Multivector Functions of a Multivector Variable*

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Abstract

In this paper we develop with considerable details a theory of multivector functions of a p-vector variable. The concepts of limit, continuity and differentiability are rigorously studied. Several important types of derivatives for these multivector functions are introduced, as e.g., the A-directional derivative (where A is a p-vector) and the generalized concepts of curl, divergence and gradient. The derivation rules for different types of products of multivector functions and for compositon of multivector functions are proved.

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1 Introduction

This is the paper VI in the present series. Here, we develop a theory of multivector functions of a p-vector variable. For these objects we investigate with details the concepts of limit and continuity, and formulate rigorously the notion of derivation. As we will see, the concept of extensor introduced in [1] (paper II on this series) plays a crucial role in our theory of differentiability. We introduce important derivative-like operators for these multivector functions, as e.g., the A-directional derivative and the generalized concepts of curl, divergence and gradient. The derivation rules for all suitable products of multivector functions of a p-vector variable and for composition of multivector functions are presented and proved.

2 Multivector Functions of a p-Vector Variable

Let $\mathbf{\Omega}^p V$ be a subset of $\bigwedge^p V$. Any mapping $F : \mathbf{\Omega}^p V \to \bigwedge V$ will be called a *multivector function of a p-vector variable over V*. In particular, $F : \mathbf{\Omega}^p V \to \bigwedge^q V$ is said to be a *q-vector function of a p-vector variable,* or a (p, q)-*function over V*, for short. For the special cases $q = 0$, $q = 1$, $q = 2, \ldots$ etc. we will employ the names of *scalar, vector, bivector,... etc. function of a p-vector variable,* respectively.

2.1 Limit Notion

We begin by introducing the concept of δ-neighborhood for a multivector A.

Take any $\delta > 0$. The set[1] $N_A(\delta) = \{X \in \Lambda V \;/\; \|X - A\| < \delta\}$ will be called a *δ-neighborhood of A.*

The set $N_A(\delta) - \{A\} = \{X \in \bigwedge V \;/\; 0 < \|X - A\| < \delta\}$ will be said to be a *reduced δ-neighborhood of A.*

We introduce now the concepts of cluster and interior points of $\mathbf{\Omega} V \subseteq \bigwedge V$.

A multivector $X_0 \in \bigwedge V$ is said to be a *cluster point of* $\mathbf{\Omega} V$ if and only if for every $N_{X_0}(\delta) : (N_{X_0}(\delta) - \{X_0\}) \cap \mathbf{\Omega} V \neq \emptyset$, i.e., all reduced δ-neighborhood of X_0 contains at least one multivector of $\mathbf{\Omega} V$.

A multivector $X_0 \in \bigwedge V$ is said to be an *interior point of* $\mathbf{\Omega} V$ if and only if there exists $N_{X_0}(\delta)$ such that $N_{X_0}(\delta) \subseteq \mathbf{\Omega} V$, i.e., any multivector of some δ-neighborhood of X_0 belongs also to $\mathbf{\Omega} V$.

It should be noted that any interior point of $\mathbf{\Omega} V$ is also a cluster point of $\mathbf{\Omega} V$.

If the set of interior points of $\mathbf{\Omega} V$ coincides with $\mathbf{\Omega} V$, then it is said to be an *open subset of* $\bigwedge V$.

Take $\mathbf{\Omega}^p V \subseteq \bigwedge^p V$ and let $F : \mathbf{\Omega}^p V \to \bigwedge V$ be a multivector function of a p-vector variable and take $X_0 \in \bigwedge^p V$ to be a cluster point of $\mathbf{\Omega}^p V$.

A multivector M is said to be the *limit of $F(X)$ for X approaching to* X_0 if and only if for every real $\varepsilon > 0$ there exists some real $\delta > 0$ such that if $X \in \mathbf{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$, then $\|F(X) - M\| < \varepsilon$. It is denoted by $\lim\limits_{X \to X_0} F(X) = M$.

In dealing with a scalar function of a p-vector variable, say Φ, the definition of $\lim\limits_{X \to X_0} \Phi(X) = \mu$ is reduced to: for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|\Phi(X) - \mu| < \varepsilon$, whenever $X \in \Omega^p V$ and $0 < \|X - X_0\| < \delta$.

Proposition 1 *Let $F : \mathbf{\Omega}^p V \to \bigwedge V$ and $G : \mathbf{\Omega}^p V \to \bigwedge V$ be two multivector functions of a p-vector variable. If there exist $\lim\limits_{X \to X_0} F(X)$ and $\lim\limits_{X \to X_0} G(X)$, then there exists $\lim\limits_{X \to X_0} (F + G)(X)$ and*

$$\lim_{X \to X_0} (F + G)(X) = \lim_{X \to X_0} F(X) + \lim_{X \to X_0} G(X). \tag{1}$$

Proof. Let $\lim\limits_{X \to X_0} F(X) = M_1$ and $\lim\limits_{X \to X_0} G(X) = M_2$. Then, we must prove that $\lim\limits_{X \to X_0} (F + G)(X) = M_1 + M_2$.

[1] We recalls that the two double bars $\| \;\|$ denotes the norm of multivectors, as defined in [2], i.e., for all $X \in \Lambda V : \|X\| = \sqrt{X \cdot X}$, where $(\cdot)$ is any fixed euclidean scalar product.

Taken a real $\varepsilon > 0$, since $\lim_{X \to X_0} F(X) = M_1$ and $\lim_{X \to X_0} G(X) = M_2$, there must be two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} \|F(X) - M_1\| &< \frac{\varepsilon}{2}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_1, \\ \|G(X) - M_2\| &< \frac{\varepsilon}{2}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_2. \end{aligned}$$

Thus, there is a real $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\|F(X) - M_1\| < \frac{\varepsilon}{2} \text{ and } \|G(X) - M_2\| < \frac{\varepsilon}{2},$$

for $X \in \mathbf{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$. Hence, by using the triangular inequality for the norm of multivectors, it follows that

$$\begin{aligned} \|(F + G)(X) - (M_1 + M_2)\| &= \|F(X) - M_1 + G(X) - M_2\| \\ &\leq \|F(X) - M_1\| + \|G(X) - M_2\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for $X \in \mathbf{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$.

Therefore, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $X \in \mathbf{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$, then $\|(F + G)(X) - (M_1 + M_2)\| < \varepsilon$. ■

Proposition 2 *Let $\Phi : \mathbf{\Omega}^p V \to \mathbb{R}$ and $F : \mathbf{\Omega}^p V \to \bigwedge V$ be a scalar function and a multivector function of a p-vector variable. If there exist $\lim_{X \to X_0} \Phi(X)$ and $\lim_{X \to X_0} F(X)$, then there exists $\lim_{X \to X_0} (\Phi F)(X)$ and*

$$\lim_{X \to X_0} (\Phi F)(X) = \lim_{X \to X_0} \Phi(X) \lim_{X \to X_0} F(X). \tag{2}$$

Proof. Let $\lim_{X \to X_0} \Phi(X) = \Phi_0$ and $\lim_{X \to X_0} F(X) = F_0$. Then, we must prove that $\lim_{X \to X_0} (\Phi F)(X) = \Phi_0 F_0$.

First, since $\lim_{X \to X_0} \Phi(X) = \Phi_0$ it can be found a $\delta_1 > 0$ such that

$$|\Phi(X) - \Phi_0| < 1, \text{ for } X \in \mathbf{\Omega}^p V \text{ and } 0 < \|X - X_0\| < \delta_1,$$

i.e.,

$$|\Phi(X)| < 1 + |\Phi_0|, \text{ for } X \in \mathbf{\Omega}^p V \text{ and } 0 < \|X - X_0\| < \delta_1.$$

Where the triangular inequality for real numbers $|\alpha| - |\beta| \leq |\alpha - \beta|$ was used.

Now, taken a $\varepsilon > 0$, since $\lim\limits_{X \to X_0} \Phi(X) = \Phi_0$ and $\lim\limits_{X \to X_0} F(X) = F_0$, they can be found a $\delta_2 > 0$ and a $\delta_3 > 0$ such that

$$\begin{aligned} |\Phi(X) - \Phi_0| &< \frac{\varepsilon}{2(1 + \|F_0\|)}, \text{ for } X \in \boldsymbol{\Omega}^p V \text{ and } 0 < \|X - X_0\| < \delta_2, \\ \|F(X) - F_0\| &< \frac{\varepsilon}{2(1 + |\Phi_0|)}, \text{ for } X \in \boldsymbol{\Omega}^p V \text{ and } 0 < \|X - X_0\| < \delta_3. \end{aligned}$$

Thus, given a real $\varepsilon > 0$ there is a real $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$\begin{aligned} |\Phi(X)| &< 1 + |\Phi_0|, \\ |\Phi(X) - \Phi_0| &< \frac{\varepsilon}{2(1 + \|F_0\|)}, \\ \|F(X) - F_0\| &< \frac{\varepsilon}{2(1 + |\Phi_0|)}, \end{aligned}$$

whenever $X \in \boldsymbol{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$. Hence, using some properties of the norm of multivectors, it follows that

$$\begin{aligned} \|(\Phi F)(X) - \Phi_0 F_0\| &= \|\Phi(X)(F(X) - F_0) + (\Phi(X) - \Phi_0)F_0\| \\ &\leq |\Phi(X)| \, \|F(X) - F_0\| + |\Phi(X) - \Phi_0| \, \|F_0\| \\ &< |\Phi(X)| \, \|F(X) - F_0\| + |\Phi(X) - \Phi_0| \, (1 + \|F_0\|) \\ &< (1 + |\Phi_0|) \frac{\varepsilon}{2(1 + |\Phi_0|)} + \frac{\varepsilon}{2(1 + \|F_0\|)} (1 + \|F_0\|) = \varepsilon, \end{aligned}$$

whenever $X \in \boldsymbol{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$.

Therefore, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $X \in \boldsymbol{\Omega}^p V$ and $0 < \|X - X_0\| < \delta$, then $\|(\Phi F)(X) - \Phi_0 F_0\| < \varepsilon$. ■

Lemma 3 *There exists* $\lim\limits_{X \to X_0} F(X)$ *if and only if there exist either* $\lim\limits_{X \to X_0} F^J(X)$ *or* $\lim\limits_{X \to X_0} F_J(X)$. *It holds*

$$\lim_{X \to X_0} F(X) = \sum_J \frac{1}{\nu(J)!} \lim_{X \to X_0} F^J(X) e_J = \sum_J \frac{1}{\nu(J)!} \lim_{X \to X_0} F_J(X) e^J. \quad (3)$$

Proof. It is an immediate consequence of eqs.(1) and (2). ■

Proposition 4 *Let $F : \mathbf{\Omega}^p V \to \bigwedge V$ and $G : \mathbf{\Omega}^p V \to \bigwedge V$ be two multivector functions of a p-vector variable. We can define the products $F * G : \mathbf{\Omega}^p V \to \bigwedge V$ such that $(F * G)(X) = F(X) * G(X)$ where $*$ holds for either* $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ *or* (Clifford product). *If there exist $\lim\limits_{X \to X_0} F(X)$ and $\lim\limits_{X \to X_0} G(X)$, then there exists $\lim\limits_{X \to X_0} (F * G)(X)$ and*

$$\lim_{X \to X_0} (F * G)(X) = \lim_{X \to X_0} F(X) * \lim_{X \to X_0} G(X). \tag{4}$$

Proof. It is an immediate consequence of eq.(3). ■

2.2 Continuity Notion

Take $\mathbf{\Omega}^p V \subseteq \bigwedge^p V$. A multivector function of a p-vector variable $F : \mathbf{\Omega}^p V \to \bigwedge V$ is said to be *continuous at $X_0 \in \mathbf{\Omega}^p V$* if and only if there exists[2] $\lim\limits_{X \to X_0} F(X)$ and

$$\lim_{X \to X_0} F(X) = F(X_0). \tag{5}$$

Lemma 5 *The multivector function $X \mapsto F(X)$ is continuous at X_0 if and only if any component scalar function, either $X \mapsto F^J(X)$ or $X \mapsto F_J(X)$ is continuous at X_0.*

Proposition 6 *Let $F : \mathbf{\Omega}^p V \to \bigwedge V$ and $G : \mathbf{\Omega}^p V \to \bigwedge V$ be two continuous functions at $X_0 \in \mathbf{\Omega}^p V$. Then, the addition $F + G : \mathbf{\Omega}^p V \to \bigwedge V$ such that $(F + G)(X) = F(X) + G(X)$ and the products $F * G : \mathbf{\Omega}^p V \to \mathbf{\Omega} V$ such that $(F * G)(X) = F(X) * G(X)$, where $*$ means either* $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ *or* (Clifford product), *are also continuous functions at X_0.*

Proof. It is an immediate consequence of eqs.(1) and (4). ■

Proposition 7 *Let $G : \mathbf{\Omega}^p V \to \bigwedge^q V$ and $F : \bigwedge^q V \to \bigwedge^r V$ be two continuous functions, the first one at $X_0 \in \mathbf{\Omega}^p V$ and the second one at $G(X_0) \in \bigwedge^q V$. Then, the composition $F \circ G : \mathbf{\Omega}^p V \to \bigwedge^r V$ such that $F \circ G(X) = F(G(X))$ is a continuous function at X_0.*

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[2] Observe that X_0 has to be cluster point of $\mathbf{\Omega}^p V$.

2.3 Differentiability Notion

Let $\mathbf{\Omega}^p V$ be a subset of $\bigwedge^p V$. A (p,q)-function over V, say F, is said to be *differentiable at* $X_0 \in \mathbf{\Omega}^p V$ if and only if there exists a (p,q)-extensor over V, say f_{X_0}, such that

$$\lim_{X \to X_0} \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} = 0, \tag{6}$$

i.e.,

$$\lim_{H \to 0} \frac{F(X_0 + H) - F(X_0) - f_{X_0}(H)}{\|H\|} = 0. \tag{7}$$

It is remarkable that if there is such a (p,q)-extensor f_{X_0}, then it must be *unique*.

Indeed, assume that there is another (p,q)-extensor $\overset{\frown}{f}_{X_0}$ which satisfies

$$\lim_{H \to 0} \frac{F(X_0 + H) - F(X_0) - \overset{\frown}{f}_{X_0}(H)}{\|H\|} = 0,$$

or equivalently,

$$\lim_{H \to 0} \frac{\left\| F(X_0 + H) - F(X_0) - \overset{\frown}{f}_{X_0}(H) \right\|}{\|H\|} = 0.$$

By using the triangular inequality which is valid for the norm of multivectors [1], it can be easily establish the following inequality

$$\begin{aligned} 0 \;\leq\; & \frac{\left\| f_{X_0}(H) - \overset{\frown}{f}_{X_0}(H) \right\|}{\|H\|} \leq \frac{\| F(X_0 + H) - F(X_0) - f_{X_0}(H) \|}{\|H\|} \\ & + \frac{\left\| F(X_0 + H) - F(X_0) - \overset{\frown}{f}_{X_0}(H) \right\|}{\|H\|}, \end{aligned}$$

which holds for all $H \neq 0$ (i.e., $X \neq X_0$).

Now, taking the limits for $H \to 0$ (i.e., $X \to X_0$) of these scalar-valued functions of a p-vector variable, we get

$$\lim_{H \to 0} \frac{\left\| f_{X_0}(H) - \overset{\frown}{f}_{X_0}(H) \right\|}{\|H\|} = 0.$$

This implies[3] that for every $A \neq 0$

$$\lim_{\lambda \to 0} \frac{\left\| f_{X_0}(\lambda A) - \overset{\frown}{f}_{X_0}(\lambda A) \right\|}{\|\lambda A\|} = 0.$$

Then, it follows that for every $A \neq 0$

$$\frac{\left\| f_{X_0}(A) - \overset{\frown}{f}_{X_0}(A) \right\|}{\|A\|} = 0,$$

i.e., $f_{X_0}(A) = \overset{\frown}{f}_{X_0}(A)$. Now, for $A = 0$ this equality trivially holds. Therefore, we have proved that $f_{X_0} = \overset{\frown}{f}_{X_0}$.

The (p, q)-extensor f_{X_0} will be called the *differential of the (p, q)-function F at X_0.*

So that, the differentiability of F at $X_0 \in \mathbf{\Omega}^p V$ implies the existence of differential of F at $X_0 \in \mathbf{\Omega}^p V$.

Lemma 8 *Associated to any (p, q)-function F, differentiable at X_0, there exists a (p, q)-function φ_{X_0}, continuous at X_0, such that*

$$\varphi_{X_0}(X_0) = 0 \tag{8}$$

and for every $X \in \mathbf{\Omega}^p V$ it holds

$$F(X) = F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \, \varphi_{X_0}(X). \tag{9}$$

Proof. Since the (p, q)-function F is differentiable at X_0, we can define an auxiliary (p, q)-function φ_{X_0} by

$$\varphi_{X_0}(X) = \begin{cases} 0 & \text{for } X = X_0 \\ \dfrac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} & \text{for } X \neq X_0 \end{cases}.$$

It satisfies $\varphi_{X_0}(X_0) = 0$ and, by taking limit of $\varphi_{X_0}(X)$ for $X \to X_0$, we have

$$\lim_{X \to X_0} \varphi_{X_0}(X) = \lim_{X \to X_0} \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

[3] In order to see that, we can use a lemma: if $\lim_{H \to 0} \Phi(H) = 0$, then $\lim_{\lambda \to 0} \Phi(\lambda A) = 0$, for all $A \neq 0$.

It follows that φ_{X_0} is continuous at X_0 and $\varphi_{X_0}(X_0) = 0$.

Recall now that for $X \neq X_0$ it follows the multivector identity

$$F(X) = F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \, \varphi_{X_0}(X),$$

which for $X = X_0$ it is trivially true. ■

As happens in the $\mathbb{R}^n$ calculus, differentiability implies continuity. Indeed, by taking limits for $X \to X_0$ on both sides of eq.(9), we get $\lim\limits_{X \to X_0} F(X) = F(X_0)$.

2.3.1 Directional Derivative

Since $\mathbf{\Omega}^p V$ is an open subset of $\bigwedge^p V$, any p-vector X_0 belonging to $\mathbf{\Omega}^p V$ is an interior point of $\mathbf{\Omega}^p V$, i.e., there is some ε-neighborhood of X_0, say $N_{X_0}^p(\varepsilon)$, such that $N_{X_0}^p(\varepsilon) \subseteq \mathbf{\Omega}^p V$.

Now, take a non-zero p-vector A and choose a real number λ such that $0 < |\lambda| < \dfrac{\varepsilon}{\|A\|}$. Then, from the obvious inequality $\|(X_0 + \lambda A) - X_0\| = |\lambda| \, \|A\| < \varepsilon$ it follows that $(X_0 + \lambda A) \in N_{X_0}^p(\varepsilon)$. Thus, $(X_0 + \lambda A) \in \mathbf{\Omega}^p V$.

There exists $\lim\limits_{\lambda \to 0} \dfrac{F(X_0 + \lambda A) - F(X_0)}{\lambda}$ and it equals $f_{X_0}(A)$.

Indeed, by using eq.(9) we have

$$\begin{aligned} \frac{F(X_0 + \lambda A) - F(X_0)}{\lambda} &= \frac{f_{X_0}(\lambda A) + \|\lambda A\| \, \varphi_{X_0}(X_0 + \lambda A)}{\lambda} \\ &= f_{X_0}(A) \pm \|A\| \, \varphi_{X_0}(X_0 + \lambda A). \end{aligned}$$

Now, by taking limits for $\lambda \to 0$ on these q-vector functions of a real variable, using the continuity of φ_{X_0} at X_0 and eq.(8), the required result follows.

The *A-directional derivative of F at X_0*, conveniently denoted by $F'_A(X_0)$, is defined to be

$$F'_A(X_0) = \lim_{\lambda \to 0} \frac{F(X_0 + \lambda A) - F(X_0)}{\lambda}, \tag{10}$$

i.e.,

$$F'_A(X_0) = \frac{d}{d\lambda} F(X_0 + \lambda A) \bigg|_{\lambda = 0}. \tag{11}$$

The above observation yields a noticeable multivector identity,

$$F'_A(X_0) = f_{X_0}(A) \tag{12}$$

which relates the A-directional derivation with the differentiation.

Hence, because of the linearity property for (p,q)-extensors it follows that the A-directional derivative of F at X_0 has the remarkable property: for any $\alpha, \beta \in \mathbb{R}$ and $A, B \in \bigwedge^p V$

$$F'_{\alpha A+\beta B}(X_0) = \alpha F'_A(X_0) + \beta F'_B(X_0). \tag{13}$$

Proposition 9 *Let $X : S \to \Lambda^q V$ be any q-vector function of a real variable derivable at $\lambda_0 \in S$. Then, X is differentiable at λ_0 and the differential of X at λ_0 is $X_{\lambda_0} \in ext_0^q(V)$ given by*

$$X_{\lambda_0}(\alpha) = X'(\lambda_0)\alpha, \tag{14}$$

where $X'(\lambda_0)$ is the derivative of X at λ_0.

Proof. We only need to prove that

$$\lim_{\lambda \to \lambda_0} \frac{X(\lambda) - X(\lambda_0) - X'(\lambda_0)(\lambda - \lambda_0)}{|\lambda - \lambda_0|} = 0.$$

Since X is derivable at λ_0, there is a q-vector function of real variable, say ξ_{λ_0}, continuous at λ_0 such that for all $\lambda \in S$

$$X(\lambda) = X(\lambda_0) + (\lambda - \lambda_0)X'(\lambda_0) + (\lambda - \lambda_0)\xi_{\lambda_0}(\lambda),$$

where $\xi_{\lambda_0}(\lambda_0) = 0$.

Hence, it follows that for all $\lambda \neq \lambda_0$

$$\frac{X(\lambda) - X(\lambda_0) - X'(\lambda_0)(\lambda - \lambda_0)}{|\lambda - \lambda_0|} = \pm\xi_{\lambda_0}(\lambda).$$

Thus, by taking limits for $\lambda \to \lambda_0$ on both sides, we get the expected result. ∎

From eqs.(12) and (14), it should be noted that the α-directional derivative of X at λ_0 is given by

$$X'_\alpha(\lambda_0) = X'(\lambda_0)\alpha. \tag{15}$$

2.3.2 Differentiation Rules

Take two open subset of $\bigwedge^p V$, say $\mathbf{\Omega}_1^p V$ and $\mathbf{\Omega}_2^p V$, such that $\mathbf{\Omega}_1^p V \cap \mathbf{\Omega}_2^p V \neq \emptyset$.

Theorem 10 *Let $F : \mathbf{\Omega}_1^p V \to \bigwedge^q V$ and $G : \mathbf{\Omega}_2^p V \to \bigwedge^q V$ be two differentiable functions at $X_0 \in \mathbf{\Omega}_1^p V \cap \mathbf{\Omega}_2^p V$. Denote the differentials of F and G at X_0 by f_{X_0} and g_{X_0}, respectively.*

The addition $F+G : \mathbf{\Omega}_1^p V \cap \mathbf{\Omega}_2^p V \to \Lambda^q V$ such that $(F+G)(X) = F(X) + G(X)$ and the products $F * G : \mathbf{\Omega}_1^p V \cap \mathbf{\Omega}_2^p V \to \bigwedge V$ such that $(F * G)(X) = F(X) * G(X)$, where $*$ means either $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ or (Clifford product), *are also differentiable functions at X_0.*

The differential of $F+G$ at X_0 is $f_{X_0} + g_{X_0}$ and the differentials of $F * G$ at X_0 are given by $A \mapsto f_{X_0}(A) * G(X_0) + F(X_0) * g_{X_0}(A)$.

Proof. We must prove that $s_{X_0} = f_{X_0} + g_{X_0} \in ext_p^q(V)$ satisfies

$$\lim_{X \to X_0} \frac{(F+G)(X) - (F+G)(X_0) - s_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

And, $p_{X_0}(A) = f_{X_0}(A) * G(X_0) + F(X_0) * g_{X_0}(A)$, more general extensors over V, verify

$$\lim_{X \to X_0} \frac{(F*G)(X) - (F*G)(X_0) - p_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

Since F and G are differentiable at X_0, there are two (p,q)-functions φ_{X_0} and ψ_{X_0}, continuous at X_0, such that for all $X \in \mathbf{\Omega}_1^p V \cap \mathbf{\Omega}_2^p V$

$$\begin{aligned} F(X) &= F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \, \varphi_{X_0}(X), \\ G(X) &= G(X_0) + g_{X_0}(X - X_0) + \|X - X_0\| \, \psi_{X_0}(X), \end{aligned}$$

where $\varphi_{X_0}(X_0) = \psi_{X_0}(X_0) = 0$.

Hence, the following multivector identities which hold for all $X \neq X_0$ can be easily deduced

$$\frac{(F+G)(X) - (F+G)(X_0) - s_{X_0}(X - X_0)}{\|X - X_0\|} = \varphi_{X_0}(X) + \psi_{X_0}(X)$$

and

$$\begin{aligned}
& \frac{(F*G)(X)-(F*G)(X_0)-p_{X_0}(X-X_0)}{\|X-X_0\|} \\
= \quad & \varphi_{X_0}(X)*G(X_0)+F(X_0)*\psi_{X_0}(X) \\
& +\varphi_{X_0}(X)*g_{X_0}(X-X_0)+f_{X_0}(X-X_0)*\psi_{X_0}(X) \\
& +\frac{f_{X_0}(X-X_0)*g_{X_0}(X-X_0)}{\|X-X_0\|}+\|X-X_0\|\,\varphi_{X_0}(X)*\psi_{X_0}(X).
\end{aligned}$$

Now, by taking limits for $X \to X_0$ on both sides of these multivector identities[4], we get the required results. ■

In accordance with eq.(12) all differentiation rule turns out to be an A-directional derivation rule.

For the addition of two differentiable functions F and G we have

$$(F+G)'_A(X_0)=(f_{X_0}+g_{X_0})(A)=f_{X_0}(A)+g_{X_0}(A),$$

i.e.,

$$(F+G)'_A(X_0)=F'_A(X_0)+G'_A(X_0). \tag{16}$$

For the products $F*G$ we get

$$(F*G)'_A(X_0)=f_{X_0}(A)*G(X_0)+F(X_0)*g_{X_0}(A),$$

i.e.,

$$(F*G)'_A(X_0)=F'_A(X_0)*G(X_0)+F(X_0)*G'_A(X_0). \tag{17}$$

Theorem 11 *Take an open subset of $\bigwedge^p V$, say $\mathbf{\Omega}^p V$. Let $G:\mathbf{\Omega}^p V\to\bigwedge^q V$ and $F:\bigwedge^q V\to\bigwedge^r V$ be two differentiable functions, the first one at $X_0\in\mathbf{\Omega}^p V$ and the second one at $G(X_0)\in\bigwedge^q V$. Denote by g_{X_0} and $f_{G(X_0)}$ the differentials of G at X_0 and of F at $G(X_0)$, respectively.*

The composition $F\circ G:\mathbf{\Omega}^p V\to\bigwedge^r V$ such that $F\circ G(X)=F(G(X))$ is also a differentiable function at X_0. The differential of $F\circ G$ at X_0 is $f_{G(X_0)}\circ g_{X_0}$.

Proof. We must prove that for $f_{G(X_0)}\circ g_{X_0}\in ext_p^r(V)$ it holds

$$\lim_{X\to X_0}\frac{F\circ G(X)-F\circ G(X_0)-f_{G(X_0)}\circ g_{X_0}(X-X_0)}{\|X-X_0\|}=0.$$

[4]For calculating some limits we have used an useful lemma. For any $f\in ext_p^q(V)$ there exists a real number $M\geq 0$ such that for every $X\in\bigwedge^p V:\|f(X)\|\leq M\|X\|$.

Since G is differentiable at X_0 and F is differentiable at $G(X_0)$, there are a (p,q)-function $X \mapsto \psi_{X_0}(X)$ and a (q,r)-function $Y \mapsto \varphi_{G(X_0)}(Y)$, the first one continuous at X_0 and the second one continuous at $G(X_0)$, such that for all $X \in \mathbf{\Omega}^p V$ and $Y \in \bigwedge^q V$

$$\begin{aligned} G(X) &= G(X_0) + g_{X_0}(X - X_0) + \|X - X_0\| \, \psi_{X_0}(X), \\ F(Y) &= F(G(X_0)) + f_{G(X_0)}(Y - G(X_0)) + \|Y - G(X_0)\| \, \varphi_{G(X_0)}(Y), \end{aligned}$$

where $\psi_{X_0}(X_0) = 0$ and $\varphi_{G(X_0)}(G(X_0)) = 0$.

Hence, it follows easily a multivector identity which holds for all $X \neq X_0$

$$\begin{aligned} &\frac{F \circ G(X) - F \circ G(X_0) - f_{G(X_0)} \circ g_{X_0}(X - X_0)}{\|X - X_0\|} \\ &= f_{G(X_0)} \circ \psi_{X_0}(X) + \frac{\|G(X) - G(X_0)\|}{\|X - X_0\|} \varphi_{G(X_0)} \circ G(X). \end{aligned}$$

Now, by taking limits for $X \to X_0$ on both sides, using the equations: $\lim_{X \to X_0} f_{G(X_0)} \circ \psi_{X_0}(X) = 0$ and $\lim_{X \to X_0} \dfrac{\|G(X) - G(X_0)\|}{\|X - X_0\|} \varphi_{G(X_0)} \circ G(X) = 0$, we get the expected result. ■

This *chain rule* for differentiation turns out to be a chain rule for A-directional derivation.

For a differentiable G at X_0 and a differentiable F at $G(X_0)$ we have

$$\begin{aligned} (F \circ G)'_A(X_0) &= f_{G(X_0)} \circ g_{X_0}(A) = f_{G(X_0)}(G'_A(X_0)), \\ (F \circ G)'_A(X_0) &= F'_{G'_A(X_0)}(G(X_0)). \end{aligned} \tag{18}$$

We study now two very important particular cases of the general chain rule for the A-directional derivation:

For $p > 0$, $q = 0$ and $r > 0$, i.e., for the A-directional derivative of the composition of $\Phi : \mathbf{\Omega}^p V \to \mathbb{R}$ with $X : \mathbb{R} \to \bigwedge^r V$ at $X_0 \in \mathbf{\Omega}^p V$, by using eq.(18) and eq.(15), we have

$$\begin{aligned} (X \circ \Phi)'_A(X_0) &= X'_{\Phi'_A(X_0)}(\Phi(X_0)), \\ (X \circ \Phi)'_A(X_0) &= X'(\Phi(X_0))\Phi'_A(X_0). \end{aligned} \tag{19}$$

For $p = 0$, $q > 0$ and $r > 0$, i.e., for the derivative of the composition of $X : S \to \bigwedge^q V$ with $F : \bigwedge^q V \to \bigwedge^r V$ at $\lambda_0 \in S$, by using eq.(15), eq.(18) and eq.(13), we have

$$\begin{aligned} (F \circ X)'(\lambda_0)\alpha &= (F \circ X)'_\alpha(\lambda_0) = F'_{X'_\alpha(\lambda_0)}(X(\lambda_0)), \\ (F \circ X)'(\lambda_0)\alpha &= F'_{X'(\lambda_0)\alpha}(X(\lambda_0)). \end{aligned} \tag{20}$$

2.3.3 Derivatives

Let $(\{e_k\}, \{e^k\})$ be a pair of reciprocal bases of V. Let $F : \mathbf{\Omega}^p V \to \bigwedge^q V$ be any differentiable function at $X_0 \in \mathbf{\Omega}^p V$. Define the set $\mathbf{\Lambda}^p V = \{X \in \mathbf{\Omega}^p V \ / \ F$ is differentiable at $X\} \subseteq \mathbf{\Omega}^p V$.

It follows that it must exist a well-defined function $F'_A : \mathbf{\Lambda}^p V \to \bigwedge^q V$ such that $F'_A(X)$ equals the A-directional derivative of F at each $X \in \mathbf{\Lambda}^p V$. It is called the *$A$-directional derivative function of F*.

Then, we can define exactly *four* derivative-like functions for F, namely, $*F' : \mathbf{\Lambda}^p V \to \bigwedge^q V$ such that

$$\begin{aligned} *F'(X) &= \frac{1}{p!}(e^{j_1} \wedge \ldots e^{j_p}) * F'_{e_{j_1} \wedge \ldots e_{j_p}}(X) \\ &= \frac{1}{p!}(e_{j_1} \wedge \ldots e_{j_p}) * F'_{e^{j_1} \wedge \ldots e^{j_p}}(X), \end{aligned} \tag{21}$$

where $*$ means either $(\wedge)$, $(\cdot)$, $(\lrcorner)$ or (*Clifford product*).

Whichever $*F'$ is a well-defined function associated to F, since $*F'(X)$ are multivectors which do not depend on the choice of $(\{e_k\}, \{e^k\})$.

We will call $\wedge F'$, $\cdot F'$, $\lrcorner F'$ and F' (i.e., $* \equiv$ (*Clifford product*)) respectively the (generalized) *curl, scalar divergence, left contracted divergence* and *gradient of F*. Sometimes the gradient of F is called the *standard derivative of F*.

On the *real vector space of differentiable (p,q)-functions over V* we can introduce exactly four derivative-like operators, namely, $F \mapsto \partial_X * F$ such that

$$\partial_X * F = *F', \tag{22}$$

i.e., for every $X \in \mathbf{\Lambda}^p V$

$$\begin{aligned} \partial_X * F(X) &= \frac{1}{p!}(e^{j_1} \wedge \ldots e^{j_p}) * F'_{e_{j_1} \wedge \ldots e_{j_p}}(X) \\ &= \frac{1}{p!}(e_{j_1} \wedge \ldots e_{j_p}) * F'_{e^{j_1} \wedge \ldots e^{j_p}}(X). \end{aligned} \tag{23}$$

The special cases $\partial_X \wedge$, $\partial_X \cdot$, $\partial_X \lrcorner$ and ∂_X (i.e., $* \equiv$ (*Clifford product*)) will be called respectively the (generalized) *curl, scalar divergence, left contracted divergence* and *gradient operator.*

For differentiable functions it is also possible to introduce a remarkable operator denoted by $A \cdot \partial_X$, and defined as follows

$$\begin{aligned} A \cdot \partial_X F(X) &= (A \cdot \frac{1}{p!} e^{j_1} \wedge \ldots e^{j_p}) F'_{e_{j_1} \wedge \ldots e_{j_p}}(X), \\ &= (A \cdot \frac{1}{p!} e_{j_1} \wedge \ldots e_{j_p}) F'_{e^{j_1} \wedge \ldots e^{j_p}}(X), \end{aligned} \tag{24}$$

i.e., by eq.(13)

$$A \cdot \partial_X F(X) = F'_A(X). \tag{25}$$

The operator $A \cdot \partial_X$ is called the A-directional derivative operator. It maps $F \to F'_A$, i.e., $A \cdot \partial_X F = F'_A$.

Now, we write out the property expressed by eq.(13) using the operator $A \cdot \partial_X$. We have,

$$(\alpha A + \beta B) \cdot \partial_X F(X_0) = \alpha A \cdot \partial_X F(X_0) + \beta B \cdot \partial_X F(X_0). \tag{26}$$

We have then a suggestive operator identity

$$(\alpha A + \beta B) \cdot \partial_X = \alpha A \cdot \partial_X + \beta B \cdot \partial_X. \tag{27}$$

We have also rules holding for the A-directional derivation of addition, products and composition of differentiable functions, and eq.(16), eq.(17) and eq.(18) can be written as:

$$A \cdot \partial_X (F + G)(X_0) = A \cdot \partial_X F(X_0) + A \cdot \partial_X G(X_0), \tag{28}$$

i.e., $A \cdot \partial_X (F + G) = A \cdot \partial_X F + A \cdot \partial_X G$.

$$A \cdot \partial_X (F * G)(X_0) = A \cdot \partial_X F(X_0) * G(X_0) + F(X_0) * A \cdot \partial_X G(X_0), \tag{29}$$

i.e., $A \cdot \partial_X (F * G) = (A \cdot \partial_X F) * G + F * (A \cdot \partial_X G)$.

If $X \mapsto G(X)$ and $Y \mapsto F(Y)$, then

$$A \cdot \partial_X (F \circ G)(X_0) = A \cdot \partial_X G(X_0) \cdot \partial_Y F(G(X_0)). \tag{30}$$

3 Conclusions

We studied in detail the concepts of limit, continuity and differentiability for multivector functions of a p-vector variable. Several types of derivatives for these objects have been introduced as, e.g., the A-directional derivative and the generalized curl, divergence and gradient. We saw that the concept of extensor plays a key role in the formulation of the notion of differentiability, it implies the existence of the differential extensor. We have proved the basic derivation rules for all suitable products of multivector functions and for composition of multivector functions. The generalization of these results towards a formulation of a general theory of multivector functions of several multivector variables can be done easily. The concept of multivector derivatives has been first introduced in [3]. We think that our presentation is an improvement of that presentation, clearing many issues.

In the following paper about multivector *functionals*, we will see that the gradient-derivative plays a key role in the formulation of derivation concepts for the so-called induced multivector functionals.

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