Trees and Reflection Groups

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Abstract

We define a reflection in a tree as an involutive automorphism whose set of fixed points is a maximal geodesic and prove that, for the case of an homogeneous tree of degree $4k$, the topological closure of the group generated by reflections has index 2 in the group of automorphisms of the tree.

1 Basic Concepts

Although many of the constructions in this work make sense in the wide context of trees, and with minor modifications even to graphs or Λ-trees, we are concerned with homogeneous trees, so that the definitions are introduced in this restricted context. All the concepts and definitions needed may be found in both [1] and [4]. We start considering the free monoid $M(X)$ over an alphabet X with $N \in \mathbb{N}$ elements. The product is just the concatenation and the empty word $\emptyset$ plays the role of the identity element. Given a word $x = x_{i_1}x_{i_2}...x_{i_k}$, we denote its *length* by $|x| = k$. A *prefix of the word* $x = x_{i_1}x_{i_2}...x_{i_k}$ is a sub-word $x = x_{i_1}x_{i_2}...x_{i_l}$, with $l \leq k$. This induces a partial order on $M(X)$:

$$x \leq y \text{ if and only if } x \text{ is a prefix of } y.$$

The tree $- := -(M)$ is just the Cayley graph of the monoid M. The set of vertices is identified with M and we say there is an edge connecting vertices x and y if and only if $d(x, y) = 1$, where $d(\cdot, \cdot)$ is the distance defined by the length function:

$$d(x, y) = |x| + |y| - 2|z|$$

where z is the maximal common prefix of x and y. Vertices connected by an edge are said to be adjacent. We denote respectively by $S(x,n)$ and $B(x,n)$ the usual metric sphere and closed ball of Γ, centered at x with radius n. In particular, $S(x,1)$ is the set of vertices adjacent to x.

Every vertex of $\Gamma(M)$ is adjacent to exactly $N+1$ other vertices, excepts for the distinguished vertex defined by the empty word.

Any morphism of the monoid M induces a morphism of the metric space (Γ, d), that is, an isometry, and vice versa.

If we consider two copies Γ and Γ' of the tree $\Gamma(M)$ and add a single edge, connecting the vertex of Γ labeled by by $\emptyset$ to the vertex of Γ' labeled by $\emptyset'$, we eliminate the distinguished role of the empty word and get an *homogeneous tree*, that is, a tree where the number $\mathrm{Degree}_\Gamma(x)$ of vertices adjacent to x is constant. Since we are assuming that $\mathrm{Degree}_\Gamma(x)$ is constant, we denote it by $\mathrm{Degree}(\Gamma)$ and call it the *degree* of Γ. Homogeneous trees with even degree, the kind we should focus at, arise naturally as the Cayley graph of a free group.

An *integer interval* is a subset of $\mathbb{Z}$ of one of the kinds $\mathbb{Z}$, $\mathbb{N}$ or $\mathrm{I}_{a,b} := \{n \in \mathbb{Z} | a \le n \le b\}$, with $a, b \in \mathbb{Z}$. A subset $\gamma(\mathrm{I}) \subset \Gamma$ is called a *geodesic, a geodesic ray or a geodesic segment* if there is a map $\gamma : \mathrm{I} \to \Gamma$ defined on an integer interval respectively of type $\mathbb{Z}, \mathbb{N}$ or $\mathrm{I}_{a,b}$ such that $d(\gamma(n), \gamma(m)) = |n-m|$ for every $n, m \in \mathrm{I}$. We call the map $\gamma : \mathrm{I} \to \Gamma$ a *parametrization* but often make no distinction between the map γ and its image $\gamma(\mathrm{I})$. Note that such subsets may be seen as sub-trees of the tree Γ. We denote by $[x, y]$ the geodesic segment joining the vertices $x, y \in \Gamma$. A path in Γ is a map $\gamma : \mathrm{I} \to \Gamma$ such that $d(\gamma(n), \gamma(n+1)) = 1$, whenever $n, n+1 \in \mathrm{I}$, where I is an integer interval. As we did before, we often make no distinction between the path and its image $\gamma(\mathrm{I})$.

2 Reflections on Trees

There are many possibilities to define a reflection on a tree. The minimal condition for a map $\phi : \Gamma \to \Gamma$ to resemble what is commonly known as a reflection in geometry, is to demand ϕ to be an involutive automorphism. Indeed, this is the definition adopted by Moran in [2]. We can get a good felling of how much this definition is minimal from the fact that it implies that every automorphism of an homogeneous tree is the product of at most two reflections ([2, theorem 4.13]). In this work, we adopt a much more restrictive definition:

Definition 1 *A* reflection *on a tree* Γ *is a automorphism* $\phi : \Gamma \longrightarrow \Gamma$, *satisfying:*

1. ϕ *is a involution, i.e.,* $\phi^2 = \mathrm{Id}$.
2. *The set of fixed points of the* ϕ *is a maximal geodesic* $\gamma \subset \Gamma$*, i.e., there is a maximal geodesic* γ *such that* $\phi(x) = x \Leftrightarrow x \in \gamma$.

   Under these conditions, we say that ϕ *is a reflection* in the geodesic γ.

A *maximal geodesic* is either a geodesic, ray or segment γ such that for every vertex $x \in \gamma$, if $\mathrm{Degree}_{\Gamma}(x) \geq 2$ then $\mathrm{Degree}_{\gamma}(x) = 2$. Equivalently, γ is not properly contained in any other geodesic, ray or segment.

From here on, we assume that Γ is an homogeneous tree. With this condition, the choice of a geodesic as fixed points of a reflection is irrelevant, as will be shown in Proposition 4. We start with some definitions:

Definition 2 *Given a maximal geodesic* γ, G_{γ} *is the set of all reflections in* γ. *We denote by* $\langle G_{\gamma} \rangle$ *the subgroup of* $\mathrm{Aut}(\Gamma)$ *generated by* G_{γ}.

The following lemma is probably know, but we could find its proof in the literature. Since the labeling of the vertices introduced in the proof will be used later again, we prefer to prove it. We note that an equivalent formulation is to say that an homogeneous tree is two-point homogeneous.

Lemma 3 *Given maximal geodesics* γ *and* β *in an homogeneous tree* Γ*, there is* $\psi \in \mathrm{Aut}(\Gamma)$ *such that* $\psi(\gamma) = \beta$.

Proof. First of all we assume that $\gamma \cap \beta$ is infinite. In this case, this intersection is a geodesic ray and we loose no generality by assuming that $\gamma(n) = \beta(n)$ if and only if $n \leq 0$. We denote $\gamma(0) = \beta(0)$ by x_0 and label the other vertices of Γ starting from this vertex. If $N = \mathrm{Degree}(\Gamma)$, there are exactly N vertices adjacent to x_0, and we label them as $x_{0,1}, x_{0,2}, ..., x_{0,N}$, assuming that $\gamma(1) = x_{0,1}$ and $\beta(1) = x_{0,N}$. Since Γ is homogeneous, each $x_{0,i}$ is adjacent to exactly N vertices, $N-1$ of them at distance 2 from x_0. We label them as $x_{0,i,1}, x_{0,i,2}, ..., x_{0,i,N-1}$, assuming that $\gamma(2) = x_{0,1,1}$ and $\beta(2) = x_{0,N,1}$. Proceeding in this way, each vertex in Γ is labeled as $x_{0,i_1,i_2,...,i_k}$, with $i_1 = 1, 2, ..., N$ and $i_j = 1, 2, ..., N-1$ if $j \geq 2$, where $k = d(x, x_0)$.

Note that the distance between two vertices can be easily determined from their labels. Let $x, y \in \Gamma$ be vertices labeled as $x = x_{0,i_1,i_2,\dots,i_k}$ and $y = x_{0,j_1,j_2,\dots,j_l}$ and define

$$r = \max\{s \leq \min\{k,l\} \, | i_t = j_t \text{ if } t \leq s\},$$

where, by definition, $i_0 = j_0 = 0$. Then, we find that

$$d(x,y) = k + l - 2r. \tag{1}$$

We denote by γ^+ and β^+ the geodesic rays $\gamma(\mathbb{N})$ and $\beta(\mathbb{N})$ respectively. Because of the choices made in the labeling, we find that the vertices of γ^+ and β^+ are labeled by the sequences

$$\begin{array}{l} \gamma^+ = x_0, x_{0,1}, x_{0,1,1}, \dots . x_{0,1,1,\dots,1}, \dots; \\ \beta^+ = x_0, x_{0,N}, x_{0,N,1}, \dots . x_{0,N,1,\dots,1}, \dots; \end{array}$$

We define a map $\psi : \Gamma \to \Gamma$ by

$$\psi(x) = \begin{cases} x_{0,N,i_2,\dots,i_k} & \text{if } x = x_{0,1,i_2,\dots,i_k} \\ x_{0,1,i_2,\dots,i_k} & \text{if } x = x_{0,N,i_2,\dots,i_k} \\ x & \text{otherwise} \end{cases}. \tag{2}$$

It follows from formula (1) that $\psi \in \mathrm{Aut}(\Gamma)$. Moreover, by construction, $\psi(\gamma^+) = \beta^+$ and $\psi|_{\gamma\cap\beta} = \mathrm{Id}$, so that $\psi(\gamma) = \beta$.

Let us assume now that $\gamma \cap \beta$ is not empty and finite. In this case this intersection is a geodesic segment (possibly containing a unique vertex) and we loose no generality by assuming $\gamma(n) = \beta(n)$ if and only if $a \leq n \leq 0$, for some $a \leq 0$. We label the vertices of Γ, starting from $x_0 := \gamma(0) = \beta(0)$, in the same way we did before, assuming again that the vertices of $\gamma^+ := \gamma(\mathbb{N})$ are labeled by $x_0, x_{0,1}, x_{0,1,1}, \dots . x_{0,1,1,\dots,1}, \dots$ and the vertices of $\beta^+ := \beta(\mathbb{N})$ by $x_0, x_{0,N}, x_{0,N,1}, \dots . x_{0,N,1,\dots,1}, \dots$. The map ψ defined as in (2), is again an automorphism and $\psi(\gamma^+) = \beta^+$. But $\psi(\gamma) \cap \beta$ is the geodesic ray $\beta(\mathrm{I})$, $\mathrm{I} = \{n \in \mathbb{Z} | n \geq a\}$ so we are in the situation of the first case.

At last, we consider the case when $\gamma \cap \beta = \emptyset$. Let α_1 be the (unique) geodesic segment joining γ to β with endpoints x_0 and y_0 in γ and β respectively. We write $\gamma = \gamma^+ \cup \gamma^-$ and $\beta = \beta^+ \cup \beta^-$ with $\gamma^+ \cap \gamma^- = x_0$ and $\beta^+ \cap \beta^- = y_0$. Then, $\alpha := \gamma^+ \cup \alpha_1 \cup \beta^+$ is a geodesic intersecting β in the ray β^+. Applying the first case, we find an automorphism ψ_1 such that $\psi_1(\alpha) = \beta$. But this implies that $\psi_1(\gamma^+) \subset \beta$, and so $\psi_1(\gamma) \cap \beta$ is a geodesic ray. Again, we find an automorphism ψ_2 such that $\psi_2 \circ \psi_1(\gamma) = \beta$. ■

Proposition 4 *Given maximal geodesics γ and β in an homogeneous tree Γ, G_β and G_γ are conjugated in* $\mathrm{Aut}(\Gamma)$.

Proof. A reflection ϕ lies in $G_\gamma \Leftrightarrow \phi(x) = x, \forall\, x \in \gamma$. By the preceding lemma, there is an automorphism ψ such that $\psi(\gamma) = \beta$, therefore $\forall x \in \beta$, there is $y \in \gamma$ such that $y = \psi(x)$. Since $\phi(x) = x$, it follows that

$$\phi\left(\psi^{-1}(y)\right) = \psi^{-1}(y) \Rightarrow \psi \circ \phi \circ \psi^{-1}(y) = y,$$

hence $(\psi \circ \phi \circ \psi^{-1})$ fixes the geodesic β pointwise. Since $(\psi \circ \phi \circ \psi^{-1})^2 = \mathrm{Id}$, we have that G_β is conjugated to G_γ by $\psi \in \mathrm{Aut}(\Gamma)$. ■

Corollary 5 *Given maximal geodesics γ and β in an homogeneous tree Γ, the subgroups $\langle G_\gamma\rangle$ and $\langle G_\beta\rangle$ are conjugated in* $\mathrm{Aut}(\Gamma)$.

Proof. Follows immediately from proposition 4. ■

Remark 6 *The choice for the fixed points of a reflection carries a certain amount of arbitrariness. Indeed, with the choice we made, there are no reflections at all in an homogeneous tree of odd degree. This situation may be avoided if, for odd degree trees, we consider the set of fixed points to be maximal sub-trees with fixed odd degree.*

3 The Index of $\overline{\langle\mathcal{R}\rangle}$

The main question we are asked to answer is whenever every automorphism of Γ may be described as a product of reflections. The following proposition asserts that not every automorphism may be produced by reflections.

Proposition 7 *Let $\phi_1, \phi_2, ..., \phi_n$ be reflections in a tree Γ and $\varphi = \phi_1 \circ \phi_2 \circ ... \circ \phi_n$. Then $d_\varphi(x) \equiv 0 \mathrm{mod} 2$ for every $x \in \Gamma$, where $d_\varphi := d(x, \varphi(x))$ is the* displacement function *of φ.*

Proof. Given a reflection ϕ in the geodesic γ and a vertex $x \in \Gamma$, the (unique) vertex $x_0 \in \gamma$ such that $d(x_0, x) = d(x, \gamma)$ is the middle point of the geodesic segment $[x, \phi(x)]$ joining x to $\phi(x)$. Since $d(x, x_0) = d(\phi(x), x_0)$ and $d(x, \phi(x)) = d(x, x_0) + d(x_0, \phi(x))$, we find that $d_\phi(x) = 2d(x, x_0) \equiv 0 \mathrm{mod} 2$. Given reflections ϕ_1 and ϕ_2, $[x, \phi_2(x)] \cup [\phi_2(x), \phi_1 \circ \phi_2(x)]$ is a path joining x to $\phi_1 \circ \phi_2(x)$. We consider parametrizations $\gamma_1 : [-n, 0] \to \Gamma$ and $\gamma_2 : [0, m] \to \Gamma$ of the segments $[x, \phi_2(x)]$ and $[\phi_2(x), \phi_1 \circ \phi_2(x)]$ respectively, with

$$\gamma_1(0) = \gamma_2(0) = \phi_2(x).$$

Since $d(x, \phi_2(x))$ and $d(\phi_2(x), \phi_1 \circ \phi_2(x))$ are even, we have $n = 2k$ and $m = 2l$. But

$$[x, \phi_2(x)] \cap [\phi_2(x), \phi_1 \circ \phi_2(x)] = \gamma_1([-r, 0]) = \gamma_2([0, r]).$$

Then, $\gamma : [-2k + r, 2l - r] \to \Gamma$ defined by

$$\gamma(i) = \begin{cases} \gamma_1(i - r) & \text{if } i \leq 0 \\ \gamma_2(i + r) & \text{if } i \geq 0 \end{cases}$$

is a geodesic ray joining x to $\phi_1 \circ \phi_2(x)$ with length $2(l + k - r)$, and it follows that

$$d(x, \phi_1 \circ \phi_2(x)) = 2(l + k - r) \equiv 0 \mathrm{mod} 2.$$

The general case can be proved by induction, in the same faction. ■

This proposition says that automorphisms with odd displacement function can not be produced by reflections. The most we can expect is to produce the automorphisms with even displacement function.

This is not a bad situation, since the subgroup

$$\mathrm{Aut}^+(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) \,|\, d_\varphi(x) \equiv 0 \mathrm{mod} 2 \text{ for every } x \in \Gamma\}$$

is a subgroup of index 2 in $\mathrm{Aut}(\Gamma)$ ([3, Proposition 1]). We will prove that this expectation is not vain: the closure of the group generated by reflections is the subgroup $\mathrm{Aut}^+(\Gamma)$, if $\mathrm{Degree}(\Gamma) \equiv 0 \mathrm{mod} 4$ (Corollary 15). Moreover, the need to consider the closure of such a group is unavoidable, and this fact gives an indication about how much our definition of reflection is restrictive, when compared to the one adopted in [2].

We start proving that, given an isometry $\psi : \Gamma \to \Gamma$ that fixes a point of Γ, its action on the vertices adjacent to the given fixed point may be produced by reflections (Proposition 9). Both in the proposition as in the lemma that precedes it, the hypothesis that $\mathrm{Degree}(\Gamma) \equiv 0 \mathrm{mod} 4$ is essential. We assume the labeling introduced in the proof of Lemma 3.

Lemma 8 *Let* Γ *be an homogenous tree with* $\text{Degree}(\Gamma) = 4k$*,* $x_0 \in \Gamma$ *and* $x_{0,1}, x_{0,2}, ..., x_{0,4k}$ *the* $4k$*-vertices of* Γ *adjacent to* x_0*. Let* ψ_{ij} *be an isometry such that, for a given pair of indices* i, j

$$\begin{aligned} \psi_{ij}(x_0) &= x_0, \\ \psi_{ij}(x_{0,i}) &= x_{0,j}, \\ \psi_{ij}(x_{0,j}) &= x_{0,i}, \\ \psi_{ij}(x_{0,n}) &= x_{0,n} \text{ for } n \neq i, j. \end{aligned}$$

Then there are reflections $\phi_0, \phi_1, \phi_2, ..., \phi_r$ *such that*

$$\psi_{ij}(x_{0,n}) = \phi_0 \circ \phi_1 \circ \phi_2 \circ ... \circ \phi_r(x_{0,n})$$

for every $n \in \{1, 2, ..., 4k\}$*.*

Proof. A reflection ϕ such that $\phi(x_0) = x_0$ acts as a permutation in the set $S(x_0, 1) := \{x_{0,1}, ..., x_{0,4k}\}$ of vertices adjacent to x_0. Since its set of fixed points is a maximal geodesic containing x_0, it must fix exactly two of the vertices in $S(x_0, 1)$. Moreover, on the remaining vertices of $S(x_0, 1)$, it acts as a product of disjoint transposition, involving all the remaining symbols. To put it explicitly, we may write $\{i_1, j_1, i_2, j_2, ..., i_{2k}, j_{2k}\} = \{1, ..., 4k\}$ in such a way that $i = i_1, j = j_1$. Moreover, for any ordering

$$\{i_1, j_1, i_2, j_2, ..., i_{2k}, j_{2k}\} = \{1, ..., 4k\}$$

of the indices, there is a reflection ϕ such that

$$\begin{aligned} \phi(x_{0,i_r}) &= x_{0,i_r},\ \phi(x_{0,j_r}) = x_{0,j_r}, \\ \phi(x_{0,i_s}) &= x_{0,j_s},\ \phi(x_{0,j_s}) = x_{0,i_s}, \text{ for } s \neq r. \end{aligned}$$

So, for given $j_1, j_2, l_1, l_2 \in \{1, ..., 4k\}$ distinct, there are reflections ϕ_j and ϕ_l that fix x_0 and satisfy

$$\begin{aligned} \phi_j(x_{0,j_1}) &= x_{0,j_1},\ \phi_j(x_{0,j_2}) = x_{0,j_2}, \\ \phi_j(x_{0,l_1}) &= x_{0,l_2},\ \phi_j(x_{0,l_2}) = x_{0,l_1}, \\ \phi_l(x_{0,j_1}) &= x_{0,j_2},\ \phi_l(x_{0,j_2}) = x_{0,j_1}, \\ \phi_l(x_{0,l_1}) &= x_{0,l_1},\ \phi_l(x_{0,l_2}) = x_{0,l_2}, \\ \phi_j(x_{0,i}) &= \phi_l(x_{0,i}) \text{ for every } i \neq j_1, j_2, l_1, l_2. \end{aligned} \tag{3}$$

Considering the restriction $\phi_j \circ \phi_l \mid_{S(x_0,1)}$, and expressing the restriction of ϕ_j and ϕ_l as product of transpositions in the symmetric group, it is immediate to verify that

$$\phi_j \circ \phi_l \mid_{S(x_0,1)} = (x_{0,j_1} x_{0,j_2})(x_{0,l_1} x_{0,l_2}),$$

that is, the product of two disjoint transpositions $(x_{0,j_1} x_{0,j_2})$ and $(x_{0,l_1} x_{0,l_2})$. Now we order de indices

$$\{i_1, j_1, i_2, j_2, ..., i_{2k}, j_{2k}\} = \{1, ..., 4k\}$$

in such a way that

$$\begin{aligned} \phi(x_{0,i_1}) &= x_{0,i_1},\ \phi(x_{0,j_1}) = x_{0,j_1;} \\ \phi(x_{0,i_l}) &= x_{0,j_l},\ \phi(x_{0,j_l}) = x_{0,i_l} \text{ if } l \neq 1 \end{aligned}$$

and choose reflections $\phi_{i_1}, ..., \phi_{i_{2k}}$ such that

$$\phi_{i_1} \circ \phi_{i_2} \circ ...\phi_{i_{2k}} \mid_{S(x_0,1)} = (x_{0,i_1} x_{0,j_1})(x_{0,i_2} x_{0,j_2}) ... (x_{0,i_{2k}} x_{0,j_{2k}}). \qquad (4)$$

This can be done by taking reflections as in 3 such that

$$\begin{aligned} \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}\left(x_{i_{2l+1}}\right) &= x_{j_{2l+1}},\ \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}\left(x_{i_{2l+2}}\right) = x_{j_{2l+2}}, \\ \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}\left(x_{j_{2l+1}}\right) &= x_{i_{2l+1}},\ \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}\left(x_{j_{2l+2}}\right) = x_{i_{2l+2}}, \\ \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}(x_{i_r}) &= x_{i_r},\ \phi_{i_{2l+1}} \circ \phi_{i_{2l+2}}(x_{j_r}) = x_{j_r} \text{ if } r \neq 2l+1, 2l+2. \end{aligned}$$

Observe that all the $\{x_{0,1}, ..., x_{0,4k}\}$ are listed one and only one time in the right side of equation 4 and this is possible only because we are assuming Degree $(\Gamma) \equiv 0 \mathrm{mod} 4$.

We take now a reflection ϕ_0 such that

$$\phi_0 \mid_{S(x_0,1)} = (x_{0,i_2} x_{0,j_2}) ... (x_{0,i_{2k}} x_{0,j_{2k}})$$

and find that

$$\begin{aligned} \phi|_{S(x_0,1)} &= \phi_0 \circ [\phi_{i_1} \circ \phi_{i_2} \circ ...\phi_{i_{2k}}] \mid_{S(x_0,1)} \\ &= [(x_{0,i_2} x_{0,j_2}) ... (x_{0,i_{2k}} x_{0,j_{2k}})] [(x_{0,i_1} x_{0,j_1})(x_{0,i_2} x_{0,j_2}) ... (x_{0,i_{2k}} x_{0,j_{2k}})] \\ &= (x_{0,i_1} x_{0,j_1}) \\ &= \psi_{ij}|_{S(x_0,1)}, \end{aligned}$$

since we are assuming $i = i_1$ and $j = j_1$. ■

Proposition 9 *In the same conditions as in the previous lemma, given an isometry* $\psi : \Gamma \longrightarrow \Gamma$, *such that* $\psi(x_0) = x_0$, *there are reflections* $\phi_1, \phi_2, ..., \phi_l$ *fixing the vertex* x_0, *such that* $\phi_1 \circ \phi_2 \circ ... \circ \phi_l (x_{0,n}) = \psi(x_{0,n})$, *for every* $n \in \{1, 2, ..., 4k\}$.

Proof. The restriction of any isometry to $S(x_0, 1) = \{x_{0,1}, x_{0,2}, ..., x_{0,4k}\}$ acts as a permutation. Since any permutation may be expressed as a product of transposition and, by the previous lemma, any transposition of $S(x_0, 1)$ may be produced by reflections, the restriction of ψ to $S(x_0, 1)$ may be produced by reflections. ■

The next lemma is the first step needed to extend the proposition 9, passing from the sphere $S(x_0, n)$ to $S(x_0, n+1)$.

Lemma 10 *Given vertices* x_0, v_1, v_2 *of a tree* Γ, *with* $\text{Degree}(\Gamma) \equiv 0 \text{mod} 4$, *such that* $d(v_1, x_0) = d(v_2, x_0) = R$, *let* $w_1^1, w_2^1, w_1^2, w_2^2$ *be vertices satisfying:*

i) $d(w_j^i, v_i) = 1$, *i.e.,* w_j^i *and* v_i *are adjacent, for* $i, j = 1, 2$;

ii) $d(w_j^i, x_0) > d(v_i, x_0)$, *i.e.,* $v_i \in [x_0, w_j^i]$ *for* $i, j = 1, 2$.

Then, there are reflections ϕ *and* ψ *such that:*

1. $\phi(x_0) = \psi(x_0) = x_0$;
2. $\phi \circ \psi(w_1^i) = w_2^i$ *and* $\phi \circ \psi(w_2^i) = w_1^i$, *for* $i = 1, 2$;
3. $\phi \circ \psi(x) = x$ *if* $d(x, x_0) \leq R + 1$ *and* $x \neq w_j^i$, *for* $i, j = 1, 2$.

Proof. Since $d(v_1, x_0) = d(v_2, x_0) = R$, it follows that $d(v_1, v_2)$ is even. Let x_1 be the middle point of the geodesic segment $[v_1, v_2]$. The geodesic segments $[x_1, v_1]$ and $[x_1, v_2]$ are defined respectively by the sequences of vertices $x_1 = x_1^1, x_2^1, ..., x_{l-1}^1, x_l^1 = v_1$ and $x_1 = x_1^2, x_2^2, ..., x_{l-1}^2, x_l^2 = v_2$, where $d(x_i^j, x_{i+1}^j) = 1$, for every $j = 1, 2, i = 1, ..., l-1$. We let γ be a geodesic containing both x_0 and x_1 but neither of x_2^1 and x_2^2. Assuming again the identification between the symmetric group and the action of an isometry in the vertices adjacent to a fixed point, we let ϕ be the reflection in γ defined as follows:

$$\phi\,|_{S(x_0,R+1)} = (\varphi)\left(x_2^1 x_2^2\right)\left(x_3^1 x_3^2\right) ... \left(x_{l-1}^1 x_{l-1}^2\right)(v_1 v_2)\left(w_1^1 w_2^2\right)\left(w_2^1 w_1^2\right)$$

where φ is any product of transpositions of the remaining vertices. In a similar way, we define ψ as a reflection such that

$$\psi\mid_{S(x_0,R+1)}=(\varphi)\left(x_2^1x_2^2\right)\left(x_3^1x_3^2\right)...\left(x_{l-1}^1x_{l-1}^2\right)(v_1v_2)\left(w_1^1w_1^2\right)\left(w_2^1w_2^2\right).$$

Observe that ϕ and ψ differ only in the last two transpositions, so that

$$\phi\circ\psi=\left(w_1^1w_2^2\right)\left(w_2^1w_1^2\right)\left(w_1^1w_1^2\right)\left(w_2^1w_2^2\right)=\left(w_2^1w_1^1\right)\left(w_2^2w_1^2\right)$$

that is,

$$\begin{aligned}\phi\circ\psi\left(w_1^i\right) &= w_2^i,\ \phi\circ\psi\left(w_2^i\right)=w_1^i;\\ \phi\circ\psi\left(x\right) &= x \text{ if } d\left(x,x_0\right)\leq R+1 \text{ and } x\neq w_j^i\end{aligned}$$

for $i=1,2$. ■

The preceding lemma assures that we can transpose vertices in a sphere, maintaining all other vertices in the closed metric ball fixed. However, those transpositions are done simultaneously in two pairs of distinct vertices. In order to extend Proposition 9 from a ball of radius R to a ball of radius $R+1$, we need to transpose only one pair of chosen vertices. This is done in the following lemma, by considering a sufficiently large ball, that contains the given ball of radius R.

Lemma 11 *In the same conditions as before, let v,x_0 be vertices of Γ such that $d\left(v,x_0\right)=R$. Let w_1 and w_2 be vertices satisfying:*

i $d\left(w_i,v\right)=1$, *i.e., w_i and v are adjacent, for $i=1,2$;*

ii $d\left(w_i,x_0\right)>d\left(v,x_0\right)$, *i.e., $v\in\left[x_0,w_i\right]$, for $i=1,2$.*

Then, there are reflections ϕ and ψ such that:

1. $\phi\circ\psi\left(w_1\right)=w_2$ *and* $\phi\circ\psi\left(w_2\right)=w_1$;
2. $\phi\circ\psi\left(x\right)=x$ *if* $d\left(x,x_0\right)\leq R+1$ *and* $x\neq w_i$, *for* $i=1,2$.

Proof. Let y_0 be a vertex of Γ such that

$$\begin{aligned}d\left(y_0,x_0\right) &= 1\\ d\left(v,y_0\right) &> d\left(v,x_0\right)\end{aligned}$$

Denote $v = v_1, w_1 = w_1^1, w_2 = w_2^1$ and let v_2 be a vertex such that y_0 is the middle point of the geodesic segment $[v_1, v_2]$. We choose now w_1^2 and w_2^2 satisfying

$$\begin{aligned} d\left(w_1^2, v_2\right) &= d\left(w_2^2, v_2\right) = 1 \\ d\left(w_i^2, y_0\right) &> d\left(v_2, y_0\right), \text{ for } i = 1, 2. \end{aligned}$$

By the previous lemma, there are reflections ϕ and ψ such that

$$\begin{aligned} \phi\left(y_0\right) &= \psi\left(y_0\right) = y_0; \\ \phi \circ \psi\left(w_1^i\right) &= w_2^i, \ \ \phi \circ \psi\left(w_2^i\right) = w_1^i \text{ for } i = 1, 2; \\ \phi \circ \psi\left(x\right) &= x \text{ if } d\left(x, y_0\right) \leq R + 2 \text{ and } x \neq w_j^i, \ i, j = 1, 2. \end{aligned}$$

By construction, we find that $B\left(x_0, R+1\right) \subset B\left(y_0, R+2\right)$ and also

$$d\left(w_i^2, x_0\right) = d\left(w_i^2, y_0\right) + d\left(y_0, x_0\right) = R + 3.$$

It follows that

$$\begin{aligned} \phi \circ \psi\left(w_1\right) &= w_2 \ , \ \phi \circ \psi\left(w_2\right) = w_1; \\ \phi \circ \psi\left(x\right) &= x \text{ if } d\left(x, x_0\right) \leq R + 1 \text{ and } x \neq w_i, \ \text{ for } i = 1, 2. \end{aligned}$$

■

We are able now to make the inductive step:

Proposition 12 *Let x_0 be a vertex of Γ and $\varphi : \Gamma \longrightarrow \Gamma$ an isometry such that $\varphi\left(x_0\right) = x_0$. Then, for any $R \geq 1$, there are reflections $\phi_1, \phi_2, ..., \phi_l$ such that $\varphi\mid_{B(x_0,R)} = \phi_1 \circ \phi_2 \circ ... \circ \phi_l \mid_{B(x_0,R)}$.*

Proof. We prove by induction on R.
The case $R = 1$ was proved in Proposition 9.
Assuming it holds for $r < R$, there are reflections $\phi_1, \phi_2, ..., \phi_m$ such that

$$\varphi\mid_{B(x_0,R-1)} = \phi_1 \circ \phi_2 \circ ... \circ \phi_m \mid_{B(x_0,R-1)} .$$

Let v be a vertex with $d\left(v, x_0\right) = R - 1$ and let $w_1, w_2, ..., w_{4k-1}$ be vertices adjacent to v with $d\left(w_i, x_0\right) = R$, for $i = 1, 2, ..., 4k - 1$. Let $v' = \varphi\left(v\right)$ and $w_i' = \varphi\left(w_i\right)$. Since $d\left(v, x_0\right) = R - 1$, we have that

$$v' = \phi_1 \circ \phi_2 \circ ... \circ \phi_m\left(v\right).$$

Moreover,

$$\phi_1 \circ \phi_2 \circ ... \circ \phi_m (w_i) = w_j^{'} \text{ for some } j \in \{1, 2, ..., 4k - 1\}.$$

To conclude the proof, we just note again that any permutation may be produced by transpositions and then apply the previous lemma. ∎

We consider on $\mathrm{Aut}(\Gamma)$ the topology of uniform convergence over compact sets. Since Γ is discrete, this is equivalent to say that a sequence $\phi_n \in \mathrm{Aut}(\Gamma)$ converges to ϕ_0 if and only if, for any finite subset K of vertices of Γ, there is an $N \geq 0$ such that, for every $x \in K$, $\phi_n(x) = \phi_0(x)$ if $n \geq N$. With this topology, for a given subgroup $G \subset \mathrm{Aut}(\Gamma)$, we define $\overline{G}$ to be the minimal subgroup of $\mathrm{Aut}(\Gamma)$ containing the topological closure of G. If $G = \langle \mathcal{R} \rangle$ is generated by a set $\mathcal{R}$, a set of generators of $\overline{\langle \mathcal{R} \rangle}$ is given by sequences $(\phi_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{R}$ such that

$$\phi(x) = \lim_{n \longrightarrow \infty} \phi_n \circ \phi_{n-1} \circ ... \circ \phi_1(x)$$

exist for every vertex $x \in \Gamma$ and such that $\phi(x) \in \mathrm{Aut}(\Gamma)$.

With those definitions and terminology, the previous proposition can be restated as follows:

Proposition 13 *Let Γ be an homogeneous tree with* $\mathrm{Degree}(\Gamma) \equiv 0 \mathrm{mod} 4$. *Given $x_0 \in \Gamma$, denote by $\mathcal{R}_{x_0}$ the set of reflections that fix the vertex x_0 and let $\mathrm{Aut}(\Gamma)_{x_0}$ be the stabilizer of x_0 in $\mathrm{Aut}(\Gamma)$. Then $\overline{\langle \mathcal{R}_{x_0} \rangle} = \mathrm{Aut}(\Gamma)_{x_0}$.*

Proof. It follows from proposition 12, since any finite set of vertices is contained in ball $B(x_0, R)$, for R sufficiently large. ∎

We remember that $\mathrm{Aut}^+(\Gamma)$ is the set of automorphisms ϕ with even displacement function d_ϕ.

Theorem 14 *Let Γ be an homogeneous tree with* $\mathrm{Degree}(\Gamma) \equiv 0 \mathrm{mod} 4$. *Let $\mathcal{R} = \{\phi \in Aut(\Gamma) \,|\phi \text{ is a reflection}\}$. Then $\overline{\langle \mathcal{R} \rangle} = \mathrm{Aut}^+(\Gamma)$.*

Proof. Since every reflection has even displacement function (Lemma 7), $\overline{\langle \mathcal{R} \rangle} \subseteq \mathrm{Aut}^+(\Gamma)$. We shall prove the other inclusion.

If $\phi \in \mathrm{Aut}^+(\Gamma)$ has a fixed point x_0, then $\phi \in \mathrm{Aut}(\Gamma)_{x_0} = \overline{\langle \mathcal{R}_{x_0} \rangle} \subset \overline{\langle \mathcal{R} \rangle}$. So, we assume that ϕ has no fixed points. Since $d_\phi(x) \equiv 0 \mathrm{mod} 2$ for any vertex x, there is a middle point p of the segment $[x, \phi(x)]$ and consequently, there is a reflection $\varphi \in \mathcal{R}_p$ such that $\varphi(\phi(x)) = x$. It follows that $\varphi \circ \phi \in \overline{\langle \mathcal{R} \rangle}$, since it has x as a fixed point. But φ also belongs to $\overline{\langle \mathcal{R} \rangle}$, and this implies that $\phi = \varphi \circ \varphi \circ \phi \in \overline{\langle \mathcal{R} \rangle}$, and we find that $\mathrm{Aut}^+(\Gamma) \subseteq \overline{\langle \mathcal{R} \rangle}$. ∎

Corollary 15 *If* Γ *is an homogeneous tree with* $\mathrm{Degree}(\Gamma) \equiv 0 \mathrm{mod} 4$*, the closure* $\overline{\langle\mathcal{R}\rangle}$ *of the group generated by reflections is a normal subgroup of index* 2 *in* $\mathrm{Aut}(\Gamma)$*.*

Proof. Follows from the facts that $\left[\mathrm{Aut}(\Gamma) : \mathrm{Aut}^{+}(\Gamma)\right] = 2$ and $\overline{\langle\mathcal{R}\rangle} = \mathrm{Aut}^{+}(\Gamma)$. ■

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