Chain transitive sets for flows on flag bundles*

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July 17, 2002

Abstract

We study the chain transitive sets and Morse decompositions of flows on fiber bundles whose fibers are compact homogeneous spaces of Lie groups. The emphasis is put on generalized flag manifolds of semi-simple (and reductive) Lie groups. In this case an algebraic description of the chain transitive sets is given. Our approach consists in shadowing the flow by semigroups of homeomorphisms to take advantage of the good properties of the semigroup actions on flag manifolds. The description of the chain components in the flag bundles generalizes the Theorem of Selgrade for projective bundles with an independent proof.

AMS 2000 subject classification: Primary: 37B05, 37B35. Secondary: 20M20, 22E46.
Key words: Fiber bundles, flows, chain transitive sets, flag manifolds, semi-simple Lie groups, semigroups, Morse decomposition.

*Research partially supported by CAPES/PROCAD, grant n^0 00186/00-7.
†Supported by CNPq grant n$^{\underline{o}}$ 301060/94-0

1 Introduction

The subject matter of this article are flows on fiber bundles whose fibers are compact homogeneous spaces of Lie groups, with emphasis to non-compact semi-simple Lie groups and their flag manifolds. The aspects of these flows to be studied are the chain transitive sets and Morse decompositions.

In our set up we start with a principal bundle $Q \to X$ with structural group G, and let ϕ_t be a flow of homeomorphisms of Q which commutes with the right action of G. If F is a homogeneous space of G we can form the associated fiber bundle $E = Q \times_G F \to X$ with typical fiber F. The flow ϕ_t on Q induces a flow on E, in whose dynamics we are interested. We shall be concerned mainly with the generalized flag bundles when G is a non-compact semi-simple (or more generally reductive) Lie group and $F = G/P$ is one of its flag manifolds, where P is a parabolic subgroup of G.

The main result describes the maximal chain transitive subsets of a flow on a flag bundle by giving an algebraic characterization of their intersections with the fibers. In fact, fixing a maximal chain transitive subset $\mathcal{X}$ in the base space we prove in Theorem 9.11 that there exists an adjoint orbit, say $\mathcal{O}_\phi$, in the Lie algebra $\mathfrak{g}$ of G and a map $x \in \mathcal{X} \mapsto H_x \in \mathcal{O}_\phi$ such that the intersection of a maximal chain transitive subset $\mathcal{M}$ with the fiber over x is given by the singularities of H_x. Precisely, if we identify the fiber over x with G/P then the intersection of $\mathcal{M}$ with the fiber is a connected component of the fixed-point set of the one-parameter group $\exp(tH_x)$ acting on G/P. These connected components are algebraic varieties that are orbits of the identity component of the centralizer of H_x in G.

The class of flows treated here forms a natural generalization of linear flows on projective bundles, which have been extensively studied in the literature (see e.g. Colonius-Kliemann [5], [6], Conley [7], Sacker-Sell [16], Salamon-Zehnder [17], Selgrade [18], and references therein). In fact, linear flows on projective bundles are obtained when we specialize $Q \to X$ to be a $\mathrm{Gl}(n, \mathbb{R})$ principal bundle and take as F the real projective space with the standard projective action of the linear group.

As a motivation to work with other fiber bundles, we note that a natural way to produce flows on projective bundles is to start with a smooth dynamical system in a manifold. Its linearization induces a flow on the projective bundle of the tangent space. Some new bundles arise in for dynamical systems which leave invariant a geometrical structure like e.g. Hamiltonian flows, flows of isometries of pseudo-Riemannian manifolds, flows of holomorphic maps in pseudo-complex manifolds etc. In such cases we can see the linearized flow as given by a right invariant flow on a reduction of the bundle of frames of the manifold. Anyway we mention that the problem of studying flows on bundles of homogeneous spaces was posed by Conley [7] (see page 83), having in mind Hamiltonian flows evolving e.g. in bundles of Lagrangian subspaces.

The study of chain recurrence and Morse decompositions of linear flows on projective bundles goes back to Selgrade [18], whose theorem shows that the chain recurrent components of a flow which covers a chain recurrent flow on the base are vector subbundles, which decompose the vector bundle in a Whitney sum (see also Salamon-Zenhder [17]). Recently, Colonius-Kliemann [6] generalized the result of Selgrade by showing the existence of a finest Morse decomposition in the bundles whose fibers are flag manifolds of subspaces of a vector bundle.

Here we extend these results to generalized flag manifolds. In fact, by the very construction of the adjoint orbit $\mathcal{O}_\phi$, mentioned above, $\mathrm{ad}\,(H_x)$ is diagonalizable with real eigenvalues. Hence when specialized to vector and projective bundles we get on each fiber a diagonalizable linear map whose eigenspaces are the chain recurrent components, recovering the results of Selgrade and Colonius-Kliemann (see Theorem 10.1). Actually, our proof does not require compactness of the bundles (or equivalently of the base space). This is why the results are stated in terms of maximal chain transitive sets, which in the compact case coincide with the chain recurrent components, providing a finest Morse decomposition.

At this point we must mention that our approach to chain recurrence requires that the set of local homeomorphisms of the base space X is locally transitive in the sense that we can map any $x \in X$ to neighboring points using "small" local homeomorphisms of X. Although restrictive this condition is weak enough so that many classes of reasonable metric spaces are allowed as base spaces, like e.g. compact Riemannian manifolds or open sets in Frechet spaces.

We explain now the method of proof, which we believe to have independent interest since it establishes a link between topological dynamics and semigroup theory. Starting with a flow ϕ we generate semigroups of local homeomorphisms $S_{\varepsilon,T}$, $\varepsilon, T > 0$, by successively composing the local homeomorphisms which are ε-close (in their domains) to some ϕ_t, $t > T$. We call $S_{\varepsilon,T}$, $\varepsilon, T > 0$, the shadowing semigroups of the flow. The orbits of $S_{\varepsilon,T}$ are related to chain attainability with the conclusion that a maximal chain transitive set for the flow is the intersection of control sets for the shadowing semigroups (see Theorem 4.7 below). Here the local transitivity assumption enters to ensure that it is possible to substitute ε, T-chains by the action of $S_{\varepsilon,T}$. The idea of looking at ε, T-chains through shadowing semigroups was already exploited by the authors in [3] to study chain control sets for semigroup actions and control systems.

After relating chain transitivity to control sets we proceed to apply the theory of semigroups to handle the maximal chain transitive sets (and hence the Morse decompositions, in the compact case). In first place the topological arguments of [4] are used to reduce the problem to a fiberwise analysis, which amounts to look at semigroup actions on homogenous spaces. This leads us into the realm of the Lie theoretic results about control sets on flag manifolds which were developed in [19], [20], [21], [23], [24], [25], [26], [27]. These results yield quite quickly the existence of a finite number of maximal chain transitive subsets for the flows on the flag bundles, and hence the existence a finest Morse decomposition in the compact case. From the control sets on flag manifolds we get also that there exists a unique attractor as well as a unique repeller chain recurrent component.

Now, a key point is the notion of parabolic type of a semigroup S with non-empty interior in a semi-simple Lie group G. There are several equivalent ways of characterizing the parabolic type of S. The most suitable for our exposition here is the one which says that the parabolic type of S is the (only) flag manifold, say $\mathbb{F}_{\Theta(S)} = G/P_{\Theta(S)}$, such that the unique invariant control set $C_{\Theta(S)}$ of S in $\mathbb{F}_{\Theta(S)}$ is contractible under iterations of elements in the interior of S. Furthermore, $\pi^{-1}\left(C_{\Theta(S)}\right)$ is the invariant control set of S in the maximal flag manifold $\mathbb{F}$, where $\pi : \mathbb{F} \to \mathbb{F}_{\Theta(S)}$ is the canonical fibration.

The parabolic type of semigroups in G yields the notion of parabolic type of semigroups of local homeomorphisms in Q, and hence of the shadowing semigroups. Using the latter we associate to a flow ϕ a specific flag bundle, say $\mathbb{E}_{\Theta(\phi)}$, which we call analogously the parabolic type of ϕ. The property of $\mathbb{E}_{\Theta(\phi)}$ that emerges is that the attractor component of the finest Morse decomposition of the flow in $\mathbb{E}_{\Theta(\phi)}$ meets each fiber in a single point. Reverting time we get the same picture for the repeller, but in a "dual" flag bundle $\mathbb{E}_{\Theta^*(\phi)}$. These are the central results for the characterization of the Morse components, since they give the attractor and repeller components in any flag bundle.

Finally, from incidence relations in the flag manifolds related to the domains of attraction of the control sets we obtain other recurrent components from the attractor and the repeller ones. This way we obtain Theorem 9.11, mentioned above, where the H_x are intimately related to the parabolic type of the flow.

2 Preliminaries

In this section we recall basic facts and concepts about flows, fiber bundles and semigroups to be used afterwards.

2.1 Flows

Regarding flows on metric spaces we refer to the the books Colonius-Kliemann [5] (Appendix B) and Conley [7]. Let (Y, d) be a metric space. Although most aspects of the theory of flows requires compactness of the state space, the basic concepts can be stated without this assumption. Hence we do not assume in advance that Y is compact.

Given a continuous-time flow $\phi : \mathbb{R} \times Y \to Y$ we write the corresponding homeomorphisms by $\phi_t(\cdot) = \phi(t, \cdot)$ or simply by $\phi_t(x) = t \cdot x$, so that $\mathbb{R} \cdot x$ stands for the orbit of x under the flow. A set $A \subset Y$ is called invariant if $t \cdot x \subset A$ for all $x \in A$. A compact subset $A \subset L$ is called isolated invariant, if it is invariant and there exists an isolating neighborhood N of A, i.e., a set N with $A \subset \mathrm{int}(N)$, such that $\mathbb{R} \cdot x \subset N$ implies $x \in A$.

For $x \in Y$, the ω-limit set of x is denoted by $\omega_\phi(x)$ or simpler by $\omega(x)$:

$$\omega(x) = \{y \in Y : \exists t_k \to +\infty,\, t_k \cdot x \to y\}.$$

Analogously, $\omega^*_\phi(x) = \omega^*(x)$ is defined for $t \to -\infty$. On the other hand for a subset $A \subset Y$ we put

$$\omega(A) = \{x \in Y : \exists x_k \in A,\, t_k \to +\infty,\; t_k \cdot x_k \to x\},$$

and define the same way $\omega^*(A)$ with $t \to -\infty$.

A Morse decomposition of the flow ϕ_t is a finite collection $\{\mathcal{M}_i : i = 1, \ldots, n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets satisfying the following conditions:

1. For all $x \in Y$ the sets $\omega(x)$ and $\omega^*(x)$ are contained in $\bigcup_{i=1}^{n} \mathcal{M}_i$.

2. Suppose there are $\mathcal{M}_{j_0}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in Y \backslash \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, l$, then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements of a Morse decomposition are called Morse sets. We say that a set A is an attractor if it admits a neighborhood N such that $\omega(N) = A$. A repeller is a compact invariant set R that has a neighborhood N^* with $\omega^*(N^*) = R$. The neighborhoods N and N^* are called attractor and repeller neighborhoods, respectively. If Y is compact, every attractor is compact and invariant, and a repeller is an attractor for the time reversed flow.

A Morse decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ is called finer than a Morse decomposition $\{\mathcal{M}'_1, \ldots, \mathcal{M}'_{n'}\}$ if for all $j \in \{1, \ldots, n'\}$ there exists $i \in \{1, \ldots, n\}$ with $\mathcal{M}_i \subset \mathcal{M}'_j$. Usually one seeks for a finest Morse decomposition which provides all other decompositions through by joining together their components.

The chain recurrence which we discuss now is a nice device for getting Morse decompositions. For $x, y \in Y$ and $\varepsilon, T > 0$ an ε, T-chain from x to y is given by points $x_0 = x, x_1, \ldots, x_n = y \in Y$ and $t_0, \ldots, t_{n-1} \geq T$, for some $n \in \mathbb{N}$, such that

$$d(t_i \cdot x_i, x_{i+1}) < \varepsilon, \qquad i = 0, 1, \ldots, n-1.$$

We denote by $\mathcal{C}_{\varepsilon,T}(x)$ the set of those $y \in Y$ such that there exists an ε, T-chain from x to y, and put $\mathcal{C}(x) = \bigcap_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(x)$. On the other hand $\mathcal{C}^*_{\varepsilon,T}(x)$ is the set of those $y \in Y$ such that there exists an ε, T-chain from y to x, and $\mathcal{C}^*(x) = \bigcap_{\varepsilon,T} \mathcal{C}^*_{\varepsilon,T}(x)$. Equivalently, $\mathcal{C}^*(x)$ is the set of those y such that for all $\varepsilon, T > 0$ there exists an ε, T-chain from x to y for the reversed flow (see [8], Theorem 3.2D).

A subset $A \subset Y$ is *chain transitive* if for all $x \in A$, $A \subset \mathcal{C}(x)$. A chain transitive subset A is maximal transitive (with respect to set inclusion) if and only if for all $x \in A$, $A = \mathcal{C}(x)$ or equivalently $A = \mathcal{C}^*(x)$.

A point $x \in Y$ is chain recurrent if $x \in \mathcal{C}(x)$. We denote by $\mathcal{R}$ the chain recurrent set, that is, the set of all chain recurrent points. Note that a connected component of $\mathcal{R}$ is chain transitive since for any $y \in \mathcal{R}$ and $\varepsilon, T > 0$ $y \in \mathrm{int}\mathcal{C}_{\varepsilon,T}(y)$, so that $\mathcal{R} \subset \mathcal{C}_{\varepsilon,T}(x)$ for every $x \in \mathcal{R}$. In the compact case the connected components of the chain recurrent set $\mathcal{R}$ indeed coincide with the maximal chain transitive subsets, although in general the connected components may approximate to each other creating maximal chain transitive subsets containing more than one connected component.

Another property of the chain recurrent set is that it contains the ω and ω^*-limit sets, since if $y \in \omega(x)$ the flow property $\phi_{t_{k+l}-t_k}(\phi_{t_k}(x)) = \phi_{t_k}(x)$ ensures that for every $\varepsilon, T > 0$, $y \in \mathcal{C}_{\varepsilon,T}(y)$. Finally, the following proposition relates the chain recurrent components with Morse decompositions.

Proposition 2.1 *If Y is compact, there exists a finest Morse decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ if and only if the chain recurrent set $\mathcal{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the connected components of the chain recurrent set $\mathcal{R}$.*

Proof: See [5], Theorem B.2.25. ∎

2.2 Fiber bundles

Our starting point is a principal bundle $\pi : Q \to X$ with structural group G. Thus G acts freely on the right on Q and its orbits are the fibers $Q_x = \pi^{-1}\{x\}$, $x \in X$ (for fiber bundles we refer to Husemoller [11] and Kobayashi-Nomizu [12]). Each fiber is diffeomorphic to G. We assume allways that $Q \to X$ is locally trivial. Often a local trivialization is realized through a local cross section $\chi : U \to Q$, $U \subset X$.

Recall that if G acts on the left on a space F we can construct the associated bundle with typical fiber F by taking in $Q \times F$ the equivalence relation $(q_1, v_1) \sim (q_2, v_2)$ if and only if there exists $g \in G$ such that $q_2 = q_1 g$ and $v_2 = g^{-1} v_1$. Let E be the quotient space by this equivalence relation and denote by $q \cdot v$ the class in E of $(q, v) \in Q \times F$. Then $q \cdot v \mapsto \pi(q)$ defines a projection $E \to X$, also denoted by π or π_E if we wish to distinguish it from the projection $\pi_Q : Q \to X$ of Q. Our notation emphasizes the fact that the map $v \in F \mapsto q \cdot v \in E$ establishes a bijection between F and the fiber above $x = \pi(q)$. We denote in a similar way the inverse of this map. Thus $q^{-1} \cdot e$, $q \in Q$, $e \in E$, stands for $v \in F$, such that $q \cdot v = e$.

The associated bundle $E \to X$ is locally trivial when this happens to $Q \to X$. In locally trivial bundles over metric spaces we use the following metric.

Proposition 2.2 *Let $\pi : E \to X$ be a locally trivial bundle with (X, d) a metric space as well as the fiber (F, d_F). Fix a covering U_α of X with*

$\pi^{-1}(U_\alpha) \approx U_\alpha \times F$. *Then there exists a metric* d_E *on* E *such that on each trivialization* $U_\alpha \times F$ *it holds*

$$d_E((x,v),(y,w)) = \max\{d(x,y), d_F(v,w)\}.$$

Also, $d(\pi e, \pi f) \leq d_E(e,f)$ *for all* $e, f \in E$.

Proof: See [5], [17]. ∎

To consider flows on fiber bundles $E \to X$ we start with a flow ϕ_t on the principal bundle $Q \to X$, which commutes with the right action of G, that is, $\phi_t(qg) = \phi_t(q)g$, for all $t \in \mathbb{R}$, $q \in Q$ and $g \in G$. This condition implies that ϕ_t interchanges the fibers of Q and thus induces a flow on X. We shall denote the flow on the base by $t \cdot x$, $t \in \mathbb{R}$, $x \in X$. On the other hand the flow induced on $E \to X$ is also denoted by ϕ_t, so that $\phi_t(q \cdot v) = \phi_t(q) \cdot v$.

Restricting a flow on $Q \to X$ to the domain of a local cross section we obtain a *local cocycle* in the following sense: Let $\chi_i : U_i \to Q$ be cross sections above $U_i \subset X$, $i = 1, 2$. If $x \in U$ and $t \in \mathbb{R}$ are such that $x \in U_1$ and $t \cdot x \in U_2$, then $\phi_t(\chi_1(x))$ belongs to the same fiber as $\chi_2(t \cdot x)$ so that there exists an element in G, say $\rho_{\chi_1,\chi_2}(t,x)$, such that

$$\phi_t(\chi_1(x)) = \chi_2(t \cdot x)\, \rho_{\chi_1,\chi_2}(t,x).$$

We call the map ρ_{χ_1,χ_2} the local cocycle defined by χ_1 and χ_2. An easy application of the flow property of ϕ together with its right invariance yields the cocycle property:

$$\rho_{\chi_1,\chi_3}(t+s,b) = \rho_{\chi_2,\chi_3}(s, t \cdot b)\rho_{\chi_1,\chi_2}(t,b),$$

if χ_3 is a cross section defined on $(t+s) \cdot x$.

Of course, taking different cross sections χ_i' defined on the same U_i, the local cocycle $\rho_{\chi_1',\chi_2'}$ may change. We note however the following simple formula: If $\chi_1' = \chi_1 a$ and $\chi_2' = \chi_2 b$ with $a, b \in G$, then $\rho_{\chi_1',\chi_2'} = b\rho_{\chi_1,\chi_2}a^{-1}$.

For another way of writing ϕ_t locally, suppose that $Q = U \times G$. Then $\phi_t(x,g) = (f_1(x), f_2(x,g))$ with $f_2(x,gh) = f_2(x,g)h$. In this case the induced map in $U \times F$ is given by $\phi_t(x,v) = (f_1(x), f_2(x,g)v)$.

2.3 Semigroup actions

By a *local homeomorphism* of a metric space Y we mean a homeomorphism $\xi : U \to V$ between open subsets of Y. We denote by $\mathrm{loc}(Y)$ the set of local homeomorphisms of Y. The set $\mathrm{loc}\,(Y)$ is a *local group* in the sense that the operations of taking inverses and compositions – when allowed – are closed in $\mathrm{loc}\,(Y)$. A subset $\mathcal{G} \subset \mathrm{loc}\,(Y)$ is a local (sub) group if it is closed under these operations. Accordingly we say that $\mathcal{S} \subset \mathrm{loc}\,(Y)$ is a *local semigroup* in case $\mathcal{S}$ is closed under the allowed compositions.

In the sequel we follow the control theory terminology and say that a local semigroup $\mathcal{S}$ satisfies the accessibility property at $x \in Y$ if $\mathrm{int}\,(Sx) \neq \emptyset$, and it satisfies the *accessibiltity property* if this holds at every $x \in Y$.

Recall that a *control set* of a local semigroup $\mathcal{S} \subset \mathrm{loc}\,(Y)$ is a subset $D \subset Y$ such that

1. $\mathrm{int} D \neq \emptyset$,
2. $D \subset \mathrm{cl}\,(Sx)$ for all $x \in D$, and
3. D is maximal with these two properties.

The control sets are ordered by $D_1 \prec D_2$ if $D_2 \subset \mathrm{cl}\,(Sx)$ for any $x \in D_1$. An *invariant control set* is a control set D which maximal with respect to this order, that is, $\mathrm{cl}\,(\mathcal{S}x) = \mathrm{cl} D$ for all $x \in D$. It is known that under the accessibility property an invariant control set is closed and has non-empty interior. On the other hand if the control set D is minimal with respect to the order then it is open. Still under the assumption of accessibility it makes sense to introduce the (possibly empty) set

$$D_0 = \{x \in D : x \in \mathrm{int}\,(\mathcal{S}x) \cap \mathrm{int}\,\left(\mathcal{S}^{-1}x\right)\},$$

where D is a control set. In view of the proposition below we call D_0 the *set of transitivity* of D (or following Albertini-Sontag [1], D_0 is the *core* of D). A control set D such that $D_0 \neq \emptyset$ is called *effective control set*. These control sets have the following properties, proved in [4], Proposition 2.2 (see also [25], Proposition 2.2).

Proposition 2.3 *Suppose $D_0 \neq \emptyset$, that is, D is an effective control set. Then*

1. $D \subset \mathrm{int}\,(\mathcal{S}^{-1}x)$ for every $x \in D_0$.
2. $D_0 = \mathrm{int}\,(\mathcal{S}^{-1}x) \cap \mathrm{int}\,(\mathcal{S}x)$ for every $x \in D_0$.

3. For every $x, y \in D_0$ there exist $g \in \mathcal{S}$ with $gx = y$.

4. D_0 is dense in D.

5. D_0 is $\mathcal{S}$-invariant inside D, i.e., $\xi(x) \in D_0$ if $\xi \in \mathcal{S}$, $x \in D_0$ and $\xi(x) \in D$.

A special case of local semigroups which will show up below is obtained through the action of a Lie group G. If Y is a homogeneous space $Y = G/H$ then G acts transitively on Y and a subsemigroup $S \subset G$ with $\mathrm{int}S \neq \emptyset$ (w.r.t the topology of G) satisfies $\mathrm{int}(Sx) \neq \emptyset$ for all $x \in Y$ because the map $g \in G \mapsto gx \in Y$ is open. In this context it is not difficult to show that $D_0 = \{x \in D : x \in (\mathrm{int}S)\,x\} = \{x \in D : x \in (\mathrm{int}S^{-1})\,x\}$. As a complement to the above proposition we have the following statement which ensures the existence of effective control sets.

Proposition 2.4 *Let $x \in M$ be such that*

$$x \in \mathrm{int}(\mathcal{S}x) \cap \mathrm{int}\left(\mathcal{S}^{-1}x\right).$$

Then there exists a unique effective control set D such that $x \in D_0$.

Proof: See [4], Proposition 2.3. ∎

3 Locally transitive groups

Our method of studying the chain recurrence consists in perturbing the flow obtaining semigroups of local homeomorphisms (shadowing semigroups) whose control sets are shrinked to the chain transitive sets. In order that this approach works we need a technical assumption on the flow which permits to compare chains of the flow with the action of the shadowing semigroups. This assumption is stated in terms of local transitivity of local groups, which we discuss in this section.

Let (Y, d) be a metric space and consider the local group $\mathrm{loc}(Y)$ of local homeomorphisms of Y. We denote by $\mathrm{dom}(\xi)$ the domain of the local homeomorphism $\xi : \mathrm{dom}(\xi) \to V$ in $\mathrm{loc}(Y)$. For $\xi, \eta \in \mathrm{loc}(Y)$ whose domains overlap put

$$d'(\xi, \eta) = \sup d(\xi(x), \eta(x))$$

where the supremum is taken over $\mathrm{dom}\,(\xi) \cap \mathrm{dom}\,(\eta)$. Note that for $\xi, \tau, \eta \in \mathrm{loc}\,(Y)$ it holds

$$d'\,(\xi\eta, \tau\eta) \leq d'\,(\xi, \tau)\,, \tag{1}$$

since the supremum in the left hand side is taken over a smaller set than in the right hand side.

Definition 3.1 *We say that a local group* $\mathcal{G} \subset \mathrm{loc}\,(Y)$ *is* locally transitive *(abbreviated* loctrans*) with parameters* $c, \rho > 0$ *if for every* $x \in Y$ *and* y *in the ball* $B_\rho\,(x)$ *there exists* $\xi \in \mathcal{G}$ *such that* $\xi\,(x) = y$ *and* $d\,(y, x) \geq cd'\,(\xi, \mathrm{id})$.

We shall prove below that some reasonable local groups are locally transitive. However, in general this condition is not satisfied even if $\mathcal{G}$ is the full local group $\mathrm{loc}\,(Y)$ of a metric space.

Exemple: In $\mathbb{R}^2$ denote by C_q the circle of radious $q \geq 0$ centered at the origin. Consider the compact metric space

$$Y = \bigcup_{n=1}^{\infty} C_{1/n} \cup \{0\},$$

with the metric inherited from the standard metric of $\mathbb{R}^2$. Any local homeomorphism $\psi : U \to V$ of Y with $0 \in U$ has the property that $\phi(0) = 0$ for otherwise ψ^{-1} would map a connected component of the meeting of V with a circle into $\{0\}$. Hence $\mathrm{loc}\,(Y)$ is not locally transitive at 0.

We can change this example by taking Y to be the union of the circles C_q, $q \in \mathbb{Q}$, $q \geq 0$. Then we can take in Y the flow where ϕ_t is the rotation by the angle t. This flow is chain recurrent. Hence the existence of a chain recurrent flow on a metric space is not enough for $\mathrm{loc}\,(Y)$ to be locally transitive.

Yet another modification of Y gives an example with connected metric space. In fact, in $\mathbb{R}^3$ put

$$Y' = (Y \times (-\infty, 1]) \cup \left(\mathbb{R}^2 \times [1, +\infty)\right).$$

Again $\mathrm{loc}\,(Y')$ is not locally transitive at the origin.

We shall now see some cases of metric spaces whose local groups are locally transitive. First, let Y be a metric space such that $\mathrm{loc}\,(Y)$ is *loctrans* with parameters c, ρ. Then for any open subset $Y' \subset Y$ endowed with the induced distance, $\mathrm{loc}\,(Y')$ is *loctrans* with the same parameters, since we can always shrink the domain of a local homeomorphism of Y to be a local homeomorphism of Y' with the effect that d' diminishes.

Now, let E be a Frechet space with translation invariant metric d. For any $v \in E$ the translation $\tau_v(x) = v + x$ satisfies $d'(\tau_v, \mathrm{id}) = d(v, 0)$. Taking $x, y \in E$, $\tau_v(x) = y$ if $v = y - x$, so that τ_v satisfies the condition for the required local homeomorphism in the definition of *loctrans*. Thus $\mathrm{loc}(E)$ is *loctrans* with parameters $c = 1$ and arbitrary $\rho > 0$. Therefore, the *loctrans* property holds at open subsets of E:

Proposition 3.2 *If Y is an open set of a Frechet space E, endowed with the induced metric, then* $\mathrm{loc}(Y)$ *is locally transitive with $c = 1$ and any ρ.*

Now take Y to be an open set in a finite dimensional vector space E. Then a similar result holds if instead of a distance coming from E we consider a Riemannian metric in Y. In this case the translations restricted to open sets are still local homeomorphisms of Y. The difference here is that we cannot take $c = 1$ trivially. However, we can prove the *loctrans* property if we ask equivalence between the Riemannian metric and the distance coming from E.

Proposition 3.3 *Given a finite dimensional vector space E with a norm $|\cdot|$ let $Y \subset E$ be a connected open set and $g(\cdot,\cdot)$ a Riemannian metric in Y. Denote by $d(\cdot,\cdot)$ the distance in Y defined by g and suppose that there are constants $k_1, k_2 > 0$ with*

$$k_1 d(x, y) < |x - y| < k_2 d(x, y) \qquad x, y \in Y.$$

Then $\mathrm{loc}(Y)$ *is locally transitive with parameters $c = k_1/k_2$ and arbitrary ρ.*

Proof: Given $x_0, x \in Y$ and $v \in E$ such that $x + v \in Y$,

$$d(x + v, x) < \frac{1}{k_1}|v| < \frac{k_2}{k_1} d(x_0 + v, x_0).$$

Thus if we take a suitable restriction of τ_v, it follows that $d'(\tau_v, \mathrm{id}) \leq (k_2/k_1)\, d(\tau_v x_0, x_0)$. This implies at once the *loctrans* property. ∎

Now, we patch together the open sets to show that $\mathrm{loc}(Y)$ is *loctrans* if Y is a compact Riemannian manifold. Recall first that a positive real λ is a Lyapunov number of a covering $\{W_i\}_{i \in I}$ of a metric space if every set of diameter $< \lambda$ is contained in some W_i. It is well known that any covering of a compact metric space admits Lyapunov numbers.

Proposition 3.4 *If Y is a compact Riemannian manifold then* $\mathrm{loc}(Y)$ *is locally transitive.*

Proof: Let (V_α, ϕ_α) be a finite atlas for Y and take a subcovering to get a finite atlas (W_β, ψ_β) such that each W_β is relatively compact in some V_α. In $Y_\beta = \psi_\beta(W_\beta)$ take the metric induced from Y by ψ_β. Since Y_β is relatively compact the corresponding distance function is equivalent to the Euclidian norm. Hence, the above proposition applies and $\mathrm{loc}(Y_\beta)$ is *loctrans* with parameters $c_\beta, \rho_\beta > 0$. Now, let λ be a Lyapunov number of the covering and take $c = \min\{c_\beta\}$ and $\rho = \min\{\rho_\beta, \lambda/2\}$. Since any ball of radious ρ is contained in some W_β the result on the charts combine to show that $\mathrm{loc}(Y)$ is *loctrans* with parameters c, ρ. ∎

For the rest of this section we specialize the discussion of the *loctrans* property to fiber bundles. Our purpose is to combine this property on the basis and on the fibers to get local transitivity on the total space. Thus given a principal bundle $\pi : Q \to X$ with structure group G denote by $\mathrm{Aut}(Q)$ the local group of the right invariant local homeomorphisms ξ of Q having domain $\mathrm{dom}(\xi) = \pi^{-1}(U)$ with U open in X. Of course, a right invariant flow on Q is just a one-parameter group $\phi_t \in \mathrm{Aut}(Q)$ of globally defined homeomorphisms.

Now, let $E \to X$ be a bundle associated to $Q \to X$ with typical fiber F where G acts on the left. Any $\xi \in \mathrm{Aut}(Q)$ induces homeomorphisms on both X and E. Usually the induced maps are also denoted by ξ. However, for the moment we shall write $e(\xi)$ and $b(\xi)$ for the local homeomorphisms in E and X, respectively. Note that the domain of $e(\xi)$ also has the form $\pi^{-1}(U)$, $U \subset X$. The maps $e : \mathrm{Aut}(Q) \to \mathrm{loc}(E)$ and $b : \mathrm{Aut}(Q) \to \mathrm{loc}(X)$ define actions of $\mathrm{Aut}(Q)$ on E and X, respectively. The images of e and b are local groups in the corresponding spaces.

In general b is not onto $\mathrm{loc}(X)$. However, we can 'lift' to $\mathrm{Aut}(Q)$ a local homeomorphism θ of X, provided $\mathrm{dom}(\theta)$ and $\mathrm{im}(\theta)$ are contained in domains of trivializations of Q. In fact, let $\chi_i : U_i \to Q$, $i = 1, 2$, be local cross sections with $\mathrm{dom}(\theta) \subset U_1$ and $\mathrm{im}(\theta) \subset U_2$. Then the map $\widetilde{\theta}(\chi_1(x) \cdot g) = \chi_2(\theta(x)) \cdot g$, $x \in \mathrm{dom}(\theta)$, is a lifting of θ to a local homeomorphism in $\mathrm{Aut}(Q)$. With this in mind we prove that $\mathrm{Aut}(Q)$ is locally transitive when this condition holds at both the fiber and the base space.

Proposition 3.5 *Let E be given with a metric d_E like in Proposition 2.2. Then the action of* $\mathrm{Aut}(Q)$ *on E is locally transitive provided*

1. $\mathrm{loc}(X)$ *and the left action of* G *on the fiber* F *are* loctrans, *and*

2. *The covering of* X *defining* d_E *admits a Lebesgue number* $\lambda > 0$.

Proof: Let $\rho_1, c_1 > 0$ and $\rho_2, c_2 > 0$ be the parameters of local transitivity of $\mathrm{loc}(X)$ and G, respectively. Put $\rho = \min\{\rho_1, \rho_2, \lambda/2\}$ and $c = \min\{c_1, c_2\}$. Take $e, f \in E$ with $d_E(e, f) < \rho$. Then e, f are contained in a domain of trivialization $\pi^{-1}(U) \approx U \times F$, so we can write $e = (x, v)$ and $f = (y, w)$. Choose $\theta \in \mathrm{loc}(X)$ such that $\theta(x) = y$ and $c_1 d'(\theta, \mathrm{id}) \leq d(x, y)$. Also take $g \in G$ with $g(v) = w$ and $c_2 d'_F(g, \mathrm{id}) \leq d_F(u, v)$. The map $\xi : U \times G \to U \times G$ defined by $\xi(z, h) = (\theta(z), gh)$ belongs to $\mathrm{Aut}(Q)$. The induced map on $U \times F$ is given by $\xi(z, u) = (\theta(z), gu)$. Hence, $\xi(x, v) = (y, w)$. We have

$$d'_E(\xi, \mathrm{id}) = \sup d_E((\theta(z), gu)) = \sup \max\{d(\theta(z), z), d_F(gu, u)\}.$$

Therefore, $d'_E(\xi, \mathrm{id}) \leq \max\{1/c_1 d(x, y), 1/c_2 d_F(v, w)\}$. By the choice of c, it follows that $c d'_E(\xi, \mathrm{id}) \leq d_E(e, f)$, concluding the proof. ∎

Corollary 3.6 *Suppose that* X *is compact. Then* $\mathrm{Aut}(Q)$ *is locally transitive on* E *if both* $\mathrm{loc}(X)$ *and the left action of* G *on* F *are locally transitive.*

Regarding the local transitivity on the fibers, we recall the following result proved in [3], Corollary 3.4.

Lemma 3.7 *Let* G/H *be a homogeneous space and suppose that there exists a compact subgroup* $K \subset G$ *acting transitively on* G/H. *Endow* G/H *with a distance* d *given by a* K*-invariant Riemannian metric. Then the action of* G *on* G/H *is locally transitive.*

4 Shadowing semigroups

In this section we introduce semigroups of local homeomorphisms of the state space Y of a flow ϕ_t by perturbing the homeomorphisms of the flow at large times. These semigroups will be called shadowing semigroups and play a central role in the study of chain transitivity. In fact, we show that if the flow ϕ_t can be embedded in a locally transitive semigroup then its chain transitive sets are obtained as intersections of control sets for the shadowing semigroups.

Given a local group $\mathcal{G}$ and $\xi \in \mathcal{G}$ we put

$$V_\varepsilon(\xi, \mathcal{G}) = \{\eta \in \mathcal{G} : d'(\xi, \eta) < \varepsilon\}$$

(or simply $V_\varepsilon(\xi)$ if $\mathcal{G}$ is understood).

Definition 4.1 *Let $\mathcal{G}$ be a local semigroup containing ϕ_t for all $t \in \mathbb{R}$. Given $\varepsilon, T > 0$ define the* shadowing semigroup $S_{\varepsilon,T}(\phi, \mathcal{G})$ *(or simply $S_{\varepsilon,T}$) to be the local subsemigroup of $\mathcal{G}$ generated by the sets $V_\varepsilon(\phi_t, \mathcal{G})$ with t running through the interval $(T, +\infty)$. The shadowing semigroups for the reversed flow ϕ^* are denoted by $S^*_{\varepsilon,T}$.*

Remark: It is tempting to think that the shadowing semigroups for the reversed flows are given by the inverses $S^{-1}_{\varepsilon,T}$ of the forward semigroups. However it is not immediate that the subsets $V_\varepsilon(\phi_t, \mathcal{G})$, $t \in (-\infty, T)$, that generate $S^*_{\varepsilon,T}$ have the form $V_{\varepsilon'}(\phi_t, \mathcal{G})^{-1}$, $t \in (T', \infty)$ for some $\varepsilon', T' > 0$. For this to happen it is required a kind of equicontinuity of $\xi^{-1}\phi_t$ for every local homeomorphisms ξ defined in the several open sets of Y. Since the relation between these semigroups is not used afterwards we does not discuss it.

Note that by the very definition $S_{\varepsilon,T} \subset S_{\varepsilon_1,T_1}$ if $\varepsilon \leq \varepsilon_1$ and $T \geq T_1$. Actually, the next lemma shows that in a certain sense $S_{\varepsilon_1,T}$ is contained in the interior of $S_{\varepsilon_2,T}$ if $\varepsilon_1 < \varepsilon_2$.

Lemma 4.2 *Let ξ be a local homeomorphism satisfying $d'(\xi, \mathrm{id}) < \delta$. Then for $\psi \in S_{\varepsilon,T}$, the composition $\xi\psi \in S_{\varepsilon+\delta,T}$.*

Proof: Write $\psi = \psi_1 \cdots \psi_k$ with $\psi_i \in V_\varepsilon(\phi_{t_i}, \mathcal{G})$, $t_i > T$, $i = 1, \ldots, k$. To prove the lemma it is enough to check that $\xi\psi_1 \in S_{\varepsilon+\delta,T}$, because $\psi_2 \cdots \psi_k \in S_{\varepsilon,T} \subset S_{\varepsilon+\delta,T}$. By inequality (1), $d'(\xi\psi_1, \psi_1) \leq d'(\xi, \mathrm{id})$, so that $d'(\xi\psi_1, \psi_1) < \delta$. However, $\psi_1 \in V_\varepsilon(\phi_{t_1}, \mathcal{G})$. Hence for any z in $\mathrm{dom}(\xi\psi_1) \cap \mathrm{dom}(\psi_1) = \mathrm{dom}(\psi_1)$ it holds,

$$\begin{aligned} d\left(\xi\psi_1(z), \phi_{t_1}(z)\right) &\leq d(\xi\psi_1(z), \psi_1(z)) + d\left(\psi_1(z), \phi_{t_1}(z)\right) \\ &< \delta + \varepsilon, \end{aligned}$$

showing that $\xi\psi_1 \in V_{\varepsilon+\delta}(\phi_{t_1}, \mathcal{G})$, concluding the proof. ∎

Given $S \subset \mathrm{loc}(Y)$ and $x \in Y$ we write

$$Sx = \{\phi(x) : \phi \in S,\ x \in \mathrm{dom}(\phi)\}$$

for the orbit of x under S. Using the previous lemma we get the following inclusion relation between the orbits of the shadowing semigroups.

Lemma 4.3 *Suppose that ϕ_t belongs to the* loctrans *local group $\mathcal{G}$ for all $t \in \mathbb{R}$. Take $x \in Y$. Then $S_{\varepsilon,T}x \subset \operatorname{int}(S_{\varepsilon_1,T}x)$ if $\varepsilon < \varepsilon_1$.*

Proof: Given $\eta \in S_{\varepsilon,T}$ let us show that $\eta x \in \operatorname{int}(S_{\varepsilon_1,T}x)$. Write $\eta = \eta_1 \cdots \eta_k$ with $\eta_i \in V_\varepsilon(\phi_{t_i}, \mathcal{G})$, $i = 1, \ldots, k$. Now, let $c, \rho > 0$ be the parameters of local transitivity of $\mathcal{G}$, and choose $\rho' \leq \min\{\rho, c(\varepsilon_1 - \varepsilon)\}$. Then for any $y \in B_{\rho'}(\eta x)$ there exists $\xi \in \mathcal{G}$ with $\xi\eta(x) = y$ and $d(\xi(\eta x), \eta x) \geq cd'(\xi, \mathrm{id})$. By Lemma 4.2, $\xi\eta \in S_{\varepsilon_1,T}$, because the choice of ρ' ensures that $d'(\xi, \mathrm{id}) \leq \varepsilon_1 - \varepsilon$. Therefore, every $y \in B_{\rho'}(\eta x)$ belongs to $S_{\varepsilon_1,T}x$, proving the lemma. ∎

Corollary 4.4 *Suppose that ϕ_t belongs to the* loctrans *local group $\mathcal{G}$ for all $t \in \mathbb{R}$. Then for every $\varepsilon, T > 0$ and $x \in Y$, $\operatorname{int}(S_{\varepsilon,T}x) \neq \emptyset$.*

Our objective is to show that points reachable by chains of the flow can be reached by the action of the shadowing semigroups and conversely. At this regard the basic fact is given by the following proposition whose proof is essentially a reformulation of [3], Proposition 3.1.

Proposition 4.5 *Keep the above notations and take $x, y \in Y$. Then*

1. *For all $\varepsilon, T > 0$, $S_{\varepsilon,T}x \subset \mathcal{C}_{\varepsilon,T}(x)$. Also, for all $\varepsilon' > \varepsilon$, $\operatorname{cl}(S_{\varepsilon,T}x) \subset \mathcal{C}_{\varepsilon',T}(x)$.*
2. *Let ϕ_t, $t \in \mathbb{R}$, be contained in the locally transitive group $\mathcal{G}$ with parameters c, ρ. Take ε with $0 < \varepsilon < \rho$ and put $\varepsilon' = \varepsilon/c$. Then $\mathcal{C}_{\varepsilon,T}(x) \in \operatorname{int}(S_{\varepsilon',T}x)$.*

Proof:

1. Take $y \in S_{\varepsilon,T}x$ and let $\psi \in S_{\varepsilon,T}$ be such that $y = \psi(x)$. Write $\psi = \psi_k \cdots \psi_1$ with $\psi_i \in V_\varepsilon(\phi_{t_i}, \mathcal{G})$, $t_i > T$, $i = 1, \ldots, k$. Then the sequence $x_0 = x$, $x_1 = \psi_1(x_0)$,..., $x_k = \psi_k(x_{k-1}) = y$ together with $t_1, \ldots, t_{n-1} > T$ determine an ε, T-chain from x to y, since

$$d\left(\phi_{t_i}(x_{i-1}), x_i\right) = d\left(\phi_{t_i}(x_{i-1}), \psi_i(x_{i-1})\right) < \varepsilon.$$

   Now, for $y \in \operatorname{cl}(S_{\varepsilon,T}x)$ take a sequence $\psi_n \in S_{\varepsilon,T}$ with $\psi_n(x) \to y$. Let n_0 be such that $d\left(\psi_{n_0}(x), y\right) < \varepsilon' - \varepsilon$. As before, there exists an ε, T-chain from x to $\psi_{n_0}(x)$. Let this chain be given by $y_1 = x$,..., $y_k =$

$\psi_{n_0}(x_0)$, $s_1, \ldots, s_{n-1} > T$. Thus $d\left(\phi_{s_i}(y_i), y_{i+1}\right) < \varepsilon$ for $i = 1, \ldots, k$. Therefore, $z_1 = x$, …, $z_{n-1} = y_{n-1}$, $z_n = y$ and $s_0, \ldots, s_{n-1} > T$ determine an ε', T-chain from x to y, since

$$d\left(\phi_{s_{n-1}}(y_{n-1}), y\right) \leq d\left(\phi_{s_{n-1}}(y_{n-1}), \psi_{n_0}(x)\right) + d\left(\psi_{n_0}(x), y\right) < \varepsilon'.$$

2. Since $d\left(\phi_{t_i}(x_i), x_{i+1}\right) < \varepsilon < \rho$, the *loctrans* property of $\mathcal{G}$ implies that there exists $\xi \in \mathcal{G}$ such that

$$d\left(\xi\left(\phi_{t_i}(x_i)\right), x_{i+1}\right) = d\left(\xi\left(\phi_{t_i}(x_i)\right), \phi_{t_i}(x_i)\right) \geq cd'(\xi, \mathrm{id})$$

for $i = 0, \ldots, n-1$. Hence $d'(\xi, \mathrm{id}) < \varepsilon/c = \varepsilon'$. Define $\eta_i = \xi\phi_{t_i}$. Then

$$d'\left(\eta_i, \phi_{t_i}\right) = d'\left(\xi\phi_{t_i}, \phi_{t_i}\right) \leq d'(\xi, \mathrm{id}) < \varepsilon'$$

because multiplication on the right diminishes d'. Therefore, $\eta_i \in V_{\varepsilon'}\left(\phi_{t_i}\right)$. On the other hand, $\eta_i(x_i) = \xi\phi_{t_i}(x_i) = x_{i+1}$, and $x_n = \eta_{n-1}\cdots\eta_0(x_0)$, concluding the proof since $\psi = \eta_{n-1}\cdots\eta_0 \in S_{\varepsilon',T}$.

∎

This proposition ensures that we can replace an ε, T-chain by the action of an element in $S_{\varepsilon,T}$. From this we get the following useful property of the control sets of the shadowing semigroups.

Lemma 4.6 *With the same assumptions as the previous proposition, take $\varepsilon_1 < \varepsilon_2$ and suppose that $D_{\varepsilon_1,T}$ and $D_{\varepsilon_2,T}$ are effective control sets for $S_{\varepsilon_1,T}$ and $S_{\varepsilon_2,T}$, respectively, such that $\left(D_{\varepsilon_1,T}\right)_0 \cap \left(D_{\varepsilon_2,T}\right)_0 \neq \emptyset$. Then $D_{\varepsilon_1,T} \subset \left(D_{\varepsilon_2,T}\right)_0$.*

Proof: Take $x \in \left(D_{\varepsilon_1,T}\right)_0 \cap \left(D_{\varepsilon_2,T}\right)_0$. Then for any $y \in \left(D_{\varepsilon_1,T}\right)_0$, $y \in S_{\varepsilon_1,T}x$ and $x \in S_{\varepsilon_1,T}y$. Since $S_{\varepsilon_1,T} \subset S_{\varepsilon_2,T}$, the maximality property in the definition of control sets ensures that $y \in D_{\varepsilon_2,T}$, and a fortiori, by Proposition 2.3, $y \in \left(D_{\varepsilon_2,T}\right)_0$. Hence, $\left(D_{\varepsilon_1,T}\right)_0 \subset \left(D_{\varepsilon_2,T}\right)_0$. To conclude the proof we show that $z \in S_{\varepsilon_2,T}x$ and $x \in S_{\varepsilon_2,T}z$. By Proposition 2.3 (1), $x \in S_{\varepsilon_1,T}z \subset S_{\varepsilon_2,T}z$. On the other hand, $D_{\varepsilon_1,T} \subset \mathrm{cl}\left(D_{\varepsilon_1,T}\right)_0$, so that any $z \in D_{\varepsilon_1,T}$ belongs to $\mathrm{cl}\left(S_{\varepsilon_1,T}x\right)$. Hence by the the second statement of the above proposition, it follows that $z \in \mathrm{int}\left(S_{\varepsilon_2,T}x\right) \subset S_{\varepsilon_2,T}x$, as we desired to show. ∎

Now we can prove the main result of this section which gives a characterization of the chain recurrent components in terms of the control sets of the shadowing semigroups.

Theorem 4.7 *Let ϕ_t be a flow on Y contained in a* loctrans *local group $\mathcal{G}$. Suppose that for each $\varepsilon, T > 0$ there exists a control set $D_{\varepsilon,T}$ of $S_{\varepsilon,T}(\phi, \mathcal{G})$ such that $\mathcal{M}' = \bigcap_{\varepsilon,T} D_{\varepsilon,T} \neq \emptyset$. Then $\mathcal{M}'$ is a maximal chain transitive subset.*

Conversely let $\mathcal{M}$ be a maximal chain transitive subset. Then for every $\varepsilon, T > 0$ there exists an effective control set $D_{\varepsilon,T}(\mathcal{M})$ of $S_{\varepsilon,T}(\phi, \mathcal{G})$ such that $\mathcal{M}$ is contained in the set of transitivity $D_{\varepsilon,T}(\mathcal{M})_0$. Furthermore,

$$\mathcal{M} = \bigcap_{\varepsilon,T} D_{\varepsilon,T}(\mathcal{M}) = \bigcap_{\varepsilon,T} D_{\varepsilon,T}(\mathcal{M})_0 . \tag{2}$$

Proof: If $x, y \in \mathcal{M}'$ then for all $\varepsilon, T > 0$, $x, y \in D_{\varepsilon,T}$, so that $y \in \mathrm{cl}(S_{\varepsilon,T}x)$. Hence by Proposition 4.5 (1) there exists an ε, T-chain from x to y. This shows that $\mathcal{M}'$ is chain transitive. The maximality follows by Proposition 4.5 (2). In fact, if $x \in \mathcal{M}'$ and for every $\varepsilon, T > 0$, $z \in \mathcal{C}_{\varepsilon,T}(x)$ and $z \in \mathcal{C}_{\varepsilon,T}(x)$ then $z \in D_{\varepsilon,T}$, so that $z \in \mathcal{M}'$.

For the second part take $x \in \mathcal{M}$. Since $\mathcal{M}$ is chain recurrent, $x \in \mathcal{C}_{\varepsilon,T}(x)$ for all ε, $T > 0$. By Proposition 4.5 (2) and Lemma 4.3, it follows that $x \in \mathrm{int}(S_{\varepsilon,T}x)$ for every $\varepsilon, T > 0$. But this implies that there exists a control set $D_{\varepsilon,T}(\mathcal{M}, x)$ of $S_{\varepsilon,T}$ such that $x \in D_{\varepsilon,T}(\mathcal{M}, x)_0$ (see Proposition 2.4). We claim that $D_{\varepsilon,T}(\mathcal{M}, x) = D_{\varepsilon,T}(\mathcal{M}, y)$ for all $x, y \in \mathcal{M}$. In fact, since $\mathcal{M}$ is chain transitive, $y \in \mathcal{C}_{\varepsilon,T}(x)$ for all $\varepsilon, T > 0$. Hence, by Proposition 4.5 (2), $y \in S_{\varepsilon,T}x$. The same way $x \in S_{\varepsilon,T}y$, showing that x and y belong to the same control set.

As to the equalities in (2), note that the second one is a consequence of Lemma 4.6. Hence it remains to prove that $\bigcap_{\varepsilon,T} D_{\varepsilon,T}(\mathcal{M}) \subset \mathcal{M}$. Pick $x \in \bigcap_{\varepsilon,T} D_{\varepsilon,T}(\mathcal{M})$. By definition of control set we have $x \in \mathrm{cl}(S_{\varepsilon,T}x)$ for every $\varepsilon, T > 0$. Using Proposition 4.5, we see that any two points $x, y \in \bigcap_{\varepsilon,T} D_{\varepsilon,T}(\mathcal{M})$ are attainable to each other by ε, T-chains, so that this intersection is indeed contained in a chain transitive set, which must be $\mathcal{M}$. ∎

Corollary 4.8 *Let the assumptions be as in Theorem 4.7. Then the shadowing semigroups $S_{\varepsilon,T}$ are transitive on Y if the flow is chain transitive on Y.*

Proof: In fact, assuming that the flow is chain transitive on Y it follows

$$Y \subset \mathcal{M} \subset D_{\varepsilon,T}(\mathcal{M})_0 \subset Y$$

for every $\varepsilon, T > 0$. Therefore, $S_{\varepsilon,T}$ is transitive on Y. ∎

The results proved so far apply without change to the reversed flow ϕ^* and its shadowing semigroups $S^*_{\varepsilon,T}$. Since the chain transitive sets for ϕ and ϕ^* are the same (see [8], Theorem 3.2D), each chain transitive set $\mathcal{M}$ is contained in a unique control set $D^*_{\varepsilon,T}(\mathcal{M})$ of $S^*_{\varepsilon,T}$. As in Theorem 4.7, $\mathcal{M}$ is the intersection of the control sets $D^*_{\varepsilon,T}(\mathcal{M})$ as well as of their sets of transitivity $D^*_{\varepsilon,T}(\mathcal{M})_0$. Clearly, intersecting the sets $D^*_{\varepsilon,T}(\mathcal{M}) \cap D_{\varepsilon,T}(\mathcal{M})$ we also get $\mathcal{M}$. For later reference we explicitate this fact.

Corollary 4.9 *With the notations and asssumptions as above,*

$$\mathcal{M} = \bigcap_{\varepsilon,T} \left(D_{\varepsilon,T}(\mathcal{M}) \cap D^*_{\varepsilon,T}(\mathcal{M}) \right) = \bigcap_{\varepsilon,T} \left(D_{\varepsilon,T}(\mathcal{M})_0 \cap D^*_{\varepsilon,T}(\mathcal{M})_0 \right).$$

As another application of the shadowing semigroup description of chains we get the domain of attraction of a chain recurrent component $\mathcal{M}$ as the intersection of the domains of attraction of the corresponding control sets.

Recall that the domain of attraction $\mathcal{A}(\mathcal{M})$ of the chain recurrent component $\mathcal{M}$ of a flow on Y is defined as the set of those $z \in Y$ for which there exists $x \in \mathcal{M}$ such that $x \in \mathcal{C}(z)$. We denote the domain of attraction of $\mathcal{M}$ for the reversed flow by $\mathcal{A}^*(\mathcal{M})$. Analogously, if D is a control set for the semigroup S, its domain of attraction $\mathcal{A}(D)$ is the set of $z \in Y$ such that there exists $\xi \in S$ with $\xi z \in D$. We note that if $D_0 \neq \emptyset$ then $\mathcal{A}(D)$ is open and $z \in \mathcal{A}(D)$ if and only if $\xi z \in D_0$ for some $\xi \in S$ (cf. [21]). In reversing the action of the semigroup it is this latter condition which is convenient, that is, we write $\mathcal{A}^*(D)$ for the set of those $z \in Y$ such that $\xi^{-1} z \in D_0$ for some $\xi \in S$. It is an immediate consequence of the definitions that $\mathcal{M} = \mathcal{A}(\mathcal{M}) \cap \mathcal{A}^*(\mathcal{M})$ and $D_0 = \mathcal{A}(D) \cap \mathcal{A}^*(D)$.

Proposition 4.10 *Let the notations and assumptions be as in Theorem 4.7. Then the domain of attraction of the chain recurrent component $\mathcal{M}$ is given by*

$$\mathcal{A}(\mathcal{M}) = \bigcap_{\varepsilon,T} \mathcal{A}(D_{\varepsilon,T}(\mathcal{M})).$$

Analogously, $\mathcal{A}^*(\mathcal{M}) = \bigcap_{\varepsilon,T} \mathcal{A}^*(D_{\varepsilon,T}(\mathcal{M}))$.

Proof: Take $z \in \mathcal{A}(\mathcal{M})$. Then, there exists $x \in \mathcal{M}$ such that $x \in \mathcal{C}_{\varepsilon,T}(z)$ for all $\varepsilon, T > 0$. By Proposition 4.5 there exist $\phi_{\varepsilon,T} \in S_{\varepsilon,T}$ such that $\phi_{\varepsilon,T}(z) = x$ for every $\varepsilon, T > 0$. Therefore, $z \in \mathcal{A}(D_{\varepsilon,T}(\mathcal{M}))$ for every $\varepsilon, T > 0$, i.e., $z \in \bigcap_{\varepsilon,T} \mathcal{A}(D_{\varepsilon,T}(\mathcal{M}))$. For the converse, assume that $z \in \bigcap_{\varepsilon,T} \mathcal{A}(D_{\varepsilon,T}(\mathcal{M}))$. Hence, there exists $\phi_{\varepsilon,T} \in S_{\varepsilon,T}$ and $x_{\varepsilon,T} \in D_{\varepsilon,T}(\mathcal{M})$ such that $\phi_{\varepsilon,T}(z) = x_{\varepsilon,T}$. Take $x \in \mathcal{M} \subset (D_{\varepsilon,T})_0$. By Proposition 4.5, $x_{\varepsilon,T} \in \mathcal{C}_{\varepsilon,T}(z)$, hence $x \in \mathcal{C}_{\varepsilon,T}(z)$. ∎

Once we have the description of the chain recurrent components in terms of control sets the strategy is to use known results about the latter in order to understand the chain transitive sets. The following statements are easy consequences of this approach.

Proposition 4.11 *Let $Q \to X$ with fiber G and $E \to X$ an associated bundle with fiber $F = G/L$. Assume that F is connected and the action of G on F leaves invariant a probability measure. Then under the* loctrans *condition a flow on E is chain recurrent if it is chain recurrent on X.*

Proof: In fact, the existence of an invariant probability measure ensures that any semigroup with non-empty interior in G acts transitively on F (see [22], Lemma 6.2). This implies that the semigroups $S_{\varepsilon,T}$ are transitive on E, and hence the chain transitivity of the flow. ∎

Cases covered by this proposition are the compact solvmanifolds and compact groups. In fact, in both cases there are invariant probability measures. If G is compact then the Haar measure induces invariant measures on its homogeneous spaces. Analogously, if G is solvable and G/L compact, then there exists an invariant probability on G/L (see Mostow [14]).

Corollary 4.12 *Let the notations and assumptions be as in the above proposition an assume furthermore that G is compact or G is solvable and G/L compact. Then a flow on E is chain recurrent if it is chain recurrent on the base X.*

To conclude this section we show two facts about chain recurrence which in some situations may be helpful to weaken the *loctrans* condition to a dense subset of the state space.

Proposition 4.13 *Let ϕ_t be a flow on a metric space (Y, d) and suppose that $Z \subset Y$ is a dense subset which is invariant by ϕ_t. Take $x, y \in Z$ and suppose that $y \in \mathcal{C}_{\varepsilon,T}(x)$. Then there exists an ε, T-chain from x to y for the flow restricted to Z.*

Proof: Let $(x_1, \ldots, x_n)$, $(t_1, \ldots, t_n)$ be a chain between x and y, and suppose that some $x_i \notin Z$. By continuity of the flow we can take $x_i' \in Z$ close enough to x_i such that $d\left(\phi_{t_{i-1}}(x_{i-1}), x_i'\right) < \varepsilon$ and $d\left(\phi_{t_i}(x_i'), x_{i+1}\right) < \varepsilon$. Substituting this way x_i by x_i' every time $x_i \notin Z$, we get a chain from x to y without leaving the invariant subset Z. ∎

Proposition 4.14 *Let ϕ_t be a flow on a compact metric space (Y, d) containing a dense invariant subset Z. Denote by $\overline{\phi}_t$ the restriction of ϕ_t to Z and suppose that $\mathcal{M}$ is a maximal chain transitive set of $\overline{\phi}_t$. Then its closure* $\mathrm{cl}\mathcal{M}$ *is a maximal chain transitive set of ϕ_t.*

Proof: Take $x \in \mathcal{M}$ and $y \in \mathrm{cl}\mathcal{M}$. Clearly, for $\varepsilon, T > 0$, $y \in \mathcal{C}_{\varepsilon,T}(x)$. Hence, by [8], Theorem 3.2D, we have also ε, T-chains from y to x, showing that $\mathrm{cl}\mathcal{M}$ is chain transitive. As to the maximality note first that by compactness of Y, $\mathrm{cl}\mathcal{M}$ is contained in a maximal chain transitive set, say $\mathcal{M}'$. Any z in $\mathcal{M}'$ is attainable by chains from $w \in \mathrm{cl}\mathcal{M}$. Take a sequence $x_n \in \mathcal{M}$ with $x_n \to w$. For n large anough and an ε, T-chain starting at w there exists an ε, T-chain starting at x_n whose end point is close enough to z. Using the above propostion we see that $z \in \mathrm{cl}\mathcal{M}$, concluding the proof. ∎

As an example where we can apply this proposition, suppose that Y is the closure of an open set O in a Frechet space, and consider a flow ϕ_t on Y which leaves O invariant. By Proposition 3.2, $\mathrm{loc}(O)$ is locally transitive. Hence, we can use the shadowing semigroup method to the flow $\overline{\phi}_t$ restricted to O. If we are able to get this way maximal chain transitive subsets of $\overline{\phi}_t$, then we get also maximal chain transitive subsets of ϕ_t.

5 Semigroups and flag manifolds

The purpose of this section is to establish notations and background results about semi-simple Lie groups, their flag manifolds and subsemigroups. We follow Borel-Tits [2], Duistermaat-Kolk-Varadarajan[9], Varadarajan [29] and Warner [28] as basic references to semi-simple Lie groups and flag manifolds. The results about semigroups to be recalled here appeared in [19], [20], [21], [23], [24], [25], [26].

5.1 Semi-simple Lie groups and flag manifolds

Given a non-compact semi-simple Lie algebra $\mathfrak{g}$ let us take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Take a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and denote by Π^+ the corresponding set of positive roots and Σ the set of simple roots. Put

$$\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \qquad \mathfrak{n}^- = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ stands for the α-root space and $\Pi^- = -\Pi^+$. The Iwasawa decomposition reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. The standard minimal parabolic subalgebra is defined by $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. On the other hand given a subset $\Theta \subset \Sigma$ denote by $\langle \Theta \rangle$ the subset of Π spanned by Θ over the integers. Put $\langle \Theta \rangle^\pm = \Pi^\pm \cap \langle \Theta \rangle$ and let $\mathfrak{n}^\pm(\Theta)$ be the subalgebra of $\mathfrak{n}^\pm$ spanned by $\mathfrak{g}_\alpha$, $\alpha \in \langle \Theta \rangle^\pm$. The standard parabolic subalgebra $\mathfrak{p}_\Theta$, associated to Θ, is given by

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}.$$

In particular, $\mathfrak{p}_\emptyset = \mathfrak{p}$.

Now let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For each $\Theta \subset \Sigma$ let the standard parabolic subgroup P_Θ of G be defined as the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \{g \in G : \mathrm{Ad}(g)\,\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\},$$

and write $\mathbb{F}_\Theta = G/P_\Theta$ for the associated flag manifold of G. The coset G/P_Θ identifies with the set of parabolic subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{p}_\Theta$, so that $\mathbb{F}_\Theta$ depends only on $\mathfrak{g}$ and not on the specific Lie group G having Lie algebra $\mathfrak{g}$. In the sequel we write simply $\mathbb{F}$ for the maximal flag manifold $\mathbb{F}_\emptyset$.

Given two subsets $\Theta_1 \subset \Theta_2 \subset \Sigma$, the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $G/P_{\Theta_1} \to G/P_{\Theta_2}$, $gP_{\Theta_1} \mapsto gP_{\Theta_2}$. Alternatively, the fibration assigns to the parabolic subalgebra $\mathfrak{q} \in \mathbb{F}_{\Theta_1}$ the unique parabolic subalgebra in $\mathbb{F}_{\Theta_2}$ containing $\mathfrak{q}$. In particular, $\mathbb{F}$ projects onto every flag manifold $\mathbb{F}_\Theta$.

We denote by $K = \exp \mathfrak{k}$, $N^\pm = \exp \mathfrak{n}^\pm$ and $A = \exp \mathfrak{a}$ the connected subgroups with corresponding Lie algebras. Analogously, we put $A^+ = \exp \mathfrak{a}^+$

for the Weyl chamber in G corresponding to $\mathfrak{a}^+$. The group K acts transitively on each $\mathbb{F}_\Theta$, allowing an identification $G/P_\Theta = K/K_\Theta$ where $K_\Theta = K \cap P_\Theta$.

Recall that a flag manifold $\mathbb{F}_\Theta$ can be embedded into the $\mathfrak{s}$ component of a Cartan decomposition. In fact, let $H_\Theta \in \mathrm{cl}\mathfrak{a}^+$ be such that $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. Then K_Θ is the centralizer of H_Θ in K so that the adjoint orbit $\mathrm{Ad}(K) H_\Theta$ identifies with $\mathbb{F}_\Theta$. Conversely, given $H \in \mathrm{cl}\mathfrak{a}^+$, $\mathrm{Ad}(K) H$ identifies with $\mathbb{F}_{\Theta(H)}$ where $\Theta(H) = \{\alpha \in \Sigma : \alpha(H) = 0\}$.

This realization is helpful in describing the Morse decomposition of the flow in $\mathbb{F}_\Theta$ induced by $\exp(tH)$, $t \in \mathbb{R}$, $H \in \mathrm{cl}\mathfrak{a}^+$. In fact, any $Z \in \mathfrak{s}$ defines a height function $f_Z : \mathrm{Ad}(K)(H_\Theta) \to \mathbb{R}$ by $f_Z(x) = \langle Z, x\rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form. Now, there exists in $\mathbb{F}_\Theta = \mathrm{Ad}(K) H_\Theta$ a K-invariant metric, say $(\cdot, \cdot)_{H_\Theta}$, depending on H_Θ such that the gradient of f_Z with respect to $(\cdot, \cdot)_{H_\Theta}$ is precisely the vector field $\widetilde{Z}$ induced by Z on $\mathbb{F}_\Theta$ (see [9]). The flow of $\widetilde{Z}$ is given by the action of $\exp(tZ)$, $t \in \mathbb{R}$, so that the finest Morse decomposition of $\exp(tZ)$ is given by the singularities of $\widetilde{Z}$.

In order to describe the singularities, denote by $\mathcal{W}$ the Weyl group of $\mathfrak{g}$, which is the group generated by the reflections with respect to the roots in Π. This group is isomorphic to M^*/M, where M^* and M are the normalizer and centralizer of A in K, respectively. The orbit of H_Θ under M^* is finite and the action of M^* on this orbit factors through $\mathcal{W}$. Thus we abuse notation and write the elements of this orbit as $b_w^\Theta = w b_\Theta^+$, $w \in \mathcal{W}$, where b_Θ^+ is the origin in $\mathbb{F}_\Theta$ (the point which identifies with H_Θ). The proof of the following lemma can be found in [9] (see Proposition 1.3 and Corollary 3.5).

Lemma 5.1 *Given $H \in \mathrm{cl}\mathfrak{a}^+$, the set of fixed points of $\exp(tH)$ in $\mathbb{F}_\Theta$ is given by the disjoint union of connected subsets*

$$\bigcup_{w \in \mathcal{W}_H \backslash \mathcal{W}} K_H^0 b_w^\Theta$$

where K_H^0 is the identity component of the centralizer K_H of H in K.

In this decomposition the component $K_H^0 b_\Theta^+$ is the only attractor, while the unique repeller is given by $K_H^0 b_\Theta^-$, where $b_\Theta^- = w_0 b_\Theta^+$ and w_0 is the principal involution of $\mathcal{W}$, that is, the element of largest length as a product of reflections with respect to the simple roots.

Let us take in particular $H \in \mathfrak{a}^+$. Then $K_H^0 = M_0$, so that $K_H^0 b_w^\Theta = b_w^\Theta$ for all $w \in \mathcal{W}$, and the fixed-points are isolated (alternatively, f_H is a Morse function). In this case the stable manifold of the fixed-point b_w^Θ is given by the orbit $N^- b_w^\Theta$, while the unstable manifold is $N^+ b_w^\Theta$. Thus there exists a unique attractor fixed-point b_Θ^+ whose stable manifold is the open and dense orbit $N^- b_\Theta^+$ and a unique repeller b_Θ^- with unstable manifold $N^+ b_\Theta^-$, which is also open and dense.

More generally, we say that $Z \in \mathfrak{g}$ is split-regular in case $Z = \mathrm{Ad}\,(g)\,(H)$ for some $g \in G$, $H \in \mathfrak{a}^+$. Analogously, $x \in G$ is said to be split-regular in case $x = ghg^{-1}$ with $h \in A^+ = \exp \mathfrak{a}^+$, that is, $x = \exp Z$, with Z split-regular in $\mathfrak{g}$. By taking conjugations we carry over the Morse decomposition for split-regular elements: If $Z = \mathrm{Ad}\,(g)\,(H)$, $H \in \mathfrak{a}^+$, then its fixed-points are gb_w^Θ with stable manifolds $gN^+ b_w^\Theta$ and unstable manifolds $gN^- b_w^\Theta$. The same picture holds for the discrete time flow x^n if $x = ghg^{-1}$ is split-regular. In the sequel we write $\mathrm{fix}_\Theta\,(x)$ for the set of fixed-points of x in $\mathbb{F}_\Theta$ and put $\mathrm{fix}_\Theta\,(x, w) = gb_w^\Theta$ and call this the *fixed-point of type* w of x. Also, we write $\mathrm{at}_\Theta\,(x) = \mathrm{fix}_\Theta\,(x, 1)$ for the attractor and $\mathrm{rp}_\Theta\,(x) = \mathrm{fix}_\Theta\,(x, w_0)$ for the repeller. The stable manifold of the attractor will be denoted by $\mathrm{st}_\Theta\,(x)$ while the unstable manifold of the repeller is $\mathrm{un}_\Theta\,(x)$. We use analogous notations for a split-regular $Z \in \mathfrak{g}$, for instance, $\mathrm{at}_\Theta\,(Z)$ is the attractor of $\exp\,(tZ)$, etc. Also, in case $\mathbb{F}_\Theta = \mathbb{F}$ is the maximal flag manifold we suppress the subscripts Θ in the notations.

Now we discuss the notion of dual flag manifolds. We refer to [23] and [24] for further details. The principal involution $w_0 \in \mathcal{W}$ maps Σ onto $-\Sigma$, so that $\iota = -w_0$ leaves Σ invariant. Thus for $\Theta \subset \Sigma$, $\Theta^* = \iota\,(\Theta) \subset \Sigma$, and we can form the flag manifold $\mathbb{F}_{\Theta^*}$, called *dual* of $\mathbb{F}_\Theta$. The diagonal action $g\,(b_1, b_2) = (gb_1, gb_2)$ of G on $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ has a unique open orbit, say $\mathcal{O}_\Theta$, which as a homogeneous space identifies with the adjoint orbit $\mathrm{Ad}\,(G)\,(H_\Theta)$, with H_Θ as above. In fact, take the pair $\left(\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^-\right) \in \mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$, where $\mathfrak{p}_\Theta^- = \mathfrak{n}^+\,(\Theta) \oplus \mathfrak{p}^-$ with

$$\mathfrak{n}^+\,(\Theta) = \sum_{\alpha \in \langle\Theta\rangle^+} \mathfrak{g}_\alpha \qquad \mathfrak{p}^- = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^-$$

(cf. [24], page 590). The isotropy subgroup of the G-action at $\left(\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^-\right)$ is the intersection of the normalizers of $\mathfrak{p}_\Theta$ and $\mathfrak{p}_\Theta^-$, which is exactly the centralizer $Z_G\,(H_\Theta)$. Hence the G-orbit of $\left(\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^-\right)$ is in bijection with $G/Z_G\,(H_\Theta)$. It is known that the orbit is open. In the sequel we say that two parabolic subalgebras $\mathfrak{q}_1 \in \mathbb{F}_\Theta$ and $\mathfrak{q}_2 \in \mathbb{F}_{\Theta^*}$ are opposed if $(\mathfrak{q}_1, \mathfrak{q}_2)$ belongs to the open G-orbit in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$.

In case $\Theta = \emptyset$, we have the maximal flag manifold, which is self-dual. Given two opposed minimal parabolic subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$, $\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a unique maximal abelian split subalgebra of $\mathfrak{g}$, $\mathfrak{a}_1 = \mathrm{Ad}\,(g)\,\mathfrak{a}$, $g \in G$. In this case there exists a unique Weyl chamber $\mathfrak{a}_1^+ \subset \mathfrak{a}_1$, such that b_1 is the attractor and b_2 the repeller of $H \in \mathfrak{a}_1^+$. Denote by $\mathfrak{a}^+(b_1, b_2)$ the chamber coming from this construction, and put $A^+(b_1, b_2) = \exp(\mathfrak{a}^+(b_1, b_2))$. The fixed point of type w for elements in $A^+(b_1, b_2)$ is denoted by $w(b_1, b_2)$.

5.2 Semigroups

We discuss now semigroups in a non-compact semi-simple Lie group G with finite center. Let $S \subset G$ be a semigroup with $\mathrm{int}S \neq \emptyset$. Then S acts on the flag manifolds $\mathbb{F}_\Theta$ of G. It was proved in [25], Theorem 6.2, that S is not transitive in $\mathbb{F}_\Theta$ unless $S = G$. Moreover, there exists just one invariant control set $C_\Theta\,(S) \subset \mathbb{F}_\Theta$. If S is proper $C_\Theta\,(S) \neq \mathbb{F}_\Theta$. We denote the set of transitivity of $C_\Theta\,(S)$ by $C_\Theta^+\,(S)$. In view of Proposition 5.2 below we call $C_\Theta^+\,(S)$ the *attractor set* of S in $\mathbb{F}_\Theta$. Replacing S by S^{-1} we get the *repeller set* $C_\Theta^-\,(S)$ which is the only minimal control set of S. In case $\mathbb{F}_\Theta = \mathbb{F}$ is the maximal flag manifold, we suppress the subscript Θ and write simply $C^\pm\,(S)$ for $C_\Theta^\pm\,(S)$, and if the semigroup is understood we put $C_\Theta^\pm = C_\Theta^\pm\,(S)$. The following statement was proved in [25].

Proposition 5.2 *The attractor set* C_Θ^+ *is given by* $\mathrm{at}_\Theta\,(h)$ *with* h *running through the split-regular elements in* $\mathrm{int}S$. *Analogously the repeller set* C_Θ^- *is formed by* $\mathrm{rp}_\Theta\,(h)$, *with* h *running through the split-regular elements in* $\mathrm{int}S$.

The semigroups in G are distinguished according to the geometry of their invariant control sets. This geometry is described by the following statements, proved in [25] (see also [21] and [23]).

Proposition 5.3 *There exists* $\Theta \subset \Sigma$ *such that* $\pi_\Theta^{-1}\,(C_\Theta\,(S))$ *is the invariant control set in the maximal flag manifold* $\mathbb{F}_\Theta$. *Among the subsets* Θ *satisfying this property there exists a unique maximal one (with respect to set inclusion).*

We denote the maximal subset by $\Theta\,(S)$ and say that it is the *parabolic type* of S. Alternatively, we say also that the parabolic type of S is the corresponding flag manifold $\mathbb{F}\,(S) = \mathbb{F}_{\Theta(S)}$ (see [23], [25], [27] for further discussions about the parabolic type of a semigroup). Given two semigroups

$S_1 \subset S_2$ with non-empty interior, their control sets satisfy $C(S_1) \subset C(S_2)$. This implies the inclusion between the parabolic types: $\Theta(S_1) \subset \Theta(S_2)$.

When $\Theta = \Theta(S)$, the invariant control set $C_{\Theta(S)}$ has the following nice property, proved in [25].

Proposition 5.4 *Let $h \in \mathrm{int} S$ be split-regular. Then $C_{\Theta(S)} \subset \mathrm{st}_{\Theta(S)}(h)$.*

The other effective control sets are given analogously as sets of fixed-points: Denote by $R(S)$ the set of split-regular elements in $\mathrm{int}(S)$. Then we have the following result of [25].

Proposition 5.5 *For each $w \in \mathcal{W}$ there exists a control set $D_\Theta(w) \subset \mathbb{F}_\Theta$ whose set of transitivity is*

$$D_\Theta(w)_0 = \{\mathrm{fix}_\Theta(h, w) : h \in R(S)\}.$$

The invariant control set is $C_\Theta = D_\Theta(1)$ and the minimal control set $C_\Theta^- = D_\Theta(w_0)$. Conversely, for any effective control set $D \subset \mathbb{F}_\Theta$ there exists $w \in \mathcal{W}$ such that $D = D_\Theta(w)$.

Note that $R(S^{-1}) = R(S)^{-1}$. Hence $D_\Theta(w)_0$ is also the set of transitivity of a control set, say $D_\Theta^-(w)$ of S^{-1} (cf. [21], Proposition 3.1).

Although the map $w \mapsto D_\Theta(w)$ is onto the effective control sets it is not in general one-to-one. To relate its level sets at the maximal flag manifold with the parabolic type of S put

$$\mathcal{W}(S) = \{w \in \mathcal{W} : D(w) = D(1)\}.$$

Then $\mathcal{W}(S)$ is the subgroup $\mathcal{W}_{\Theta(S)} = \left(M^* \cap P_{\Theta(S)}\right)/M$, and $D(w_1) = D(w_2)$ if and only if $\mathcal{W}(S)\, w_1 = \mathcal{W}(S)\, w_2$ (see [25]). Hence the number of effective control sets in $\mathbb{F}$ is $|\mathcal{W}| / |\mathcal{W}(S)|$. On the other hand the control sets in $\mathbb{F}_\Theta$ are the image of those in $\mathbb{F}$ under the projection $\pi_\Theta : \mathbb{F} \to \mathbb{F}_\Theta$.

For later reference we record the following fact proved in [23], Proposition 6.3.

Proposition 5.6 *Take $b_1 \in C_{\Theta(S)}^+$ and $b_2 \in C_{\Theta^*(S)}^-$ and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the corresponding parabolic subalgebras, respectively. Then $\mathfrak{p}_1$ is opposed to $\mathfrak{p}_2$.*

5.3 Reductive groups

For applications to flows on flag bundles it is convenient to consider also reductive groups besides the semi-simple ones. We have in mind, for instance, the reductive non-connected group $\mathrm{Gl}(n,\mathbb{R})$, which appears when studying flows on vector bundles. The point is that control sets for semigroups in reductive Lie groups are determined only by the action of the semi-simple component so that we can develop our results in the semi-simple setting and get for free the same results for reductive groups.

To discuss this extension let R be a reductive Lie group with Lie algebra $\mathfrak{r} = \mathfrak{g} \oplus \mathfrak{z}$, with $\mathfrak{g}$ semi-simple and $\mathfrak{z}$ the center of $\mathfrak{r}$. We assume that R has a finite number of connected components. Denote by Z_R the center of R which is a closed normal subgroup of R. A parabolic subgroup, say P_R, of R is defined like in the semi-simple case (cf. [28], page 85 ff), namely, $P_R = \mathrm{N}_R(\mathfrak{p})$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. The Lie algebra of P_R is $\mathfrak{p} \oplus \mathfrak{z}$. Let R_0 be the identity component of R, and put $G = R_0/(Z_R \cap R_0)$. It follows that G is actually the identity component of $\mathrm{Aut}(\mathfrak{g})$, and thus a semi-simple Lie group.

Put $P = \mathrm{N}_G(\mathfrak{p})$ for the parabolic subgroup of G corresponding to P_R.

Lemma 5.7 *The coset R/P_R is a union of copies of G/P, and $R/P_R = G/P$ if R/P_R is connected, that is, if P_R meets every component of R.*

Proof: Since $Z_R \subset P_R$, any $z \in Z_R$ acts as identity on R/P_R. This implies that the action of R on R/P_R factors through the semi-simple group R/Z_R in the sense that $R/P_R = (R/Z_R)/(P_R/Z_R)$. Hence, $P = (P_R \cap R_0)/Z_R$ and

$$R_0/(P_R \cap R_0) = G/((P_R \cap R_0)/Z_R) = G/P.$$

Furthermore, given a connected component K of R the set $\{gP_R : g \in K\}$ is in bijection with $R_0/(P_R \cap R_0)$. The coset spaces R/P_R and G/P are equal if and only if P_R meets every component of R. ∎

Now, let $\overline{S} \subset R$ be a semigroup with non-empty interior, and write S for the image of $\overline{S}$ under the canonical homomorphism $R \to R/Z_R$. Then the action of $\overline{S}$ on R/P_R depends solely on the action of S, in particular, the control sets of $\overline{S}$ coincide with the control sets of S. Clearly S has non-empty interior in the (possibly non-connected) semi-simple Lie group R/Z_R. Next we compare non-connected semi-simple Lie groups with their connected components.

Proposition 5.8 *Let $\overline{G}$ be a semi-simple Lie group with a finite number of connected componets. Suppose that a parabolic subgroup $P_{\overline{G}}$ meets every component of $\overline{G}$. Then for any semigroup $\overline{S} \subset \overline{G}$, with $\mathrm{int}\overline{S} \neq \emptyset$, the effective control sets of $\overline{S}$ in $\overline{G}/P_{\overline{G}}$ coincides with those of $S = \overline{S} \cap \overline{G}_0$.*

Proof: Put $P^0 = P_{\overline{G}} \cap \overline{G}_0$, so that $\overline{G}/P^0$ is a union of copies of $\overline{G}/P_{\overline{G}}$, each copy is the image of a connected component of $\overline{G}$ under the projection $\overline{G} \to \overline{G}/P^0$. Form the canonical bundle map

$$\pi : \overline{G}/P^0 \longrightarrow \overline{G}/P_{\overline{G}}.$$

The control sets of $\overline{S}$ in $\overline{G}/P^0$ project onto the control sets in $\overline{G}/P_{\overline{G}}$. Also, if $D \subset \overline{G}/P_{\overline{G}}$ is a control set and $x \in D_0$, then any point $y \in \pi^{-1}\{x\}$ belongs to the set of transitivity of a control set, say $\overline{D} \subset \overline{G}/P^0$. Now, y and gy, $g \in \overline{G}$ belong to the same component, then $g \in \overline{G}_0$. In particular, $\overline{D} \subset \mathrm{cl}Sy$, so that $\overline{D}$ is also a S-control set. Since the control sets of S are contained in the control sets of $\overline{S}$, the result follows. ∎

Corollary 5.9 *Let R be a reductive Lie group with a finite number of connected components. Suppose that a parabolic subgroup $P_R \subset R$ meets every component of R. Then for any semigroup $\overline{S} \subset R$, with $\mathrm{int}\overline{S} \neq \emptyset$, the effective control sets of $\overline{S}$ in G/P_R coincides with those of $S = \left(\overline{S} \cap R_0\right) / \left(Z_R \cap R_0\right)$.*

6 Sequences in G

Let g_k be a sequence in the semi-simple Lie group G. In order to see the pointwise limit of the action of g_k on the flag manifolds let us fix a polar decomposition $G = K\mathrm{cl}\,(A^+)\,K$, and write $g_k = u_k h_k v_k$ with $u_k, v_k \in K$ and $h_k \in \mathrm{cl}\,(A^+)$.

For a root $\alpha \in \Pi$ and $h \in A$, put $\lambda_\alpha(h) = \exp(\alpha(\log h))$. We say that g_k is admissible if $u_k \to u$, $v_k \to v$, $u, v \in K$ and the sequence $\lambda_\alpha(h_k)$ are convergent for all negative roots α. Note that for every negative root α, $\lambda_\alpha\,(h_k) \in (0, 1]$, so that any sequence has an admissible subsequence. The numbers $\lambda_\alpha\,(h_k)$ together with 1 are the eigenvalues of $\mathrm{Ad}\,(h_k)$. Hence, for an admissible sequence the restriction of $\mathrm{Ad}\,(h_k)$ to $\mathfrak{n}^-$ converges to a linear map $\tau : \mathfrak{n}^- \to \mathfrak{n}^-$ (cf. [10] and [26], Proposition 2.5).

Now take a flag manifold $\mathbb{F}_\Theta$ and denote by b_0 the origin corresponding to the standard parabolic subgroup defined by A^+. Also, put $\sigma = N^- b_0$ for the open Bruhat cell. Then

$$g_k v^{-1} (\exp Y) b_0 \to u \exp (\tau Y) b_0$$

for any $Y \in \mathfrak{n}^-$ (cf. [26], Proposition 2.5). Hence $g_k x$ has a limit for any $x \in v^{-1}\sigma_0$ and the limit belongs to $u (\exp (\mathrm{im}\tau)) b_0$.

In the sequel we write $\mathrm{dom}_\Theta (g_k) = v^{-1}\sigma_0$ and $\mathrm{im}_\Theta (g_k) = u (\exp (\mathrm{im}\tau)) b_0$ and refer to these sets as the *principal domain* and *principal image* in $\mathbb{F}_\Theta$, respectively.

Both sets $\mathrm{dom}_\Theta (g_k)$ and $\mathrm{im}_\Theta (g_k)$ are connected and the principal image reduces to a point if and only if τ anihilates on $\mathfrak{n}^-_\Theta$, that is, $\lambda_\alpha (h_k) \to 0$ for the negative roots $\alpha \notin \langle \Theta \rangle$. In this case the sequence is said to be contracting with respect to $\mathbb{F}_\Theta$ (cf. [10]).

The next lemma about the inverses g_k^{-1} of contracting sequences will be essential in the study of flows on flag bundles.

Lemma 6.1 *Let $g_k = v_k h_k u_k$ be a contractible sequence with respect to $\mathbb{F}_\Theta$ with $u_k \to 1$ and $v_k \to v$. Suppose that $C \subset \sigma$ is a compact subset and $b \neq v b_0$. Then there exists $k_0 > 0$ such that $g_k^{-1} b \notin C$ if $k \geq k_0$.*

Proof: Recall that $\sigma = N^-_\Theta \cdot b_0$ where $N^-_\Theta = \exp \mathfrak{n}^-_\Theta$ and $\mathfrak{n}^-_\Theta$ is the nilpotent Lie algebra spanned by the root spaces $\mathfrak{g}_\alpha$, $0 < \alpha \notin \langle \Theta \rangle$. The adjoint $\mathrm{Ad}(h)$ of $h \in A$ restricted to $\mathfrak{n}^-_\Theta$ is diagonal with eigenvalues $\exp (\alpha (\log h))$, $0 < \alpha \notin \langle \Theta \rangle$. The action of h on $\mathfrak{n}^-_\Theta$ is equivalent to the action on N^-_Θ. Take a basis of $\mathfrak{n}^-_\Theta$ formed by root vectors and endow $\mathfrak{n}^-_\Theta$ with the corresponding sup-norm

$$||Z|| = \max |a_i|$$

where a_i is the coordinate with respect to the i-th basic vector. By the contractibility assumption $\exp \left(\alpha \left(\log h_k^{-1}\right)\right) \to \infty$ for every negative root $\alpha \notin \langle \Theta \rangle$. Hence, $\left|\left|h_k^{-1} \cdot Z\right|\right| \to \infty$ if $Z \in \mathfrak{n}^-_\Theta$ is not zero. Denote also by $||\cdot||$ the function on σ obtained through the diffeomorphism with $\mathfrak{n}^-_\Theta$. Since $C \subset \sigma$ is compact $||\cdot||$ attains a maximum c on C.

With these preparations we can prove that for large k, $g_k^{-1} b$ stays outside the ball of radious c if $b \neq v b_0$. Since $v_k^{-1} b \to v^{-1} b \neq b_0$, there exists k_1 such that

$$m = \inf\{\left|\left|v_k^{-1} b\right|\right| : v_k b \in \sigma,\, k > k_1\} > 0.$$

We write $m = \infty$ if $v_k^{-1}b \notin \sigma$ for all $k > k_1$. Applying h_k^{-1} it follows that for large k, $h_k^{-1}v_k^{-1}b$ is outside a neighborhood $O \supset C$. In fact, if $||Z|| = m < \infty$ then $\left|\left|h_k^{-1}Z\right|\right| \to \infty$ and if $m = \infty$ then $h_k^{-1}v_k^{-1}b$ belongs to the complement of σ. Finally, by continuity in the compact-open topology, the assumption that $u_k \to 1$ ensures that for large k, $u_kC \subset O$, so that $g_kb = u_k^{-1}h_k^{-1}v_k^{-1}b \notin C$. ∎

7 Domain of attraction

The domains of attraction of control sets in flag manifolds were given algebraic descriptions in [21]. For later use in the study of flows on flag bundles we shall recall here some results of [21] and prove aditional related facts.

Let $D(w)$ be an effective control set for the semigroup $S \subset G$ in the maximal flag manifold $\mathbb{F}$. In [21] it was proved that the domain of attraction $\mathcal{A}(D(w))$ is a union of Schubert cells as follows: Fix a simple system of roots Σ and for a finite sequence $\alpha_1, \ldots, \alpha_n$ in Σ let $s_1, \ldots, s_n$ be the reflections with respect to these roots, and denote by $P_i = P_{\{\alpha_i\}}$ the parabolic subgroup defined by $\Theta = \{\alpha_i\}$. The corresponding flag manifold is denoted by $\mathbb{F}_i = G/P_i$. Associated with $\mathbb{F}_i$ there is the canonical fibration $\pi_i : \mathbb{F} \to \mathbb{F}_i$. Now, given $i = 1, \ldots, n$ let γ_i stand for the operation of exhausting a subset of $\mathbb{F}$ with the fibers of π_i, that is, if $X \subset \mathbb{F}$ then

$$\gamma_i(X) = \pi_i^{-1}\pi_i(X) = \bigcup_{x \in X} \mathbb{F}_x,$$

with $\mathbb{F}_x$ standing for the fiber through x of $\pi_i : \mathbb{F} \to \mathbb{F}_i$. Before proceeding we note that the simple system of roots Σ is used merely to label the flag manifolds and the maps γ_i, since these maps are independent of the choice of Σ, as happens to the fibrations $\mathbb{F} \to \mathbb{F}_i$. The following statement was proved in [21], Theorem 6.3.

Proposition 7.1 *The domain of attraction of $D(w)$ is given by*

$$\mathcal{A}(D(w)) = \gamma_1 \cdots \gamma_n \left(C^-\right),$$

where C^- is the repeller set of S in $\mathbb{F}$. Here the sequence γ_i is chosen in such a way that $w_0 w = s_n \cdots s_1$ is a reduced expression of $w_0 w$ as a product of simple roots, where w_0 is the principal involution of $\mathcal{W}$.

Applying this result to S^{-1} we get the repeller domain of $D(w)$:

Proposition 7.2 *Let $D^-(w)$ be the control set of S^{-1} having the same set of transitivity as $D(w)$. Denote by $\mathcal{A}^*(D^-(w))$ its attractor (reppeller of $D(w)$). Then*

$$\mathcal{A}^*\left(D^-(w)\right) = \gamma_1' \cdots \gamma_m' \left(C^+\right),$$

where $C^+ \subset \mathbb{F}$ is the attractor set of S. The sequence γ_i' corresponds to the reflections obtained by a reduced expression $w = r_m \cdots r_1$.

Proof: Follows from the above proposition and [21], Proposition 3.1. ∎

By [21], Theorem 5.3 and Corollary 5.4, it follows that $\gamma_1 \cdots \gamma_n\{b\}$ is a Schubert cell in $\mathbb{F}$, for any $b \in \mathbb{F}$. Our next objective is to describe the intersection of a pair of such cells in terms of the exhausting maps. Denote by P the minimal parabolic subgroup corresponding to Σ and let b^+ be the origin in G/P. Now, take $w \in \mathcal{W}$ with reduced expression $w = r_m \cdots r_1$, and write γ_i' for the corresponding exausting maps. On the other hand we put γ_i for such maps corresponding to a reduced expression $w_0 w = s_n \cdots s_1$.

Lemma 7.3 $\gamma_1' \cdots \gamma_m'\{b^+\} \cap \gamma_1 \cdots \gamma_n\{b^-\} = \{wb^+\}$.

Proof: By [21], Corollary 5.4, $\gamma_1 \cdots \gamma_n\{b^-\} = \mathrm{cl}\,(N^- w b^+)$. To find an analogous expression for $\gamma_1' \cdots \gamma_m'\{b^+\}$ we recall [21], Theorem 5.3, which shows that $\mathrm{cl}\left(N^{w^{-1}} b^+\right) = \gamma_1' \cdots \gamma_m'\{w^{-1}b^+\}$ where $N^{w^{-1}} = w^{-1}N^+w$. Applying w to both sides of this equallity we get

$$\gamma_1' \cdots \gamma_m'\{b^+\} = \mathrm{cl}\left(N^+ w b^+\right).$$

But it is well known that the cells $\mathrm{cl}\,(N^+wb^+)$ and $\mathrm{cl}\,(N^-wb^+)$ meet transversally exactly at wb^+, concluding the proof. ∎

We can think this lemma as a method of obtaining the whole set of fixed points from the attractor and repeller ones. In fact, take a split regular $h = \exp(H)$, $H \in \mathfrak{a}^+$. Then b^+ is the attractor of h, while b^- is the repeller and the other fixed points are wb^+, $w \in \mathcal{W}$. Thus the above lemma reconstructs the fixed points from $b^\pm$ and the exausting maps. The next lemma generalizes this construction for non-regular $H \in \mathrm{cl}\mathfrak{a}^+$.

Lemma 7.4 $\gamma_1' \cdots \gamma_m'\,(K_H^0 b^+) \cap \gamma_1 \cdots \gamma_n\,(K_H^0 b^-) = K_H^0 b_w$.

Proof: Take x in the left hand side and $u \in K_H^0$. We have $x \in \gamma_1' \cdots \gamma_m' (v_1 b^+) \cap \gamma_1 \cdots \gamma_n (v_2 b^-)$ for some $v_1, v_2 \in K_H^0$. Using the equivariance of the exhausting maps we get

$$ux \in \gamma_1' \cdots \gamma_m' \left(uv_1 b^+\right) \cap \gamma_1 \cdots \gamma_n \left(uv_2 b^-\right),$$

so that the entire orbit $K_H^0 x$ is contained in $\gamma_1' \cdots \gamma_m' (K_H^0 b^+) \cap \gamma_1 \cdots \gamma_n (K_H^0 b^-)$. Combining this with the previous lemma we conclude that the right hand side is contained in the left one. For the reverse inclusion, take $x \in \gamma_1' \cdots \gamma_m' (v_1 b^+) \cap \gamma_1 \cdots \gamma_n (v_2 b^-)$. Proceeding as in the proof of the previous lemma, we obtain $\gamma_1' \cdots \gamma_m' (v_1 b^+) = \mathrm{cl}\,(v_1 N^+ b_w)$ and $\gamma_1 \cdots \gamma_n (v_2 b^-) = \mathrm{cl}\,(v_2 N^- b_w)$. Now, for any $z \in \mathrm{cl}\,(N^+ b_w)$, the limit $\lim_{t \to -\infty} \exp(tH)\, z$ belongs to a component, say $K_H^0 b_{w_1}$, bigger than $K_H^0 b_w$ (in fact, $N^+ b_w$ contains the unstable manifold of the fixed-point set $K_H^0 b_w$, cf. [9]). Hence, $\lim_{t \to -\infty} \exp(tH)\, x$ belongs to $K_H^0 b_{w_1}$, since v_1 commutes with $\exp(tH)$. Symmetrically, $\lim_{t \to +\infty} \exp(tH)$ belongs to a component $K_H^0 b_{w_2}$ smaller than $K_H^0 b_w$, because $x \in \mathrm{cl}\,(v_2 N^- b_w)$. Combining the two limits and using the fact that $\bigcup_{w \in \mathcal{W}_H \backslash \mathcal{W}} K_H^0 b_w$ is a Morse decomposition we conclude that $K_H^0 b_w = K_H^0 b_{w_1} = K_H^0 b_{w_2}$, and hence $x \in K_H^0 b_w$. ∎

By taking conjugations we carry over this lemma to the fixed-point set of $\exp(tA)$ if A belongs to an adjoint orbit crossing $\mathrm{cl}\mathfrak{a}^+$. In fact, for any $g \in G$ and $b \in \mathbb{F}$, $g\gamma_1 \cdots \gamma_n (b) = \gamma_1 \cdots \gamma_n (gb)$, and the fixed point set of $\exp(t\mathrm{Ad}\,(g)\, H)$ is the image under g of the fixed point set of $\exp(tH)$. For later reference we state this fact.

Corollary 7.5 *Take $A \in \mathrm{Ad}\,(G)\, H$, $H \in \mathrm{cl}\mathfrak{a}^+$ and let $\mathrm{fix}\,(Z)$ be the set of fixed points of $\exp(tA)$ in $\mathbb{F}$. Then there exists a map $w \in \mathcal{W} \mapsto \mathrm{fix}\,(A, w)$ onto the set of connected components of $\mathrm{fix}\,(A)$ such that $\mathrm{fix}\,(A, 1)$ is the unique attractor, $\mathrm{fix}\,(A, w_0)$ is the unique repeller and*

$$\mathrm{fix}\,(A, w) = \gamma_1' \cdots \gamma_m' \left(\mathrm{fix}\,(A, 1)\right) \cap \gamma_1 \cdots \gamma_n \left(\mathrm{fix}\,(A, w_0)\right),$$

with γ_i' and γ_i given by reduced expressions of w and $w_0 w$, respectively. Furthermore, $\mathrm{fix}\,(A, w_1) = \mathrm{fix}\,(A, w_2)$ if and only if $\mathcal{W}_H w_1 = \mathcal{W}_H w_2$.

We conclude this section with an application of the above results to the control sets of a semigroup S. Let b_1 and b_2 be two points in the maximal flag manifold $\mathbb{F}$ with isotropy subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively. We say that b_1 and b_2 are opposed if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are opposed.

Proposition 7.6 *Let $C^{\pm} \subset \mathbb{F}$ be the attractor and repeller set of S, respectively. Take opposed $b_1 \in C^+$ and $b_2 \in C^-$. Then $w(b_1, b_2) \in D(w)_0$ (see the notation at the end of subsection 5.1).*

Proof: By Lemma 7.3, $w(b_1, b_2) = \gamma_1' \cdots \gamma_m' \{b_1\} \cap \gamma_1 \cdots \gamma_n \{b_2\}$. On the other hand Propositions 7.1 and 7.2 show that $\gamma_1' \cdots \gamma_m' \{b_1\} \cap \gamma_1 \cdots \gamma_n \{b_2\}$ belongs to $D(w)_0$. ∎

8 Flag bundles

In this section we construct the maximal chain transitive subsets of a flow in a flag bundle with the aid of the shadowing semigroups. It will produce that, analogously to the control sets on flag manifolds, the maximal chain transitive sets are parametrized by the Weyl group. Hence there is a finite number of such components, which for compact metric spaces implies the existence of a finest Morse decomposition of the flow.

8.1 Control sets

Before applying the shadowing semigroup method we must have a description of the control sets on the flag bundles. This will be done by improving the results of [4] with the inclusion of the algebraic characterizations discussed in Section 5 above.

To start with let $Q \to X$ be a principal bundle whose structure group G is semi-simple and non-compact. As before let $\mathbb{F}_\Theta$ be a flag manifold of G and put $\mathbb{E}_\Theta \to X$ for the associated bundle $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$, having typical fiber $\mathbb{F}_\Theta$. For the maximal flag manifold $\mathbb{F}$ we write the associated bundle simply by $\mathbb{E} \to X$. Recall that when $\Theta_1 \subset \Theta_2$ there exists a natural fibration $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ inducing a fibration $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$. In particular, we have the fiber bundle $\mathbb{E} \to \mathbb{E}_\Theta$ for any $\Theta \subset \Sigma$.

Let S be a local subsemigroup of $\mathrm{Aut}(Q)$. To look at the control sets of S on the flag bundles we assume that S satisfies the accessibility property on Q and its action on X is transitive. By the results of [4] the control sets for the action of S on $\mathbb{E}$ are built fiberwise from control sets in $\mathbb{F}$. We sketch the main construction: Given $q \in Q$ define

$$S_q = \{g \in G : \exists \phi \in S,\ \phi(q) = q \cdot g\}, \qquad (3)$$

then S_q is a subsemigroup of G and the accessibility assumption on Q implies that $\mathrm{int}_G(S_q) \neq \emptyset$ (see [4], Lemma 3.2). Let D_q be an effective control set of S_q on $\mathbb{F}$. According to our notation, $q \cdot D_q$ is a subset of the fiber $\mathbb{E}_x$ of $\mathbb{E}$ above $x = \pi(q)$. Actually the sets $q \cdot D_q$ are independent of $q \in Q_x$. In fact, if $p = q \cdot a$, $a \in G$ is in the same fiber as q, then $S_{q \cdot a} = a^{-1} S_q a$, so that $a^{-1}D$ is a control set for S_p. Therefore,

$$p \cdot D_p = (q \cdot a) \cdot \left(a^{-1} D_q\right) = q \cdot D_q.$$

By Theorem 3.5 of [4], the intersection of the set of transitivity of an effective control set of S in $\mathbb{E}$ with a fiber has the form $q \cdot (D_q)_0$. Conversely, given an effective control set D_q, the set $q \cdot (D_q)_0$ is the intersection of the set of transitivity of a unique effective control set in $\mathbb{E}$ with the fiber above $x = \pi(q)$ (see [4], Proposition 3.6). These results yields immediately the finiteness of control sets on the flag bundles.

Proposition 8.1 *Suppose S satisfies the accessibility property on Q and is transitive on the base space X. Then the effective control sets in a flag bundle $\mathbb{E}_\Theta \to X$ is in bijection with the effective control sets of S_q on $\mathbb{F}_\Theta$. Hence, the number of S-effective control sets on $\mathbb{E}_\Theta$ is finite.*

Proof: Fix $x \in X$. The transitivity of S on X ensures that any effective control set E meets the fiber above x. By [4], Theorem 3.5, the intersection of E_0 with the fiber has the form $q \cdot (D_q)_0$. Thus we have a well defined map which associates an effective control set in $\mathbb{E}_\Theta$ to an effective control set of S_q. Since different control sets are disjoint, this map is one-to-one. On the other hand the map is onto by [4], Proposition 3.6, concluding the proof. ∎

Using the bijection of this proposition we can label the control sets in $\mathbb{E}_\Theta$ by the Weyl group $\mathcal{W}$. Thus let $D_q^\Theta(w)$ be the control set of S_q on $\mathbb{F}_\Theta$ whose set of transitivity is formed by fixed-points of type w of the elements of S_q (cf. Section 5). The set $q \cdot D_q^\Theta(w)$ is independent of q in the fiber over $x = \pi(q)$. We put $F_\Theta^x(w) = q \cdot D_q^\Theta(w)$, $F_\Theta^x(w)_0 = q \cdot D_q^\Theta(w)_0$, and let $E_\Theta^x(w)$ be the control set of S in $\mathbb{E}_\Theta$ which contains $F_\Theta^x(w)_0$.

Our objective is to relate the control sets $E_\Theta^x(w)$ for different $x \in X$. In the general framework of [4] this was done only for invariant control sets. Here however we take advantage of the algebraic properties of the fibers $\mathbb{F}_\Theta$. First we consider the maximal flag bundle.

Lemma 8.2 *Given $x \in X$ there exists an open neigborhhood U of x such that $E^y(w) = E^x(w)$ for all $y \in U$.*

Proof: Take $x \in X$ and a trivializing neighborhood V of x, so that $\pi_Q^{-1}(V) \approx V \times G$ and $\pi_E^{-1}(V) \approx V \times \mathbb{F}$. Write S_x for the semigroup $S_{(x,1)}$.

Using the transitivity of S on X, we apply Theorem 4.4 of [4] to conclude that S has a unique invariant control set $C(S_x) \subset \mathbb{E}$ as well as a unique minimal control set $C^-(S_x) \subset \mathbb{E}$. Both control sets project onto X, and by the same result $C(S_x) \cap \pi^{-1}\{x\}$ is the invariant control set of S_x in $\mathbb{F}$ while $C^-(S_x) \cap \pi^{-1}\{x\}$ is the minimal control set. As before we put $C^+(S_x)$ for the attractor set of S_x.

Now, we can choose $(x, b_1) \in C^+$ and $(x, b_2) \in C^-$ such that b_1 and b_2 are opposite to each other (see Section 5). By Proposition 7.6 we have $w(b_1, b_2) \in D_x(w)_0$, so that $(x, w(b_1, b_2)) \in (E^x(w))_0$. Since $(E^x(w))_0$ is open there exists a neigborhood U_1 of x in X such that $U_1 \times \{w(b_1, b_2)\} \subset E^x(w)$. Also, there exists a neigborhood U_2 of x such that $U_2 \times \{b_1\} \subset C^+$ and $U_2 \times \{b_2\} \subset C^-$. Applying again Proposition 7.6, it follows that for every $y \in U_2$, $(y, w(b_1, b_2)) \in E^y(w)$. Then $U = U_1 \cap U_2$ satisfies the condition of the lemma since $(y, w(b_1, b_2)) \in E^y(w) \cap E^x(w)$ for all $y \in U$, so that $E^y(w) = E^x(w)$ because these control sets overlap. ∎

For the sake of simplicity in the notations we proved the above lemma only for the maximal flag bundle, but the same result holds for any other bundle $\mathbb{E}_\Theta$, because the control sets in $\mathbb{E}_\Theta$ are projections of the control sets in $\mathbb{E}$. Hence if we use subscript Θ for control sets in $\mathbb{E}_\Theta$ we arrive at the following picture.

Corollary 8.3 *Take a connected component κ of X and $w \in \mathcal{W}$. Then the control sets $E_\Theta^x(w)$ are independent of $x \in \kappa$.*

Proof: The relation $x \sim y$ if $E^x(w) = E^y(w)$ is an equivalence relation on X. By Lemma 8.2 the equivalence classes are open sets, and hence union of connected components of X. ∎

Therefore, fixing a connected component κ we get a well defined control set $E_\Theta^\kappa(w)$ in $\mathbb{E}$ for each $w \in \mathcal{W}$. We do not know whether the control sets $E_\Theta^\kappa(w)$ are independent of the connected component κ. Note however that any effective control set has the form $E_\Theta^\kappa(w)$, hence the control sets are labelled by $w \in \mathcal{W}$, once κ is given.

For the sake of completeness let us discuss what happens in case κ is changed into another connected component κ_1 of X. There exists a map $\tau : \mathcal{W} \to \mathcal{W}$ such that $E_\Theta^{\kappa_1}(w) = E_\Theta^{\kappa}(\tau(w))$. Since any effective control set has the form $E_\Theta^{\kappa}(w)$, it follows that τ is a bijection. Furthermore, the map τ is increasing with respect to the Borel-Chevalley order $\preceq$ in $\mathcal{W}$ (cf. [21]). In fact, it was proved in [21] that $D_q^\Theta(w_1) \preceq D_q^\Theta(w_2)$ if and only if $w_2 \preceq w_1$, so that $E_\Theta^{\kappa_1}(w_1) = E_\Theta^{\kappa}(\tau(w_1))$ is smaller than $E_\Theta^{\kappa_1}(w_2) = E_\Theta^{\kappa}(\tau(w_2))$ if and only if $w_2 \preceq w_1$, implying that $\tau(w_2) \preceq \tau(w_1)$ if $w_2 \preceq w_1$.

8.2 Chain transitive sets

We proceed now to apply the above results to the shadowing semigroups of a flow ϕ on a flag bundle. For this we assume that the local group $\operatorname{loc}(X)$ of the base space is locally transitive, implying that the shadowing semigroups $S_{\varepsilon,T}$ in $\operatorname{Aut}(Q)$ are locally transitive by Corollary 3.6 and Lemma 3.7. In particular, $S_{\varepsilon,T}$ satisfies the accessibility property for all $\varepsilon, T > 0$.

Let us fix once and for all a maximal chain transitive subset $\mathcal{X}$ of the flow on the base space X and denote by $\mathcal{X}_{\varepsilon,T}$ the control set of the shadowing semigroup $S_{\varepsilon,T}$ containing $\mathcal{X}$ (see Theorem 4.7). Let $\mathcal{X}_{\varepsilon,T}^0$ be the set of transitivity of $\mathcal{X}_{\varepsilon,T}$. Clearly, $S_{\varepsilon,T}$ acts transitively on $\mathcal{X}_{\varepsilon,T}^0$. Hence, the previous results apply if we restrict the action of $S_{\varepsilon,T}$ on a flag bundle to the open set above $\mathcal{X}_{\varepsilon,T}^0$.

To avoid cumbersome notation we write in the sequel the control sets of $S_{\varepsilon,T}$ above $\mathcal{X}_{\varepsilon,T}^0$ without any further reference to this restriction. Hence a control set of $S_{\varepsilon,T}$ in $\mathbb{E}_\Theta$ should be understood as a control set for the restriction of the action of this semigroup to the bundle $\mathbb{E}_\Theta \to \mathcal{X}_{\varepsilon,T}^0$. Also, we denote by $\mathcal{E}_\Theta$ the restriction of a flag bundle $\mathbb{E}_\Theta$ above $\mathcal{X}$, and for $e \in \mathcal{E}_\Theta$ we write $\mathcal{C}_r(e) = \mathcal{C}(e) \cap \mathcal{E}_\Theta$ and $\mathcal{C}_r^*(e) = \mathcal{C}^*(e) \cap \mathcal{E}_\Theta$.

Now for $w \in \mathcal{W}$, there exists an effective control set $E_{\varepsilon,T}^\Theta(w) \subset \mathbb{E}_\Theta$ of $S_{\varepsilon,T}$ and for every effective control set $E \subset \mathbb{E}_\Theta$ of $S_{\varepsilon,T}$ there exists $w \in \mathcal{W}$ such that $E = E_{\varepsilon,T}^\Theta(w)$.

Lemma 8.4 *Let $\varepsilon_1, T_1 > 0$ and $\varepsilon_2, T_2 > 0$ be given such that $\varepsilon_1 \leq \varepsilon_2$ and $T_1 \geq T_2$. Then for any $w \in \mathcal{W}$, $E_{\varepsilon_1,T_1}^\Theta(w) \subset E_{\varepsilon_2,T_2}^\Theta(w)$.*

Proof: Since for any $q \in Q$, $S_{\varepsilon_1,T_1} \subset S_{\varepsilon_2,T_2}$ it follows that $S_q^{\varepsilon_1,T_1} \subset S_q^{\varepsilon_2,T_2}$ (with obvious notation). Hence the control sets of $S_q^{\varepsilon_1,T_1}$ are contained in those of $S_q^{\varepsilon_2,T_2}$. Thus the lemma follows by the fiberwise construction of the control sets on $\mathbb{E}_\Theta$. ∎

Therefore, to be able to apply Theorem 4.7 it remains to check that $\bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(w) \neq \emptyset$. We consider first the invariant control sets.

Lemma 8.5 *$\bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(1) \neq \emptyset$. Furthermore, $\bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(1) = \bigcap_{e\in\mathcal{E}_\Theta} \mathcal{C}_r(e)$.*

Proof: The control sets $E^{\Theta}_{\varepsilon,T}(1)$ are closed and due to transitivity on the basis the invariant control sets meet every fiber in a non-empty compact set. By the inclusion $E^{\Theta}_{\varepsilon_1,T_1}(1) \subset E^{\Theta}_{\varepsilon_2,T_2}(1)$ if $\varepsilon_1 \leq \varepsilon_2$ and $T_1 \geq T_2$, it follows that for each $x \in \mathcal{X}$ the family $E^{\Theta}_{\varepsilon,T}(1)_x$ satisfies the finite intersection property. Hence by compacteness of the fiber we conclude that $\bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(1) \neq \emptyset$. The last equality is a consequence of Proposition 4.5 and the fact that $E^{\Theta}_{\varepsilon,T}(1) \subset S_{\varepsilon,T}e$ for every $e \in \mathcal{E}_\Theta$. ∎

Now we consider the minimal control sets. For this we apply the above lemma to the shadowing semigroups $S^*_{\varepsilon,T}$ for the reversed flow. The corresponding invariant control sets have a non-empty intersection, which equals $\bigcap_{e\in E} \mathcal{C}^*(e)$.

Lemma 8.6 *$\bigcap_{e\in\mathcal{E}_\Theta} \mathcal{C}^*_r(e) = \bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(w_0)$.*

Proof: Take $f \in \bigcap_{e\in\mathcal{E}_\Theta} \mathcal{C}^*_r(e)$. By Proposition 4.5, for all $\varepsilon, T > 0$, $S_{\varepsilon,T}f = \mathbb{E}$. Hence f belongs to the minimal control set of $S_{\varepsilon,T}$, that is, $E^{\Theta}_{\varepsilon,T}(w_0)$. This implies that $\bigcap_{e\in\mathcal{E}_\theta} \mathcal{C}^*_r(e) \subset \bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(w_0)$. The reverse inclusion is due to the fact that both sets are maximal chain transitive, by Theorem 4.7. ∎

To get non-empty intersection for the other control sets we apply the results about domains of attraction of Section 7. Since there the statements are made for the maximal flag manifold we shall work out here the case of $\mathbb{E}$ and afterwards project down to the other flag bundles. Thus fix $q \in Q$, let $x = \pi(q) \in \mathcal{X}$ and write a subscript x for intersections of subsets of $\mathbb{E}$ with the fiber through x. For example, the sets $q^{-1}\cdot E_{\varepsilon,T}(w_0)_x$ and $q^{-1}\cdot E_{\varepsilon,T}(1)_x$ are the minimal and invariant control set of $S^{\varepsilon,T}_q$, respectively.

Now, take $e^{-} \in \bigcap_{\varepsilon,T} E_{\varepsilon,T}(w_0)_x$ and $e^{+} \in \bigcap_{\varepsilon,T} E_{\varepsilon,T}(1)_x$. Put $w(e^{+}, e^{-}) = q \cdot w(q^{-1} \cdot e^{+}, q^{-1} \cdot e)$ (see the notations of Proposition 7.6). By Proposition 7.6, $w(q^{-1} \cdot e^{+}, q^{-1} \cdot e)$ belongs to the w-control set of $S_q^{\varepsilon,T}$ for every $\varepsilon, T > 0$. Hence for $\varepsilon, T > 0$, $w(e^{+}, e^{-}) \in E_{\varepsilon,T}(w)_x$, showing that $\bigcap_{\varepsilon,T} E_{\varepsilon,T}(w) \neq \emptyset$.

Lemma 8.7 *For any $w \in \mathcal{W}$, $\bigcap_{\varepsilon,T} E^{\Theta}_{\varepsilon,T}(w) \neq \emptyset$.*

Proof: We showed above that $\bigcap_{\varepsilon,T} E_{\varepsilon,T}(w) \neq \emptyset$. Since $E^{\Theta}_{\varepsilon,T}(w)$ is the projection of $E_{\varepsilon,T}(w)$ the lemma follows. ∎

Thus we have proved one of the main results of this paper.

Theorem 8.8 *Suppose that* $\operatorname{loc}(X)$ *is locally transitive. Let ϕ_t be a right invariant flow on Q and take a maximal chain transitive subset $\mathcal{X} \subset X$. Then the associated flow on a flag bundle $\mathcal{E}_\Theta \to \mathcal{X}$ satifies:*

1. *For each $w \in \mathcal{W}$ there exists a maximal chain transitive set $\mathcal{M}_{\Theta,\phi}(w)$ (or simply $\mathcal{M}_\Theta(w)$).*
2. *If $\mathcal{M} \subset \mathcal{E}_\Theta$ is a maximal chain transitive set then $\mathcal{M} = \mathcal{M}_\Theta(w)$ for some $w \in \mathcal{W}$.*
3. *$\mathcal{M}_\Theta(1)$ is the only attractor while $\mathcal{M}_\Theta(w_0)$ is the only repeller, where w_0 is the principal involution of $\mathcal{W}$.*

In the sequel we put $\mathcal{M}^{+}_{\Theta} = \mathcal{M}_\Theta(1)$, $\mathcal{M}^{-}_{\Theta} = \mathcal{M}_\Theta(w_0)$, and supress the subscripts when $\mathbb{E} = \mathbb{E}_\emptyset$ is the maximal flag manifold.

Clearly, in the compact case the maximal chain transitive subsets coincide with the connected components of the chain recurrent set, giving rise to the finest Morse decomposition.

Corollary 8.9 *In the situation of the above theorem, suppose furthermore that $\mathcal{X}$ is compact. Then the flow on a flag bundle $\mathcal{E}_\Theta$ admits a finest Morse decomposition with components $\mathcal{M}_\Theta(w)$.*

8.3 Parabolic type

As happens to the control sets on the flag manifolds the map $w \mapsto \mathcal{M}(w)$ of Theorem 8.8 is not injective. Analogously to the semigroup case the level sets of this map are described by the parabolic type of the flow, a concept which we shall introduce below based on the parabolic type of semigroups.

Our first task is to check that the semigroups S_q, $q \in Q$, defined above, have the same parabolic type. For this fix $q \in Q$, let $\Theta = \Theta(S_q)$ be the parabolic type of S_q, and form the flag bundle $\mathbb{E}_\Theta$. Then there exists a natural fibration $\pi : \mathbb{E} \to \mathbb{E}_\Theta$ whose fiber coincides with that of $\pi_\Theta : \mathbb{F} \to \mathbb{F}_\Theta$. Since Θ is the parabolic type of S_q, it follows that the invariant control set in $\mathbb{F}$, $C^q = C(S_q)$, is given by $C^q = \pi_\Theta^{-1}(C_\Theta^q)$, where $C_\Theta^q = C_\Theta(S_q)$ is the invariant control set in $\mathbb{F}_\Theta$. By [4], Theorem 3.5, the subset $q \cdot C^q \subset E(1)$. The same way $q \cdot C_\Theta^q \subset E_\Theta(1)$. Hence for every $e \in q \cdot C_\Theta^q$, $\pi^{-1}\{e\} \subset E(1)$. Applying [4], Proposition 3.7, we conclude that $\pi^{-1}(E_\Theta(1))$ is contained in $E(1)$. This shows that for any $p \in Q$ the parabolic type of S_p, $\Theta_p \subset \Theta_q$. Since q is arbitrary the claim follows. Thus we have proved the

Proposition 8.10 *Let $S \subset \mathrm{Aut}(Q)$ be a local semigroup which satisfies the accessibility property and is transitive on the base X. Then the parabolic type of S_q is independent of $q \in Q$.*

In view of this proposition it makes sense to talk about the parabolic type of a local semigroup $S \subset \mathrm{Aut}(Q)$.

Definition 8.11 *Let $S \subset \mathrm{Aut}(Q)$ be a semigroup satisfying the accessibility property. The* parabolic type *of S is the common parabolic type of S_q, $q \in Q$.*

In particular a shadowing semigroup $S_{\varepsilon,T} = S_{\varepsilon,T}(\phi, \mathrm{Aut}(Q))$ of a flow ϕ has a parabolic type, which we denote by $\Theta_{\varepsilon,T}$. If $\varepsilon_1 < \varepsilon$ and $T_1 > T$ then the control sets of S_{ε_1,T_1} are contained in those of $S_{\varepsilon,T}$ (see Lemma 4.6). Thus the definition of the parabolic type implies that $\Theta_{\varepsilon_1,T_1} \subset \Theta_{\varepsilon,T}$. Also, note that the number of possible parabolic types is finite. Hence the intersection $\bigcap_{\varepsilon,T} \Theta_{\varepsilon,T}$, which is possible empty, is well defined.

Definition 8.12 *The* parabolic type *of the flow on Q is defined to be*

$$\Theta(\phi) = \bigcap_{\varepsilon,T} \Theta_{\varepsilon,T}$$

where $\Theta_{\varepsilon,T}$ is the parabolic type of the shadowing semigroup $S_{\varepsilon,T}$.

Analogous to the case of control sets the parabolic type of a flow is intimately related to the geometry of the attractor maximal chain transitive subset. In fact, the results about control sets of the shadowing semigroups yield immediately the following properties of the parabolic type of ϕ.

Proposition 8.13 *The fibers of $\mathcal{M}_{\Theta(\phi)}(1)$ are contained in open cells. Also, $\pi^{-1}\mathcal{M}_{\Theta(\phi)}(1) = \mathcal{M}(1)$.*

Proposition 8.14 *The number of maximal chain transitive subsets in $\mathbb{E}_\Theta$ equals the number of orbits of $\mathcal{W}_\Theta$ in $\mathcal{W}/\mathcal{W}_{\Theta(\phi)}$. In particular, in $\mathbb{E}$ this number is $|\mathcal{W}| / |\mathcal{W}_{\Theta(\phi)}|$.*

For the parabolic type of the reversed flow ϕ^* we must look at the invariant control sets of the shadowing semigroups $S^*_{\varepsilon,T}$. The repeller maximal chain transitive subset is the intersection of the invariant control sets of $S^*_{\varepsilon,T}$ as well as the intersection of the minimal control sets of $S_{\varepsilon,T}$. From this we get the reversed parabolic type of ϕ:

Proposition 8.15 *Denote by $\Theta^*_{\varepsilon,T}$ the parabolic type of $S^*_{\varepsilon,T}$ and by $\Theta^-_{\varepsilon,T}$ the parabolic type of $S^{-1}_{\varepsilon,T}$. Then*

$$\Theta(\phi^*) = \bigcap_{\varepsilon,T} \Theta^*_{\varepsilon,T} = \bigcap_{\varepsilon,T} \Theta^-_{\varepsilon,T}.$$

Proof: By Proposition 8.14 a fiber of $\mathbb{E} \to \mathbb{E}_{\Theta(\phi^*)}$ is contained in the minimal control set of every $S^{-1}_{\varepsilon,T}$, so that $\Theta(\phi^*) \subset \bigcap_{\varepsilon,T} \Theta^-_{\varepsilon,T}$. Since the repeller maximal chain transitive subset in $\mathbb{E}$ is the intersection of the minimal control sets of $S_{\varepsilon,T}$. The reverse inclusion follows the same way. ∎

According to [23] the parabolic type of the inverse S^{-1} of a semigroup in G is given by the dual flag manifold of the parabolic type of S. This implies that the parabolic type of ϕ^* corresponds to the dual flag manifold of the parabolic type of ϕ. In view of this we conform to the notation of [23] and write $\Theta^*(\phi)$ for $\Theta(\phi^*)$.

9 Algebraic description

In this section we look at maximal chain transitive sets more carefully. Our objective is to prove Theorem 9.11, which gives an algebraic description of these sets. The main lemma in this direction is Lemma 9.3, which ensures that the flow on the bundle $\mathbb{E}_{\Theta(\phi)}$, corresponding to the parabolic type of ϕ, is such that fibers of the attractor maximal chain transitive set reduces to a single point. Here contrary to the previous section we must ask for the existence of ω-limits on the base space, an assumption which is automatic in the compact case.

By reverting the flow the same result holds for the repeller on $\mathbb{E}_{\Theta^*(\phi)}$. This gives at once the description of the attractor and repeller maximal chain transitive sets on every flag bundle. The other chain transitive sets will be determined by the extremal ones and the domains of attraction.

We keep assuming that the flow on the base space X is chain recurrent and $\mathrm{loc}\,(X)$ is locally transitive. As before denote by $\mathcal{M}_\Theta^+$ the attractor maximal chain transitive set of the flow on $\mathbb{E}_\Theta$. Let $x, y \in X$ be such that $t_k \cdot x \to y$ for a sequence $t_k \to +\infty$, and take local cross sections $\chi_i : U_i \subset X \to Q$, $i = 1, 2$, with $x \in U_1$ and $y \in U_2$. Writing $\rho = \rho_{\chi_1,\chi_2}$ for the corresponding local cocycle we obtain the sequence $g_k = \rho\,(t_k, x)$ in G. Taking a subsequence if necessary we shall assume that g_k is admissible, so that it makes sense to consider its principal image $\mathrm{im}_\Theta\,(g_k)$ and principal domain $\mathrm{dom}_\Theta\,(g_k)$. The following lemma relates $\mathrm{im}_\Theta\,(g_k)$ with $\mathcal{M}_\Theta^+$. It is crucial in the proof of Lemma 9.3.

Lemma 9.1 *Let the notations and assumptions be as above. Then the principal image $\chi_2\,(y) \cdot \mathrm{im}_\Theta\,(g_k)$ is contained in $\mathcal{M}_\Theta^+$.*

Proof: First we prove that the principal image meets $\mathcal{M}_\Theta^+$. For this fix $\varepsilon, T > 0$, denote as before $E_{\varepsilon,T}^\Theta\,(1)$ the invariant control set of $S_{\varepsilon,T}$ in $\mathbb{E}_\Theta$ and put $N = E_{\varepsilon,T}^\Theta\,(1)_0$, the set of transitivity of $E_{\varepsilon,T}^\Theta\,(1)$. The latter has non-empty interior and projects onto X. Hence, N intercepts $\chi_1\,(x) \cdot \mathrm{dom}_\Theta\,(g_k)$, which is dense in the fiber above x. But if $b \in \chi_1\,(x) \cdot \mathrm{dom}_\Theta\,(g_k)$ then $\phi_{t_k}\,(b)$ converges to a point in the principal image. Therefore, for any $b \in N \cap \chi_1\,(x) \cdot \mathrm{dom}_\Theta\,(g_k)$, $\lim \phi_{t_k}\,(b)$ belongs to $\mathcal{M}_\Theta^+ \cap (\chi_2\,(y) \cdot \mathrm{im}_\Theta\,(g_k))$, showing that this intersection is not empty. However, any point of $\chi_2\,(y) \cdot \mathrm{im}_\Theta\,(g_k)$ belongs to $\omega\,(c)$ for some $c \in \mathbb{E}_\Theta$, and hence to the chain recurrent set $\mathcal{R}$. Since $\mathrm{im}_\Theta\,(g_k)$ is connected, it follows that $\chi_2\,(y) \cdot \mathrm{im}_\Theta\,(g_k)$ is contained in a connected component of $\mathcal{R}$, which in turn is contained in a unique maximal chain transitive set. By the first part of the proof, the principal image meets $\mathcal{M}_\Theta^+$, implying the lemma. ∎

When we specialize this lemma to the case $\Theta = \Theta\,(\phi)$, the parabolic type of the flow, we see that the principal image $\mathrm{im}_{\Theta(\phi)}\,(g_k)$ reduces to a single point. In fact, for this specific bundle the attractor set $\mathcal{M}_{\Theta}^{+}$ is contained in open Bruhat cells, that is, the set $\chi(y)^{-1}\cdot\left(\mathcal{M}_{\Theta}^{+}\cap\mathbb{E}_{\Theta(\phi)}\right)$ is contained in some open Bruhat cell of $\mathbb{F}_{\Theta(\phi)}$. Thus Lemma 9.1 implies that $\mathrm{im}_{\Theta(\phi)}(g_k)$ is contained in an open cell. But the only possibility for this ocurrence is when g_k is contractible with respect to $\Theta(\phi)$, that is $\mathrm{im}_{\Theta(\phi)}(g_k)$ is a point.

Corollary 9.2 *Keep the notations and assumptions as above. Then* $\mathrm{im}_{\Theta(\phi)}(g_k)$ *reduces to a single point.*

Now we can prove the main lemma about the structural property of the attractor maximal chain transitive sets in the flag bundles.

Lemma 9.3 *Let* $\mathcal{M}_{\Theta(\phi)}^{+}\subset\mathbb{E}_{\Theta(\phi)}$ *be the attractor maximal chain transitive set in the flag bundle corresponding to the parabolic type of* ϕ*. Suppose that* $x\in X$ *is such that* $\omega(x)\neq\emptyset$*. Then* $\mathcal{M}_{\Theta(\phi)}^{+}$ *meets the fiber* $\left(\mathbb{E}_{\Theta(\phi)}\right)_x$ *over* x *in a single point.*

Proof: Write $\mathcal{A}=\chi(x)^{-1}\cdot\mathcal{M}_{\Theta(\phi)}^{+}$ and fix $b_0\in\mathcal{A}$. We shall take a polar decomposition of G adapted to b_0 and $\mathcal{A}$ as follows: Choose a Weyl chamber $A^{+}\subset G$ so that b_0 is the attractor of A^{+} in $\mathbb{F}_{\Theta(\phi)}$ and the corresponding stable manifold (open cell) σ contains $\mathcal{A}$ (e.g. take A^{+} meeting a shadowing semigroup $S_{\varepsilon,T}$ for small enough $\varepsilon>0$ and large T). This Weyl chamber determines a maximal compact subgroup $K\subset G$ and the polar decomposition $G=KA^{+}K$.

For $y\in\omega(x)$ let $t_k\to+\infty$ be a sequence with $t_k\cdot x\to y$. Take local cross sections $\chi_i:U_i\to Q$, $i=1,2$, around x and y, respectively, and let $\rho=\rho_{\chi_1,\chi_2}$ be the corresponding local cocycle. Put $g_k=\rho(t_k,x)$ and assume without loss of generality that g_k is admissible.

Now, write $g_k=v_kh_ku_k$ with $v_k,u_k\in K$ and $h_k\in A^{+}$ with $u_k\to u$, $v_k\to v$. By the above corollary g_k is contractible in $\mathbb{F}_{\Theta(\phi)}$, so that $\mathrm{im}_{\Theta(\phi)}(g_k)=vb_0$. Changing, if necessary, the cross section χ_1 with $\chi'=\chi_1\cdot u$, $u\in K$, we can assume that $u_k\to 1$. Then by Lemma 6.1 we conclude that $g_k^{-1}b$ is outside the compact subset $\mathcal{A}\subset\sigma$ if $b\neq vb_0$. However,

$$\phi_{-t_k}\left(\chi(t_k\cdot x)\,b\right)=\chi(x)\cdot\left(\rho(t_k,x)^{-1}b\right)=\chi(x)\cdot\left(g_k^{-1}b\right).$$

Since for large k, $g_k^{-1}b\notin\mathcal{A}=\chi(x)^{-1}\cdot\mathcal{M}_{\Theta(\phi)}^{+}$, it follows that $\chi(t_k\cdot x)\cdot b\notin\mathcal{M}_{\Theta(\phi)}^{+}$ if $vb\neq b_0$. Therefore, for large values of k the fiber of $\mathcal{M}_{\Theta(\phi)}^{+}$ above $t_k\cdot x$ reduces to the point $\chi(t_k\cdot x)\cdot(v^{-1}b_0)$. This implies that the fiber above x is also a single point, since ϕ_{t_k} settles a bijection between the fibers $\left(\mathbb{E}_{\Theta(\phi)}\right)_x \to \left(\mathbb{E}_{\Theta(\phi)}\right)_{t_k \cdot x}$. ∎

Clearly, reverting time this proof yields an analogous result for the repeller component, as soon as we consider the flag $\mathbb{E}_{\Theta^*(\phi)}$ corresponding to the parabolic type of the reversed flow.

Corollary 9.4 *Let $\mathcal{M}^-_{\Theta^*(\phi)} \subset \mathbb{E}_{\Theta^*(\phi)}$ be the repeller maximal chain transitive set in the flag bundle corresponding to the reversed parabolic type of ϕ. Suppose that $x \in X$ is such that $\omega^*(x) \neq \emptyset$. Then $\mathcal{M}^-_{\Theta^*(\phi)}$ meets the fiber $\left(E_{\Theta^*(\phi)}\right)_x$ over x in a single point.*

Of course, the conditions about ω and ω^*-limits are satisfied in case the base space X is compact.

Corollary 9.5 *In the situation of Lemma 9.3, assume furthermore that the base space is compact. Then the maximal chain transitive sets $\mathcal{M}^+_{\Theta(\phi)} \subset \mathbb{E}_{\Theta(\phi)}$ and $\mathcal{M}^-_{\Theta^*(\phi)} \subset \mathbb{E}_{\Theta^*(\phi)}$ meet the fibers in singletons.*

Corollary 9.6 *The bundles $\mathbb{E}_{\Theta(\phi)} \to X$ and $\mathbb{E}_{\Theta^*(\phi)} \to X$ are trivial if $\omega(x), \omega^*(x) \neq \emptyset$ for all $x \in X$.*

Proof: Define $\chi : X \to \mathbb{E}_{\Theta(\phi)}$ by the requirement $\mathcal{M}^+ \cap \left(\mathbb{E}_{\Theta(\phi)}\right)_x = \{\chi(x)\}$. Then χ is a global cross section of $\mathbb{E}_{\Theta(\phi)} \to X$. It remains only to check that χ is continuous. But this follows by local trivialization and the elementary fact that a map between metric spaces is continuous provided its graph is closed and the target space is compact. The proof for $\mathbb{E}_{\Theta^*(\phi)} \to X$ is similar. ∎

In order to have specific notations for the cross sections in this corollary we write $\Omega : X \to \mathbb{E}_{\Theta(\phi)}$ and $\Omega^* : X \to \mathbb{E}_{\Theta^*(\phi)}$ with $\{\Omega(x)\} = \mathcal{M}^+_{\Theta(\phi)} \cap \left(\mathbb{E}_{\Theta(\phi)}\right)_x$ and $\{\Omega^*(x)\} = \mathcal{M}^-_{\Theta^*(\phi)} \cap \left(\mathbb{E}_{\Theta^*(\phi)}\right)_x$.

Now, we encode the cross sections Ω and Ω^* into a global cross section of a bundle whose fiber is an adjoint orbit of G. For this let $f : Q \to \mathbb{F}_{\Theta(\phi)}$ and $f^* : Q \to \mathbb{F}_{\Theta^*(\phi)}$ be the functions corresponding to Ω and Ω^*, respectively. Explicitly,

$$f(q) = q^{-1} \cdot \Omega(\pi(q)) \quad \text{and} \quad f^*(q) = q^{-1} \cdot \Omega^*(\pi(q)).$$

Note that for every $\varepsilon, T > 0$, $f(q)$ belongs to the set of transitivity of the invariant control set in $\mathbb{F}_{\Theta(\phi)}$ of $S^q_{\varepsilon,T}$, while $f^*(q)$ belongs to the minimal control set in $\mathbb{F}_{\Theta^*(\phi)}$.

Hence, the pair $(f(q), f^*(q))$ belongs to the generic G-orbit $\mathcal{O}_{\Theta(\phi)} \subset \mathbb{F}_{\Theta(\phi)} \times \mathbb{F}_{\Theta^*(\phi)}$, which as homogeneous space is $\mathcal{O}_{\Theta(\phi)} = G/Z_G(H_{\Theta(\phi)})$ where $H_{\Theta(\phi)} \in \mathrm{cl}\mathfrak{a}^+$ satisfies $\alpha(H_{\Theta(\phi)}) = 0$ if and only if $\alpha \in \langle \Theta(\phi) \rangle$. Thus we have a map $h : Q \to G/Z_G(H_{\Theta(\phi)})$ which is equivariant in the sense that $h(q \cdot g) = g^{-1} \cdot h(q)$. Therefore h defines a cross section of the associated bundle whose typical fiber is $\mathrm{Ad}(G) H_{\Theta(\phi)}$.

Note that the identification of $\mathcal{O}_{\Theta(\phi)}$ with $\mathrm{Ad}(G) H_{\Theta(\phi)}$ is made in such a way that A in the adjoint orbit corresponds to the pair $(b_1, b_2) \in \mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ with b_1 the attractor of $\exp A$ in $\mathbb{F}_\Theta$ and b_2 the repeller of $\exp A$ in $\mathbb{F}_{\Theta^*}$. For later reference we record this construction in the following statement.

Proposition 9.7 *Let the notations and assumptions be as in Lemma 9.3. Let $\mathbb{A}_{\Theta(\phi)} \to X$ be the associated bundle, having typical fiber the adjoint orbit $\mathrm{Ad}(G) H_{\Theta(\phi)}$. Then there exists a cross section $\zeta : X \to \mathbb{A}$ with corresponding map $h : Q \to \mathrm{Ad}(G) H_{\Theta(\phi)}$, such that $f(q)$ is the attractor of $h(q)$ in $\mathbb{F}_{\Theta(\phi)}$ and $f^*(q)$ is the repeller of $h(q)$ in $\mathbb{F}_{\Theta^*(\phi)}$.*

Once we have the attractor and repeller components in the finest Morse decomposition (and the cross section given in Proposition 9.7), the other components are easily obtained through intersections of the attracting and repelling domains of the control sets. Presently we shall use the results of [21] (cf. Section 7 above) to describe an arbitrary component $\mathcal{M}$ from the extremal ones $\mathcal{M}^\pm$.

In the maximal flag bundle $\mathbb{E} \to X$ let $\mathcal{M}^\pm$ be the attractor and repeller maximal chain transitive sets, respectively. If $\mathcal{M}$ is another maximal chain transitive set

$$\mathcal{M} = \mathcal{A}(\mathcal{M}) \cap \mathcal{A}^*(\mathcal{M})$$

and by Proposition 4.10, $\mathcal{A}(\mathcal{M}) = \bigcap_{\varepsilon,T} \mathcal{A}(D_{\varepsilon,T}(\mathcal{M}))$ while $\mathcal{A}^*(\mathcal{M}) = \bigcap_{\varepsilon,T} \mathcal{A}^*(D_{\varepsilon,T}(\mathcal{M}))$.

Hence,

$$\mathcal{M} = \bigcap_{\varepsilon,T} \left(\mathcal{A}(D_{\varepsilon,T}(\mathcal{M})) \cap \mathcal{A}^*(D_{\varepsilon,T}(\mathcal{M})) \right).$$

Combining Propositions 4.10 and 7.1, we get the domain of attraction of the chain transitive set $\mathcal{M}(w)$. To state the result we use the same notations used before for projections between flag bundles. Thus, fix a simple system of roots Σ, and for a finite sequence $\alpha_1, \ldots, \alpha_n$ in Σ we let $s_1, \ldots, s_n$ be the reflections with respect to these roots. Then we write $\mathbb{E}_i \to X$ for the flag bundle with fiber $\mathbb{F}_i = \mathbb{F}_{\{\alpha_i\}}$ and put $\pi_i : \mathbb{E} \to \mathbb{E}_i$ for the canonical projection. Accordingly, we write $\gamma_i = \pi_i^{-1}\pi_i$ for the exhausting map.

Before proceeding recall that by Corollary 4.9,

$$\mathcal{M} = \bigcap_{\varepsilon,T} \left(D_{\varepsilon,T}(\mathcal{M}) \cap D^*_{\varepsilon,T}(\mathcal{M})\right)$$

for every maximal chain transitive set $\mathcal{M}$. This implies the following lemma which will be used in the description of $\mathcal{M}(w)$ to be given below.

Lemma 9.8 *Take sequences $\varepsilon_n \to 0$ and $T_n \to +\infty$, and suppose that a sequence*

$$b_n \in D_{\varepsilon_n,T_n}\left(\mathcal{M}^-\right) \cap D^*_{\varepsilon_n,T_n}\left(\mathcal{M}^-\right)$$

converges to b. Then $b \in \mathcal{M}^-$.

Proof: For any $\varepsilon, T > 0$, $b_n \in D^*_{\varepsilon,T}(\mathcal{M}^-)$ if n is large enough. But the control set $D^*_{\varepsilon,T}(\mathcal{M}^-)$ is closed, so that $b \in D^*_{\varepsilon,T}(\mathcal{M}^-)$, showing the lemma. ∎

Proposition 9.9 *The domain of attraction of $\mathcal{M}(w)$ is given by*

$$\mathcal{A}(\mathcal{M}(w)) = \gamma_1 \cdots \gamma_n \left(\mathcal{M}^-\right), \tag{4}$$

where $\gamma_1, \ldots, \gamma_n$ is taken from a reduced expression $w_0 w = s_n \cdots s_1$.

Proof: After taking local cross sections we see that it is enough to prove that

$$\bigcap_{\varepsilon,T} \gamma_1 \cdots \gamma_n \left(C^-_{\varepsilon,T}\right) = \gamma_1 \cdots \gamma_n \left(\bigcap_{\varepsilon,T} C^-_{\varepsilon,T}\right),$$

where $C^-_{\varepsilon,T} = E_{\varepsilon,T}(w_0)$ stands for the minimal control set of $S_{\varepsilon,T}$ in $\mathbb{E}$. The inclusion of the second hand side into the first is immediate. For the converse, take $x \in \bigcap_{\varepsilon,T} \gamma_1 \cdots \gamma_n \left(C^-_{\varepsilon,T}\right)$ and sequences $\varepsilon_k \to 0$, $T_k \to +\infty$ and $b_k \in C_{\varepsilon_k,T_k}$. We can assume that $b_k \to b$, so that by Lemma 9.8, $b \in \mathcal{M}^- = \bigcap_{\varepsilon,T} C^-_{\varepsilon,T}$.

Now any converging sequence $y_k \in \gamma_1 \cdots \gamma_n\{b_k\}$ has limit in $\gamma_1 \cdots \gamma_n\{b\}$. In particular the constant sequence $y_k = x$ belongs to $\gamma_1 \cdots \gamma_n\{b\}$, concluding the proof. ∎

The same result can be applied to the reversed flow to get $\mathcal{A}^*(\mathcal{M}(w))$. We must only take care with the labelling of the control sets by the elements of the Weyl group to pick the right sequence $\gamma_1 \cdots \gamma_n$. When working with $\mathcal{A}(D(w))$ we are tacitly assuming that the map $w \mapsto D(w)$ is defined in such a way that $D(1)$ is the invariant control set while $D(w_0)$ is the minimal control set. Hence for the reversed flow we must choose another set of simple roots (corresponding to a reduced expression) in order to write down a formula like (4) for $\mathcal{A}^*(D(w))$. According to [21], Proposition 3.1, we must take a reduction expression for $w = w_0(w_0 w)$. In fact, if we label the control sets of S^{-1}, say as $D^-(w)$, in such a way that $\mathrm{cl}C^- = D^-(1)$ and $C_0^+ = D^-(w_0)$ then $D(w)$ and $D^-(w_0 w)$ have the same set of transitivity. Thus we get,

Proposition 9.10 *The repelling domain of $\mathcal{M}(w)$ is given by*

$$\mathcal{A}^*(\mathcal{M}(w)) = \gamma_1 \cdots \gamma_m(\mathcal{M}^+), \tag{5}$$

where $\gamma_1, \ldots, \gamma_m$ is taken from a reduced expression $w = s_m \cdots s_1$.

Now we can give the full picture of the chain recurrent components.

Theorem 9.11 *Let the notations and assumptions be as in Lemma 9.3. Consider the map $h : Q \to \mathrm{Ad}(G)H$ of Proposition 9.7, where H is any element of the "partial chamber" $\mathfrak{a}^+(\Theta(\phi))$. Then the chain recurrent components in the full flag bundle $\mathbb{E}$ are given by the fixed points of $h(q)$ as follows:*

$$\mathcal{M}(w)_{\pi(q)} = q \cdot \mathrm{fix}(h(q), w).$$

Proof: Follows immediately from Corollary 7.5 and the above two propositions. ∎

Remark: If H is like in the above theorem, then the vector field induced by H on a flag manifold $\mathbb{F}_\Theta$ is gradient with respect to a certain Riemannian metric on $\mathbb{F}_\Theta$. Thus it might be expected that the gradient-like functions for the flow on a bundle $\mathbb{E}_\Theta \to X$ could be built from the cross section $h(q)$ (cf. Conley [8]).

10 Examples and special cases

10.1 Vector bundles

Given an n-dimensional real vector bundle $V \to X$ let ϕ_t be a flow on V which is linear on fibers. This flow can be put in our principal bundle set up by taking the bundle of frames $Q = BV \to X$ of V. The elements of BV are the invertible linear maps $p : \mathbb{R}^n \to V_x$ where V_x is the fiber of V above $x \in X$, and the structural group of BV is $G = \mathrm{Gl}(n, \mathbb{R})$ which acts on the right on BV by $pg = p \circ g$, $p \in BV$, $g \in \mathrm{Gl}(n, \mathbb{R})$. The vector bundle is recovered from BV as the associated bundle obtained by the standard linear action of $\mathrm{Gl}(n, \mathbb{R})$ in $\mathbb{R}^n$.

The linear flow ϕ_t on V lifts to a flow, also denoted by ϕ_t, on BV by putting $\phi_t(p) = \phi_t \circ p$, which clearly satisfies $\phi_t(pg) = \phi_t(p)\, g$, $g \in G$. Conversely, a right invariant flow on BV induces a linear flow on the associated bundle V, showing that flows on $\mathrm{Gl}(n, \mathbb{R})$-bundles are equivalent to linear flows on vector bundles.

The flag manifolds of $\mathrm{Gl}(n, \mathbb{R})$ are the usual manifolds of flags of subspaces of $\mathbb{R}^n$. Hence the associated flag bundles are precisely the bundles over X which are built from $V \to X$ by taking flags of subspaces of V_x, $x \in X$. We specialize our results to these bundles. Here the semi-simple component of the Lie algebra of $\mathrm{Gl}(n, \mathbb{R})$ is $\mathfrak{sl}(n, \mathbb{R})$. We take the Lie algebra $\mathfrak{a}$ of zero trace diagonal matrices (with respect to a basis fixed in advance). A Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ is given by the set of diagonal matrices $\mathrm{diag}\{a_1, \ldots, a_n\}$ satisfying $a_1 > \cdots > a_n$, so that $\mathrm{cl}\mathfrak{a}^+$ is the set of zero trace diagonal matrices with $a_1 \geq \cdots \geq a_n$. With these choices, the adjoint orbit $\mathrm{Ad}(G)\, H$ of $H \in \mathrm{cl}\mathfrak{a}^+$ is the set of zero trace diagonalizable matrices with the same eigenvalues as H.

To label the parabolic type of a flow recall that the roots of $\mathfrak{a}$ are the functionals $\alpha_{ij}(\mathrm{diag}\{a_1, \ldots, a_n\}) = a_i - a_j$, $i \neq j$, and the simple system of roots corresponding to $\mathfrak{a}^+$ is $\Sigma = \{\alpha_i = \alpha_{i,i+1} : i = 1, \ldots, n-1\}$. Note that for a subset $\Theta \subset \Sigma$ a matrix $\mathrm{diag}\{a_1, \ldots, a_n\}$ is anihilated by Θ if and only if $a_i = a_{i+1}$ when $\alpha_i \in \Theta$. Thus if $\Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, a matrix $\mathrm{diag}\{a_1, \ldots, a_n\}$ belongs to the partial chamber $\mathfrak{a}^+(\Theta)$ if and only if

$$a_1 > a_2 > \cdots > a_{i_1} = a_{i_1+1} > \cdots,$$

that is, Θ determines a set $k_\Theta = \{k_1, \ldots, k_s\}$ such that the matrices in $\mathfrak{a}^+(\Theta)$

are given in diagonal blocks as

$$\begin{pmatrix} \lambda_1 \mathrm{id}_{k_1} & & \\ & \ddots & \\ & & \lambda_s \mathrm{id}_{k_s} \end{pmatrix}$$

with $\lambda_1 > \cdots > \lambda_s$.

Now, according to Theorem 9.11, the Morse decomposition of a flow satisfying our conditions is given as the set of fixed points of $h(p)$ where $h : BV \to \mathrm{Ad}(G) H$ is an equivariant map into the adjoint orbit of some $H \in \mathfrak{a}^+(\Theta(\phi))$. Since the elements of $\mathrm{Ad}(G) H$ are linear maps in $\mathbb{R}^n$ we can transfer $h(p)$ through $p : \mathbb{R}^n \to V_x$ to the linear map $H_x = p \circ h(p) \circ p^{-1} : V_x \to V_x$. Hence, Theorem 9.11 restates as:

Theorem 10.1 *Let ϕ_t be a flow on the vector bundle $V \to X$. If the assumptions of Theorem 9.11 are satisfied then for each $x \in X$ there exists a diagonalizable linear map $H_x : V_x \to V_x$ such that the Morse sets of ϕ_t in a flag bundle are given fiberwise by the connected components of the fixed point set of* $\exp(tH_x)$. *Furthermore, the map $x \mapsto H_x$ is continuous and the spectra of H_x is constant along X.*

Let us specialize this description to some flag bundles. First we recover the Theorem of Selgrade [18] about flows on the projective bundle $\mathbb{P}(V) \to X$, whose fibers are the projective spaces $\mathbb{P}_x$ of V_x, $x \in X$. In this case the fixed-points of $\exp(tH_x)$ in $\mathbb{P}_x$ are the eigenvectors of H_x, and the connected components of the set of fixed-points are given by the eigenspaces of H_x. Since H_x is diagonalizable and $x \mapsto H_x$ is continuous we conclude that the maximal chain transitive subsets are given by $\bigcup_{x \in X} \mathbb{P}\left(V_{\lambda_i}^x\right)$, $i = 1, \ldots, s$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ are the common eigenvalues of H_x, $x \in X$, and $V_{\lambda_i}^x$ is the λ_i-eigenspace of H_x. This recovers the Theorem of Selgrade [18].

Note that the parabolic type of the flow corresponds to the flag manifold containing flags whose subspaces have dimensions $\dim(V_{\lambda_1})$, $\dim(V_{\lambda_1}) + \dim(V_{\lambda_2})$ etc. This relates the parabolic type of the flow and the Selgrade subbundles.

Corollary 10.2 *Let ϕ be a linear flow on the vector bundle $V \to X$, and assume that* $\mathrm{loc}(X)$ *is locally transitive. Then the parabolic type of ϕ corresponds to the flag manifold containing flags whose vector spaces have the* *same dimension as the flag*

$$V_1 \subset V_1 \oplus V_2 \subset \cdots \subset V_1 \oplus \cdots \oplus V_s$$

where $V_1, \ldots, V_s$ are the Selgrade subbundles given in decreasing order.

Morse decompositions on flag bundles were studied by Colonius-Kliemann [6] exploiting the successive fibrations between the flag manifolds and the theorem on projective bundles. As a result it is proved the existence of a finest Morse decomposition in the full flag manifold with at most $n!$ components on the fibers (see [6], Theorem 5). From the above theorem we get in fact that the number of chain recurrent components on the fibers is $|\mathcal{W}| / |\mathcal{W}_{\Theta(\phi)}|$ which is in fact less than $n! = |\mathcal{W}|$. On the other hand [6] describes the following Morse decomposition in the Grassmann bundle $\mathrm{Gr}_k(V)$ of k-dimensional subspaces of $V \to X$: Let

$$V = V^1 \oplus \cdots \oplus V^s$$

be the decomposition of V into the subbundles given by chain recurrent components in $\mathbb{P}(V)$. For $x \in X$ and a multi-index $\kappa = (k_1, \ldots, k_s)$ with $k_i \geq 0$ and $k_1 + \cdots + k_s = k$ define the set

$$M_x^\kappa = \{U \in \mathrm{Gr}_k(V)_x : \dim\left(U \cap V_x^i\right) = k_i\}$$

and form $\mathcal{M}^k = \bigcup_{x \in X} M_x^\kappa$. Then the sets $\mathcal{M}^\kappa$, with κ running through the multi-indices is a Morse decomposition (see [6], Theorem 6). It follows from Theorem 10.1 that the sets $\mathcal{M}^\kappa$ actually constitute the finest Morse decompostion. Indeed it is easy to see that the fixed point set of the action of $\exp(tH_x)$ in the Grasmannian $\mathrm{Gr}_k(V_x)$ is M_x^κ, since the subspaces V_x^i are the eigenspaces of H_x.

10.2 Representations

Linear flows on vector bundles arise if we start with a principal bundle $Q \to X$ with structural group G, and take a representation ρ of G in a vector space U. Then the associated bundle $V = Q \times_G U \to X$ obtained by the action of G on U is a vector bundle and right invariant flows on Q induce linear flows on V.

For a flow on $V \to X$ we can take the Morse decomposition on $\mathbb{P}(V) = \bigcup_i \mathbb{P}(V_i)$, given by a Whitney sum $V = \bigoplus_i V_i$. However, it happens in most of the cases that the action of G on the projective space $\mathbb{P}(U)$ has a compact proper orbit yielding the existence of a closed subbundle E of $\mathbb{P}(V)$ invariant under the flow. It might be interesting to look at the Morse decomposition of the flow restricted to E. Of course the intersections with E of the Morse components $\mathbb{P}(V_i)$ provides a Morse decomposition for the restricted flow. But the embedding of E into $\mathbb{P}(V)$ can be in such a complicated way so that it is hard, if feasible, to see what happens to $\mathbb{P}(V_i) \cap E$. Thus it is more sensible to study the restricted flow intrinsically, according to our set up.

We already encountered examples of this situation above: A linear flow ϕ_t on the vector bundle $V \to X$ induces a flow $\phi_t^{\wedge k}$ on the k-fold exterior product $\bigwedge^k V$ of V. The bundle $\mathbb{P}\left(\bigwedge^k V\right)$ contains as a subbundle the Grassmann bundle $\mathrm{Gr}_k(V)$, given by the set of decomposable vectors. The finest Morse decomposition in $\mathrm{Gr}_k(V)$ was described before, while it is not clear how to obtain it from decompositions of the flow on the whole $\mathbb{P}\left(\bigwedge^k V\right)$.

10.3 Linearized flows

A flow ϕ_t of diffeomorphisms of an n-dimensional manifold M lifts to a right invariant flow on the bundle of frames BM by defining $(t,p) \mapsto d\phi_t \circ p$ where $p : \mathbb{R}^n \to T_xM$ is a frame in BM. The study of this "linearized" flow is one of the motivations for considering flows on fiber bundles. Clearly a flow on BM is a special case of the flow considered above on general vector bundles. However, there are interesting flows whose symmetry allows to consider subbundles of BM (geometric structures) and thus flows on bundles with groups different from $\mathrm{Gl}(n,\mathbb{R})$. Our general set up is adapted to an intrinsic approach to such flows. Below we list some cases.

1. Let M be an orientable manifold endowed with a volume element ν. The bundle BM admits a reduction to the $\mathrm{Sl}(n,\mathbb{R})$-bundle Vol formed by the frames $p : \mathbb{R}^n \to T_xM$ such that $p^*\nu$ is the standard volume element in $\mathbb{R}^n$. The lifting of a flow ϕ_t on M leaves invariant Vol if ϕ_t, $t \in \mathbb{R}$, is volume preserving, that is $\phi_t^*\nu = \nu$. Although Vol is a subbundle of BM, the situation here is not much different from BM itself, since the $\mathrm{Gl}(n,\mathbb{R})$ and $\mathrm{Sl}(n,\mathbb{R})$ flag manifolds coincide, and the actions of $\mathrm{Gl}(n,\mathbb{R})$ factor through $\mathrm{Sl}(n,\mathbb{R})$. We observe nevertheless that if M is compact then ϕ_t is chain recurrent, due to the recurrence theorem.

2. Let M be a $2n$-dimensional manifold endowed with a symplectic form ω. The symplectic structure defines a reduction of the bundle of frames to a subbundle $\mathrm{Sp} \subset BM$ composed of the frames $p : \mathbb{R}^{2n} \to T_xM$ such that $p^*\omega = \omega_0$ where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$:

$$\omega_0(u,v) = v^T J u, \qquad J = \begin{pmatrix} 0 & -\mathrm{id}_{n\times n} \\ \mathrm{id}_{n\times n} & 0 \end{pmatrix}.$$

The structure group of the bundle $\mathrm{Sp} \to M$ is the symplectic group $\mathrm{Sp}(n,\mathbb{R}) = \{g : g^T J g = J\}$, which is a simple Lie group. Its flag manifolds are the submanifolds of the general flag manifolds formed by flags of subspaces of $\mathbb{R}^{2n}$ which are Lagrangian with respect to the standard symplectic form ω_0 (a subspace U is Lagrangian if the restriction of ω_0 to U is identically zero). Thus the associated flag bundles are built analogously from subspaces of T_xM which are Lagrangian with respect to ω.

Right invariant flows on $\mathrm{Sp} \to M$ are obtained e.g. by lifting to BM Hamiltonian vector fields on M. Any such lifting leaves invariant Sp and thus induces flows on the Lagrangian flag bundles.

3. There are further examples on manifolds endowed with different geometric structures. For instance: 1) Flows of isometries of a pseudo-Riemannian manifold where the structure group is $\mathrm{SO}(p,q)$. 2) Flows of holomorphic diffeomorphisms on a complex manifold where the structure group is $\mathrm{Gl}(n,\mathbb{C}) \subset \mathrm{Gl}(2n,\mathbb{R})$.

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