ALMOST SUMMING MAPPINGS

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ABSTRACT. We introduce a general definition of almost p-summing mappings and give several concrete examples of such mappings. Some known results are considerably generalized and we present various situations in which the space of almost p-summing multilinear mappings coincides with the whole space of continuous multilinear mappings.

1. INTRODUCTION

The rapid development of the theory of absolutely summing linear mappings has lead to the study of innumerous new classes of multilinear mappings and polynomials between Banach spaces (see [10],[7],[3],[1]). Recently, Botelho [3] and Botelho-Braunss-Junek [2] introduced the concept of almost p-summing multilinear mappings and gave the first examples and properties of such mappings. The recent work of Matos [8], concerning absolutely summing arbitrary mappings, turns natural to ask whether it is possible to follow the same line of thought with almost p-summing mappings. In this paper we will present a more general definition of almost p-summing mappings, several examples and a natural version of a Dvoretzky-Rogers Theorem for this kind of applications. It will be shown that almost p-summing multilinear mappings are much more common than it was known until now. For example, we prove that every continuous n-linear mapping from $C(K)\times ...\times C(K)$ into a Banach space F is almost 2-summing, generalizing a recent result obtained in [2]. This paper also analyzes the connections of almost p-summing mappings and type/cotype and provides various examples of analytic almost p-summing mappings.

2. ABSOLUTELY SUMMING MAPPINGS

Throughout this paper $E, E_1, ..., E_n, F$ will stand for Banach spaces. For $p\in [1,\infty[$, the linear space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that

$$\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty}\|x_j\|^p)^{\frac{1}{p}} < \infty$$

will be denoted by $l_p(E)$. We will denote by $l_p^w(E)$ the linear subspace of $l_p(E)$ formed by the sequences $(x_j)_{j=1}^{\infty}$ in E such that $(<\varphi, x_j>)_{j=1}^{\infty}\in l_p(\mathbb{K})$, for every continuous linear functional $\varphi : E\to\mathbb{K}$. We also define $\|.\|_{w,p}$ in $l_p^w(E)$ by

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} := \sup_{\varphi\in B_{E'}} (\sum_{j=1}^{\infty} |<\varphi, x_j>|^p)^{\frac{1}{p}}.$$

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The linear subspace of $l_p^w(E)$ of all sequences $(x_j)_{j=1}^{\infty} \in l_p^w(E)$, such that

$$\lim_{m\to\infty} \|(x_j)_{j=m}^{\infty}\|_{w,p} = 0$$

will be denoted by $l_p^u(E)$. The sequences in $l_p^u(E)$ are called unconditionally p-summable.

The multilinear theory of absolutely summing mappings was first sketched by Pietsch in [14] and has been broadly explored (see [11], [10], [6]). The next definition can be found in [10].

Definition 1. *A multilinear mapping $T : E_1\times...\times E_n \to F$ is absolutely $(p; q_1, ..., q_n)$-summing if*

$$(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^{\infty} \in l_p(F)$$

for every $(x_j^{(s)})_{j=1}^{\infty} \in l_{q_s}^w(E)$, $s = 1, ..., n$. An n-homogeneous polynomial $P : E \to F$ is absolutely $(p; q)$-summing if

$$(P(x_j))_{j=1}^{\infty} \in l_p(F)$$

whenever $(x_j)_{j=1}^{\infty} \in l_q^w(E)$.

It is worth observing that, in Definition 1, there is no difference if we consider $l_{q_s}^u(E)$ $(l_q^u(E))$ instead of $l_{q_s}^w(E)$ $(l_q^w(E))$ (see [10, Proposition 2.4] for polynomials, and the multilinear case is analogous).

The following well known characterization can be found in [4, Theorem 1.2(ii)], and is sometimes useful.

Theorem 1. *Let $T : E_1 \times ... \times E_n \to F$ be a multilinear mapping. The following statements are equivalent:*

(1) T is absolutely $(p; q_1, ..., q_n)$-summing.

(2) There exists $L > 0$ such that for every natural k and any $x_j^{(l)} \in E_l$,

$$(\sum_{j=1}^{k} \|T(x_j^{(1)}, ..., x_j^{(n)})\|^p)^{\frac{1}{p}} \le L\|(x_j^{(1)})_{j=1}^{k}\|_{w,q_1}...\|(x_j^{(n)})_{j=1}^{k}\|_{w,q_n}. \tag{2.1}$$

The least $L > 0$ for which inequality (2.1) always holds defines a norm for the space of absolutely $(p; q_1, ..., q_n)$-summing multilinear mappings. This norm will be denoted by $\|.\|_{as(p;q)}$. A characterization for n-homogeneous polynomials is analogous.

Inspired on the work of Matos [9], we introduce the following concept, which generalizes Definition 1, as we will see later.

Definition 2. *An arbitrary mapping f:$E \to F$ is absolutely (p, q)-summing at a if there exist $M_a > 0$, $\delta_a > 0$ and $r_a > 0$ so that*

$$\sum_{j=1}^{k} \|f(a + x_j) - f(a)\|^p \le M_a\|(x_j)_{j=1}^{k}\|_{w,q}^{r_a}$$

for all k and $\|(x_j)_{j=1}^{k}\|_{w,q} < \delta_a$.

Theorem 2. *If F has cotype q, E is an $\mathcal{L}_{\infty,\lambda}$ space and $f : E \to F$ is analytic at a, then f is absolutely $(q; 2)$-summing at a.*

Proof. Since f is analytic at a, there are $C \geq 0$ and $c > 0$ such that

$$\|\frac{1}{k!}\overset{\wedge}{d^k} f(a)\| \leq Cc^k \text{ for every } k.$$

A recent result of D. Perez (see [13]) states that whenever each E_j is an $\mathcal{L}_{\infty,\lambda_j}$ space, every continuous n-linear ($n \geq 2$) mapping T, from $E_1 \times ... \times E_n$ into $\mathbb{K}$, is absolutely $(1; 2, ..., 2)$-summing and

$$\|T\|_{as(1;2,...,2)} \leq K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j. \tag{2.2}$$

Using the polynomial version of this result, it is not hard to prove that (see [12, Theorem 4]) whenever F has finite cotype q, every bounded n-homogeneous ($n \geq 2$) polynomial $P : E \to F$ is absolutely $(q; 2)$-summing and $\|P\|_{as(q;2)} \leq C_q(F) K_G 3^{\frac{n-2}{2}} \|P\| \lambda^n$, where $C_q(F)$ and K_G are the cotype's constant of F and Grothendieck's constant, respectively.

For $n = 1$, we still have $\mathcal{L}(E; F) = \mathcal{L}_{as(q;2)}(E; F)$, which is a particular case of a result due to Dubinsky-Pełczyński-Rosenthal (case $q = 2$) and Maurey (case $q > 2$) (see [5, Theorem 11.14 (a) and (b)]). So, for every natural n, there exist positive D and D_1 so that

$$\|\frac{1}{k!}\overset{\wedge}{d^k} f(a)\|_{as(q;2)} \leq D_1 D^k \|\frac{1}{k!}\overset{\wedge}{d^k} f(a)\|.$$

Hence, if δ_a is the radius of convergence of f around a, then, whenever $(x_j)_{j=1}^m$ is such that $\|(x_j)_{j=1}^m\|_{w,1} \leq \min\{\frac{1}{2D}, \delta_a\}$, we have

$$\begin{aligned}
(\sum_{j=1}^{m} \|f(a + x_j) - f(a)\|^q)^{\frac{1}{q}} &= \sum_{j=1}^{m}(\|\sum_{k=1}^{\infty} \frac{1}{k!}\overset{\wedge}{d^k} f(a)(x_j)\|^q)^{\frac{1}{q}} \\
&\leq \sum_{k=1}^{\infty}[\sum_{j=1}^{m} \|\frac{1}{k!}\overset{\wedge}{d^k} f(a)(x_j)\|^q]^{\frac{1}{q}} \\
&\leq \sum_{k=1}^{\infty} \|\frac{1}{k!}\overset{\wedge}{d^k} f(a)\|_{as(q;2)} \|(x_j)_{j=1}^m\|_{w,2}^k \\
&\leq D_1 \|(x_j)_{j=1}^m\|_{w,2} \sum_{k=1}^{\infty} \frac{D^k}{2^{k-1} D^{k-1}} = 2DD_1 \|(x_j)_{j=1}^m\|_{w,2}.
\end{aligned}$$

□

Several other results concerning absolutely summing analytic mappings can be found in [6] and [12].

Proposition 1. *If $f : E \to F$ is absolutely $(p; q)$-summing at a, then f is so that $(f(a + x_j) - f(a))_{j=1}^\infty \in l_p(F)$ whenever $(x_j)_{j=1}^\infty$ is unconditionally q-summable.*

Proof. Let f be $(p; q)$-summing at a. For any $(x_j)_{j=1}^\infty \in l_p^u(E)$, we have

$$\lim_{k,m\to\infty}(\sum_{j=k}^{m} \|f((a + x_j) - f(a)\|^p)^{\frac{1}{p}} \leq \lim_{k,m\to\infty} C_a \|(x_j)_{j=k}^m\|_{w,p}^{r_a} = 0$$

and, by the completeness of $l_p(F)$, we obtain $(f(a + x_j) - f(a))_{j=1}^\infty \in l_p(F)$.□

An immediate outcome of Proposition 1 is that Definition 2 applied for n-homogeneous polynomials and the usual definition of absolutely (p, q)-summing polynomials coincides at $a = 0$. In order to prove that Definition 2 for n-linear mappings actually generalizes the standard definition (Definition 1) of absolutely $(p; q_1, ..., q_n)$-summing multilinear mappings for $q_1 = ... = q_n = q$, we need the following Lemma, which is a simple consequence of the Open Mapping Theorem.

Lemma 1. *$l_q^u(E_1 \times ... \times E_n)$ is isomorphic to $l_q^u(E_1) \times .... \times l_q^u(E_n)$.*

Proposition 2. *An n-linear mapping T is $(p; q, ..., q)$-summing in the usual sense if, and only if, it is absolutely $(p; q)$-summing at the origin in the sense of Definition 2.*

Proof. Consider an absolutely $(p; q)$-summing (in the sense of Definition 2, at the origin) n-linear mapping, $T : E_1 \times ... \times E_n \to F$. Then, given $(x_j^{(1)})_{j=1}^{\infty} \in l_q^u(E_1), ..., (x_j^{(n)})_{j=1}^{\infty} \in l_q^u(E_n)$, we have $(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^{\infty} \in l_q^u(E_1 \times .... \times E_n)$. Hence, by Proposition 1, $(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^{\infty} \in l_p(F)$. Thus, by the usual definition, it follows that T is absolutely $(p; q, ..., q)$-summing .

Conversely, consider an absolutely $(p; q, ..., q)$-summing n-linear mapping T in the usual meaning. Then, if $x_1^{(l)}, ..., x_k^{(l)} \in E_l, l = 1, ..., n$, we have

$$(\sum_{j=1}^{k} \|(T(x_j^{(1)}, ..., x_j^{(n)})\|^p)^{\frac{1}{p}} \leq C\|(x_j^{(1)})_{j=1}^k\|_{w,q} ....\|(x_j^{(n)})_{j=1}^k\|_{w,q}.$$

Therefore, since $l_q^u(E_1 \times .... \times E_n)$ is isomorphic to $l_q^u(E_1) \times .... \times l_q^u(E_n)$, it follows that there exists $C_1 > 0$ so that, for every k,

$$\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,q} \geq C_1(\|(x_j^{(1)})_{j=1}^k\|_{w,q} + ... + \|(x_j^{(n)})_{j=1}^k\|_{w,q})$$

and

$$\begin{aligned}\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,q}^n &\geq C_1^n(\|(x_j^{(1)})_{j=1}^k\|_{w,q} + ... + \|(x_j^{(n)})_{j=1}^k\|_{w,q})^n \\ &\geq C_1^n(\|(x_j^{(1)})_{j=1}^k\|_{w,q} ...\|(x_j^{(n)})_{j=1}^k\|_{w,q}) \\ &\geq \frac{C_1^n}{C}(\sum_{j=1}^{k} \|(T(x_j^{(1)}, ..., x_j^{(n)})\|^p)^{\frac{1}{p}}\end{aligned}$$

and so T is absolutely $(p; q)$-summing in the sense of Definition 2. □

3. Almost summing mappings

Considering the Rademacher functions $(r_j(t))_{j=1}^{\infty}$, we say that the sequence $(x_j)_{j=1}^{\infty}$ of points of E is almost unconditionally summable if $\sum_{j=1}^{\infty} r_j(t)x_j \in L_p([0, 1], E)$ for some, and then for all p, $0 < p < \infty$.

Definition 3. *(Botelho [3]) An n-linear mapping $T : E_1 \times ... \times E_n \to F$ is said to be almost $(p_1, ..., p_n)$-summing if there exists $C \geq 0$ such that*

$$(\int_0^1 \|\sum_{j=1}^{k} T(x_j^{(1)}, ..., x_j^{(n)})r_j(t)\|^2 dt)^{\frac{1}{2}} \leq C\|(x_j^{(1)})_{j=1}^k\|_{w,p_1} ...\|(x_j^{(n)})_{j=1}^k\|_{w,p_n}$$

for every k and any $x_j^{(l)}$ in $E_l, l = 1, ..., n$ and $j = 1, ..., k$. An n-homogeneous polynomial $P : E \to F$ is said almost p-summing when $\overset{\vee}{P}$ is almost $(p, ..., p)$-summing. The space of all almost p-summing polynomials is denoted by $\mathcal{P}_{al,p}(^nE; F)$.

Theorem 3. *([2, Theorem 3.3])For $1 \leq p \leq 2n$ and $P \in \mathcal{P}_{al,p}({}^nE;F)$, the following properties are equivalent:*

(i) P is almost p-summing.

(ii) P maps unconditionally p-summable sequences in E into almost unconditionally summable sequences in F.

The following definition is a natural generalization of Definition 3 and allows us to give examples of analytic almost p-summing mappings.

Definition 4. *A mapping $f : E \to F$ is said to be almost p-summing at $a \in E$ if there exist $C_a > 0$, $\epsilon_a > 0$ and $r_a > 0$ such that*

$$(\int_0^1 \| \sum_{j=1}^k (f(a+x_j) - f(a))r_j(t)\|^2 dt)^{\frac{1}{2}} \leq C_a \|(x_j)_{j=1}^k\|_{w,p}^{r_a}$$

for every natural k, any $x_1, ..., x_k$ in E and $\|(x_j)_{j=1}^k\|_{w,p} < \epsilon_a$. If f is almost p-summing at every $a \in E$, we say that f is almost p-summing everywhere.

It is worth observing that if f is almost p-summing at a, then f is continuous at a. The space of all polynomials from E into F which are almost p-summing everywhere will be denoted by $\mathcal{P}_{al,p(E)}({}^nE;F)$.

Proposition 3. *If $f : E \to F$ is almost p-summing at a, then f is so that $(f(a+x_j) - f(a))_{j=1}^\infty$ is almost unconditionally summable whenever $(x_j)_{j=1}^\infty$ is unconditionally p-summable.*

Proof. Analogous to the proof of Proposition 1.

An immediate outcome of Theorem 3 and Proposition 3 is that Definitions 4 and 3 coincides for n-homogeneous polynomials and $a = 0$. The proof that Definition 4, for $a = 0$, generalizes Definition 3, for multilinear mappings and $p_1 = ... = p_n = p$, is similar to the proof of Proposition 2.

Proposition 4. *If $P \in \mathcal{P}({}^nE;F)$, then $P \in \mathcal{P}_{al,p(E)}({}^nE;F) \Leftrightarrow \overset{\vee}{P} \in \mathcal{L}_{al,p(E)}({}^nE,F)$.*

Proof. Suppose that $P \in \mathcal{P}_{al,p(E)}({}^nE;F)$. Then, by the polarization formula,

$$\overset{\vee}{P}(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - \overset{\vee}{P}(a_1, ..., a_n) =$$

$$= [\frac{1}{n!2^n} \sum_{e_i=1,-1} e_1...e_n P(e_1(a_1 + x_j^{(1)}) + ... + e_n(a_n + x_j^{(n)})] -$$

$$- [\frac{1}{n!2^n} \sum_{e_i=1,-1} e_1...e_n P(e_1a_1 + ... + e_na_n)]$$

$$= \frac{1}{n!2^n} \sum_{e_i=1,-1} e_1...e_n [P((e_1a_1 + ... + e_na_n) + (e_1x_j^{(1)} + ... + e_nx_j^{(n)})) -$$

$$- P(e_1a_1 + ... + e_na_n)].$$

For any $(x_j^{(1)})_{j=1}^k, ..., (x_j^{(n)})_{j=1}^k$, in order to simplify notation, we will write

$$A = (\int_0^1 \| \sum_{j=1}^k (\overset{\vee}{P}(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - \overset{\vee}{P}(a_1, ..., a_n))r_j(t)\|^2 dt)^{\frac{1}{2}}.$$

Lemma 1 asserts that there exists $D > 0$ so that

$$(\|(x_j^{(1)})_{j=1}^k\|_{w,p} + ... + \|(x_j^{(n)})_{j=1}^k\|_{w,p}) \leq D\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,p}$$

for every k. Now suppose

$$\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,p} < \frac{1}{D} \min_{e_i=-1,1}\{1, \epsilon_{e_1a_1+...+e_na_n}\},$$

where the $\epsilon_{e_1a_1+...+e_na_n}$ are given by Definition 4 applied to P. Then, for any choice of -1 and 1 for e_j, we have

$$\|(e_1x_j^{(1)} + ... + e_nx_j^{(n)})_{j=1}^k\|_{w,p} < \min_{e_i=-1,1}\{1, \epsilon_{e_1a_1+...+e_na_n}\}.$$

Therefore,

$$A = (\int_0^1 \|\sum_{j=1}^k \frac{1}{n!2^n} \sum_{e_i=1,-1} e_1...e_n[P((e_1a_1 + ...e_na_n) + (e_1x_j^{(1)} + ... + e_nx_j^{(n)}))-$$

$$- P(e_1a_1 + ... + e_na_n)]r_j(t)\|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{1}{n!2^n} \sum_{e_i=1,-1} (\int_0^1 \|\sum_{j=1}^k e_1...e_n[P((e_1a_1 + ... + e_na_n) + (e_1x_j^{(1)} + ... + e_nx_j^{(n)}))-$$

$$- P(e_1a_1 + ... + e_na_n)]r_j(t)\|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{1}{n!2^n} \sum_{e_i=1,-1} C_{e_1a_1+...+e_na_n}\|(e_1x_j^{(1)} + ... + e_nx_j^{(n)})_{j=1}^k\|_{w,p}^{r(e_1a_1+...+e_na_n)}$$

$$\leq \frac{1}{n!2^n} \sum_{e_i=1,-1} C_{e_1a_1+...+e_na_n}(\|(x_j^{(1)})_{j=1}^k\|_{w,p} + ... + \|(x_j^{(n)})_{j=1}^k\|_{w,p})^{r(e_1a_1+...+e_na_n)}$$

$$\leq \frac{1}{n!2^n} \sum_{e_i=1,-1} C_{e_1a_1+...+e_na_n} D^{r(e_1a_1+...+e_na_n)} \|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,p}^{r(e_1a_1+...+e_na_n)}$$

$$\leq D_1\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,p}^{\min\{r(e_1a_1+...+e_na_n)\}}$$

if $\|(x_j^{(1)}, ..., x_j^{(n)})_{j=1}^k\|_{w,p} < \delta = \frac{1}{D} \min_{e_i=-1,1}\{1, \epsilon_{e_1a_1+...+e_na_n}\}$.The converse is obvious. □

Naturally, the concepts of type and cotype give us the next Proposition.

Proposition 5. *If F has type q, then every absolutely $(q;p)$-summing mapping (at a) is almost p-summing at a. On the other hand, if F has finite cotype r, then every almost p-summing mapping (at a) is $(r;p)$-summing at a.*

Corollary 1. *If F is a Hilbert space and E is an $\mathcal{L}_\infty$ space, then every $f : E \to F$, analytic at a, is almost 2-summing at a. In particular, under the same hypothesis, every entire mapping $f : E \to F$ is almost 2-summing everywhere.*

Proof. Since $\cot F = 2$, by Proposition 2, f is absolutely $(2;2)$-summing at a. Besides, since F has type 2, then f is almost 2-summing at a, by Proposition 5.□

In order to give the other examples of analytic almost summing mappings, the next Proposition will be useful.

Proposition 6. *If f is such that there exist C, $\delta, r > 0$ so that*

$$\|(f(a + x_j) - f(a))_{j=1}^k\|_{w,1} \leq C\|(x_j)_{j=1}^k\|_{w,p}^r$$

for any natural k, every $x_1,...,x_k$ in E and $\|(x_j)_{j=1}^k\|_{w,p} < \delta$, then f is almost p-summing at a.

Proof.

$$\int_0^1 \|\sum_{j=1}^k (f(a+x_j)-f(a))r_j(t)\|^2 dt)^{\frac{1}{2}} \leq \sup_{t\in[0,1]} \|\sum_{j=1}^k (f(a+x_j)-f(a))r_j(t)\| =$$

$$= \sup_{t\in[0,1]} \sup_{\varphi\in B_{E'}} |< \varphi, \sum_{j=1}^k (f(a+x_j)-f(a))r_j(t) >|$$
$$\leq \|(f(a+x_j)-f(a))_{j=1}^k\|_{w,1} \leq C\|(x_j)_{j=1}^k\|_{w,p}^r$$

for $\|(x_j)_{j=1}^k\|_{w,p} < \delta$.□

In [3, Corollary 6.3] it is stated that regardless of the positive integer n, every absolutely $(1;2)$-summing n-homogeneous polynomial is almost 2-summing. It is worth remarking that, when f is a polynomial, $a=0$ and $p=2$, Proposition 6 is a significant improvement of [3, Corollary 6.3], since in Proposition 6 we just need a weak estimate whereas in [3, Corollary 6.3] we need a norm estimate. As we will see later in Corollary 3, the aforementioned Proposition is the key of innumerous new Coincidence Theorems which will generalize the few Coincidence Theorems known until now (see [3, Proposition 7.1],[2, Proposition 5.1]). The next Corollary give other examples of almost p-summing analytic mappings.

Corollary 2. *Let E be an $\mathcal{L}_{\infty,\lambda}$ space and F be an arbitrary Banach space. Every mapping $g: E\to F$, analytic at a, such that $dg(a)=0$ is almost 2-summing at a.*

Proof. Let C and c be such that

$$\|\frac{1}{k!}\hat{d}^k g(a)\| \leq Cc^k \text{ for every } k\geq 1.$$

Then, for any bounded linear functional φ, defined on F, we obtain

$$\|\frac{1}{k!}\hat{d}^k \varphi g(a)\| = \|\varphi\frac{1}{k!}\hat{d}^k g(a)\| \leq Cc^k\|\varphi\| \text{ for every } k\geq 1.$$

By (2.2) we have

$$\|\frac{1}{k!}\hat{d}^k \varphi g(a)\|_{as(1;2)} \leq K_G 3^{\frac{k-2}{2}}\lambda^k Cc^k\|\varphi\| \text{ for every } k\geq 2.$$

Therefore, defining δ_a as the radius of convergence of g around a, if we assume $(x_j)_{j=1}^m$ such that

$$\|(x_j)_{j=1}^m\|_{w,2} \leq \delta = \min\{\frac{1}{(2\sqrt{3}\lambda c)}, \delta_a\},$$

we obtain

$$\sum_{j=1}^{m} \mid \varphi g(a+x_j) - \varphi g(a) \mid \leq \sum_{k=2}^{\infty} \| \frac{1}{k!} \overset{\wedge}{d^k} \varphi g(a) \|_{as(1;2)} \| (x_j)_{j=1}^m \|_{w,2}^k$$

$$= \|(x_j)_{j=1}^m\|_{w,2} \sum_{k=2}^{\infty} \| \frac{1}{k!} \overset{\wedge}{d^k} \varphi g(a) \|_{as(1;2)} \| (x_j)_{j=1}^m \|_{w,2}^{k-1}$$

$$\leq \|(x_j)_{j=1}^m\|_{w,2} \sum_{k=2}^{\infty} \frac{K_G 3^{\frac{k-2}{2}} \lambda^k C c^k \|\varphi\|}{(2\sqrt{3}\lambda c)^{k-1}} \leq D \|(x_j)_{j=1}^m\|_{w,2}$$

for every $\varphi \in B_{F^{\prime}}$ and every m. Therefore,

$$\|(g(a+x_j) - g(a))_{j=1}^m\|_{w,1} \leq D \|(x_j)_{j=1}^m\|_{w,2}$$

regardless of the $\|(x_j)_{j=1}^m\|_{w,p} < \delta$, and $x_1, ..., x_m$. Now, Proposition 6 yields the result. □

In [2, Proposition 5.1] it is shown that if E is an $\mathcal{L}_\infty$ space then $\mathcal{L}(^2E;\mathbb{K}) = \mathcal{L}_{al,2}(^2E;\mathbb{K})$. Next corollary shows that the aforementioned result is still valid for vector valued n-linear mappings, for every $n \geq 2$.

Corollary 3. *If E is an $\mathcal{L}_\infty$ space and $n \geq 2$, then for every Banach space F we have*

$$\mathcal{P}_{al,2}(^nE;F) = \mathcal{P}(^nE;F) \text{ and } \mathcal{L}(^nE;F) = \mathcal{L}_{al,2}(^nE;F). \tag{3.1}$$

Proof. Since every scalar valued n-linear ($n \geq 2$) mapping defined on $\mathcal{L}_\infty$ spaces is absolutely $(1;2,...,2)$-summing, it is not hard to prove, using (2.2), that if E is an $\mathcal{L}_{\infty,\lambda}$ space, then, regardless of the Banach space F, we have

$$\|(T(x_j^{(1)},...,x_j^{(n)})_{j=1}^m\|_{w,1} \leq \lambda^n K_G 3^{\frac{n-2}{2}} \|T\| \; \|(x_j^{(1)})_{j=1}^m\|_{w,2} ... \|(x_j^{(n)})_{j=1}^m\|_{w,2} \tag{3.2}$$

for every continuous n-linear mapping $T: E \times ... \times E \to F$. Then, using the estimates of Proposition 6, we have

$$\int_0^1 \| \sum_{j=1}^m T(x_j^{(1)},...,x_j^{(n)}) r_j(t) \|^2 dt)^{\frac{1}{2}} \leq \|(T(x_j^{(1)},...,x_j^{(n)})_{j=1}^m\|_{w,1}$$

and by Definition 3 and (3.2), the proof is done. The polynomial case is analogous.□

4. A Dvoretzky-Rogers Theorem for almost p-summing polynomials

The Theorem of Dvoretzky-Rogers for absolutely summing linear operators has natural versions for absolutely summing multilinear mappings and polynomials (see [9]). A linear Dvoretzky-Rogers Theorem for almost p-summing mappings can be found in [2, Ex 4.1] and tells us that if $p > 1$, then $\mathcal{L}_{al,p}(E;E) \neq \mathcal{L}(E;E)$ for every infinite dimensional Banach space E. In this section, we will show that we also have multilinear and polynomial versions for this result.

Lemma 2. *If $P \in \mathcal{P}_{al,p(E)}(^nE;F)$ then, regardless of the $a \in E$, $dP(a)$ is almost p-summing at the origin.*

Proof. (Adaptation of Lemma 6.1 of [9]).We have the following estimates for $dP(a)(x)$:

$$dP(a)(x) = \frac{n}{n!2^n} \sum_{(e_i=1,-1),i=1,...,n} e_1 e_2 ... e_n P(e_1 x + (e_2 + ... + e_n)a)$$

$$= \frac{n}{n!2^n} \sum_{(e_i=1,-1),i=2,\ldots,n} (e_2...e_nP(x+(e_2+...+e_n)a) - (e_2...e_nP(-x+(e_2+...+e_n)a))$$

$$= \frac{n}{n!2^n}(\sum_{(e_i=1,-1),i=2,\ldots,n} e_2...e_n[P(x+(e_2+...+e_n)a) - P((e_2+...+e_n)a)])-$$

$$- \frac{n}{n!2^n}(\sum_{(e_i=1,-1),i=2,\ldots,n} e_2...e_n[P(-x+(e_2+...+e_n)a) - P((e_2+...+e_n)a)])$$

Therefore, defining $Q_{e_2\ldots e_n}(x) = e_2...e_n[P(x+(e_2+...+e_n)a) - P((e_2+...+e_n)a)]$ we have

$$\int_0^1 \|\sum_{j=1}^k dP(a)(x_j)r_j(t)\|^2 dt)^{\frac{1}{2}} =$$

$$= (\int_0^1 \|\sum_{j=1}^k \frac{n}{n!2^n} \sum_{(e_i=1,-1),i=2,\ldots,n} (Q_{e_2\ldots e_n}(x_j) - Q_{e_2\ldots e_n}(-x_j))r_j(t)\|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{n}{n!2^n} \sum_{(e_i=1,-1),i=2,\ldots,n} (\int_0^1 \|\sum_{j=1}^k (Q_{e_2\ldots e_n}(x_j) - Q_{e_2\ldots e_n}(-x_j))r_j(t)\|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{n}{n!2^n}\{ \sum_{(e_i=1,-1),i=2,\ldots,n} [(\int_0^1 \|\sum_{j=1}^k Q_{e_2\ldots e_n}(x_j)r_j(t)\|^2 dt)^{\frac{1}{2}} +$$

$$+ (\int_0^1 \|\sum_{j=1}^k Q_{e_2\ldots e_n}(-x_j)r_j(t)\|^2 dt)^{\frac{1}{2}}]\}$$

$$\leq \frac{n}{n!2^n} \sum_{(e_i=1,-1),i=2,\ldots,n} 2C_{(e_2+\ldots+e_n)a} \|(x_j)_{j=1}^k\|_{w,p}^{r_{(e_2+\ldots+e_n)a}}$$

$$\leq D\|(x_j)_{j=1}^k\|_{w,p}^{\min\{r_{(e_2\ldots e_n)a}\}}$$

for $\|(x_j)_{j=1}^k\|_{w,p} < \delta$ and $0 < \delta < \min\{1; \epsilon_{(e_2+\ldots+e_n)a}\}$. □

Theorem 4. *(Dvoretzky-Rogers for almost p-summing polynomials) If* $\dim E < \infty$*, then for* $p \leq 2$ *we have*

$$\mathcal{P}_{al,p(E)}(^nE;E) = \mathcal{P}(^nE;E).$$

If $\dim E = \infty$ *and* $p > 1$*, then* $\mathcal{P}_{al,p(E)}(^nE;E) \neq \mathcal{P}(^nE;E)$*. The multilinear version is also valid.*

Proof. If $\dim E < \infty$, let us consider $\{e_1, ..., e_n\}$ and $\{\varphi_1, ..., \varphi_n\}$ basis for E and E' so that $\varphi_j(e_k) = \delta_{jk}$. Given an n-homogeneous polynomial P from E into E, we have

$$P(x) = \overset{\vee}{P}(\sum_{j=1}^m \varphi_j(x)e_j)^n = \sum_{j_1,\ldots,j_n=1}^m \varphi_{j_1}(x)...\varphi_{j_n}(x)\overset{\vee}{P}(e_{j_1}, ..., e_{j_n}).$$

Since every finite type n-homogeneous bounded polynomial is almost p-summing (at zero) for $p \leq 2n$ (see [2, Proposition 3.1 (ii)]), it is not hard to prove that P is almost p-summing everywhere, for $p \leq 2$.

On the other hand, suppose that E is an infinite dimensional Banach space. It suffices to consider the case $1 < p \leq 2$. Choose a non null continuous linear functional $\varphi \in E'$ and $a \notin Ker\varphi$. Define

$$P(x) = \varphi(x)^{n-1}x.$$

If we had P almost p-summing everywhere, we would have, by Lemma 2, $dP(a)$ almost p-summing (at zero). Since φ is almost p-summing and

$$dP(a)(x) = (n-1)\varphi(a)^{n-2}\varphi(x)a + \varphi(a)^{n-1}x,$$

we would have $\varphi(a)^{n-1}x$ almost p-summing. Since $\varphi(a) \neq 0$, we would have that id_E is almost p-summing, and it is a contradiction. $\square$

Example 1. *It is worth observing that by Corollary 3, for $n \geq 2$, we have*

$$\mathcal{P}_{al,2}(^n c_0; c_0) = \mathcal{P}(^n c_0; c_0)$$

whereas Theorem 4 asserts that $\mathcal{P}_{al,2(c_0)}(^n c_0; c_0) \neq \mathcal{P}(^n c_0; c_0)$.

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