Multivector Functionals*

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Abstract

In this paper we introduce the concept of *multivector functionals.* We study some possible kinds of derivative operators that can act in interesting ways on these objects such as, e.g., the A-directional derivative and the generalized concepts of curl, divergence and gradient. The derivation rules are rigorously proved. Since the subject of this paper has not been developed in previous literature, we work out in details several examples of derivation of multivector functionals.

Contents

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1 Introduction

This is the last paper (VII) of series of papers dealing with the theory of multivector and extensor functions and multivector functionals. It is dedicated to the introduction of a key concept, that of *multivector functionals* and the study of their properties. Particularly important is the concept of *induced multivector functionals.* Several kinds of derivatives of multivector functionals, such as A-directional derivative and generalized concepts of curl, divergence and gradient are defined. Since the subject of the present paper has not been explored in the literature[1], we present in section 3 several examples worked in detail of calculations of different types of derivatives for multivector functionals. Multivector functionals are fundamental for the formulation of the Lagrangian field theory of multivector and extensor fields on an arbitrary manifold, a subject that will be studied in a new series of papers.

2 Multivector Functionals

Any mapping which sends general extensors over V into multivectors over V will be called a *general multivector functional over V.*

In particular, the general functionals with image-values belonging to $\bigwedge^r V$ are said to be *r-vector functionals of general extensor.* For the cases $r = 0$, $r = 1$, $r = 2, \ldots$ and $r = n$ we speak about *scalar, vector, bivector,*$\ldots$ and *pseudoscalar functionals*, respectively.

[1]For the best of our knowldge the only place where the concept has been rudimentary used was in [1]. The concept has been used also in ([2],[3]).

For the applications we have in mind we shall need only some particular cases of these general functionals for which we will give special names.

Any mapping $\mathcal{F} : ext_p^q(V) \to \bigwedge^r V$ will be called a *r-vector functional of a (p,q)-extensor.* In accordance to what was said above, the cases for which $\mathcal{F}[t]$ belongs to $\mathbb{R}$, V, $\bigwedge^2 V, \ldots$ and $\bigwedge^n V$ will be named respectively as *scalar, vector, bivector,...* and *pseudoscalar functionals of a (p,q)-extensor.*

2.1 Induced Multivector Functionals

Let $F : \underbrace{\bigwedge^q V \times \cdots \times \bigwedge^q V}_{k \text{ factors}} \to \bigwedge^r V$ be any r-vector function of k q-vector variables. Take some k-uple of p-vectors $(A^1, \ldots, A^k)$.

Associated to F and with respect to $(A^1, \ldots, A^k)$ it is possible to construct a r-vector functional of a (p,q)-extensor, say $\mathcal{F}_{(A^1,\ldots,A^k)}$, given by

$$ext_p^q(V) \ni t \mapsto \mathcal{F}_{(A^1,\ldots,A^k)}[t] \in \bigwedge^r V \text{ such that}$$

$$\mathcal{F}_{(A^1,\ldots,A^k)}[t] = F[t(A^1), \ldots, t(A^k)]. \tag{1}$$

It will be called the *r-vector functional of a (p,q)-extensor induced by F, relative to $(A^1, \ldots, A^k)$.*

If F is differentiable on $\underbrace{\bigwedge^q V \times \cdots \times \bigwedge^q V}_{k \text{ factors}}$, then $\mathcal{F}_{(A^1,\ldots,A^k)}$ is said to be *differentially-induced by F with respect to $(A^1, \ldots, A^k)$.*

In this way if $\mathcal{F}_{(A^1,\ldots,A^k)}$ is differentially-induced, then there must exist the standard derivatives of F with respect to each p-vector variable $X^1, \ldots$ and X^k (the so-called *partial derivatives of F*), i.e., $\partial_{X^1} F, \ldots$ and $\partial_{X^k} F$, (see [6]).

Associated to $\partial_{X^1} F, \ldots$ and $\partial_{X^k} F$ with respect to $(A^1, \ldots, A^k)$ we can define the following multivector functionals of a (p,q)-extensor:

$$\begin{aligned} ext_p^q(V) \ni t \quad &\mapsto \quad \partial_{X^1} F[t(A^1), \ldots, t(A^k)] \in \bigwedge V, \\ &\cdots \end{aligned}$$

and

$$ext_p^q(V) \ni t \mapsto \partial_{X^k} F[t(A^1), \ldots, t(A^k)] \in \bigwedge V. \tag{2}$$

We see that they are induced by the partial derivatives of F with respect to $(A^1, \ldots, A^k)$.

2.1.1 Directional Derivative

Take an arbitrary p-vector A. We introduce the *A-directional derivative* of the differentially-induced r-vector functional $\mathcal{F}_{(A^1,\ldots,A^k)}$ as being the multivector functional $\mathcal{F}'_{(A^1,\ldots,A^k)A}$ given by

$$ext_p^q(V) \ni t \mapsto \mathcal{F}'_{(A^1,\ldots,A^k)A}[t] \in \Lambda V \text{ such that}$$

$$\mathcal{F}'_{(A^1,\ldots,A^k)A}[t] = \sum_{i=1}^{k} A \cdot A^i \partial_{X^i} F[t(A^1), \ldots, t(A^k)]. \tag{3}$$

Note that the algebraic object just defined associated to $\mathcal{F}_{(A^1,\ldots,A^k)}$ has the property of linearity with respect to the direction, i.e., for any $\alpha, \beta \in \mathbb{R}$ and $A, B \in \bigwedge^p V$

$$\mathcal{F}'_{(A^1,\ldots,A^k)\alpha A+\beta B}[t] = \alpha \mathcal{F}'_{(A^1,\ldots,A^k)A}[t] + \beta \mathcal{F}'_{(A^1,\ldots,A^k)B}[t], \tag{4}$$

as expected to hold for a well-defined A-directional derivative of $\mathcal{F}_{(A^1,\ldots,A^k)}$.

2.1.2 Derivatives

Let $(\{e_k\}, \{e^k\})$ be a pair of arbitrary reciprocal bases of V. It is also possible to introduce *four* derivatives-like operators for the differentially-induced r-vector functional $\mathcal{F}_{(A^1,\ldots,A^k)}$ as the following multivector functionals $*\mathcal{F}'_{(A^1,\ldots,A^k)}$ defined by

$$\begin{aligned} *\mathcal{F}'_{(A^1,\ldots,A^k)}[t] &= \frac{1}{p!}(e^{j_1} \wedge \ldots e^{j_p}) * \mathcal{F}'_{(A^1,\ldots,A^k)e_{j_1}\wedge\ldots e_{j_p}}[t] && (5)\\ &= \frac{1}{p!}(e_{j_1} \wedge \ldots e_{j_p}) * \mathcal{F}'_{(A^1,\ldots,A^k)e^{j_1}\wedge\ldots e^{j_p}}[t], && (6) \end{aligned}$$

where $*$ means either $(\wedge)$, $(\cdot)$, $(\lrcorner)$ or (*Clifford product*).

It should be noted that $*\mathcal{F}'_{(A^1,\ldots,A^k)}$ are well-defined multivector functionals of (p,q)-extensor *only* associated with $\mathcal{F}_{(A^1,\ldots,A^k)}$ since, by taking into account eq.(4), $*\mathcal{F}'_{(A^1,\ldots,A^k)}[t]$ are multivectors which do not depend on the choice of $(\{e_k\}, \{e^k\})$.

Recall also that a straightforward calculation gives with the use eq.(3) that

$$
\begin{aligned}
*\mathcal{F}'_{(A^1,\ldots,A^k)}[t] &= \frac{1}{p!}(e^{j_1}\wedge\ldots e^{j_p}) * (\sum_{i=1}^{k}(e_{j_1}\wedge\ldots e_{j_p})\cdot A^i\partial_{X^i}F[\ldots]) \\
&= (\sum_{i=1}^{k}\frac{1}{p!}(e_{j_1}\wedge\ldots e_{j_p})\cdot A^i e^{j_1}\wedge\ldots e^{j_p}) * \partial_{X^i}F[\ldots] \\
*\mathcal{F}'_{(A^1,\ldots,A^k)}[t] &= \sum_{i=1}^{k} A^i * \partial_{X^i}F[t(A^1),\ldots,t(A^k)]. \qquad (7)
\end{aligned}
$$

Eq.(7) shows explicitly that $*\mathcal{F}'_{(A^1,\ldots,A^k)}$ can be intrinsically defined without using any pair of reciprocal bases of V.

The special cases: $\wedge\mathcal{F}'_{(A^1,\ldots,A^k)}$, $\cdot\mathcal{F}'_{(A^1,\ldots,A^k)}$, $\lrcorner\mathcal{F}'_{(A^1,\ldots,A^k)}$ and $\mathcal{F}'_{(A^1,\ldots,A^k)}$ (i.e., $*\equiv$*Clifford product*) will be called respectively the *curl, scalar divergence, left contracted divergence* and *gradient* of $\mathcal{F}_{(A^1,\ldots,A^k)}$. Sometimes, $\mathcal{F}'_{(A^1,\ldots,A^k)}$ will be called the *standard derivative* of $\mathcal{F}_{(A^1,\ldots,A^k)}$.

We introduce now on the *real vector space of differentially-induced r-vector functionals of (p,q)-extensor* the following *four* derivative-like operators ∂_t* as follows

$$
\partial_t * \mathcal{F}_{(A^1,\ldots,A^k)}[t] = *\mathcal{F}'_{(A^1,\ldots,A^k)}[t], \qquad (8)
$$

i.e., by eq.(7)

$$
\partial_t * \mathcal{F}_{(A^1,\ldots,A^k)}[t] = \sum_{i=1}^{k} A^i * \partial_{X^i}F[t(A^1),\ldots,t(A^k)]. \qquad (9)
$$

The special cases: $\partial_t\wedge$, $\partial_t\cdot$, $\partial_t\lrcorner$ and ∂_t (i.e., $*\equiv$*Clifford product*) will be called respectively the (functional) *curl, scalar divergence, left contracted divergence* and *gradient operator.* Sometimes, we will say that ∂_t is the *standard derivative operator with respect to t.*

$\partial_t\wedge\mathcal{F}_{(A^1,\ldots,A^k)}[t]$, $\partial_t\cdot\mathcal{F}_{(A^1,\ldots,A^k)}[t]$, $\partial_t\lrcorner\mathcal{F}_{(A^1,\ldots,A^k)}[t]$ and $\partial_t\mathcal{F}_{(A^1,\ldots,A^k)}[t]$ (i.e., $*\equiv$*Clifford product*) will be named respectively as the *curl, scalar divergence, left contracted divergence* and *gradient* of $\mathcal{F}_{(A^1,\ldots,A^k)}$. The gradient of $\mathcal{F}_{(A^1,\ldots,A^k)}$ will be called the *standard derivative* of $\mathcal{F}_{(A^1,\ldots,A^k)}$ *with respect to t*

It is still possible to define the noticeable derivative-like operator $A\cdot\partial_t$ as follows

$$A\cdot\partial_t\mathcal{F}_{(A^1,\ldots,A^k)}[t] = (A\cdot\frac{1}{p!}e^{j_1}\wedge\ldots e^{j_p})\mathcal{F}'_{(A^1,\ldots,A^k)e_{j_1}\wedge\ldots e_{j_p}}[t] \quad (10)$$

$$= (A\cdot\frac{1}{p!}e_{j_1}\wedge\ldots e_{j_p})\mathcal{F}'_{(A^1,\ldots,A^k)e^{j_1}\wedge\ldots e^{j_p}}[t], \quad (11)$$

i.e., by eq.(4)

$$A\cdot\partial_t\mathcal{F}_{(A^1,\ldots,A^k)}[t] = \mathcal{F}'_{(A^1,\ldots,A^k)A}[t]. \quad (12)$$

Eq.(12) means that $A\cdot\partial_t$ is the *A-directional derivative operator* which maps $\mathcal{F}_{(A^1,\ldots,A^k)}\mapsto\mathcal{F}'_{(A^1,\ldots,A^k)A}$.

It is often convenient when doing calculations to employ some abuses of notation for simplifying the handle of the fundamental formulas. Thus, eqs.(3) and (9) will be usually written

$$A\cdot\partial_t F[t(A^1),\ldots,t(A^k)] = \sum_{i=1}^k A\cdot A^i\partial_{t(A^i)}F[t(A^1),\ldots,t(A^k)], \quad (13)$$

$$\partial_t * F[t(A^1),\ldots,t(A^k)] = \sum_{i=1}^k A^i * \partial_{t(A^i)}F[t(A^1),\ldots,t(A^k)]. \quad (14)$$

No confusion arises since $A\cdot\partial_t$ and ∂_t* denote derivation of r-vector functional with respect to (p,q)-extensor t, and $\partial_{t(A^i)}$ holds for derivation of r-vector function with respect to q-vector $t(A^i)$.

It should be noted that by employing the abused notation we can re-write eqs.(5) and (6) as

$$\partial_t * F[\ldots] = \frac{1}{p!}(e^{j_1}\wedge\ldots e^{j_p}) * (e_{j_1}\wedge\ldots e_{j_p})\cdot\partial_t F[\ldots] \quad (15)$$

$$= \frac{1}{p!}(e_{j_1}\wedge\ldots e_{j_p}) * (e^{j_1}\wedge\ldots e^{j_p})\cdot\partial_t F[\ldots]. \quad (16)$$

2.1.3 A-Directional Derivation Rules

Proposition 1 *Take a real λ and a multivector M. If $t\mapsto F[t(A^1),\ldots,t(A^k)]$ is any differentially-induced r-vector functional of a (p,q)-extensor, then*

$$A\cdot\partial_t(\lambda F[\ldots]) = \lambda A\cdot\partial_t F[\ldots], \quad (17)$$

$$A\cdot\partial_t(F[\ldots]M) = (A\cdot\partial_t F[\ldots])M. \quad (18)$$

Proof. It follows directly from eq.(13) by using the derivation formulas: $\partial_{X^i}(\lambda F(\ldots)) = \lambda \partial_{X^i} F(\ldots)$ and $\partial_{X^i}(F(\ldots)M) = (\partial_{X^i} F(\ldots))M$. ■

Theorem 2 *Let $t \mapsto F[t(A^1), \ldots, t(A^k)]$ and $t \mapsto G[t(A^1), \ldots, t(A^k)]$ be any two differentially-induced r-vector functionals of a (p, q)-extensor.*

The addition $t \mapsto (F + G)[t(A^1), \ldots, t(A^k)]$ is a differentially-induced r-vector functional of a (p, q)-extensor and the following rule holds

$$A \cdot \partial_t (F + G)[\ldots] = A \cdot \partial_t F[\ldots] + A \cdot \partial_t G[\ldots]. \tag{19}$$

Proof. As we can see, it is an immediate consequence of the derivation rule $\partial_{X^i}(F + G)(\ldots) = \partial_{X^i}(F)(\ldots) + \partial_{X^i} G(\ldots)$. ■

Theorem 3 *Let $t \mapsto \Phi[t(A^1), \ldots, t(A^k)]$ and $t \mapsto G[t(A^1), \ldots, t(A^k)]$ be any differentially-induced scalar and r-vector functional of a (p, q)-extensor, respectively.*

The scalar multiplication $t \mapsto (\Phi G)[t(A^1), \ldots, t(A^k)]$ is also a differentially-induced r-vector functional of a (p, q)-extensor and we have

$$A \cdot \partial_t (\Phi G)[\ldots] = (A \cdot \partial_t \Phi[\ldots]) G[\ldots] + \Phi[\ldots] A \cdot \partial_t G[\ldots]. \tag{20}$$

It is rightly a Leibnitz-like rule.

Proof. As the reader can easily prove, eq.(20) is an immediate consequence of the derivation rule $\partial_{X^i}(\Phi G)(\ldots) = (\partial_{X^i} \Phi(\ldots)) G(\ldots) + \Phi(\ldots) \partial_{X^i} G(\ldots)$. ■

Theorem 4 *Let $t \mapsto \Psi[t(A^1), \ldots, t(A^k)]$ and $\lambda \mapsto \phi(\lambda)$ be any differentially-induced scalar functional and a derivable ordinary real function, respectively. Then, $t \mapsto \phi(\Psi[t(A^1), \ldots, t(A^k)])$ is a differentially-induced scalar functional and the following rule holds*

$$A \cdot \partial_t \phi(\Psi[\ldots]) = \phi'(\Psi[\ldots]) A \cdot \partial_t \Psi[\ldots]. \tag{21}$$

It is an interesting and useful chain-like rule for A-directional derivation of a special type of scalar functionals.

Proof. Eq.(21) follows easily from eq.(13) by taking into account the derivation rule $\partial_{X^i} \phi \circ \Psi(\ldots) = \phi' \circ \Psi(\ldots) \partial_{X^i} \Psi(\ldots)$. ■

3 Examples

Example 5 *Let* $h \in ext_1^1(V)$ *and take* $a, b, c \in V$. *Then,*

$$
\begin{aligned}
a \cdot \partial_h (h(b) \cdot h(c)) &= a \cdot b \partial_{h(b)} (h(b) \cdot h(c)) + a \cdot c \partial_{h(c)} (h(b) \cdot h(c)) \\
&= a \cdot b h(c) + a \cdot c h(b), \\
a \cdot \partial_h (h(b) \cdot h(c)) &= h(a \cdot bc + a \cdot cb). \qquad (22)
\end{aligned}
$$

Also,

$$
\begin{aligned}
a \cdot \partial_h (h(b) \wedge h(c)) &= a \cdot b \partial_{h(b)} (h(b) \wedge h(c)) + a \cdot c \partial_{h(c)} (h(b) \wedge h(c)) \\
&= a \cdot b(n-1) h(c) - a \cdot c(n-1) h(b) \\
&= (n-1) h(a \cdot bc - a \cdot cb), \\
a \cdot \partial_h (h(b) \wedge h(c)) &= (n-1) h(a \lrcorner (b \wedge c)). \qquad (23)
\end{aligned}
$$

In eqs.(22) and (23) we have used the derivative formulas $\partial_x (x \cdot y) = y$ and $\partial_x (x \wedge y) = (n-1) y$, where n is the dimension of V.

The second formula developed in this example has an interesting and useful generalization, which is:

The a-derivative of the k-vector functional $ext_1^1(V) \ni h \mapsto \underline{h}(a^1 \wedge \ldots a^k) \in \bigwedge^k V$, with $a^1, \ldots, a^k \in V$, is given by

$$
a \cdot \partial_h \underline{h}(a^1 \wedge \ldots a^k) = (n - k + 1) \underline{h}(a \lrcorner (a^1 \wedge \ldots a^k)). \qquad (24)
$$

Example 6 *Let* $h \in ext_1^1(V)$ *and take* $b \in V$.

We shall calculate $a \cdot \partial_h h(b)$ *and* $a \cdot \partial_h h^\dagger(b)$. *And, also* $\partial_h * h(b)$ *and* $\partial_h * h^\dagger(b)$.

First, we have

$$
\begin{aligned}
a \cdot \partial_h h(b) &= a \cdot b \partial_{h(b)} h(b) = (a \cdot b) n, \\
a \cdot \partial_h h(b) &= n(a \cdot b), \qquad (25)
\end{aligned}
$$

were we used the derivative formula $\partial_x x = n$. Thus,

$$
\partial_h * h(b) = e^j * e_j \cdot \partial_h h(b) = e^j * n(e_j \cdot b) = b * n,
$$

i.e.,

$$\begin{aligned}\partial_h \wedge h(b) &= \partial_h h(b) = nb,\\ \partial_h \cdot h(b) &= \partial_h \lrcorner h(b) = 0.\end{aligned}$$

Now, by employing a trick we have

$$a \cdot \partial_h h^\dagger(b) = a \cdot \partial_h (h^\dagger(b) \cdot e^j e_j) = a \cdot \partial_h (b \cdot h(e^j) e_j).$$

Thus, by using eq.(18)

$$\begin{aligned}a \cdot \partial_h h^\dagger(b) &= (\sum_{i=1}^{n} a \cdot e^i \partial_{h(e^i)} b \cdot h(e^j))e_j = \sum_{i=1}^{n} a \cdot e^i b \delta_i^j e_j,\\ a \cdot \partial_h h^\dagger(b) &= ba, \qquad (26)\end{aligned}$$

were we used the derivative formula $\partial_x (b \cdot x) = b$. Thus,

$$\partial_h * h^\dagger(b) = e^j * e_j \cdot \partial_h h^\dagger(b) = e^j * (be_j).$$

It follows that

$$\begin{aligned}\partial_h \wedge h^\dagger(b) &= e^j \wedge (b \cdot e_j) + e^j \wedge (b \wedge e_j) = b.\\ \partial_h \cdot h^\dagger(b) &= e^j \cdot (b \cdot e_j) + e^j \cdot (b \wedge e_j) = 0.\\ \partial_h \lrcorner h^\dagger(b) &= e^j \lrcorner (b \cdot e_j) + e^j \lrcorner (b \wedge e_j) = (e^j \cdot b)e_j - (e^j \cdot e_j)b = (1-n)b.\\ \partial_h h^\dagger(b) &= e^j (2e_j \cdot b - e_j b) = (2-n)b.\end{aligned}$$

Example 7 *Let $t \in ext_1^1(V)$. The trace of t, i.e., $t \mapsto tr[t] = t(e^j) \cdot e_j$, is a scalar functional and the bivector of t, i.e., $t \mapsto biv[t] = t(e^j) \wedge e_j$, is a bivector functional, both of them associated to t. We shall calculate $a \cdot \partial_t tr[t]$ and $a \cdot \partial_t biv[t]$. And, also $\partial_t * tr[t]$ and $\partial_t * biv[t]$.*

First, we have

$$\begin{aligned}a \cdot \partial_t tr[t] &= \sum_{i=1}^{n} a \cdot e^i \partial_{t(e^i)} (t(e^j) \cdot e_j) = \sum_{i=1}^{n} a \cdot e^i \delta_i^j e_j,\\ a \cdot \partial_t tr[t] &= a. \qquad (27)\end{aligned}$$

We have used once again the derivative formula $\partial_x (x \cdot y) = y$. Hence,

$$\partial_t * tr[t] = e^j * e_j \cdot \partial_t tr[t] = e^j * e_j,$$

i.e.,

$$\begin{aligned}\partial_t \wedge tr[t] &= 0,\\ \partial_t \cdot tr[t] &= \partial_t \lrcorner tr[t] = \partial_t tr[t] = n.\end{aligned}$$

Now, we have also

$$\begin{aligned}a \cdot \partial_t biv[t] &= \sum_{i=1}^{n} a \cdot e^i \partial_{t(e^i)}(t(e^j) \wedge e_j) = \sum_{i=1}^{n} a \cdot e^i (n-1)\delta_i^j e_j,\\ a \cdot \partial_t biv[t] &= (n-1)a, \qquad (28)\end{aligned}$$

were we have used once again the derivative formula $\partial_x(x \wedge y) = (n-1)y$. Hence,

$$\partial_t * biv[t] = (n-1)e^j * e_j,$$

i.e.,

$$\begin{aligned}\partial_t \wedge biv[t] &= 0,\\ \partial_t \cdot biv[t] &= \partial_t \lrcorner biv[t] = \partial_t biv[t] = (n-1)n.\end{aligned}$$

Example 8 *Let $h \in ext_1^1(V)$ and take a non-zero $I \in \bigwedge^n V$. We shall calculate the a-directional derivative of the pseudoscalar functional $h \mapsto \underline{h}(I)$, i.e., $a \cdot \partial_h \underline{h}(I)$.*

By employing one of the expansion formulas for pseudoscalars (see [4]), eq.(17) and eq.(24) we have

$$\begin{aligned}a \cdot \partial_h \underline{h}(I) &= a \cdot \partial_h I \cdot (e_1 \wedge \ldots e_n)\underline{h}(e^1 \wedge \ldots e^n)\\ &= I \cdot (e_1 \wedge \ldots e_n) a \cdot \partial_h \underline{h}(e^1 \wedge \ldots e^n)\\ &= I \cdot (e_1 \wedge \ldots e_n)\underline{h}(a \lrcorner (e^1 \wedge \ldots e^n)),\\ a \cdot \partial_h \underline{h}(I) &= \underline{h}(a \lrcorner I) = h(aI). \qquad (29)\end{aligned}$$

Example 9 *Let $h \in ext_1^1(V)$ and take a non-zero $I \in \Lambda^n V$. The determinant of h, i.e., $h \mapsto \det[h]$ such that $\underline{h}(I) = \det[h]I$, is a characteristic scalar functional of h. We shall calculate $a \cdot \partial_h \det[h]$ and $\partial_h * \det[h]$.*

By employing eq.(18) and eq.(29) we have

$$a \cdot \partial_h \det[h] = (a \cdot \partial_h \underline{h}(I))I^{-1} = \underline{h}(aI)I^{-1}.$$

But, by taking into account the extensor formula $h^{-1}(a) = \det{}^{-1}[h]\underline{h}^{\dagger}(aI)I^{-1}$ (see[5]) and recalling that $\det[h^{\dagger}] = \det[h]$ and $h^{*} = (h^{\dagger})^{-1} = (h^{-1})^{\dagger}$ we get

$$a \cdot \partial_h \det[h] = \det[h] h^{*}(a). \tag{30}$$

Hence, it follows that

$$\partial_h * \det[h] = e^j * e_j \cdot \partial_h \det[h] = \det[h] e^j * h^{*}(e_j),$$

i.e.,

$$\begin{aligned} \partial_h \wedge \det[h] &= -\det[h] h^{*}(e_j) \wedge e^j = \det[h] biv[h^{-1}]. \\ \partial_h \cdot \det[h] &= \partial_h \lrcorner \det[h] = \det[h] h^{-1}(e^j) \cdot e_j = \det[h] tr[h^{-1}]. \\ \partial_h \det[h] &= \det[h] e^j h^{*}(e_j) = \det[h](tr[h^{-1}] + biv[h^{-1}]). \end{aligned}$$

3.1 An Enlightening Discussion

Let us consider for example a differentially-induced scalar functional of $(1,1)$-extensor $t \mapsto \Phi[t(a^1)]$. We have the possibility for constructing a differentiable scalar function of $n \times n$ real variables $(t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}) \mapsto \widehat{\Phi}(t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn})$, defined by

$$\widehat{\Phi}(t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}) = \Phi[t_{ij}(a^1 \cdot e^i) e^j] \tag{31}$$

where $t_{ij} = t(e_i) \cdot e_j$ are the $n \times n$ matrix elements of t with respect to $\{e_k\}$.

Eq.(31) shows that all information just contained into the *classical real function* $(t_{11}, \ldots, t_{nn}) \mapsto \widehat{\Phi}(t_{11}, \ldots, t_{nn})$ whose real variables are t_{pq}, is also codified into the scalar functional $t \mapsto \Phi[t(a^1)]$.

We shall search for the relationship which exists between the *ordinary partial derivatives* of $\widehat{\Phi}(\ldots)$ with respect to each *tensor covariant component*[2] t_{pq} and the a-directional derivative of $\Phi[\ldots]$.

By using $\partial_{\lambda_i} \Phi(x(\lambda_1, \ldots, \lambda_k)) = \partial_{\lambda_i} x(\lambda_1, \ldots, \lambda_k) \cdot \partial_x \Phi(x(\lambda_1, \ldots, \lambda_k))$, a chain-like derivation rule, we may write

$$\begin{aligned} \frac{\partial \widehat{\Phi}}{\partial t_{pq}}(t_{11}, \ldots, t_{nn}) &= \partial_{t_{pq}}(t_{ij}(a^1 \cdot e^i) e^j) \cdot \partial_x \Phi[t_{ij}(a^1 \cdot e^i) e^j] \\ &= \delta_{ij}^{pq}(a^1 \cdot e^i) e^j \cdot \partial_{t(a^1)} \Phi[t(a^1)], \\ \frac{\partial \widehat{\Phi}}{\partial t_{pq}}(t_{11}, \ldots, t_{nn}) &= (a^1 \cdot e^p) e^q \cdot \partial_{t(a^1)} \Phi[t(a^1)]. \end{aligned} \tag{32}$$

[2]They are the $n \times n$ covariant components of a 2-tensor T in biunivocal correspondence with the $(1,1)$-extensor t, see [5], i.e., $T_{pq} \equiv T(e_p, e_q) = t(e_p) \cdot e_q \equiv t_{pq}$.

Now, Clifford multiplication by $(a\cdot e_p)e_q$ (and summing over p,q) on both sides of eq.(32) yields

$$\begin{aligned}(a\cdot e_p)e_q\frac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn}) &= (a\cdot a^1)e_q e^q\cdot\partial_{t(a^1)}\Phi[t(a^1)]\\ &= a\cdot a^1\partial_{t(a^1)}\Phi[t(a^1)],\\ (a\cdot e_p)e_q\frac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn}) &= a\cdot\partial_t\Phi[t(a^1)]. \end{aligned}\tag{33}$$

That is the required result relating both $\dfrac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn})$ and $a\cdot\partial_t\Phi[t(a^1)]$.

It is still possible to find a relationship between $\dfrac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn})$ and the $*$-derivatives of $\Phi[t(a^1)]$. From eq.(32) we have

$$\begin{aligned}e_p*(e_q\frac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn})) &= a^1*(e_q e^q\cdot\partial_{t(a^1)}\Phi[t(a^1)])\\ &= a^1*\partial_{t(a^1)}\Phi[t(a^1)],\\ e_p*(e_q\frac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn})) &= \partial_t*\Phi[t(a^1)]. \end{aligned}\tag{34}$$

That is the expected identity which relates both $\dfrac{\partial\widehat{\Phi}}{\partial t_{pq}}(t_{11},\ldots,t_{nn})$ and $\partial_t*\Phi[t(a^1)]$.

4 Conclusions

In this paper we introduced the key concepts of a theory of multivector functionals. We studied several aspects of the notion of derivative that can be applied to these objects, as e.g., the A-directional derivatives and the generalized concepts of curl, divergence and gradient. We worked in details several examples where we calculate different types of derivatives for multivector functionals. It is worth to said once again that these objects play a decisive role in the development of a Lagrangian formalism for extensor fields as it will be seen in two future series of papers: *geometric theories of gravitation and Lagrangian formulation of the multivector and extensor fields theory.*

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