S-Convex Fuzzy Processes

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Abstract. We introduce the notion of s-convex fuzzy processes. We study their properties and we give some applications.

1. Introduction

In 1967, Rockafellar [1] introduced the notion of convex processes (see also [2]). These are set-valued maps whose graphs are closed convex cones. For instance, they can be see as the set-valued version of a continuous linear operator. Derivatives of some set-valued maps are closed convex processes, which is a desirable property for a derivative (see [3]). An important property of convex processes is that it is possible transpose closed convex processes and use the benefits of duality theory. And as it is well known, these facts are very useful in optimization theory (see for instance [4], [5], [6], [7], [8]).

The extension of this notion to the fuzzy framework was done by Matłoka [9]. Recently, Syan, Low and Wu [10] observed that Matłoka definition is very strict. Therefore, they give other definition that extend the Matłoka definition. In 2000 was introduced by the authors the concept M-convex fuzzy mapping [11], we observe that 1-convex fuzzy mapping is coincident with definition of convex process given in [10] (see Theorem 3.4, p. 195 in [10]) for the case m=1.

In 1978, Breckner introduced s-convex functions as a generalization of convex functions [12], and in 1993 studied the set-valued version [13]. We observe that convex processes are one particular case of s-convex set-valued maps. Also, in that one work Breckner proved the fact important that the set-valued map is s-convex

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if only if her support function is s-convex function. Other works relationated are [16], [17], [18].

In this work, we introduce the fuzzy version of the Breckner definition, and we will call this generalization s-convex fuzzy process. Moreover we will prove the equivalence with the s-convexity of the fuzzy support function and we study some properties.

The plan of the paper is as follows. In Section 2, we introduce the notations, definitions and preliminaries results used throughout the paper. In Section 3 we establish the main results and finally in Section 4 we show some algebraic properties and the connection with the fuzzy integral mean for fuzzy set-valued map.

2. Preliminaries

Let $\mathbb{R}^n$ be denote the n-dimensional Euclidean space. Let $s \in]0,1]$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that for all $a \in [0,1]$ and for all $x, y \in \mathbb{R}^n$ the following inequality holds

$$f((1-a)x + ay) \leq (1-a)^s f(x) + a^s f(y). \tag{2.1}$$

Theses functions are called s-convex and have been introduced by Breckner [12], where also it is possible to find examples of s-convex functions.

Let $P(\mathbb{R}^n)$ be denote the set of all nonempty subsets of $\mathbb{R}^n$, in [13] Breckner generalized the notion of s-convexity for a set-valued mapping $F : \mathbb{R}^m \to P(\mathbb{R}^n)$, he say that F is s-convex if the following relation is verified

$$(1-a)^s F(x) + a^s F(y) \subseteq F((1-a)x + ay) \tag{2.2}$$

for all $a \in [0,1]$ and all $x, y \in \mathbb{R}^m$. We denote by $\mathcal{K}(\mathbb{R}^m)$ the subset of $P(\mathbb{R}^m)$ whose elements are compact nonempty and by $\mathcal{K}_c(\mathbb{R}^m)$ the subset $\mathcal{K}(\mathbb{R}^m)$ whose elements are convex. We recall that if $A \in \mathcal{K}(\mathbb{R}^m)$, the support function $\sigma(A, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is defined as

$$\sigma(A, \psi) = \sup_{a \in A} < \psi, a >, \ \forall \psi \in \mathbb{R}^m.$$

It is important to remark that if $A, B \in \mathcal{K}_c(\mathbb{R}^m)$ then, as a direct consequence of the separation Hahn-Banach theorem, we obtain: $\sigma(A, \cdot) = \sigma(B, \cdot) \Leftrightarrow A = B$.

The generalization (2.2) is based on the s-convexity of the function $\sigma(F(\cdot), \psi)$, that is, F is s-convex if and only if $\sigma(F(\cdot), \psi)$ is s-convex for all $\psi \in \mathbb{R}^n$. The notion of s-convex set-valued mapping was studied by several authors, including Trif [14].

Now, we will give the extensions of the above results to the fuzzy context. A fuzzy subset of $\mathbb{R}^n$ is a function $u : \mathbb{R}^n \to [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of all nonempty fuzzy sets in $\mathbb{R}^n$. A fuzzy set u is called convex [15] if

$$u(\lambda y_1 + (1-\lambda)y_2) \geq \min\{u(y_1), u(y_2)\},$$

for all $y_1, y_2 \in$ supp $(u) = \overline{\{y \ / \ u(y) > 0\}}$ and $\lambda \in]0,1[$.

We shall define addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by the usual extension principle:

$$(u+v)(y) = \sup_{y_1, y_2 : y_1 + y_2 = y} \min\{u(y_1), v(y_2)\}$$

and

$$(\lambda u)(y) = \begin{cases} u(\frac{y}{\lambda}) & if \quad \lambda \neq 0 \\ \chi_{\{0\}}(y) & if \quad \lambda = 0. \end{cases}$$

We can define an order $\subseteq$ on $\mathcal{F}(\mathbb{R}^n)$ by setting

$$u \subseteq v \Leftrightarrow u(y) \leq v(y), \quad \forall y \in \mathbb{R}^n.$$

We define the intersection of two fuzzy sets u, v, denoted by $u \wedge v$, by

$$(u \wedge v)(y) = \min\{u(y), v(y)\}.$$

Let $u \in \mathcal{F}(\mathbb{R}^n)$. For $0 < \alpha \leq 1$, we denote by $[u]^\alpha = \{y \in \mathbb{R}^n \ / \ u(y) \geq \alpha\}$ the α-level of u. $[u]^0 =$ supp $(u) = \overline{\{y \in \mathbb{R}^n \ / \ u(y) > 0\}}$ is called the support of u.

A fuzzy set $u : \mathbb{R}^n \to [0,1]$ is called fuzzy compact set if $[u]^\alpha$ is compact for all $\alpha \in [0,1]$. If $u \in \mathcal{F}(\mathbb{R}^n)$ is convex, then $[u]^\alpha$ is convex for all $\alpha \in [0,1]$.

We denote by $\mathcal{F}_C(\mathbb{R}^n)$ the space of all fuzzy compact convex sets. Given $u, v \in \mathcal{F}_C(\mathbb{R}^n)$, it is satisfied that

(a) $u \subseteq v \Leftrightarrow [u]^\alpha \subseteq [v]^\alpha \ \ \forall \alpha \in [0,1]$

(b) $[\lambda u]^\alpha = \lambda [u]^\alpha \ \ \forall \lambda \in \mathbb{R}, \ \forall \alpha \in [0,1]$

(c) $[u+v]^\alpha = [u]^\alpha + [v]^\alpha \ \ \forall \alpha \in [0,1]$.

Any application $F : \mathbb{R}^m \to \mathcal{F}_C(\mathbb{R}^n)$ will be call a fuzzy process. For each $\alpha \in [0,1]$ we define the set-valued mapping $F_\alpha : \mathbb{R}^m \to \mathcal{K}(\mathbb{R}^n)$ by

$$F_\alpha(x) = [F(x)]^\alpha.$$

For any $u \in \mathcal{F}_C(\mathbb{R}^n)$ the support function of u, $S(u,(\cdot,\cdot)) : [0,1] \times \mathbb{R}^m \to \mathbb{R}$, is defined as

$$S(u,(\alpha,\psi)) = \sigma([u]^\alpha, \psi).$$

For the details about support functions see [19].

A fuzzy process $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called convex if satisfies the following relation

$$F((1-a)x_1 + ax_2)(y) \geq \sup_{y_1,y_2:(1-a)y_1+ay_2=y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}, \tag{2.3}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$. This notion of convex fuzzy processes was recently introduced in [10].

Next we introduce the definition of s-convex fuzzy processes. This definition is a generalization of the notion of s-convexity of a set-valued mapping.

Definition 2.1. *Let $s \in]0,1]$. A fuzzy process $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called s-convex fuzzy process, if for all $a \in]0,1[$ and for all $x, y \in \mathbb{R}^m$ it satisfies the condition*

$$(1-a)^s F(x) + a^s F(y) \subseteq F((1-a)x + ay).$$

Remark 1. *Usually 1-convex fuzzy processes are simply called convex fuzzy processes (see [10]).*

Example 2.2. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a s-convex function. Consider $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R})$ defined by*

$$F(x) := \chi_{[f(x),\infty[},$$

where χ_A denotes the characteristic function of A.

Since f is s-convex , we have that

$$[f((1-a)x + ay), \infty[\supseteq [(1-a)^s f(x), \infty[\quad + [a^s f(y), \infty[$$

for all $a \in]0,1[$ and $x, y \in \mathbb{R}^m$. Consequently,

$$\begin{aligned} F((1-a)x + ay) &= \chi_{[f(((1-a)x+ay),\infty[} \\ &\supseteq \chi_{\{(1-a)^s[f(x),\infty[\}} + \chi_{\{a^s[f(y),\infty[\}} \\ &= (1-a)^s \chi_{[f(x),\infty[} + a^s \chi_{[f(y),\infty]} \\ &= (1-a)^s F(x) + a^s F(y). \end{aligned}$$

Thus, F is s-convex fuzzy process.

3. Main results

In this Section, we present some properties of a s-convex fuzzy process and we give two characterizations: the first is using the membership and the second is given using the concept of support function of a fuzzy set.

Theorem 3.1. *Let $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy process. F is a s-convex fuzzy process if and only if*

$$F(ax_1 + (1-a)x_2)(y) \geq \sup_{y_1,y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\} \tag{3.1}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$.

Proof. Suppose that F is a s-convex fuzzy process. Let $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$ arbitrary. Then, from the Definition 2.1, from the addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$, we have that

$$\begin{aligned}
& F(ax_1 + (1-a)x_2)(y) \\
\geq\ & (a^s F(x_1) + (1-a)^s F(x_2))(y) \\
=\ & \sup_{y_1,y_2 : y_1 + y_2 = y} \min\{a^s F(x_1)(y_1), (1-a)^s F(x_2)(y_2)\} \\
=\ & \sup_{y_1,y_2 : y_1 + y_2 = y} \min\left\{F(x_1)\left(\frac{y_1}{a^s}\right), F(x_2)\left(\frac{y_2}{(1-a)^s}\right)\right\} \\
=\ & \sup_{y_1,y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}.
\end{aligned}$$

Consequently, (3.1) is satisfied. Reciprocally, let us suppose that (3.1) is satisfied. Then, for all $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
& F(ax_1 + (1-a)x_2)(y) \\
\geq\ & \sup_{y_1,y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\} \\
=\ & \sup_{y_1,y_2 : y_1 + y_2 = y} \min\left\{F(x_1)\left(\frac{y_1}{a^s}\right), F(x_2)\left(\frac{y_2}{(1-a)^s}\right)\right\} \\
=\ & (a^s F(x_1) + (1-a)^s F(x_2))(y),
\end{aligned}$$

which implies that F is s-convex.□

Proposition 3.2. *Let $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy process such that*

(1) $F(x_1 + x_2) \supseteq F(x_1) + F(x_2)\ \forall x_1, x_2 \in \mathbb{R}^m$;

(2) $F(ax) = a^s F(x)\ \forall a > 0,\ \forall x \in \mathbb{R}^m$.

Then, F is a s-convex fuzzy process.

Proof. Let $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$ arbitrary. Then, from the addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$, and from the conditions (1) and (2), we have that

$$\begin{aligned}
& F(ax_1 + (1-a)x_2)(y) \\
\geq\ & (F(ax_1) + F((1-a)x_2))(y) \\
=\ & \sup_{y_1, y_2 : y_1 + y_2 = y} \min\{F(ax_1)(y_1), F((1-a)x_2)(y_2)\} \\
=\ & \sup_{y_1, y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(ax_1)(a^s y_1), F((1-a)x_2)((1-a)^s y_2)\} \\
=\ & \sup_{y_1, y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{(a^s F(x_1))(a^s y_1), ((1-a)^s F(x_2))((1-a)^s y_2)\} \\
=\ & \sup_{y_1, y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}.
\end{aligned}$$

Consequently,

$$F(ax_1 + (1-a)x_2)(y) \geq \sup_{y_1, y_2 : a^s y_1 + (1-a)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$,i.e. F satisfies the condition (3.1) of the Theorem 3.1. Therefore F is a s-convex fuzzy process.□

Proposition 3.3. *A fuzzy process $F : \mathbb{R}^m \to \mathcal{F}_C(\mathbb{R}^n)$ is s-convex if and only if F_α is s-convex for all $\alpha \in [0,1]$.*

Proof. It is a consequence of (a), (b) and (c).□

Theorem 3.4. *Let $F : \mathbb{R}^m \to \mathcal{F}_C(\mathbb{R}^n)$. F is a s-convex fuzzy process if and only if $S(F(\cdot), (\alpha, \psi))$ is s-convex for all (α, ψ).*

Proof. Suppose that F is a s-convex fuzzy process. Let $(\alpha, \psi) \in [0,1] \times \mathbb{R}^m$, $x_1, x_2 \in \mathbb{R}^n$ and $a \in]0,1[$ arbitrary. Then, from the Proposition 3.3 and properties of the support function, we have that

$$\begin{aligned} S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) &= \sigma(F_\alpha(ax_1 + (1-a)x_2), \psi) \\ &\geq \sigma(a^s F_\alpha(x_1) + (1-a)^s F_\alpha(x_2), \psi) \\ &= a^s \sigma(F_\alpha(x_1), \psi) + (1-a)^s \sigma(F_\alpha(x_2), \psi). \end{aligned}$$

Consequently,

$$S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) \geq a^s S(F(x_1), (\alpha, \psi)) + (1-a)^s S(F(x_2), (\alpha, \psi)).$$

Therefore, $S(F(\cdot), (\alpha, \psi))$ is s-convex. To prove the converse it is enough to show that

$$S(F(ax_1 + (1-a)x_2), (\alpha, \psi)) \geq S(a^s F(x_1) + (1-a)^s F(x_2), (\alpha, \psi))$$

for all $(\alpha, \psi) \in [0,1] \times \mathbb{R}^n$, which is a consequence of the properties of the support functions of a fuzzy sets.□

4. Applications

In this section, we present some results on operations of s-convex fuzzy processes and study the s-convexity of fuzzy integral mean.

Definition 4.1. *Let $F_1, F_2 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ two fuzzy process.*

(1) *The intersection of F_1 and F_2, denoted by $F_1 \cap F_2 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$, is defined by*

$$(F_1 \cap F_2)(x) = F_1(x) \wedge F_2(x).$$

(2) *The addition of F_1 and F_2, denoted by $F_1 + F_2 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$, is defined by*

$$(F_1 + F_2)(x) = F_1(x) + F_2(x).$$

(3) *The multiplication by scalar λ, denoted by $\lambda F_1 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$, is defined by*

$$(\lambda F)(x) = \lambda(F(x)).$$

Proposition 4.2. *Let $F_1, F_2 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be two s-convex fuzzy processes. Then, $F_1 \cap F_2$ is a s-convex fuzzy process.*

Proof. Let $x_1, x_2 \in \mathbb{R}^m$, $a \in]0,1[$ and $y \in \mathbb{R}^n$ arbitrary. Then,

$$\begin{aligned}
& ((F_1 \cap F_2)(ax_1 + (1-a)x_2))\,(y) \\
= \;& (F_1(ax_1 + (1-a)x_2) \wedge F_2(ax_1 + (1-a)x_2))\,(y) \\
= \;& \min\{F_1(ax_1 + (1-a)x_2)(y), F_2(ax_1 + (1-a)x_2)(y)\} \\
\geq \;& \min\left\{ \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{F_1(x_1)(y_1), F_1(x_2)(y_2)\}, \right. \\
& \left. \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{F_2(x_1)(y_1), F_2(x_2)(y_2)\} \right\} \\
\geq \;& \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{\min\{F_1(x_1)(y_1), F_1(x_2)(y_2)\}, \min\{F_2(x_1)(y_1), F_2(x_2)(y_2)\}\} \\
= \;& \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{F_1(x_1)(y_1), F_1(x_2)(y_2), F_2(x_1)(y_1), F_2(x_2)(y_2)\} \\
= \;& \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{\min\{F_1(x_1)(y_1), F_2(x_1)(y_1)\}, \min\{F_1(x_2)(y_2), F_2(x_2)(y_2)\}\} \\
= \;& \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{(F_1(x_1) \wedge F_2(x_1))(y_1), (F_1(x_2) \wedge F_2(x_2))(y_2)\} \\
= \;& \sup_{a^s y_1 + (1-a)^s y_2 = y} \min\{(F_1 \cap F_2)(x_1)(y_1), (F_1 \cap F_2)(x_2)(y_2)\}
\end{aligned}$$

From the Theorem 3.1 we obtain that F is a s-convex fuzzy process.□

Proposition 4.3. *Let $F_1, F_2 : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ be two s-convex fuzzy processes and $\lambda \geq 0$. Then, $F_1 + \lambda F_2$ is a s-convex fuzzy process.*

Proof. Let $x_1, x_2 \in \mathbb{R}^m$ and $a \in]0,1[$ arbitrary. Then,

$$\begin{aligned}
& (F_1 + \lambda F_2)(ax_1 + (1-a)x_2) \\
= \;& F_1(ax_1 + (1-a)x_2) + \lambda F_2(ax_1 + (1-a)x_2) \\
\supseteq \;& (a^s F_1(x_1) + (1-a)^s F_1(x_2)) + \lambda\,(a^s F_2(x_1) + (1-a)^s F_2(x_2)) \\
= \;& (a^s F_1(x_1) + a^s \lambda F_2(x_1)) + ((1-a)^s F_1(x_2) + (1-a)^s \lambda F_2(x_2)) \\
= \;& a^s(F_1(x_1) + \lambda F_2(x_1)) + (1-a)^s(F_1(x_2) + \lambda F_2(x_2)),
\end{aligned}$$

which implies that

$$(F_1 + \lambda F_2)(ax_1 + (1-a)x_2) \supseteq a^s(F_1 + \lambda F_2)(x_1) + (1-a)^s(F_1 + \lambda F_2)(x_2).$$

Therefore, $F_1 + \lambda F_2$ is a s-convex fuzzy process.□

Remark 2. *From above proposition, we have that the family of the s-fuzzy convex processes is a cone.*

In the following we will study the s-convexity of a fuzzy integral mean of F. For definition and properties see [11].

Definition 4.4. *[11] Let $F : [0, b] \to \mathcal{F}(\mathbb{R}^n)$ an integrably bounded f.r.v. Then the fuzzy mapping $M_F : (0, b] \to \mathcal{F}(\mathbb{R}^n)$ defined by*

$$M_F(x) = \frac{1}{x}\int_0^x F(t)dt \ , \ \forall x \in (0, b],$$

is called the fuzzy integral mean of F.

Remark 3. *Observe that taking $t = xs$ in the previous definition, the α-level of M_F can be written as $[M_F(x)]^\alpha = \int_0^1 F_\alpha(xs)ds$, i.e., $M_F(x) = \int_0^1 F(xs)ds$.*

Theorem 4.5. *Let $F : [0, b] \to \mathcal{F}_C(\mathbb{R}^n)$ be an integrably bounded f.r.v. If F is s-convex, then so is M_F.*

Proof. Let F be s-convex, $x_1, x_2 \in [0, b]$ and $a \in]0, 1[$. Then, using Remark 3 and Proposition 3.3, we obtain

$$\begin{aligned}
[M_F(ax_1 + (1-a)x_2)]^\alpha &= \int_0^1 F_\alpha(ax_1 s + (1-a)x_2 s)ds \\
&\supseteq \int_0^1 (a^s F_\alpha(x_1 s) + (1-a)^s F(x_2 s))\, ds \\
&= a^s \int_0^1 F_\alpha(x_1 s)ds + (1-a)^s \int_0^1 F(x_2 s)ds \\
&= a^s\, [M_F(x_1)]^\alpha + (1-a)^s\, [M_F(x_2)]^\alpha
\end{aligned}$$

for all $\alpha \in [0, 1]$. Thus, M_F is s-convex.□

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