These orbits have codimension one in $\mathcal{S}$, and the tangent space $T_{a1}\mathcal{O}(a)$ is the subspace of matrices with trace zero. Note that the line of scalar matrices complements $T_{a1}\mathcal{O}(a)$ in $\mathcal{S}$. Similarly, one checks easily that

$$\mathcal{S} = T_s\mathcal{O}(a) \oplus [s] \tag{2}$$

where $[s]$ stands for the line spanned by $s \in \mathcal{O}(a)$. From this decomposition we obtain the following connection ∇ on $\mathcal{O}(a)$:

$$(\nabla_X Y)(s) = \mathrm{pr}_s\left((dY)_s (X(s))\right). \tag{3}$$

Here $\mathrm{pr}_s : \mathcal{S} \to T_s\mathcal{O}(a)$ is the projection coming from the decomposition in (2), and X, Y are vector fields in $\mathcal{O}(a)$ with Y viewed as a mapping $Y : \mathcal{O}(a) \to \mathcal{S}$ so that $(dY)_s$ stands for its differential at s. The definition of ∇ is analogous to the Levi-Civita connection of the Riemannian metric induced in an immersed submanifold of an Euclidean space. However here the projection is not orthogonal with respect to $\mathrm{tr}(AB)$, since the line $[s^{-1}]$ is orthogonal to $T_s\mathcal{O}(a)$ so that pr_s is orthogonal if and only if $s = a \cdot 1$. Each orbit $\mathcal{O}(a)$, $a \neq 0$ is diffeomorphic to $\mathrm{Sl}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$. Hence we have a family of connections ∇^a in $\mathrm{Sl}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$. It was checked in [4] that ∇^a is a α-connection for each a.

In discussing geodesics we simplify matters and take $a = 1$. The other cases follow analogously. Thus we consider the orbit $\mathcal{O}(1)$ and put $\nabla = \nabla^1$. Due to invariance its sufficient to find the geodesics going through the origin. Let $s = s(t)$, $\det s = 1$, be a geodesic of ∇. Then $\nabla_{\dot{s}}\dot{s} = 0$ and therefore by the above description of ∇ the projection of $\ddot{s}$ is annihilates. This means that $\ddot{s} = es$ for some constant e. We compute this constant by taking the second derivative of the equality $\det s = 1$. Using a well knonw formula for the derivative of the determinat we get

$$\mathrm{tr}\left(s^{-1}\dot{s}\right)\det s = 0,$$

hence $\operatorname{tr}(s^{-1}\dot{s})\det s = 0$. Taking another derivative and using $(s^{-1})' = -s^{-1}\dot{s}s^{-1}$, we get

$$\left(-\operatorname{tr}\left(s^{-1}\dot{s}s^{-1}\dot{s}\right) + \operatorname{tr}\left(s^{-1}\ddot{s}\right)\right)\det s + \operatorname{tr}\left(s^{-1}\dot{s}\right)^2 \det s = 0.$$

But $\operatorname{tr}(s^{-1}\dot{s}) = 0$ and $\det(s) = 1$. Hence,

$$\operatorname{tr}\left(s^{-1}\ddot{s}\right) = \operatorname{tr}\left(s^{-1}\dot{s}s^{-1}\dot{s}\right).$$

Now, $\ddot{s} = es$, so that $\operatorname{tr}(s^{-1}\ddot{s}) = e\operatorname{tr}(1) = ne$. Therefore, the equation satisfied by the geodesics through the identity is

$$\ddot{s} = \frac{\operatorname{tr}(s^{-1}\dot{s}s^{-1}\dot{s})}{n}s \tag{4}$$

Since this equation looks hard to integrate explicitly we shall give a geometric description of the trace of the geodesics, and then write down a reparameterization of them.

Proposition 2.1 *The traces of the geodesic of ∇ in $\mathcal{S}_1$ are the subsets*

$$\mathcal{S}_1 \cap V$$

where $V \subset \mathcal{S}$ is a 2-dimensional subespace which has non-empty intersection with $\mathcal{S}_1$.

Proof: We check first that $\mathcal{S}_1 \cap V$ is a curve, that is, a 1-dimensional submanifold in case the intersection is not empty. In fact, denote by P the restriction to V of det. Then P is a polinomial function on V. Take $s \in \mathcal{S}_1 \cap V$. By the well known formula for the differencial of det,

$$dP_s(s) = d(\det)_s(s) = \operatorname{tr}\left(ss^{-1}\right)\det(s) = n \neq 0.$$

This shows that every $s \in \mathcal{S}_1 \cap V$ is a regular point of P. Now, $\mathcal{S}_1 \cap V$ is a connected component of a level set of P. Hence the intersection is indeed a one-dimensional submanifold.

Next we verify that $\mathcal{S}_1 \cap V$ can be entirely parametrized by a curve $s(t)$ such that its second derivative $\ddot{s}$ is a multiple of s, and hence satisfies equation (4). For this let $\tilde{\nabla}$ de be the connection $\mathcal{S}_1 \cap V$ defined analogously to ∇ by projecting onto the tangent space along the line spanned by $s \in \mathcal{S}_1 \cap V$. Note that this is possible because $dP_s(s) \neq 0$, so that the line spanned by s is transversal to the tangent space of $\mathcal{S}_1 \cap V$ at s. Now, let $s : (\alpha, \omega) \in \mathbb{R} \to \mathcal{S}_1 \cap V$ be a geodesic of $\tilde{\nabla}$. Then $\ddot{s}(t)$ is a multiple of $s(t)$ for all t, by definition of $\tilde{\nabla}$. Hence, s satisfies (4), so that it is also a geodesic of ∇. The trace of this geodesic s is the whole $\mathcal{S}_1 \cap V$. In fact, suppose that $\lim_{t\to\omega} s(t) = s_\infty \in \mathcal{S}_1 \cap V$. Then by the usual argument we can extend s with a geodesic going through s_∞, concluding the proof. ∎

Now, we shall obtain parametrizations of the geodesic curves $\mathcal{S}_1 \cap V$. We restrict attention to those subspaces V containing the identity 1, having in mind that that the other subspaces are obtained by translation. In fact, if the 2-dimensional subspace V meets $\mathcal{S}_1$, then for some $g \in \mathrm{Sl}(n, \mathbb{R})$, gV contains 1, and we can use the equality

$$g(\mathcal{S}_1 \cap V) = \mathcal{S}_1 \cap gV$$

Thus let V be such that $1 \in V$ and $\dim V = 2$. As before let $\mathfrak{s}$ be the subespace of matrices with zero trace. Then there exists $A \in \mathfrak{s}$ such that V is spanned by $\{1, A\}$. If we take conjugation by an element of $\mathrm{SO}(n, \mathbb{R})$ we can assume that A is diagonal, that is,

$$A = \mathrm{diag}\{x_1, \ldots, x_n\}$$

with $x_1 + \cdots + x_n = 0$. In this case $\mathcal{S}_1 \cap V$ becomes the subset of diagonal matrices

$$\mathrm{diag}\{tx_1 + s, \ldots, tx_n + s\}$$

satisfying

$$(tx_1 + s) \cdots (tx_n + s) = 1, \qquad tx_i + s > 0. \tag{5}$$

To get a parametrization of this curve note that the matrices

$$\mathrm{diag}\{1/n + tx_1, \ldots, 1/n + tx_n\}, \qquad 1/n + tx_i > 0,$$

belong to the interior of the simplex

$$\Delta = \{(y_1, \ldots, y_n) : y_1 + \cdots + y_n = 1, y_i \geq 0\}.$$

On the other hand, the map $(y_1, \ldots, y_n) \in \mathrm{int}\Delta$

$$(y_1, \ldots, y_n) \in \mathrm{int}\Delta \mapsto \frac{1}{\sqrt[n]{y_1 \cdots y_n}} (y_1, \ldots, y_n)$$

is a bijection between Δ and the set of diagonal symmetric matrices with $\det = 1$. Thus a parametrization of our curve is given by

$$t \mapsto \frac{1}{\sqrt[n]{(1+tx_1)\cdots(1+tx_n)}}\left(1+tx_1,\ldots,1+tx_n\right). \tag{6}$$

Its domain is the largest interval such that $1+tx_i > 0$ for all i. At this point we re-order if necessary the basis so that the matrix $A = \operatorname{diag}(x_1,\ldots,x_n)$ satisfies $x_1 \geq \cdots \geq x_n$. In this case $x_1 > 0$ and $x_n < 0$ and the domain of definition of the above parametrization becomes

$$\left(-\frac{1}{x_1}, -\frac{1}{x_n}\right).$$

Now we use the parametrization (6) to write down some further equations related to the geodesic of the given curve. Another parametrization of (6) is obtained by writing $t = \phi(u)$. The reparametrization is a geodesic if and only if the second derivative is a multiple of the curve. Thus we write $s(u)$ as

$$s = \frac{1}{R}\left(1+\phi x_1,\ldots,1+\phi x_n\right),$$

where $R = \sqrt[n]{(1+\phi x_1)\cdots(1+\phi x_n)}$. In order to perform the computations we write

$$l_i = \log s_i$$

where s_i, $i = 1,\ldots,n$ are the coordinates of s. We have

$$l_i' = \frac{s_i'}{s_i} \qquad l_i'' = \frac{s_i''}{s_i} - \left(\frac{s_i'}{s_i}\right)^2. \tag{7}$$

Now the condition $\ddot{s} = cs$ for s to be a geodesic means that s_i''/s_i is independent of the index i. By the expressions (7) this happens if and only if $l_i'' + (l_i')^2$ does not depends on $i = 1,\ldots,n$. A straighforward computation yields:

- $l_i = \log(1+\phi x_i) - \dfrac{1}{n}\left(\log(1+\phi x_1) + \cdots + \log(1+\phi x_n)\right)$
- $l_i' = \phi'\left(\dfrac{x_i}{1+\phi x_i} - \dfrac{1}{n}\left(\dfrac{x_1}{1+\phi x_1} + \cdots + \dfrac{x_n}{1+\phi x_n}\right)\right)$

- $l_i'' = \phi'' \left(\frac{x_i}{1+\phi x_i} - \frac{1}{n} \left(\frac{x_1}{1+\phi x_1} + \cdots + \frac{x_n}{1+\phi x_n} \right) \right)$
$+ (\phi')^2 \left(-\frac{x_i^2}{(1+\phi x_i)^2} + \frac{1}{n} \left(\frac{x_1^2}{(1+\phi x_1)^2} + \cdots + \frac{x_n^2}{(1+\phi x_n)^2} \right) \right)$

- $(l_i')^2 = (\phi')^2 \left(\begin{array}{c} \frac{x_i^2}{(1+\phi x_i)^2} - \frac{2}{n} \frac{x_i}{1+\phi x_i} \left(\frac{x_1}{1+\phi x_1} + \cdots + \frac{x_n}{1+\phi x_n} \right) \\ + \frac{1}{n^2} \left(\frac{x_1}{1+\phi x_1} + \cdots + \frac{x_n}{1+\phi x_n} \right)^2 \end{array} \right)$

Looking at these expressions we see that the term of $l_i'' + (l_i')^2$ which depends explicitly on i is given by

$$\frac{x_i}{1+\phi x_i} \left(\phi'' - \frac{2(\phi')^2}{n} \left(\frac{x_1}{1+\phi x_1} + \cdots + \frac{x_n}{1+\phi x_n} \right) \right).$$

Therefore the re-parametrization ϕ turns the curve into a geodesic if and only if it satisfies the second order differential equation

$$\phi'' = \frac{2(\phi')^2}{n} \left(\frac{x_1}{1+\phi x_1} + \cdots + \frac{x_n}{1+\phi x_n} \right). \tag{8}$$

Taking logarithms this equation is written as

$$(\log \phi')' = \left(\frac{2}{n} \log (1+\phi x_1) \cdots (1+\phi x_n) \right)',$$

Hence (8) is equivalent to

$$\phi' = ((1+\phi x_1) \cdots (1+\phi x_n))^{2/n} + c, \tag{9}$$

where the constant c accounts for the initial condition in the second derivative in (8). If we choose ϕ so that $\phi(0) = 0$ and $\phi'(0) = 1$ we arrive at the equation for the geodesics.

Proposition 2.2 *The geodesics of the connection* $\nabla = \overset{(1/4)}{\nabla}$ *starting at the identity matrix* 1 *in the direction of the matrix*

$$A = \mathrm{diag}\{x_1, \ldots, x_n\} \qquad x_1 + \cdots + x_n = 0$$

is given by

$$\gamma\left(u\right)=\frac{1}{\sqrt{\phi'\left(u\right)}}\left(1+\phi\left(u\right)x_{1},\ldots,1+\phi\left(u\right)x_{n}\right).$$

where ϕ *is the solution of the first order differential equation*

$$\phi'=\left(\left(1+\phi x_{1}\right)\cdots\left(1+\phi x_{n}\right)\right)^{2/n}$$

with $\phi\left(0\right)=0$.

It is convenient to make the following remark about the domain of definition of the first order equation (9): First if some $x_i=0$ then the i-th term does not appear, hence we assume that $x_i\neq 0$ for every i. In this case the equation is not Lipschitz in ϕ when $\phi=-1/x_i$. Therefore, if we take $x_1\geq\cdots\geq x_n$ the the domain of definition of the equation is $(-1/x_1,-1/x_n)$, which is precisely the domain of the original parametrization.

3 The curvature tensor

Given a differentiable manifold M with an affine connection ∇, a *tensor of the type* (r,s) is a map

$$T:\underbrace{\chi\left(M\right)\times\cdots\times\chi\left(M\right)}_{\mathrm{r}\times}\rightarrow\underbrace{\chi\left(M\right)\times\cdots\times\chi\left(M\right)}_{\mathrm{s}\times}$$

that is linear in each component $\chi\left(M\right)$ considered as module on $C^{\infty}\left(M\right)$. The *covariant derivative of* T, ∇T is the tensor of type $(r+1,s)$ defined by

$$\begin{aligned}\left(\nabla T\right)\left(A_{1},\ldots,A_{r}\right) &= \left(\nabla_{A}T\right)\left(A_{1},\ldots,A_{r}\right)\\ &= \nabla_{A}\left(T\left(A_{1},\ldots,A_{r}\right)\right)\\ &= \sum_{i=1}^{r}T\left(A_{1},\ldots,\nabla_{A}\,A_{i},\ldots,A_{r}\right),\end{aligned}$$

for $A_1,\ldots,A_r,\in\ \chi(M)$. The *second covariant derivative of* T, $\nabla^2T=\nabla\left(\nabla T\right)$, is then a tensor of the type $(r+2,s)$ given by

$$\left(\nabla^{2}T\right)\left(A_{1},\ldots,A_{r},B\right)=\left(\nabla_{B}\left(\nabla T\right)\right)\left(A_{1},\ldots,A_{r}\right),$$

$A_1, \ldots, A_r, B \in \chi(M)$. In general, the *m-th covariant derivative* , $\nabla^m T$, is inductively defined by $\nabla\left(\nabla^{m-1}T\right)$.

Now, using the usual formula for the curvature

$$R(A,B,C) = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C, \qquad A,B,C \in \mathfrak{s},$$

a direct computation yields for $R(A,B,C)$ the following expression

$$\frac{1}{n}\left(\operatorname{tr}(AC)\,B - \operatorname{tr}(BC)\,A + \operatorname{tr}\left([A,B]C\right)I_n\right)\right) - \frac{[A,B]C}{4} - C[A,B]$$

Note that if we restrict R to the totally geodesic submanifold of the diagonal matrices of M, then

$$R(A,B,C) = \frac{1}{n}(\operatorname{tr}(AC)B - \operatorname{tr}(BC)A). \tag{10}$$

Our next objective it is enough to compute the covariant derivatives of R in this submanifold, that is, we want to compute $\nabla^m R(A,B,C)$, for $A,B,C \in \mathfrak{h}$. For this we introduce the tensor of the type $(r,0)$, $T_r : \mathfrak{s} \times \cdots \times \mathfrak{s} \to C^\infty(M)$, defined by

$$T_r(A_1,\ldots,A_r) = \operatorname{tr}(A_1 \cdots A_r).$$

An easy computation shows that

$$\begin{aligned} \nabla T_r\left(A_1,\ldots,A_{r+1}\right) &= -r\; T_{r+1}(A_1,\ldots,A_{r+1}) + \\ &\quad \frac{1}{n}\sum_{i=1}^{r}\left(T_2 \otimes T_{r-1}\right)\left(A_i, A_{r+1}, A_1, \ldots, \widehat{A}_i, A_r\right). \end{aligned}$$

Also, if we put $S\left(A_1,\ldots,A_r,A_{r+1}\right) = T_r\left(A_1,\ldots,A_r\right)A_{r+1}$, then we get

$$\left(\nabla T_r\right)\left(A_1,\ldots,A_r,A_{r+2}\right)A_{r+1} = \left(\nabla S\right)\left(A_1,\ldots,A_{r+1},A_{r+2}\right).$$

Using these notations we arrive at the following formulas:

- $R(A,B,C) = \dfrac{1}{n}\left(T_2(A,C)\;B - T_2(B,C)\;A\right)$.
- $(\nabla R)(A,B,C,D) = -\dfrac{2}{n}\left(T_3(A,C,D)\;B - T_3(B,C,D)\;A\right)$.

- $-\frac{n}{6}\left(\nabla^2 R\right)(A,B,C,D,E)$ is given by

  $$\left\{\left(T_4-\frac{1}{3n}U\right)(A,C,D,E)\right\}B\;+\;\left\{\left(T_4-\frac{1}{3n}U\right)(B,C,D,E)\right\}A,$$

  where $U(A,C,D,E)$ is the tensor

  $$T_2(A,E)\,T_2(C,D)+T_2(D,E)\,T_2(A,C)+T_2(C,E)\,T_2(A,D)$$

We can proceed successively and compute the covariant derivatives of any order. We shall refrain ourselves to develop a general formula for these derivatives. But it is clear from these formulas that the following statement holds.

Proposition 3.1 *If A and B are zero trace diagonal matrices then the covariant derivatives $\nabla^m R$ belong to the subspace spanned by A and B, for all $m\in\mathbb{N}$, where $\nabla^0 R=R$.*

In the sequel we shall obtain some applications of the formulas obtained so far.

First let us consider the *Ricci tensor*. For a general connection this is the $(2,0)$-tensor defined by

$$\mathrm{Ric}(A,B)=\mathrm{tr}\left(C\mapsto R(A,B,C)\right),\quad A,B,C\in\chi(M).$$

In our case if $A,B,C\in\mathfrak{s}$ then the map $C\mapsto R(A,B,C)$ is an element of $\mathfrak{gl}(\mathfrak{s})$, having trace zero. Hence, $\mathrm{Ric}\equiv 0$.

As a second application we ask weather there exists a Riemannian metric g *compatible* with the affine connection ∇. Recall that this holds if

$$Ag(B,C)=g(\nabla_A B,C)+g(B,\nabla_A C),\quad A,B,C\in\chi(M).$$

It is know that for a connection compatible with a given metric, the Lie algebra of the holonomy group in a point of the manifold is a subalgebra of $\mathfrak{so}(n)$ (see [6]). On the other hand, such Lie algebra is spanned by

$$(\nabla^m R)(A,B,C_1,\ldots,C_m),\quad A,B,C_1,\ldots,C_m\in\chi(M),\ \ m=0,1,2,\ldots$$

We shall use these facts to prove that ∇ is not compatible with any metric.

For this choose $A, B, C, D, E \in \mathfrak{h}$ satisfying $\mathrm{tr}\,(BC) = \mathrm{tr}\,(BE) = 0$, $DB = A$. Then by Proposition 3.1, we have

$$\langle (\nabla R)\,(A, B, C, D)\,, E\rangle = \langle (\nabla R)\,(A, B, E, D)\,, C\rangle.$$

Equivalently, $(\nabla R)\,(A, B, C)$ is a non-zero self-adjoint operator of $\mathfrak{h}$. Hence, it has real eigen-values, showing that this operator cannot belong to $\mathfrak{so}\,(n)$. This is enough to prove that ∇ is not compatible with a Riemannian metric.

We note that by (10)

$$\langle R\,(A, B, C)\,, D\rangle = \langle C, R\,(A, B, D)\rangle,$$

that is, $R\,(A, B) \in \mathfrak{so}\,(n)$, so that we in fact need the covariant derivative of the curvature.

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