Weighted norm inequalities for vector-valued singular integrals on homogeneous spaces

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Abstract

Let X be an homogeneous space and let E be an UMD Banach space with a normalized unconditional basis $(e_j)_{j\geq 1}$. Given an operator T from $L_c^\infty(X)$ in $L^1(X)$, we consider the vector-valued extension $\widetilde{T}$ of T given by $\widetilde{T}(\sum_j f_j e_j) = \sum_j T(f_j)e_j$. We prove a weighted integral inequality for the vector-valued extension of the Hardy-Littlewood maximal operator and a weighted Fefferman-Stein inequality between the vector-valued extensions of the Hardy-Littlewood and the sharp maximal operators, in the context of Orlicz spaces. We give sufficient conditions on the kernel of a singular integral operator to have the boundedness of the vector-valued extension of this operator on $L^p(X, W d\mu; E)$ for $1 < p < \infty$ and for a weight W in the Muckenhoupt's class $A_p(X)$. Applications to singular integral operators on the unit sphere S^n and on a finite product of local fields $\mathbb{K}^n$ are given. The versions of all these results for vector-valued extensions of operators of functions defined in a homogeneous space X and with values in an UMD Banach lattice are also given.

1 Introduction

The UMD property for Banach spaces plays a central role in the development of Vector-Valued Fourier Analysis. In spite of having been extensively studied (see e.g. [4, 2, 3, 19, 18, 10]), we point out that all the maximal

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operators and singular integral operators considered in these studies, are for functions defined in the euclidian space $\mathbb{R}^n$ or in the torus T^n.

J. Bourgain extended in [2] a result of vector-valued singular integral operators due to Benedek, Caldern and Panzone, to the context of UMD Banach spaces. The main goal of this paper is to prove a weighted extension of the result of Bourgain for vector-valued singular integral operators of functions defined in a homogeneous space X (Theorem 1.4).

In Section 2 we study weighted integral estimates for vector-valued extensions of maximal operators from Martingale Theory in the context of Orlicz spaces, which we apply in the proofs of Theorems 1.1 and 1.2 given in Section 3.

C. Fefferman and E. M. Stein introduced in [7] a technique to study the Hardy-Littlewood maximal operator. The dyadic decomposition of $\mathbb{R}^n$ is used as a fundamental tool in this technique. The idea is to obtain an integral estimate for the dyadic maximal operator and then, by a transference method, to obtain an integral estimate for the Hardy-Littlewood maximal operator. This technique was applied to study integral estimates for vector-valued extensions of this operator (see e.g. [7, 2, 25]) and to study weighted integral estimates for others maximal operators (see e.g. [21, 22, 26]).

In Section 3 we apply the technique by Fefferman and Stein for homogeneous spaces and we prove a weighted integral inequality for a vector-valued extension of the Hardy-Littlewood maximal operator (Theorem 1.1) and a weighted Fefferman-Stein inequality between vector-valued extensions of the Hardy-Littlewood and the sharp maximal operators (Theorem 1.2), in the context of Orlicz spaces.

In Section 4 we study singular integral operators. The proofs of Theorems 1.3, Theorem 1.4 and Corollary 1.1 are in Section 4.

In this section we give the statements of the main results of this paper.

Corollaries 1.1 and 1.2 are applications to vector-valued singular integral operators of functions defined in the unit sphere S^n and in a finite product of local fields $\mathbb{K}^n$, respectively.

In Theorems 1.5, 1.6 and 1.7 we consider vector-valued extensions of operators for functions defined in a homogeneous space X and with values in a UMD Banach lattice.

Let G be a locally compact Hausdorff topological group with unit element e, H a compact subgroup of G and $\pi : G \to G/H$ the canonical map. Let dg denote a left Haar measure on G, which we assume to be normalized in the case of G to be compact. If A is a Borel subset of G, we will denote by $|A|$ the Haar measure of A. The homogeneous space $X = G/H$ is the set of all left cosets $\pi(g) = gH, g \in G$, provided with the quotient topology. The Haar measure dg induces a measure μ on the Borel σ-field on X. For $f \in L^1(X)$,

$$\int_X f(x)d\mu(x) = \int_G f \circ \pi(g)dg.$$

The measure μ on X is invariable on the action of G, that is, if $f \in L^1(X)$, $g \in G$ and $R_g f(x) = f(g^{-1}x)$, then

$$\int_X f(x)d\mu(x) = \int_X R_g f(x)d\mu(x).$$

A quasi-distance on X is a map $d : X \times X \to [0, \infty)$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(gx, gy) = d(x, y)$ for all $g \in G,\ x, y \in X$;

(iv) there exists a constant $\eta \geq 1$ such that, for all $x, y, z \in X$,

$$d(x, y) \leq \eta[d(x, z) + d(z, y)];$$

(v) the balls $B(x, \ell) = \{y \in X : d(x, y) < \ell\},\ x \in X,\ \ell > 0$, are relatively compact and measurable, and the balls $B(\mathbb{1}, \ell), \ell > 0$, form a basis of neighborhoods of $\mathbb{1} = \pi(e)$;

(vi) (doubling condition) there exists a constant $A \geq 1$ such that, for all $\ell > 0$ and $x \in X$,

$$\mu(B(x, 2\ell)) \leq A\mu(B(x, \ell)).$$

Given a quasi-distance d on X, there exists a distance ρ on X and a positive real number γ such that d is equivalent to ρ^γ (see [16]). Therefore the family of d-balls is equivalent to the family of ρ^γ-balls and ρ^γ-balls are open sets. We can show that $\mu(B(x, \ell)) > 0$ for $x \in X$, $\ell > 0$, and that X is separable.

In this paper X will denote a homogeneous space provided with a quasi-distance d.

Given a Banach space E with norm $\|\cdot\|$ and a positive locally integrable function W on X, we denote by $L^p(X, W d\mu; E)$ or $L^p_E(W)$, $1 \le p < \infty$, the Bochner-Lebesgue space consisting of all E-valued (strongly) measurable functions f defined in X such that

$$\|f\|_{L^p_E(W)} = \left(\int_X \|f(x)\|^p W(x) d\mu(x)\right)^{1/p} < \infty.$$

We write $L^p_E(W) = L^p(W)$ when $E = \mathbb{R}$ and $L^p_E(W) = L^p_E(X) = L^p_E$ when $W = 1$.

Throughout this paper (except in Theorems 1.5, 1.6 and 1.7) E will denote a Banach space with the UMD property (for the definition see e.g. [4, 2, 3, 19]) and with a normalized unconditional basis $(e_j)_{j\ge1}$, and Φ will denote a non-decreasing continuous function on $[0,\infty)$ with $\Phi(0) = 0$ and satisfying the $\triangle_2$-condition, that is, there exists a constant $c > 0$ such that

$$\Phi(2\lambda) \le c\Phi(\lambda), \quad \lambda > 0. \tag{1.1}$$

We put $\Phi(\infty) = \lim_{\lambda\to\infty} \Phi(\lambda)$.

Let W be a positive locally integrable function on X and let $1 < p < \infty$. If there exists a constant C such that

$$\left(\frac{1}{\mu(B)}\int_B W d\mu\right)\left(\frac{1}{\mu(B)}\int_B W^{-1/(p-1)} d\mu\right)^{(p-1)} \le C, \tag{1.2}$$

for all ball $B = B(x,\ell)$, $\ell > 0$, $x \in X$, we say that W is a weight in the Muckenhoupt's class $A_p(X)$. If $W \in A_p(X)$, we denote by $C(p, W)$ the smallest constant C that satisfies (1.2). The class $A_\infty(X)$ is defined as the union of the classes $A_p(X)$, for $1 < p < \infty$.

Let f be a real-valued locally integrable function on X. The Hardy-Littlewood maximal operator M and the sharp maximal operator $M^\sharp$ are defined at f respectively by

$$Mf(x) = \sup_B \frac{1}{\mu(B)}\int_B |f(y)| d\mu(y)$$

and

$$M^\sharp f(x) = \sup_B \frac{1}{\mu(B)}\int_B |f(y) - f_B| d\mu(y),$$

where

$$f_B = \frac{1}{\mu(B)}\int_B f(y) d\mu,$$

and where the supremum is taken over all balls B, such that $x \in B$.

The following theorem extends results for the Hardy-Littlewood maximal operator given in [2, 25].

Theorem 1.1 *Let $W \in A_\infty(X)$ and suppose that Φ is a convex function. Then there exists a constant C, depending only on E, Φ, X and W such that,*

$$\int_X \Phi\left(\sup_{k\geq 1} \|\sum_{j=1}^{k} Mf_j(x)e_j\|\right) W(x)d\mu(x) \leq C\int_X \Phi\left(M(\|f\|)(x)\right) W(x)d\mu(x), \tag{1.3}$$

for all $f = \sum_j f_j e_j \in L^1_E$. Moreover, if $1 < p < \infty$, $W \in A_p(X)$ and $f \in L^p_E(W)$, then $\sum_j Mf_j e_j$ converges in $L^p_E(W)$ to a function $\widetilde{M}f$ and the operator $\widetilde{M}$ is bounded on $L^p_E(W)$.

There is an intimate relation between the Hardy-Littlewood maximal operator and the sharp maximal operator. This relation is contained in the inequality $\|Mf\|_p \leq C\|M^\sharp f\|_p$, $f \in L^{p_0}(\mathbb{R}^n)$, $0 < p_0 \leq p < \infty$. This inequality is known as the Fefferman-Stein inequality and it was proved in [8]. A weighted extension of this inequality and an unweighted extension for functions defined in a space of homogeneous type (in particular in a homogeneous space) are well known. The following theorem gives a weighted vector-valued extension of the Fefferman-Stein inequality for functions defined in a homogeneous space X.

Theorem 1.2 *Let $W \in A_\infty(X)$ and suppose that Φ is a convex function. Then there exists a constant C, depending only on E, Φ, X and W such that, for all $f = \sum_j f_j e_j \in \cup_{p>1} L^p_E$,*

$$\int_X \Phi\left(\|\sum_{j=1}^{\infty} Mf_j(x)e_j\|\right) W(x)d\mu(x) \leq C\int_X \Phi\left(\|\sum_{j=1}^{\infty} M^\sharp f_j(x)e_j\|\right) W(x)d\mu(x). \tag{1.4}$$

We say that a linear operator T defined in $L^\infty_c(X)$ and with values in the space of all measurable functions, is a singular integral operator if the following conditions hold:

(i) T has a bounded extension on $L^r(X)$ for some $r, 1 < r \leq \infty$;

(ii) there exists a kernel $K \in L^1_{loc}(X \times X \setminus \triangle)$, $\triangle = \{(x,x) : x \in X\}$, such that
$$Tf(x) = \int_X K(x,y)f(y)d\mu(y),$$
for all $f \in L^\infty_c(X)$ and almost all $x \notin \text{supp } f$.

Let T be a singular integral operator with a kernel K. We say that K satisfies the condition (H_∞) if

$$|K(x,y) - K(x,\mathbb{1})| \leq C\frac{d(y,\mathbb{1})}{d(x,\mathbb{1})\mu(B(\mathbb{1},d(x,\mathbb{1}))}$$

whenever $d(x,\mathbb{1}) > 2d(y,\mathbb{1})$, $\mathbb{1} = \pi(e)$. If $K^{'}(x,y) = K(y,x)$ satisfies (H_∞) we say that K satisfies $(H^{'}_\infty)$.

The following theorem is proved in Section 4.

Theorem 1.3 *Let $1 < p < \infty$, $W \in A_p(X)$ and let $(T_j)_{j\geq 1}$ be a sequence of operators from $L^p(W)$ in $L^p(W)$ such that, for every $r > 1$, there exists a constant C_r such that*

$$M^\sharp(T_jf)(x) \leq C_rM_rf(x), \;\; f \in L^\infty_c(X), \;\; j \geq 1. \tag{1.5}$$

Then for all $f = \sum_j f_je_j \in L^p_E(W)$ we have that $\sum_j T_jf_je_j$ converges in $L^p_E(W)$ and there exists a positive constant C_p such that

$$\|\sum_{j=1}^\infty T_jf_je_j\|_{L^p_E(W)} \leq C_p\|\sum_{j=1}^\infty f_je_j\|_{L^p_E(W)}. \tag{1.6}$$

It is easy to see that the condition (H_∞) for the kernel K of a singular integral operator implies the Hrmander's condition (H_1):

$$\int_{d(x,\mathbb{1})>2d(y,\mathbb{1})} |K(x,y) - K(x,\mathbb{1})|d\mu(x) \leq C < \infty.$$

The Hormander's condition was studied by R. R. Coifman and G. Weiss [6], by A. Korányi and S. Vági [14] and by B. Bordin and D. L. Fernandez [1]. It was proved that, if the kernel K satisfies (H_1) and $(H^{'}_1)$ then the singular integral operator is bounded on $L^p(X)$ for $1 < p < \infty$. The next result follows immediately from Lemma 4.2 in Section 4 and Theorem 1.3.

Theorem 1.4 *Let $1 < p < \infty$, $W \in A_p(X)$ and let T be a singular integral operator. Assume that the kernel K of T satisfies (H_∞), (H'_∞) and $K(gx, gy) = K(x, y)$ for all $x, y \in X$, $g \in G$. Then for all $f = \sum_j f_j e_j \in L^p_E(W)$ we have that $\sum_j T f_j e_j$ converges in $L^p_E(W)$ and there exists a positive constant C_p such that*

$$\| \sum_{j=1}^{\infty} T f_j e_j \|_{L^p_E(W)} \leq C_p \| \sum_{j=1}^{\infty} f_j e_j \|_{L^p_E(W)}. \tag{1.7}$$

The Theorem 1.4 for the euclidian space $\mathbb{R}^n$ and $W = 1$ was proved by Bourgain [2] and it was also studied in [19]. For $W = 1$ and $E = l^q$, $1 < q < \infty$ but for more general spaces X (spaces of homogeneous type) it was proved in [1, 20]. The Theorem 1.3 for $X = \mathbb{R}^n$ and $W = 1$ was proved in [19].

Let us consider the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \ |x| = 1\}$ provided with the Lebesgue measure $d\sigma$ and with the euclidian distance $d(x, y) = |x - y|$ and let $\mathbb{1} = (1, 0, \ldots, 0)$. A kernel $K \in L^1_{loc}(S^n \times S^n \setminus \triangle)$ satisfies the condition (H_∞) if there exists a constant C such that for $x, y \in S^n$ with $|x - \mathbb{1}| > 2|y - \mathbb{1}|$ we have

$$|K(x, y) - K(x, \mathbb{1})| \leq C \frac{|y - \mathbb{1}|}{|x - \mathbb{1}|^{n+1}}.$$

For $0 \leq r \leq 1$, $i, j \in \{1, 2, ..., n + 1\}$ and $x, y \in S^n$ ($x \neq y$ for $r = 1$), we define the kernels s_r, t_r, $K^r_{i,j}$ and K by

$$s_r(x, y) = \frac{2}{\omega_n} \frac{y - (y \cdot x)x}{|y - rx|^{n+1}},$$

$$t_r(x, y) = \frac{n - 2}{2r} \int_0^r s_\varrho(x, y) d\varrho,$$

$$K^r_{i,j}(x, y) = \frac{x_i y_j - x_j y_i}{|y - rx|^{n+1}}$$

and

$$K(x, y) = - \int_0^1 P_r(x, y) dr,$$

where $P_r(x, y)$ denote the Poisson kernel

$$P_r(x, y) = \frac{1}{\omega_n} \frac{1 - r^2}{|y - rx|^{n+1}}.$$

Let $q_r = s_r + t_r, 0 \leq r \leq 1$. For $f \in L^\infty(S^n)$ we define the operators R_r, $R^r_{i,j}$ and Λ, $0 \leq r \leq 1$ and $i, j \in \{1, 2, ..., n+1\}$, by

$$R_r f(x) = \int_{S^n} q_r(x,y) f(y) d\sigma(y),$$

$$R^r_{i,j} f(x) = \int_{S^n} K^r_{i,j}(x,y) f(y) d\sigma(y),$$

$$\Lambda f(x) = \int_{S^n} K(x,y) f(y) d\sigma(y),$$

with $x \in S^n$ if $0 \leq r < 1$ and $x \notin \operatorname{supp} f$ if $r = 1$.

The operator $R = R_1$ is called the Riesz transform on S^n and it was proved in Korányi-Vági [14, p. 636] that: $\lim_{r\to 1} R_r f = Rf$ there exists a.e. and in $L^p(S^n)$, $1 < p < \infty$; the operators R_r are uniformly bounded on $L^p(S^n)$, and $q_r(gx, gy) = q_r(x, y)$ for all $x, y \in S^n$, $g \in SO(n+1)$. The operators $R^r_{i,j}$ were considered in Coifman-Weiss [6, p. 76]. They are uniformly bounded on $L^2(S^n)$ and $K^r_{i,j}(gx, gy) = K^r_{i,j}(x, y)$ for all $x, y \in S^n$, $g \in SO(n+1)$. The operator Λ was studied in Levine [15, p. 508] where it was proved that: it is bounded on $L^p(S^n)$ for $1 \leq p \leq \infty$; if Y_k is a spherical harmonic of degree k then $\Lambda Y_k = -Y_k/(k+1)$, and $K(gx, gy) = K(x, y)$ for all $x, y \in S^n$, $g \in SO(n+1)$.

In Section 4 we prove the following result.

Corollary 1.1 *Let $1 < p < \infty$, $W \in A_p(S^n)$ and $T \in \{R_r, R^r_{i,j}, \Lambda : 0 \leq r \leq 1, 1 \leq i, j \leq n+1\}$. Then there exists a constant C_p such that,*

$$\|\sum_{j=1}^{\infty} T f_j e_j\|_{L^p_E(W)} \leq C_p \|\sum_{j=1}^{\infty} f_j e_j\|_{L^p_E(W)}, \tag{1.8}$$

for all $f = \sum_j f_j e_j \in L^p(S^n, W d\sigma; E)$.

A local field is any locally compact, non-discrete and totally disconnected field. Let $\mathbb{K}$ be a fixed local field and dx be a Haar measure of the additive group $\mathbb{K}^+$ of $\mathbb{K}$. The measure of a measurable set A of $\mathbb{K}$ with respect to dx we denote by $|A|$. Let m be the modular function for $\mathbb{K}^+$, that is, $m(\lambda)|A| = |\lambda A|$ for $\lambda \in \mathbb{K}$ and $A \subset \mathbb{K}$ measurable. We also denote $|x| = m(x)$. The sets

$$\mathbb{D} = \{x \in \mathbb{K} : |x| \leq 1\} \text{ and } \mathbb{B} = \{x \in \mathbb{K} : |x| < 1\}$$

are the ring of integers of $\mathbb{K}$ and the unique maximal ideal of $\mathbb{D}$, respectively. Let $q = p^c$ (p prime) be the order of the finite field $\mathbb{D}/\mathbb{B}$ and let π be a fixed element of maximum absolute value of $\mathbb{B}$. The Haar measure dx is normalized such that $|\mathbb{D}| = 1$ and thus $|\pi| = |\mathbb{B}| = q^{-1}$.

A local field $\mathbb{K}$ has a natural sequence of partitions by balls satisfying the conditions (i) and (ii) of Lemma 3.1 in Section 3, when we consider the distance $d(x, y) = |x - y|$. It follows from this remark that the Theorems 1.1 and 1.2 hold without the hypothesis of Φ being a convex function. The extension of these results for a finite product of local fields is an immediate consequence of a M. H. Taibleson's theorem (see [23, p. 548-549]).

A kernel $K \in L^1_{loc}(\mathbb{K}^n \times \mathbb{K}^n \setminus \triangle)$ satisfies the condition (H_∞) if for $x, y \in \mathbb{K}^n$ with $|x| > |y|$ we have

$$|K(x, y) - K(x, 0)| \leq C \frac{|y|}{|x|^{n+1}}.$$

Let $\omega(x)$ be a function defined on $\mathbb{K}^n$ and satisfying:

$$\omega(x) = \omega(\pi^j x), \ j \text{ integer}, \ \mathrm{x} \in \mathbb{K}^{\mathrm{n}};$$

$$\int_{|x|=1} \omega(x) dx = 0;$$

$$|\omega(x - \pi^j y) - \omega(x)| \leq C q^{-j}, \quad j \geq 1, \quad |x| = |y| = 1.$$

Then the kernel $\Psi(x, y) = \Psi(x - y)$ where

$$\Psi(x) = \frac{\omega(x)}{|x|^n}, \quad x \in \mathbb{K}^n \setminus \{0\},$$

satisfies (H_∞) and (H'_∞). For $f \in L^\infty(\mathbb{K}^n)$ and $x \in \mathbb{K}^n, x \notin \operatorname{supp} f$ we define

$$Uf(x) = \int_{\mathbb{K}^n} \Psi(x - y) f(y) dy.$$

The operator U was studied in Phillips-Taibleson [17] and it was proved that U is bounded on $L^p(\mathbb{K}^n)$ for $1 < p < \infty$. Therefore the next corollary follows from Theorem 1.4.

Corollary 1.2 *Let $1 < p < \infty$, $W \in A_p(\mathbb{K}^n)$. Then there exists a constant C_p such that,*

$$\|\sum_{j=1}^{\infty} U f_j e_j\|_{L^p_E(W)} \leq C_p \|\sum_{j=1}^{\infty} f_j e_j\|_{L^p_E(W)}, \tag{1.9}$$

for all $f = \sum_j f_j e_j \in L^p(\mathbb{K}^n, W dy; E)$.

Now let E be an UMD Banach lattice of real-valued measurable functions on a σ-finite measure space $(Y, \mathcal{B}, \nu)$. The absolute value of $h \in E$ is given by $|h|(y) = |h(y)|$, $y \in Y$. We identify a function $f \in L^p_E(W)$ with a function defined in the product $X \times Y$ setting $f(x)(y) = f(x, y)$. We denote by $L^p(W) \otimes E$ the set of all vector-valued functions f of the type $f = \sum_{j=1}^k a_j f_j$, for $a_j \in E$, $f_j \in L^p(W)$ and for a integer k, $k \geq 1$. This set is a dense subspace of $L^p_E(W)$ for $1 \leq p < \infty$ and any weight W. Given an operator T in $L^p(W)$, we define its extension $\overline{T}$ in $L^p(W) \otimes E$ (see Rubio de Francia [18]) in the following form:

$$\overline{T} f(x, y) = T(f(\cdot, y))(x), \ (x, y) \in X \times Y.$$

A characterization of UMD Banach lattice in terms of the extension $\overline{M}$ of the Hardy-Littlewood maximal operator, when $X = \mathbb{R}^n$, was given by Bourgain [2] (see also [18]). The Bourgain's characterization says that E has the UMD property, if and only if, $\overline{M}$ is bounded on $L^p_E(\mathbb{R}^n)$ and on $L^{p'}_{E'}(\mathbb{R}^n)$ for some p, $1 < p < \infty$, where p' is the conjugate exponent of p and X' is the dual space of E. The maximal operator $\overline{M}$ and others maximal operators of the same type were studied in [10, 12, 11], for $X = \mathbb{R}^n$. In [10] are given new characterizations of UMD Banach lattice in terms of maximal operators.

In Section 2 we consider the maximal operators $Nf = f^*$ and $N^\sharp f = f^\sharp$ from martingale theory and their vectorial extensions $\widetilde{N}(\sum_j f_j e_j) = \sum_j f_j^* e_j$, $\widetilde{N^\sharp}(\sum_j f_j e_j) = \sum_j f_j^\sharp e_j$. The analogous of Theorem 2.4 in Section 2 for the operator $\overline{N}$ was proved in [25]. By the same way we can prove the analogous of Theorem 2.7 for the operators $\overline{N}$ and $\overline{N}^\sharp$. Proceeding as in Section 3, we can apply the inequalities obtained for $\overline{N}$ and $\overline{N}^\sharp$ and prove the following theorems.

Theorem 1.5 *Let $W \in A_\infty(X)$ and suppose that Φ is a convex function. Then there exists a constant C, depending only on E, Φ, X and W such that, for all $f \in L^1(W) \otimes E$,*

$$\int_X \Phi\left(\|\overline{M} f(x)\|\right) W(x) d\mu(x) \leq C \int_X \Phi\left(M(\|f\|)(x)\right) W(x) d\mu(x). \quad (1.10)$$

Theorem 1.6 *Let $W \in A_\infty(X)$ and suppose that Φ is a convex function. Then there exists a constant C, depending only on E, Φ, X and W such that, for all $f \in L^1(W) \otimes E$,*

$$\int_X \Phi\left(\|\overline{M} f(x)\|\right) W(x) d\mu(x) \leq C \int_X \Phi\left(\|\overline{M}^\sharp f(x)\|\right) W(x) d\mu(x). \quad (1.11)$$

If $\Phi(t) = t^p$, $1 < p < \infty$, then we can extend the operators $\overline{M}$ and $\overline{M}^\sharp$ by a limit process to all $L^p_E(W)$ and the above theorems will hold for these extensions. Proceeding as in Section 4 we can apply Theorems 1.5 and 1.6 and prove the following analogous of the Theorem 1.4 for Banach lattice.

Theorem 1.7 *Let $1 < p < \infty$, $W \in A_p(X)$ and let T be a singular integral operator. Assume that the kernel K of T satisfies (H_∞), (H'_∞) and $K(gx, gy) = K(x, y)$ for all $x, y \in X$, $g \in G$. Then there exists a positive constant C_p such that, for all $f \in L^p(W) \otimes E$ we have that*

$$\|\overline{T}f\|_{L^p_E(W)} \leq C_p \|f\|_{L^p_E(W)}. \tag{1.12}$$

Moreover, the operator $\overline{T}$ can be continuously extended to all $L^p_E(W)$ and the above inequality holds for its extension and for all $f \in L^p_E(W)$.

Applying Theorem 1.7 we obtain the analogous of Corollaries 1.1 and 1.2 for the case of UMD Banach lattice.

2 Maximal Operators in Martingale Theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and for each $k = 0, 1, 2, ...$ let $\mathcal{A}_k$ be a partition of Ω by elements of $\mathcal{F}$ satisfying: $\mathbb{P}(Q) > 0$ for all $Q \in \mathcal{A}_k$; the σ-field $\mathcal{F}$ is generated by the union $\mathcal{A} = \cup_{k=0}^{\infty} \mathcal{A}_k$; the partition $\mathcal{A}_{k+1}$ is a refinement of $\mathcal{A}_k$, that is, for each $Q \in \mathcal{A}_k$, there exists an integer $n_Q \geq 1$ and $Q_1, ..., Q_{n_Q} \in \mathcal{A}_{k+1}$ such that $Q = Q_1 \cup ... \cup Q_{n_Q}$. We will denote by $\mathcal{F}_k$ the σ-field generated by $\mathcal{A}_k$ and we will always assume that the sequence $(\mathcal{A}_k)_{k \geq 0}$ is regular with respect to $\mathbb{P}$, that is, there exists an absolute constant $\theta \geq 1$ such that

$$\mathbb{P}(Q_1) \leq \theta \mathbb{P}(Q_2), \tag{2.1}$$

for all $Q_1 \in \mathcal{A}_k$ and $Q_2 \in \mathcal{A}_{k+1}$ with $Q_2 \subset Q_1, k \geq 0$.

Given a E-valued integrable function $f : \Omega \mapsto E$ we will also denote by f the martingale $(f_k)_{k \geq 0}$ where $f_k = E[f|\mathcal{F}_k]$ is the conditional expectation of the function f with respect to the σ-field $\mathcal{F}_k$. A stopping time is a function $T : \Omega \to \{0, 1, ..., \infty\}$ such that $\{T \leq k\} \in \mathcal{F}_k$ for all $k \geq 0$. For a stopping time T we denote by $\mathcal{F}_T$ the σ-field of all sets $A \in \mathcal{F}$ such that $A \cap \{T \leq k\} \in \mathcal{F}_k$, for all $k \geq 0$. The martingale transform "f

stopped at T" is defined by $f^T = (f_k^T)_{k\geq 0}, f_k^T(\omega) = f_{T(\omega)\wedge k}(\omega)$ and we write $f_T(\omega) = f_{T(\omega)}(\omega)$. We can show that

$$E[I(A)\{f - f_T\} \,|\mathcal{F}_k] = I(A)(f_k - f_k^T) \tag{2.2}$$

for all integrable functions $f : \Omega \to E$, all stopping times T, all $k \geq 0$ and all $A \in \mathcal{F}_T$, where $I(A)$ is the indicator function of the set A.

For a real-valued integrable function f we define the maximal functions

$$f^*(x) = \sup_{k\geq 0} |f_k(x)| = \sup_{\substack{x\in Q\\ Q\in\mathcal{A}}} \frac{1}{I\!P(Q)} \left| \int_Q f dI\!P \right|;$$

$$f^\sharp(x) = \sup_{k\geq 0} E[|f - f_k| \,|\mathcal{F}_k](x) = \sup_{\substack{x\in Q\\ Q\in\mathcal{A}}} \frac{1}{I\!P(Q)} \int_Q |f - f_Q| dI\!P$$

where

$$f_Q = \frac{1}{I\!P(Q)} \int_Q f dI\!P.$$

For an integer $n \geq 0$ we define $f_n^* = (f_n)^*, f_n^\sharp = (f_n)^\sharp$.

It is well known (see [9]) that

$$\|f^*\|_p \leq C_p \|f^\sharp\|_p \ , \ 1 < p < \infty, \ f \in L^p(\Omega, \mathcal{F}, I\!P). \tag{2.3}$$

We can prove (2.3) using the method known as the Calderón-Zygmund decomposition (see [8, Theorem 5, p. 153]), replacing the dyadic cubes of $\mathbb{R}^n$ by the elements of $\mathcal{A}$.

Given a positive integrable function W on Ω, we denote by $L_E^p(W)$ or $L^p(\Omega, \mathcal{F}, W dI\!P; E)$, $1 \leq p < \infty$, the Bochner-Lebesgue space consisting of all E-valued (strongly) measurable functions f defined in Ω such that

$$\|f\|_{L_E^p(W)} = \left(\int_\Omega \|f(\omega)\|^p W(\omega) dI\!P(\omega) \right)^{1/p} < \infty.$$

We write $L_E^p(W) = L^p(W)$ when $E = \mathbb{R}$ and $L_E^p(W) = L_E^p(\Omega) = L_E^p$ when $W = 1$.

Let W be a positive integrable function on Ω and let $1 < p < \infty$. If there exists a constant C such that

$$\left(\frac{1}{I\!P(Q)} \int_Q W dI\!P \right) \left(\frac{1}{I\!P(Q)} \int_Q W^{-1/(p-1)} dI\!P \right)^{(p-1)} \leq C, \tag{2.4}$$

for all $Q \in \mathcal{A}$, we say that W is a weight in the class $A_p(\mathcal{A})$. The class $A_\infty(\mathcal{A})$ is defined as the union of the classes $A_p(\mathcal{A})$ for $1 < p < \infty$.

Let U be an operator on L^1_E such that, for each $f \in L^1_E$ it associates a nonnegative process $(U_k f)_{k \geq 0}$ with $U_0 f = 0$ and $U_k f$ $\mathcal{F}_k$-measurable, $k \geq 0$. For a stopping time T we denote by U^*_T the maximal operator defined by

$$U^*_T f(\omega) = \sup_{k \leq T(\omega)} U_k f(\omega).$$

We write $U^* f = U^*_\infty f$.

Theorem 2.1 ([24]) *Let $W \in A_\infty(\mathcal{A})$ and let U and V be two operators on L^1_E as above. Suppose that*

$$U^*_{T \wedge S} f = U^*_T f^S, \quad V^*_{T \wedge S} f = V^*_T f^S$$

for all stopping times T and S and all $f \in L^1_E$. If there exists a constant C such that

$$E[\{U^*_k f - U^*_{T \wedge k} f\}^2 \ |\mathcal{F}_T] \leq C E[\{V^*_k f\}^2 \ |\mathcal{F}_T]$$

for all $k \geq 1$, all stopping times T and for all $f \in L^1_E$, then there exists a constant C such that

$$\int_\Omega \Phi(U^* f) W dI\!P \leq C \int_\Omega \Phi(V^* f) W dI\!P,$$

for all $f \in L^1_E$. The constant C depends only on W, θ, Φ and E, where θ is the constant in (2.1).

Theorem 2.2 ([25]) *Let U and V be two operators such that, for each real-valued integrable function on Ω they associate nonnegative $\mathcal{F}$-measurable functions. Suppose that for any $Z \in A_\infty(\mathcal{A})$ there exists a constant C_Z, depending only on Z, such that*

$$\int_\Omega U(h) Z dI\!P \leq C_Z \int_\Omega V(h) Z dI\!P \ ,$$

for all $h \in \cup_{k=0}^\infty L^1(\Omega, \mathcal{F}_k, I\!P)$. Then for all $1 < p < \infty$, there exists a constant C_p such that

$$\| \sum_{j=1}^\infty U f_j e_j \|_{L^p_E} \leq C_p \| \sum_{j=1}^\infty V f_j e_j \|_{L^p_E}$$

for all $f = \sum_j f_j e_j \in \cup_{k=0}^\infty L^p(\Omega, \mathcal{F}_k, I\!P; E)$.

Theorem 2.3 ([13]) *Let W be a positive integrable function and let $1 < p < \infty$. Then $W \in A_p(\mathcal{A})$ if and only if the operator $f \mapsto f^*$ is bounded on $L^p(W)$.*

Theorem 2.4 ([25]) *Let $W \in A_\infty(\mathcal{A})$. Then there exists a constant C, depending only on E, Φ and W, such that, for all $f = \sum_j f_j e_j \in L^1_E$,*

$$\int_\Omega \Phi\left(\sup_{k\geq 1} \|\sum_{j=1}^k f_j^* e_j\|\right) W\,d\mathbb{P} \leq C \int_\Omega \Phi(\|f\|^*) W\,d\mathbb{P}. \tag{2.5}$$

Lemma 2.1 *There exists an absolute constant C such that, for all stopping times T, all $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and all integers $n \geq 0$,*

$$E[\{(f - f^T)_n^*\}^2 \,|\mathcal{F}_T] \leq CE[\{(f - f^T)_n^\sharp\}^2 \,|\mathcal{F}_T]. \tag{2.6}$$

Proof. Let us fix T, f, n and $A \in \mathcal{F}_T$ and let us consider the martingale $g = (g_k)_{k\geq 0}$, $g_k = E[I(A)\{f_n - f_{T\wedge n}\} \,|\mathcal{F}_k]$. From (2.2) it follows that

$$g_k = I(A)(f_k^n - f_k^{T\wedge n})$$

and hence

$$g_n^* = I(A)(f - f^T)_n^* \tag{2.7}$$

and

$$|g_n - g_k| = I(A)|(f - f^T)_n - (f - f^T)_k|, \quad 1 \leq k \leq n.$$

Since $A \cap \{T \leq k\} \in \mathcal{F}_k$ we have that

$$\begin{aligned} E[|g_n - g_k| \,|\mathcal{F}_k] &= E[I(A \cap \{T \leq k\})|(f - f^T)_n - (f - f^T)_k| \,|\mathcal{F}_k] \\ &+ E[I(A \cap \{T > k\})|(f - f^T)_n - (f - f^T)_k| \,|\mathcal{F}_k] \\ &= I(A)E[|(f - f^T)_n - (f - f^T)_k| \,|\mathcal{F}_k] \end{aligned}$$

and hence

$$g_n^\sharp = I(A)(f - f^T)_n^\sharp. \tag{2.8}$$

Then from (2.7), (2.8) and (2.3) for $p = 2$ we obtain

$$\begin{aligned} \int_A \{(f - f^T)_n^*\}^2 d\mathbb{P} &= \|g_n^*\|_2 \\ &\leq C\|g_n^\sharp\|_2 \\ &= C\int_A \{(f - f^T)_n^\sharp\}^2 d\mathbb{P}. \end{aligned}$$

Since the above inequality is true for all $A \in \mathcal{F}_T$, we obtain (2.6). ■

Theorem 2.5 *If $W \in A_\infty(\mathcal{A})$ then there exists a constant C such that*

$$\int_\Omega \Phi(f^*) W d\mathbb{P} \leq C \int_\Omega \Phi(f^\sharp) W d\mathbb{P} \tag{2.9}$$

for all $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. The constant C depends only on W, θ and Φ, where θ is the constant in (2.1).

Proof. Let us fix $f \in L^1$, a stopping time T and an integer $n \geq 0$. Since $g \mapsto g^*$ and $g \mapsto g^\sharp$ are sublinear then

$$0 \leq f_n^* - f_{T\wedge n}^* \leq (f - f^T)_n^* \tag{2.10}$$

and

$$(f - f^T)_n^\sharp \leq f_n^\sharp + f_{T\wedge n}^\sharp \leq 2 f_n^\sharp. \tag{2.11}$$

Therefore by (2.6)

$$\begin{aligned} E[\{f_n^* - f_{T\wedge n}^*\}^2 \, |\mathcal{F}_T] &\leq E[\{(f - f^T)_n^*\}^2 \, |\mathcal{F}_T] \\ &\leq CE[\{(f - f^T)_n^\sharp\}^2 \, |\mathcal{F}_T] \\ &\leq 4CE[\{f_n^\sharp\}^2 \, |\mathcal{F}_T]. \end{aligned}$$

It is easy to see that $f_{T\wedge S}^* = (f^S)_T^*$ and $f_{T\wedge S}^\sharp = (f^S)_T^\sharp$ for all stopping times T and S. Then applying Theorem 2.1 we obtain (2.9). ■

Theorem 2.6 *Let $1 < p < \infty$. If $f = \sum_j f_j e_j \in L_E^p$ then $\sum_j f_j^* e_j$ and $\sum_j f_j^\sharp e_j$ converge in L_E^p and*

$$\| \sum_{j=1}^\infty f_j^* e_j \|_{L_E^p} \leq C_p \| \sum_{j=1}^\infty f_j^\sharp e_j \|_{L_E^p} \tag{2.12}$$

where C_p is a constant depending only on p, θ and E.

Proof. Let $\Phi(t) = t$ and $Z \in A_\infty(\mathcal{A})$. Then by Theorem 2.5 there exists a constant C_Z such that

$$\int_\Omega f^* Z d\mathbb{P} \leq C_Z \int_\Omega f^\sharp Z d\mathbb{P},$$

for all $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, from Theorem 2.2 there exists a constant C_p depending only on p, θ and E such that (2.12) is true for all $f \in \cup_{k=0}^\infty L^p(\Omega, \mathcal{F}_k, \mathbb{P}; E)$.

It follows by Theorem 2.4 for $\Phi(t) = t^p$ and $W = 1$ and by Theorem 2.3 that the operator $\widetilde{N}(\sum_j f_j e_j) = \sum_j f_j^* e_j$ is well defined and is bounded on L^p_E. Since $f_j^\sharp \leq 2f_j^*$, then the operator $\widetilde{N}^\sharp(\sum_j f_j e_j) = \sum_j f_j^\sharp e_j$ is also well defined and is bounded on L^p_E. But $\bigcup_{k=0}^\infty L^p(\Omega, \mathcal{F}_k, I\!P; E)$ is dense in L^p_E and hence we obtain (2.12) for all $f \in L^p_E$. ■

Theorem 2.7 *Let $W \in A_\infty(\mathcal{A})$. Then there exists a constant C, depending only on W, θ, Φ and E such that, for all $f = \sum_j f_j e_j \in \cup_{p>1} L^p_E$,*

$$\int_\Omega \Phi(\|\sum_{j=1}^\infty f_j^* e_j\|) W \, dI\!P \leq C \int_\Omega \Phi(\|\sum_{j=1}^\infty f_j^\sharp e_j\|) W \, dI\!P. \tag{2.13}$$

Proof. We observe that E is a Banach lattice with absolute value $|\sum_j x_j e_j| = \sum_j |x_j| e_j$.

Let $1 < p < \infty$ and $f = \sum_j f_j e_j \in L^p_E$. By the proof of Theorem 2.6, $\widetilde{N} f = \sum_j f_j^* e_j$ and $\widetilde{N}^\sharp f = \sum_j f_j^\sharp e_j$ are well defined as functions in L^p_E. We define $Uf = \|\widetilde{N} f\|$, $Vf = \|\widetilde{N}^\sharp f\|$ and $U_n f = U(E[f \mid \mathcal{F}_n]), V_n f = V(E[f \mid \mathcal{F}_n])$. Since $(U_n f)_{n \geq 0}$ is an increasing sequence and $U_n f \to Uf$ in L^p when $n \to \infty$, then it follows that $U^* f = \sup_{n \geq 0} U_n f = Uf$. By the same way $V^* f = Vf$.

If T is a stopping time, we obtain from the inequality (2.12) for $p = 2$, as in the proof of Lemma 2.1, that there exists a constant C independent of f, T and n, such that

$$E[U_n^2(f - f^T) \mid \mathcal{F}_T] \leq C E[V_n^2(f - f^T) \mid \mathcal{F}_T].$$

¿From the inequalities (2.10) and (2.11) we obtain

$$|U_n f - U_{T \wedge n} f| \leq U_n(f - f^T),$$

$$V_n(f - f^T) \leq 2 V_n f$$

and hence

$$E[\{U_n f - U_{T \wedge n} f\}^2 \mid \mathcal{F}_T] \leq 4 C E[V_n^2 f \mid \mathcal{F}_T].$$

Now, since $(f_j)^*_{T \wedge S} = (f_j^S)^*_T$ and $(f_j)^\sharp_{T \wedge S} = (f_j^S)^\sharp_T$, then it follows that $U_{T \wedge S} f = U_T f^S$ and $V_{T \wedge S} f = V_T f^S$ for all stopping times T and S. Therefore we can apply Theorem 2.1 and to obtain (2.13). ■

3 Maximal Operators on Homogeneous Spaces

Lemma 3.1 *([22], Lemma 3.21, p. 852) Let b be a positive integer and let $\lambda = 8\eta^5$. Then for each integer k, $-b \le k \le b$, there exist an enumerable Borel partition $\mathcal{A}_k^b$ of X and a positive constant C depending only on X, such that:*

(i) for all $Q \in \mathcal{A}_k^b, -b \le k \le b$, there exists $x_Q \in Q$ such that $B(x_Q, \lambda^k) \subset Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q, \lambda^{k+1})) \le C\mu(Q)$;

(ii) if $-b \le k < b$, $Q_1 \in \mathcal{A}_{k+1}^b$, $Q_2 \in \mathcal{A}_k^b$ and $Q_1 \cap Q_2 \ne \emptyset$, then $Q_2 \subset Q_1$, and $0 < \mu(Q_1) \le C\mu(Q_2)$.

For a real-valued locally integrable function f on X we define

$$M_d^b f(x) = \sup_{\substack{x\in Q \\ Q\in\mathcal{A}^b}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y),$$

$$M_d^{b,\sharp} f(x) = \sup_{\substack{x\in Q \\ Q\in\mathcal{A}^b}} \frac{1}{\mu(Q)} \int_Q |f(y) - f_Q| d\mu(y),$$

$$M^b f(x) = \sup_B \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

and

$$M^{b,\sharp} f(x) = \sup_B \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y),$$

where the supremum is taken over all balls $B = B(a, r)$, such that $x \in B$ and $\lambda^{-b-1} \le r < \lambda^b$, and $\mathcal{A}^b = \bigcup_{-b\le k\le b} \mathcal{A}_k^b$.

Lemma 3.2 *Let $W \in A_\infty(\mathcal{A}^b)$. Then there exists a constant C, depending only on E, Φ, X and W, such that, for all $f = \sum_j f_j e_j \in L_E^1$,*

$$\int_X \Phi\left(\sup_{k\ge1} \|\sum_{j=1}^k M_d^b f_j(x) e_j\|\right) W(x) d\mu(x) \le C \int_X \Phi\left(M_d^b(\|f\|)(x)\right) W(x) d\mu(x). \tag{3.1}$$

Proof. Let $\mathcal{A}_b^b = \{Q_i^b : i \in I_b\}$, $I_b \subset \mathbb{N}$, and consider the probability measure μ_i^b on the Borel subsets of Q_i^b given by $\mu_i^b(A) = \mu(A)/\mu(Q_i^b)$. Given

$f=\sum_j f_je_j \in L^1_E$ we have that $M^b_d f_j(x) = \left(|f_j|_{|Q^b_i}\right)^*(x)$, for $x \in Q^b_i$, and hence by Lemma 3.1(ii) and Theorem 2.4,

$$\begin{aligned}&\int_X \Phi\left(\sup_{k\geq 1}\|\sum_{j=1}^k M^b_d f_j(x)e_j\|\right)W(x)d\mu(x)\\ =&\sum_{i\in I_b}\mu(Q^b_i)\int_{Q^b_i}\Phi\left(\sup_{k\geq 1}\|\sum_{j=1}^k\left(|f_j|_{|Q^b_i}\right)^*(x)e_j\|\right)W_{|Q^b_i}(x)d\mu^b_i(x)\\ \leq& \; C\int_X\Phi\left(M^b_d(\|f\|)(x)\right)W(x)d\mu(x). \blacksquare\end{aligned}$$

Lemma 3.3 *Let $W \in A_\infty(\mathcal{A}^b)$. Then there exists a constant C, depending only on E, Φ, X and W, such that, for all $f=\sum_j f_je_j \in \bigcup_{p>1} L^p_E$,*

$$\begin{aligned}&\int_X \Phi\left(\|\sum_{j=1}^\infty M^b_d f_j(x)e_j\|\right)W(x)d\mu(x)\\ &\leq C\int_X \Phi\left(\|\sum_{j=1}^\infty M^{b,\sharp}_d f_j(x)e_j\|\right)W(x)d\mu(x).\end{aligned}\tag{3.2}$$

Proof. Let us consider μ^b_i, $i \in I_b$, as in the proof of Theorem 3.1. Given $f=\sum_j f_je_j \in \bigcup_{p>1} L^p_E$ we have that $M^b_d f_j(x) = \left(|f_j|_{|Q^b_i}\right)^*(x)$, for $x \in Q^b_i$, and $M^{b,\sharp}_d(|f_j|)(x) \leq 2M^{b,\sharp}_d f_j(x)$, for $x \in X$. Therefore by Lemma 3.1(ii) and Theorem 2.7,

$$\begin{aligned}&\int_X \Phi\left(\|\sum_{j=1}^\infty M^b_d f_j(x)e_j\|\right)W(x)d\mu(x)\\ =&\sum_{i\in I_b}\mu(Q^b_i)\int_{Q^b_i}\Phi\left(\|\sum_{j=1}^\infty\left(|f_j|_{|Q^b_i}\right)^*(x)e_j\|\right)W_{|Q^b_i}(x)d\mu^b_i(x)\\ \leq&\; C\sum_{i\in I_b}\mu(Q^b_i)\int_{Q^b_i}\Phi\left(\|\sum_{j=1}^\infty\left(|f_j|_{|Q^b_i}\right)^\sharp(x)e_j\|\right)W_{|Q^b_i}(x)d\mu^b_i(x)\\ =&\; C\int_X\Phi\left(\|\sum_{j=1}^\infty M^{b,\sharp}_d(|f_j|)(x)e_j\|\right)W(x)d\mu(x)\\ \leq&\; C'\int_X\Phi\left(\|\sum_{j=1}^\infty M^{b,\sharp}_d f_j(x)e_j\|\right)W(x)d\mu(x). \blacksquare\end{aligned}$$

Lemma 3.4 *Let C be the constant in Lemma 3.1. Then, for all $1 < p \leq \infty$, all real-valued locally integrable function f*

and $x \in X$, we have

$$A_p(X) \subset A_p(\mathcal{A}^b), \tag{3.3}$$

$$M_d^b f(x) \leq C M^b f(x), \tag{3.4}$$

$$M_d^{b,\sharp} f(x) \leq 2C M^{b,\sharp} f(x). \tag{3.5}$$

Proof. Let $1 < p < \infty$, $W \in A_p(X)$, $Q \in \mathcal{A}_k^b$, $-b \leq k \leq b$ and $x \in Q$. By Lemma 3.1(i) there exist $x_Q \in Q$ and $C > 0$ such that $Q \subset B = B(x_Q, \lambda^{k+1})$ and $\mu(B) \leq C\mu(Q)$. Therefore it follows by (1.2) that

$$\left(\frac{1}{\mu(Q)} \int_Q W d\mu\right) \left(\frac{1}{\mu(Q)} \int_Q W^{-1/(p-1)} d\mu\right)^{p-1} \leq C^p C(p, W).$$

Now for a real-valued locally integrable function f we have that

$$\frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y) \leq \frac{C}{\mu(B)} \int_B |f(y)| d\mu(y)$$

and

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |f(y) - f_Q| d\mu(y) &\leq \frac{1}{\mu(Q)} \int_Q |f(y) - f_B| d\mu(y) + |f_B - f_Q| \\ &\leq \frac{2C}{\mu(B)} \int_B |f(y) - f_B| d\mu(y) \\ &\leq 2C M^\sharp f(x). \end{aligned}$$

Thus we obtain (3.3), (3.4) and (3.5). ■

The following lemma is the analogous of a result by R. Wheeden [26] for the fractional maximal operator and for X with a group structure.

Lemma 3.5 *Let b be a positive integer. Then there exists a constant C, depending only on X, such that, for all real-valued locally integrable function f on X and all $x \in B(\mathbb{1}, \lambda^b)$, $\mathbb{1} = \pi(e)$, we have*

$$M^b f(x) \leq \frac{C}{|\mathcal{G}_b|} \int_{\mathcal{G}_b} M_d^{b,g} f(x) dg, \tag{3.6}$$

where

$$\mathcal{G}_b = \{g \in G \ : \ d(g\mathbb{1}, \mathbb{1}) < \lambda^{b+3}\}$$

and $M_d^{b,g} f(x) = R_{g^{-1}} M_d^b R_g f(x)$, $g \in G$, $x \in X$.

Proof. First we observe that $|\mathcal{G}_b| = \mu(B(\mathbb{1}, \lambda^{b+3})) > 0$. Let us fix $x \in B(\mathbb{1}, \lambda^b)$. From the definition of $M^b f(x)$, there exists a ball $B = B(a, r)$ such that $x \in B$, $\lambda^{-b-1} \le r < \lambda^b$ and

$$M^b f(x) \le \frac{2}{\mu(B)} \int_B |f(y)| d\mu(y). \tag{3.7}$$

Let $-b \le k \le b$ such that $\lambda^{k-1} \le r < \lambda^k$. We denote by Ω the set

$$\Omega = \left\{ g \in \mathcal{G}_b \ : \ \text{there exists } Q \in \mathcal{A}^b_{k+1} \text{ such that } B \subset g^{-1}Q \right\}.$$

Given $g \in \Omega$, let $Q \in \mathcal{A}^b_{k+1}$ such that $B \subset g^{-1}Q$. By Lemma 3.1(i) there exists $x_Q \in Q$ such that $B(x_Q, \lambda^{k+1}) \subset Q \subset B(x_Q, \lambda^{k+2})$ and hence $g^{-1}Q \subset B(g^{-1}x_Q, \lambda^{k+2})$. If s is the integer such that $2^{s-1} < \lambda^3 \le 2^s$, then by the doubling condition we have $\mu(B(g^{-1}x_Q, \lambda^{k+2})) \le A^s \mu(B)$ and thus

$$\frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \le \frac{A^s}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)| d\mu(y).$$

Therefore from (3.7) we get

$$M^b f(x) \le 2A^s M_d^{b,g} f(x), \quad g \in \Omega.$$

Now suppose that there exists a positive constant α such that $|\Omega| \ge \alpha |\mathcal{G}_b|$ for all positive integers b. Then integrating both sides of the above inequality with respect to the Haar measure dg and on Ω, we get (3.6) for $C = 2A^s \alpha^{-1}$.

We will prove that there exists a positive constant α, depending only on X, such that $|\Omega| \ge \alpha |\mathcal{G}_b|$. Given $y \in X$ we denote by g_y an element in G such that $y = g_y \mathbb{1}$.

Let $z \in g_{x_Q} \mathcal{G}_{k-3} g_x^{-1}$. Then $zx \in B(x_Q, \lambda^k)$ and hence for $y \in B$,

$$\begin{aligned} d(zy, x_Q) &\le \eta(d(zy, zx) + d(zx, x_Q)) \\ &\le \eta[\eta(d(y, a) + d(a, x)) + \lambda^k] \\ &\le \lambda^{k+1}. \end{aligned}$$

Therefore $y \in z^{-1}Q$ and hence

$$B \subset z^{-1}Q, \quad z \in g_{x_Q} \mathcal{G}_{k-3} g_x^{-1}. \tag{3.8}$$

Let us denote by Γ the set

$$\Gamma = \left\{ Q \in \mathcal{A}^b_{k+1} \ : \ Q \cap B(x, \lambda^{b+2}) \ne \emptyset \right\}.$$

Fix $Q \in \Gamma$ and let $u \in Q \cap B(x, \lambda^{b+2})$, $g \in g_{x_Q} \mathcal{G}_{k-3}$. Then $g\mathbb{1} \in B(x_Q, \lambda^k)$ and

$$\begin{aligned} d(g\mathbb{1}, \mathbb{1}) &\leq \eta(d(g\mathbb{1}, x_Q) + d(x_Q, \mathbb{1})) \\ &\leq \eta[\lambda^k + \eta(d(x_Q, u) + d(u, \mathbb{1}))] \\ &\leq \eta\{\lambda^k + \eta[\lambda^{k+2} + \eta(d(u, x) + d(x, \mathbb{1}))]\} \\ &\leq 4\eta^3 \lambda^{b+2} \end{aligned}$$

and hence

$$\begin{aligned} d(gg_x^{-1}\mathbb{1}, \mathbb{1}) &\leq \eta(d(g_x g^{-1}\mathbb{1}, g_x\mathbb{1}) + d(x, \mathbb{1})) \\ &\leq \eta(d(g\mathbb{1}, \mathbb{1}) + \lambda^b) \\ &< \lambda^{b+3}. \end{aligned}$$

Thus $g \in \mathcal{G}_b g_x$ and hence

$$g_{x_Q} \mathcal{G}_{k-3} g_x^{-1} \subset \mathcal{G}_b, \quad Q \in \Gamma.$$

Therefore from (3.8)

$$\bigcup_{Q \in \Gamma} g_{x_Q} \mathcal{G}_{k-3} g_x^{-1} \subset \Omega. \tag{3.9}$$

If $Q, Q' \in \mathcal{A}^b_{k+1}$ and $Q \neq Q'$ then $B(x_Q, \lambda^k) \cap B(x_{Q'}, \lambda^k) = \emptyset$ and hence

$$g_{x_Q} \mathcal{G}_{k-3} g_x^{-1} \cap g_{x_{Q'}} \mathcal{G}_{k-3} g_x^{-1} = \emptyset.$$

Then, since G is unimodular (see [14, p. 578]), it follows by (3.9) and by the doubling condition that

$$\begin{aligned} |\Omega| &\geq |\bigcup_{Q \in \Gamma} g_{x_Q} \mathcal{G}_{k-3} g_x^{-1}| \\ &= \sum_{Q \in \Gamma} |g_{x_Q} \mathcal{G}_{k-3}| \\ &\geq \sum_{Q \in \Gamma} A^{-s} \mu(B(x_Q, \lambda^{k+2})) \\ &\geq A^{-s} \mu \left(\bigcap_{Q \in \Gamma} Q \right) \\ &\geq A^{-s} \mu(B(x, \lambda^{b+2})) \\ &\geq A^{-2s} |\mathcal{G}_b|. \ \blacksquare \end{aligned}$$

Proof of Theorem 1.1: Let us denote by C the greatest constant among the constants C in (3.1), (3.4) and (3.6), and let s be the integer satisfying $2^{s-1} < C \leq 2^s$. Let $f = \sum_{j=1}^k f_j e_j \in L^1_E$. Since $W \in A_\infty(X)$, we can choose $1 < p < \infty$ such that $W \in A_p(X)$. Then, it follows by (1.2) that $R_g W \in A_p(X)$ and $C(p, R_g W) = C(p, W)$ for all $g \in G$. Therefore by (1.1), (3.1), (3.3), (3.4), (3.6), by Jensen's inequality and Fubini's theorem we have that,

$$
\begin{aligned}
&\int_{B(\mathbb{1},\lambda^b)} \Phi\left(\|\sum_{j=1}^k M^b f_j(x) e_j\|\right) W(x) d\mu(x) \\
\leq\ & \int_{B(\mathbb{1},\lambda^b)} \Phi\left(\frac{C}{|\mathcal{G}_b|}\int_{\mathcal{G}_b} \|\sum_{j=1}^k M_d^{b,g} f_j(x) e_j\| dg\right) W(x) d\mu(x) \\
\leq\ & \sup_{g\in\mathcal{G}_b} c^s \int_X \Phi\left(\|\sum_{j=1}^k M_d^b (R_g f_j)(y) e_j\|\right) R_g W(y) d\mu(y) \\
\leq\ & \sup_{g\in\mathcal{G}_b} c^s C \int_X \Phi\left(M_d^b(\|R_g f\|)(y)\right) R_g W(y) d\mu(y) \\
\leq\ & \sup_{g\in\mathcal{G}_b} c^{2s} C \int_X \Phi\left(M^b(\|R_g f\|)(gx)\right) W(x) d\mu(x) \\
\leq\ & c^{2s} C \int_X \Phi\left(M(\|f\|)(x)\right) W(x) d\mu(x),
\end{aligned}
$$

since $M(\|R_g f\|)(gx) = M(\|f\|)(x)$. Now, let $f = \sum_{j=1}^\infty f_j e_j$ and $f^k = \sum_{j=1}^k f_j e_j$, $k \geq 1$. Since the above inequality is true for all $f^k, k \geq 1$, it follows by the Monotone Convergence Theorem that

$$
\begin{aligned}
&\int_{B(\mathbb{1},\lambda^b)} \Phi\left(\sup_{k\geq 1} \|\sum_{j=1}^k M^b f_j(x) e_j\|\right) W(x) d\mu(x) \\
&\qquad \leq c^{2s} C \int_X \Phi\left(M^b(\|f\|)(x)\right) W(x) d\mu(x).
\end{aligned}
$$

Letting $b \to \infty$ on both sides of the above inequality we obtain (1.3).

Finally, let $1 < p < \infty$, $\Phi(t) = t^p$, $W \in A_p(X)$ and $f = \sum_{j=1}^\infty f_j e_j \in L^p_E(W) \cap L^1_E$. By (1.3) and since the operator M is bounded on $L^p(W)$ (see [5]),

$$
\|\sum_{j=\ell}^{\ell+m} M f_j e_j\|_{L^p_E(W)} \leq C^{1/p} \|M(\|\sum_{j=\ell}^{\ell+m} f_j e_j\|)\|_{L^p_{\mathbb{R}}(W)}
$$

$$\leq \quad C' \| \sum_{j=\ell}^{\ell+m} f_j e_j \|_{L^p_E(W)}.$$

¿From the above inequality we can conclude that $\sum_{j=1}^{\infty} M f_j e_j$ converges in $L^p_E(W)$ to a function $\widetilde{M} f$ and

$$\|\widetilde{M} f\|_{L^p_E(W)} \leq C' \|f\|_{L^p_E(W)}.$$

Now let $f = \sum_j f_j e_j \in L^p_E(W)$ such that $f_j \geq 0$, for all $j \geq 1$. For each j, let $(f_j^k)_{k \in \mathbb{N}}$ be a sequence of simple functions such that $0 \leq f_j^k \uparrow f_j$ a.e., $k \to \infty$. Then $M f_j^k \uparrow M f_j$ and for $f^k = \sum_j f_j^k e_j \in L^p_E(W) \cap L^1_E$ we have $\widetilde{M} f^k \uparrow \widetilde{M} f$ a.e. Then

$$\begin{aligned} \|\widetilde{M} f\|_{L^p_E(W)} &= \lim_{k \to \infty} \|\widetilde{M} f^k\|_{L^p_E(W)} \\ &\leq \lim_{k \to \infty} C' \|f^k\|_{L^p_E(W)} \\ &= C' \|f\|_{L^p_E(W)}. \ \blacksquare \end{aligned}$$

Proof of Theorem 1.2: It follows by Theorem 1.1 that the operator $\widetilde{M}(\sum_j f_j e_j) = \sum_j M f_j e_j$ is well defined and is bounded on L^p_E. Since $M^\sharp f_j \leq 2 M f_j$, then the operator $\widetilde{M^\sharp}(\sum_j f_j e_j) = \sum_j M^\sharp f_j e_j$ is also well defined and is bounded on L^p_E.

Let us denote by C the greatest constant among the constants C in (3.1), (3.5) and (3.6), and let s be the integer satisfying $2^{s-1} < C \leq 2^s$. Since $W \in A_\infty(X)$, we can choose $1 < p < \infty$ such that $W \in A_p(X)$. Then, it follows by (1.2) that $R_g W \in A_p(X)$ and $C(p, R_g W) = C(p, W)$ for all $g \in G$. Therefore by (1.1), (3.2), (3.3), (3.5), (3.6), by Jensen's inequality and Fubini's theorem we have that,

$$\begin{aligned} & \int_{B(\mathbb{1},\lambda^b)} \Phi \left(\| \sum_{j=1}^{\infty} M^b f_j(x) e_j \| \right) W(x) d\mu(x) \\ \leq \ & \int_{B(\mathbb{1},\lambda^b)} \Phi \left(\frac{C}{|\mathcal{G}_b|} \int_{\mathcal{G}_b} \| \sum_{j=1}^{\infty} M_d^{b,g} f_j(x) e_j \| dg \right) W(x) d\mu(x) \\ \leq \ & \sup_{g \in \mathcal{G}_b} c^s \int_X \Phi \left(\| \sum_{j=1}^{\infty} M_d^b (R_g f_j)(y) e_j \| \right) R_g W(y) d\mu(y) \\ \leq \ & \sup_{g \in \mathcal{G}_b} c^s C \int_X \Phi \left(\| \sum_{j=1}^{\infty} M_d^{b,\sharp} (R_g f_j)(y) e_j \| \right) R_g W(y) d\mu(y) \end{aligned}$$

$$\leq \sup_{g\in\mathcal{G}_b} c^{2s+1}C\int_X \Phi\left(\|\sum_{j=1}^{\infty} M^{b,\sharp}(R_g f_j)(gx)e_j\|\right) W(x)d\mu(x)$$

$$\leq c^{2s+1}C\int_X \Phi\left(\|\sum_{j=1}^{\infty} M^{\sharp} f_j(x)e_j\|\right) W(x)d\mu(x),$$

since $M^\sharp(R_g f_j)(gx) = M^\sharp f_j(x)$. Letting $b\to\infty$ we obtain (1.4). ■

4 Singular Integral Operators

In the proof of the following lemma we use the potential-type construction by Bourgain [2].

Lemma 4.1 *Let $1 < p < \infty$ and $W \in A_p(X)$. Then there exist positive constants C_p and r, $r > 1$, depending only on p, W, X and E, such that*

$$\|\sum_{j=1}^{\infty} M_r f_j e_j\|_{L^p_E(W)} \leq C_p\|\sum_{j=1}^{\infty} f_j e_j\|_{L^p_E(W)}, \tag{4.1}$$

for all $f = \sum_j f_j e_j \in L^p_E(W)$, where $M_r g = (M(|g|^r))^{1/r}$.

Proof. Let $1 < p < \infty$, $W \in A_p(X)$, $\Phi(t) = t^p$, let C be the constant in (1.3) and let $g = \sum_j g_j e_j \in L^p_E(W)$. For each $j \geq 1$ we define

$$\psi_j = \sum_{i=0}^{\infty} (2C^{1/p})^{-i} M^{(i)} g_j,$$

where $M^{(i)}g_j$ is defined inductively by $M^{(0)}g_j = |g_j|$, $M^{(i+1)}g_j = M(M^{(i)}g_j)$. We have that

$$M\psi_j \leq 2C^{1/p}\psi_j$$

and hence the weights $\psi_j, j\geq 1$, are by definition, uniformly in the class $A_1(X)$. It follows by the Reverse Hölder's Inequality (see Calderón [5]) that there exist positive constants C' and r, $r > 1$, depending only on p and C, such that

$$\left(\frac{1}{\mu(B)}\int_B \psi_j^r d\mu\right)^{1/r} \leq \frac{C'}{\mu(B)}\int_B \psi_j d\mu$$

for all balls B and all $j \geq 1$. Therefore

$$M_r g_j(x) \leq M_r \psi_j(x) \leq C' M\psi_j(x) \leq 2C^{1/p} C' \psi_j(x).$$

But by Theorem 1.1,

$$\begin{aligned} \|\sum_{j=1}^{\infty} \psi_j e_j\|_{L^p_E(W)} &\leq \sum_{i=0}^{\infty} (2C^{1/p})^{-i} \|\sum_{j=1}^{\infty} M^{(i)} g_j e_j\|_{L^p_E(W)} \\ &\leq 2\|g\|_{L^p_E(W)} \end{aligned}$$

and hence

$$\begin{aligned} \|\sum_{j=1}^{\infty} M_r g_j e_j\|_{L^p_E(W)} &\leq 2C^{1/p} C' \|\sum_{j=1}^{\infty} \psi_j e_j\|_{L^p_E(W)} \\ &\leq 4C^{1/p} C' \|g\|_{L^p_E(W)}. \ \blacksquare \end{aligned}$$

Lemma 4.2 *Let T be a singular integral operator bounded on $L^r(X)$ for some r, $1 < r < \infty$. Assume that the kernel K of T satisfies (H'_∞) and $K(gx, gy) = K(x, y)$ for all $x, y \in X$ and $g \in G$. Then there exists a constant C_r such that*

$$M^\sharp(Tf)(x) \leq C_r M_r f(x), \quad f \in L^\infty_c(X).$$

Proof. Let us fix $x_0 \in X$, $\ell > 0$ and let $B = B(x_0, \ell)$, $B^2 = B(x_0, 2\ell)$. For $f \in L^\infty_c(X)$ we set $g = f\chi_{B^2}, h = f - g$. Since T is bounded on $L^r(X)$, then for all $z \in B$,

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |Tg(x) - (Tg)_B| d\mu(x) &\leq \frac{2}{\mu(B)} \int_B |Tg(x)| d\mu(x) \\ &\leq C_r \left(\frac{1}{\mu(B)} \int_{B^2} |g(x)|^r d\mu(x) \right)^{1/r} \\ &\leq C_r A^{1/r} M_r g(z). \end{aligned}$$

Now let $x \in B$, $g \in G$ such that $gx_0 = \mathbb{1}$, $\bar{x} = gx$ and

$$S_j(\bar{x}) = \left\{ t : 2^j d(\bar{x}, \mathbb{1}) < d(t, \mathbb{1}) \leq 2^{j+1} d(\bar{x}, \mathbb{1}) \right\}.$$

Then by the (H'_∞) condition, for all $z \in B$,

$$
\begin{aligned}
& |Th(x) - Th(x_0)| \\
\leq \; & \int_{X\setminus B^2} |K(x,y) - K(x_0,y)|\; |h(y)| d\mu(y) \\
\leq \; & \int_{d(t,\mathbb{1})>2d(\bar{x},\mathbb{1})} |K'(t,\bar{x}) - K'(t,\mathbb{1})|\; |R_g h(t)| d\mu(t) \\
\leq \; & C \sum_{j=1}^{\infty} \int_{S_j(\bar{x})} \frac{d(\bar{x},\mathbb{1})}{d(t,\mathbb{1})\mu(B(\mathbb{1},d(t,\mathbb{1})))} |R_g h(t)| d\mu(t) \\
\leq \; & AC \sum_{j=1}^{\infty} \frac{2^{-j}}{\mu(B(x_0, 2^{j+1}d(x,x_0)))} \int_{gB(x_0,2^{j+1}d(x,x_0))} |R_g h(t)| d\mu(t) \\
= \; & AC \sum_{\substack{1\leq j<\infty \\ 2^j d(x,x_0)>\ell}}^{\infty} \frac{2^{-j}}{\mu(B(x_0, 2^{j+1}d(x,x_0)))} \int_{B(x_0,2^{j+1}d(x,x_0))} |h(y)| d\mu(y) \\
\leq \; & ACM_r h(z)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{1}{\mu(B)} \int_B |Th(x) - (Th)_B| d\mu(x) \; & \leq \; \frac{2}{\mu(B)} \int_B |Th(x) - Th(x_0)| d\mu(x) \\
& \leq \; 2ACM_r h(z).
\end{aligned}
$$

Thus for all $z \in X$

$$
\begin{aligned}
& M^\sharp(Tf)(z) \\
\leq \; & \sup_{B\ni z} \frac{1}{\mu(B)} \int_B |Tg(x) - (Tg)_B| d\mu(x) + \sup_{B\ni z} \frac{1}{\mu(B)} \int_B |Th(x) - (Th)_B| d\mu(x) \\
\leq \; & C'_r M_r f(z). \blacksquare
\end{aligned}
$$

Proof of Theorem 1.3: Let us fix $1 < p < \infty$, $W \in A_p(X)$ and let r and C_p be the constants in Lemma 4.1. Then it follows by (1.4) for $\Phi(t) = t^p$, by (1.5) and (4.1) that, for all $f = \sum_j f_j e_j$, $f_j \in L_c^\infty(X)$ for $j \geq 1$, and all positive integers ℓ and m,

$$
\begin{aligned}
\|\sum_{j=\ell}^{\ell+m} T_j f_j e_j\|_{L_E^p(W)} \; & \leq \; \|\sum_{j=\ell}^{\ell+m} M(T_j f_j) e_j\|_{L_E^p(W)} \\
& \leq \; C^{1/p} \|\sum_{j=\ell}^{\ell+m} M^\sharp(T_j f_j) e_j\|_{L_E^p(W)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C^{1/p} C_r \| \sum_{j=\ell}^{\ell+m} M_r f_j e_j \|_{L^p_E(W)} \\
&\leq C^{1/p} C_r C_p \| \sum_{j=\ell}^{\ell+m} f_j e_j \|_{L^p_E(W)}.
\end{aligned}
$$

The above inequality implies that the sequence of partial sums of the series $\sum_j T_j f_j e_j$ is a Cauchy sequence in $L^p_E(W)$ and hence it converges in $L^p_E(W)$. Putting $\ell = 1$ and letting $m \to \infty$ on both sides of this inequality we obtain (1.6). ■

Proof of Corollary 1.1: For all $0 \leq r \leq 1$ and all $x, y \in S^n$,

$$|y - (y \cdot x)x| \leq 2|y - x|, \tag{4.2}$$

$$|y - \mathbb{1} - [(y - \mathbb{1}) \cdot x]x| \leq 2|y - \mathbb{1}|, \tag{4.3}$$

$$|(x \cdot \mathbb{1})\mathbb{1} - (x \cdot y)y| \leq 2|y - \mathbb{1}|, \tag{4.4}$$

$$|y - x| \leq 2|y - rx| \tag{4.5}$$

and for all $0 \leq r \leq 1$ and all $x, y \in S^n$ such that $|x - \mathbb{1}| > 2|y - \mathbb{1}|$,

$$\frac{1}{2}|\mathbb{1} - rx| < |y - rx| < 2|\mathbb{1} - rx|, \tag{4.6}$$

$$\frac{1}{2}|x - r\mathbb{1}| < |x - ry| < 2|x - r\mathbb{1}|. \tag{4.7}$$

Now fix $0 \leq r \leq 1$ and $x, y \in S^n$ such that $|x - \mathbb{1}| > 2|y - \mathbb{1}|$. Then by (4.6)

$$
\begin{aligned}
& \left| \, |\mathbb{1} - rx|^{n+1} - |y - rx|^{n+1} \right| \\
\leq \; & (|\mathbb{1} - rx|^n + |\mathbb{1} - rx|^{n-1}|y - rx| + \cdots + |y - rx|^n) \\
\leq \; & C|y - \mathbb{1}| \; |\mathbb{1} - rx|^n
\end{aligned}
$$

and hence by (4.2), (4.3), (4.5) and (4.6) we obtain

$$
\begin{aligned}
|s_r(x, y) - s_r(x, \mathbb{1})| &\leq \frac{2}{\omega_n}|y - (y \cdot x)x| \, \frac{\left| \, |\mathbb{1} - rx|^{n+1} - |y - rx|^{n+1} \right|}{|y - rx|^{n+1}|\mathbb{1} - rx|^{n+1}} \\
&+ \frac{2}{\omega_n} \frac{|y - \mathbb{1} - [(y - \mathbb{1}) \cdot x]x|}{|\mathbb{1} - rx|^{n+1}} \\
&\leq C_1 \frac{|y - \mathbb{1}|}{|x - \mathbb{1}|^{n+1}},
\end{aligned}
$$

$$
\begin{aligned}
|t_r(x,y)-t_r(x,\mathbb{1})| &\le \frac{n-2}{2r}\int_0^r |s_\varrho(x,y)-s_\varrho(x,\mathbb{1})|d\varrho \\
&= \frac{(n-2)C_1}{2}\frac{|y-\mathbb{1}|}{|x-\mathbb{1}|^{n+1}},
\end{aligned}
$$

$$
\begin{aligned}
|K^r_{i,j}(x,y)-K^r_{i,j}(x,\mathbb{1})| &\le |x_iy_j-x_jy_i|\frac{|\,|\mathbb{1}-rx|^{n+1}-|y-rx|^{n+1}|}{|y-rx|^{n+1}|\mathbb{1}-rx|^{n+1}} \\
&+ |x_i(y_j-\mathbb{1}_j)-x_j(y_i-\mathbb{1}_i)|\frac{1}{|\mathbb{1}-rx|^{n+1}} \\
&\le C_2\frac{|y-\mathbb{1}|}{|x-\mathbb{1}|^{n+1}}.
\end{aligned}
$$

Since $|\mathbb{1}-rx|\ge 1-r$,

$$
\begin{aligned}
|P_r(x,y)-P_r(x,\mathbb{1})| &\le \frac{1-r^2}{\omega_n}\frac{|\,|\mathbb{1}-rx|^{n+1}-|y-rx|^{n+1}|}{|y-rx|^{n+1}|\mathbb{1}-rx|^{n+1}} \\
&\le C_3\frac{|y-\mathbb{1}|}{|x-\mathbb{1}|^{n+1}}
\end{aligned}
$$

and hence

$$|K(x,y)-K(x,\mathbb{1})|\le C_3\frac{|y-\mathbb{1}|}{|x-\mathbb{1}|^{n+1}}.$$

Therefore the kernels s_r, t_r, $K^r_{i,j}$ and K satisfy the condition (H_∞) uniformly for all $0\le r\le 1,\ i,j\in\{1,2,...,n+1\}$. By the same way we can use (4.2), (4.4), (4.5) and (4.7) to show that s_r, t_r, $K^r_{i,j}$ and K satisfy $(H^{'}_\infty)$ uniformly for all r,i,j. The conclusion of this corollary follows from the remark given above of the statement of Corollary 1.1 and from Theorem 1.4. ■

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