Multivector Functions of a Real Variable*

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Abstract

This paper is an introduction to the theory of multivector functions of a real variable. The notions of limit, continuity and derivative for these objects are given. The theory of multivector functions of a real variable, even being similar to the usual theory of vector functions of a real variable, has some subtle issues which make its presentation worhtwhile.We refer in particular to the derivative rules involving exterior and Clifford products, and also to the rule for derivation of a composition of an ordinary scalar function with a multivector function of a real variable.

Contents

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1 Introduction

This is paper V of a series of seven. Here, we develop a theory of multivector functions of a real variable following analogous steps to the elementary theory of vector functions of a real variable. We introduce the notions of limit and continuity, and study the concept of derivative. There are *subtle* points that are emphasized whose understanding is crucial for the development of a theory of multivector functions of a multivector variable (as constructed in paper VI of the present series). We give the complete proofs for the derivative rules involving all the suitable products of multivector functions of a real variable, and for the composition of an ordinary scalar function with a multivector function of a real variable.

2 Multivector Functions of a Real Variable

Any mapping which takes real numbers of $S \subseteq \mathbb{R}$ to multivectors of $\bigwedge V$ will be called a *multivector function of a real variable over* V. In particular, $X : S \to \bigwedge^p V$ is said to be a *p-vector function of a real variable.* And, the special cases $p = 0$, $p = 1$, $p = 2, \ldots,$ etc. are named as a *scalar, vector, bivector,. . . , etc. function of a real variable,* respectively.

2.1 Limit Notion

We begin by recalling the fundamental concept of δ-neighborhood for a real number λ_0.

Take any real $\delta > 0$. The set[1] $N_{\lambda_0}(\delta) = \{\lambda \in \mathbb{R}/\ |\lambda - \lambda_0| < \delta\}$, clearly a subset of $\mathbb{R}$, is usually called a δ*-neighborhood of* λ_0. The set $N'_{\lambda_0}(\delta) =$

[1] The symbol $|\ |$ denotes as usual the *absolute value* (or, *module*) function.

$N_{\lambda_0}(\delta) - \{\lambda_0\}$, i.e., $N'_{\lambda_0}(\delta) = \{\lambda \in \mathbb{R}/\ 0 < |\lambda - \lambda_0| < \delta\}$, is said to be a *reduced δ-neighborhood of* λ_0.

We recall now the important concept of cluster point and interior point of $S \subseteq \mathbb{R}$.

A real number λ_0 is said to be a *cluster point of* S if and only if for every $N_{\lambda_0}(\delta) : N'_{\lambda_0}(\delta) \cap S \neq \emptyset$, i.e., all reduced δ-neighborhood of λ_0 contains at least one real number of S.

A real number λ_0 is said to be a *interior point of* S if and only if there exists $N_{\lambda_0}(\delta)$ such that $N_{\lambda_0}(\delta) \subseteq S$, i.e., any real number of some δ-neighborhood of λ_0 belongs also to S.

Note that all interior point of S is also cluster point of S.

If the set of interior point of S coincides with S, i.e., all real number of S is also interior point of S, then S is said to be an *open subset of* $\mathbb{R}$.

Next we introduce the concept of *norm* of a multivector X.

Assume that $\bigwedge V$ has been endowed with an *euclidean scalar product* $(\cdot)$, as e.g., by taking any fixed basis $\{b_k\}$ for V and its dual basis $\{\beta^k\}$ for V^*, etc. See paper I of this series [1]. As we already know, the euclidean scalar product is always *definite positive,* i.e., for all $X \in \bigwedge V : X \cdot X \geq 0$ and $X \cdot X = 0$ if and only if $X = 0$.

This property of the scalar product permit us to introduce the *norm of a multivector* X as being the non-negative real number $\|X\|$ given by

$$\|X\| = \sqrt{X \cdot X}. \tag{1}$$

We read $\|X\|$ as the *norm of* X.

The norm of multivectors satisfies the following two usual inequalities:

n1 The Cauchy-Schwarz inequality, i.e., for all $X, Y \in \bigwedge V$

$$|X \cdot Y| \leq \|X\| \|Y\|. \tag{2}$$

n2 The triangular inequality, i.e., for all $X, Y \in \bigwedge V$

$$\|X + Y\| \leq \|X\| + \|Y\|. \tag{3}$$

The first inequality follows from the fact that $(\cdot)$ is positive definite. The second one is an immediate consequence of the first one.

Take $S \subseteq \mathbb{R}$. Let $X : S \to \bigwedge V$ be any multivector function of a real variable and take $\lambda_0 \in S$ to be a cluster point of S.

A multivector L is said to be the *limit of* $X(\lambda)$ *for* λ *approaching to* λ_0 if and only if for every real $\varepsilon > 0$ there exists some real $\delta > 0$ such that if for all $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$, then $\|X(\lambda) - L\| < \varepsilon$. It is denoted by $\lim_{\lambda \to \lambda_0} X(\lambda) = L$.

In particular, a scalar function of a real variable is just an ordinary real function and, as we can see, the above definition of limit is reduced to the ordinary definition of limit which appears in real analysis.

Proposition 1 *Let $X : S \to \bigwedge V$ and $Y : S \to \bigwedge V$ be two multivector functions of a real variable. If there exist $\lim_{\lambda \to \lambda_0} X(\lambda)$ and $\lim_{\lambda \to \lambda_0} Y(\lambda)$, then there exists $\lim_{\lambda \to \lambda_0} (X + Y)(\lambda)$ and*

$$\lim_{\lambda \to \lambda_0} (X + Y)(\lambda) = \lim_{\lambda \to \lambda_0} X(\lambda) + \lim_{\lambda \to \lambda_0} Y(\lambda). \tag{4}$$

Proof. Let $\lim_{\lambda \to \lambda_0} X(\lambda) = L_1$ and $\lim_{\lambda \to \lambda_0} Y(\lambda) = L_2$. Then, we must prove that $\lim_{\lambda \to \lambda_0} (X + Y)(\lambda) = L_1 + L_2$.

Given an arbitrary real $\varepsilon > 0$, since $\lim_{\lambda \to \lambda_0} X(\lambda) = L_1$ and $\lim_{\lambda \to \lambda_0} Y(\lambda) = L_2$, there are two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} \|X(\lambda) - L_1\| &< \frac{\varepsilon}{2}, \text{ for } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_1, \\ \|Y(\lambda) - L_2\| &< \frac{\varepsilon}{2}, \text{ for } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_2. \end{aligned}$$

Thus, there is a real $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\|X(\lambda) - L_1\| < \frac{\varepsilon}{2} \text{ and } \|Y(\lambda) - L_2\| < \frac{\varepsilon}{2},$$

for $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$. Hence, by using eq.(3) it follows that

$$\begin{aligned} \|(X + Y)(\lambda) - (L_1 + L_2)\| &= \|X(\lambda) - L_1 + Y(\lambda) - L_2\| \\ &\leq \|X(\lambda) - L_1\| + \|Y(\lambda) - L_2\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$.

Therefore, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$, then $\|(X + Y)(\lambda) - (L_1 + L_2)\| < \varepsilon$. ■

Proposition 2 *Let* $\phi : S \to \mathbb{R}$ *and* $X : S \to \bigwedge V$ *be an ordinary real function and a multivector function of a real variable. If there exist* $\lim_{\lambda \to \lambda_0} \phi(\lambda)$ *(the ordinary limit) and* $\lim_{\lambda \to \lambda_0} X(\lambda)$*, then there exists* $\lim_{\lambda \to \lambda_0} (\phi X)(\lambda)$ *and*

$$\lim_{\lambda \to \lambda_0} (\phi X)(\lambda) = \lim_{\lambda \to \lambda_0} \phi(\lambda) \lim_{\lambda \to \lambda_0} X(\lambda). \tag{5}$$

Proof. Let $\lim_{\lambda \to \lambda_0} \phi(\lambda) = \phi_0$ and $\lim_{\lambda \to \lambda_0} X(\lambda) = X_0$. Then, we must prove that $\lim_{\lambda \to \lambda_0} (\phi X)(\lambda) = \phi_0 X_0$.

First, since $\lim_{\lambda \to \lambda_0} \phi(\lambda) = \phi_0$ it can be found a $\delta_1 > 0$ such that

$$|\phi(\lambda) - \phi_0| < 1, \text{ whenever } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_1,$$

i.e.,

$$|\phi(\lambda)| < 1 + |\phi_0|\,, \text{ whenever } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_1.$$

Where the triangular inequality for real numbers $|\alpha| - |\beta| \leq |\alpha - \beta|$ was used.

Now, taken an arbitrary $\varepsilon > 0$, since $\lim_{\lambda \to \lambda_0} \phi(\lambda) = \phi_0$ and $\lim_{\lambda \to \lambda_0} X(\lambda) = X_0$, they can be found a $\delta_2 > 0$ and a $\delta_3 > 0$ such that

$$\begin{aligned} |\phi(\lambda) - \phi_0| &< \frac{\varepsilon}{2(1 + \|X_0\|)}, \text{ whenever } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_2, \\ \|X(\lambda) - X(\lambda_0)\| &< \frac{\varepsilon}{2(1 + |\phi_0|)}, \text{ whenever } \lambda \in S \text{ and } 0 < |\lambda - \lambda_0| < \delta_3. \end{aligned}$$

Thus, given an arbitrary $\varepsilon > 0$ there is a $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$\begin{aligned} |\phi(\lambda)| &< 1 + |\phi_0|\,, \\ |\phi(\lambda) - \phi_0| &< \frac{\varepsilon}{2(1 + \|X_0\|)}, \\ \|X(\lambda) - X_0)\| &< \frac{\varepsilon}{2(1 + |\phi_0|)}, \end{aligned}$$

for $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$. Hence, it follows that

$$\begin{aligned} \|(\phi X)(\lambda) - \phi_0 X_0\| &= \|\phi(\lambda)(X(\lambda) - X_0) + (\phi(\lambda) - \phi_0) X_0\| \\ &\leq |\phi(\lambda)|\,\|X(\lambda) - X_0\| + |\phi(\lambda) - \phi_0|\,\|X_0\| \\ &< |\phi(\lambda)|\,\|X(\lambda) - X_0\| + |\phi(\lambda) - \phi_0|\,(1 + \|X_0\|) \\ &< (1 + |\phi_0|) \frac{\varepsilon}{2(1 + |\phi_0|)} + \frac{\varepsilon}{2(1 + \|X_0\|)} (1 + \|X_0\|) = \varepsilon, \end{aligned}$$

for $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$. In the proof above we use some properties of the norm of multivectors.

Therefore, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\lambda \in S$ and $0 < |\lambda - \lambda_0| < \delta$, then $\|(\phi X)(\lambda) - \phi_0 X_0\| < \varepsilon$. ■

Lemma 3 *There exists* $\lim_{\lambda \to \lambda_0} X(\lambda)$ *if and only if there exist any one of the ordinary limits, either* $\lim_{\lambda \to \lambda_0} X^J(\lambda)$ *or* $\lim_{\lambda \to \lambda_0} X_J(\lambda)$. *It holds*

$$\lim_{\lambda \to \lambda_0} X(\lambda) = \sum_J \frac{1}{\nu(J)!} \lim_{\lambda \to \lambda_0} X^J(\lambda) e_J = \sum_J \frac{1}{\nu(J)!} \lim_{\lambda \to \lambda_0} X_J(\lambda) e^J. \tag{6}$$

Proof. It is an immediate consequence of eqs.(4) and (5). ■

Proposition 4 *Let* $X : S \to \bigwedge V$ *and* $Y : S \to \bigwedge V$ *be two multivector functions of a real variable. We can define the products* $X * Y : S \to \bigwedge V$ *such that* $(X * Y)(\lambda) = X(\lambda) * Y(\lambda)$ *where* $*$ *holds for either* $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ *or* (Clifford product). *If there exist* $\lim_{\lambda \to \lambda_0} X(\lambda)$ *and* $\lim_{\lambda \to \lambda_0} Y(\lambda)$, *then there exists* $\lim_{\lambda \to \lambda_0} (X * Y)(\lambda)$ *and*

$$\lim_{\lambda \to \lambda_0} (X * Y)(\lambda) = \lim_{\lambda \to \lambda_0} X(\lambda) * \lim_{\lambda \to \lambda_0} Y(\lambda). \tag{7}$$

Proof. It is an immediate consequence of eq.(6). ■

2.2 Continuity Notion

Take $S \subseteq \mathbb{R}$. A multivector function of a real variable $X : S \to \bigwedge V$ is said to be *continuous at* $\lambda_0 \in S$ if and only if there exists[2] $\lim_{\lambda \to \lambda_0} X(\lambda)$ and

$$\lim_{\lambda \to \lambda_0} X(\lambda) = X(\lambda_0). \tag{8}$$

Lemma 5 *The multivector function* $\lambda \mapsto X(\lambda)$ *is continuous at* λ_0 *if and only if whichever component scalar function either* $\lambda \mapsto X^J(\lambda)$ *or* $\lambda \mapsto X_J(\lambda)$ *is continuous at* λ_0.

[2] See that λ_0 has to be cluster point of S.

Proposition 6 *Let* $X : S \to \bigwedge V$ *and* $Y : S \to \bigwedge V$ *be two continuous functions at* $\lambda_0 \in S$.

The addition $X + Y : S \to \bigwedge V$ *such that* $(X+Y)(\lambda) = X(\lambda) + Y(\lambda)$ *and the products* $X * Y : S \to \bigwedge V$ *such that* $(X * Y)(\lambda) = X(\lambda) * Y(\lambda)$, *where* $*$ *means either* $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ *or* (Clifford product), *are also continuous functions at* λ_0.

Proof. It is an immediate consequence of eqs.(4) and (7). ■

Proposition 7 *Let* $\phi : S \to \mathbb{R}$ *and* $X : \mathbb{R} \to \bigwedge V$ *be two continuous functions, the first one at* $\lambda_0 \in S$ *and the second one at* $\phi(\lambda_0) \in R$.

The composition $X \circ \phi : S \to \bigwedge V$ *such that* $X \circ \phi(\lambda) = X(\phi(\lambda))$ *is a continuous function at* λ_0.

2.3 Derivative

Take $S \subseteq \mathbb{R}$ be an open set of $\mathbb{R}$. A multivector function of a real variable $X : S \to \bigwedge V$ is said to be *derivable at* $\lambda_0 \in S$ if and only if there exists $\lim\limits_{\lambda \to \lambda_0} \dfrac{X(\lambda) - X(\lambda_0)}{\lambda - \lambda_0}$. This *multivector-limit* is usually called the *derivative of* X *at* $\lambda_0 \in S$, and often denoted by $X'(\lambda_0)$, i.e.,

$$X'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{X(\lambda) - X(\lambda_0)}{\lambda - \lambda_0}. \tag{9}$$

So that, the derivability of X at λ_0 means the existence of derivative of X at λ_0.

Lemma 8 *Associated to any multivector function* X, *derivable at* λ_0, *there exists a multivector function* ξ_{λ_0}, *continuous at* λ_0, *such that*

$$\xi_{\lambda_0}(\lambda_0) = 0 \tag{10}$$

and for all $\lambda \in S$ *it holds*

$$X(\lambda) = X(\lambda_0) + (\lambda - \lambda_0)X'(\lambda_0) + (\lambda - \lambda_0)\xi_{\lambda_0}(\lambda). \tag{11}$$

Proof. Since X is derivable at λ_0 we can define ξ_{λ_0} by

$$\xi_{\lambda_0}(\lambda) = \begin{cases} 0 & \text{for } \lambda = \lambda_0 \\ \dfrac{X(\lambda) - X(\lambda_0)}{\lambda - \lambda_0} - X'(\lambda_0) & \text{for } \lambda \neq \lambda_0 \end{cases}.$$

We see that $\xi_{\lambda_0}(\lambda_0) = 0$ and by taking limit of $\xi_{\lambda_0}(\lambda)$ for $\lambda \to \lambda_0$ we have

$$\lim_{\lambda \to \lambda_0} \xi_{\lambda_0}(\lambda) = \lim_{\lambda \to \lambda_0} (\frac{X(\lambda) - X(\lambda_0)}{\lambda - \lambda_0} - X'(\lambda_0)) = X'(\lambda_0) - X'(\lambda_0) = 0.$$

It follows that ξ_{λ_0} is continuous at λ_0 and so the first statement holds.

On another way, for $\lambda \neq \lambda_0$ we get the multivector identity

$$X(\lambda) = X(\lambda_0) + (\lambda - \lambda_0)X'(\lambda_0) + (\lambda - \lambda_0)\xi_{\lambda_0}(\lambda)$$

but, for $\lambda = \lambda_0$ it is trivially true. Thus, the second statement holds. ■

As happens in real analysis, derivability implies continuity. Indeed, by taking limits for $\lambda \to \lambda_0$ on both sides of eq.(11) we get $\lim_{\lambda \to \lambda_0} X(\lambda) = X(\lambda_0)$.

2.3.1 Derivation Rules

Take two open subset of $\mathbb{R}$, say S_1 and S_2, such that $S_1 \cap S_2 \neq \emptyset$.

Theorem 9 *Let* $S_1 \ni \lambda \mapsto X(\lambda) \in \bigwedge V$ *and* $S_2 \ni \lambda \mapsto Y(\lambda) \in \bigwedge V$ *be two derivable functions at* $\lambda_0 \in S_1 \cap S_2$.

The addition $S_1 \cap S_2 \ni \lambda \mapsto (X+Y)(\lambda) \in \Lambda V$ *such that* $(X+Y)(\lambda) = X(\lambda) + Y(\lambda)$ *and the products* $S_1 \cap S_2 \ni \lambda \mapsto (X * Y)(\lambda) \in \Lambda V$ *such that* $(X * Y)(\lambda) = X(\lambda) * Y(\lambda)$, *where* $*$ *means either* $(\wedge)$, $(\cdot)$, $(\lrcorner\llcorner)$ *or (Clifford product), are also derivable functions at* λ_0.

The derivatives of $X+Y$ *and* $X*Y$ *at* λ_0 *are given by*

$$(X+Y)'(\lambda_0) = X'(\lambda_0) + Y'(\lambda_0) \tag{12}$$

and

$$(X * Y)'(\lambda_0) = X'(\lambda_0) * Y(\lambda_0) + X(\lambda_0) * Y'(\lambda_0). \tag{13}$$

Proof. We only need to verify that

$$\lim_{\lambda \to \lambda_0} \frac{(X+Y)(\lambda) - (X+Y)(\lambda_0)}{\lambda - \lambda_0} = X'(\lambda_0) + Y'(\lambda_0)$$

and that

$$\lim_{\lambda \to \lambda_0} \frac{(X*Y)(\lambda) - (X*Y)(\lambda_0)}{\lambda - \lambda_0} = X'(\lambda_0) * Y(\lambda_0) + X(\lambda_0) * Y'(\lambda_0).$$

First, we set the following multivector identities which hold for all $\lambda \neq \lambda_0$

$$\frac{(X+Y)(\lambda)-(X+Y)(\lambda_0)}{\lambda-\lambda_0} = \frac{X(\lambda)-X(\lambda_0)}{\lambda-\lambda_0} + \frac{Y(\lambda)-Y(\lambda_0)}{\lambda-\lambda_0}$$

and

$$\frac{(X*Y)(\lambda)-(X*Y)(\lambda_0)}{\lambda-\lambda_0} = \frac{X(\lambda)-X(\lambda_0)}{\lambda-\lambda_0} * Y(\lambda_0) + X(\lambda) * \frac{Y(\lambda)-Y(\lambda_0)}{\lambda-\lambda_0}.$$

Now, by taking limits for $\lambda \to \lambda_0$ on both sides of these multivector identities, using the equation[3]: $\lim\limits_{\lambda\to\lambda_0} X(\lambda) = X(\lambda_0)$, we get the expected results. ■

Theorem 10 *Let $\phi : S \to \mathbb{R}$ and $X : \mathbb{R} \to \bigwedge V$ be two derivable functions, the first one at $\lambda_0 \in S$ and the second one at $\phi(\lambda_0) \in \mathbb{R}$.*

The composition $X \circ \phi : S \to \bigwedge V$ such that $X \circ \phi(\lambda) = X(\phi(\lambda))$ is a derivable function at λ_0 and its derivative at λ_0 is given by

$$(X \circ \phi)'(\lambda_0) = \phi'(\lambda_0) X'(\phi(\lambda_0)). \tag{14}$$

Proof. We must prove that

$$\lim_{\lambda\to\lambda_0} \frac{X\circ\phi(\lambda) - X\circ\phi(\lambda_0)}{\lambda-\lambda_0} = \phi'(\lambda_0) X'(\phi(\lambda_0)).$$

Since X is derivable at $\phi(\lambda_0)$, there is a multivector function $\mu \mapsto \xi_{\phi(\lambda_0)}(\mu)$, continuous at $\phi(\lambda_0)$, such that for all $\mu \in \mathbb{R}$

$$X(\mu) = X(\phi(\lambda_0)) + (\mu - \phi(\lambda_0)) X'(\phi(\lambda_0)) + (\mu - \phi(\lambda_0)) \xi_{\phi(\lambda_0)}(\mu),$$

where $\xi_{\phi(\lambda_0)}(\phi(\lambda_0)) = 0$.

Now, the following multivector identity (as can be easily shown) holds for all $\lambda \neq \lambda_0$,

$$\frac{X\circ\phi(\lambda) - X\circ\phi(\lambda_0)}{\lambda-\lambda_0} = \frac{\phi(\lambda)-\phi(\lambda_0)}{\lambda-\lambda_0} X'(\phi(\lambda_0)) + \frac{\phi(\lambda)-\phi(\lambda_0)}{\lambda-\lambda_0} \xi_{\phi(\lambda_0)} \circ \phi(\lambda).$$

Now, by taking limits for $\lambda \to \lambda_0$ on both sides, using the equation[4]: $\lim\limits_{\lambda\to\lambda_0} \xi_{\phi(\lambda_0)} \circ \phi(\lambda) = 0$, we get the required result. ■

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[3]We have used the fact that for X, derivability implies in continuity.

[4]It was used that composition of ϕ with $\xi_{\phi(\lambda_0)}$, where ϕ is continuous at λ_0 and $\xi_{\phi(\lambda_0)}$ is continuous at $\phi(\lambda_0)$, is continuous at λ_0.

3 Conclusions

In this paper we introduced the concept of multivector functions of a real variable, and the notions of limit and continuity for them, and studied the concept of derivative of these objects. Although our theory of multivector functions of a real variable parallels the theory of vector functions of a real variable, we believe that our presentation is worthwhile, since it treats some subtle points as, e.g., the derivative rules involving all the suitable products of the multivector functions of a real variable. The generalization of these ideas towards a general theory of multivector functions of several real variables can be done without great difficulty.

The results developed in this paper are essential ingredients for papers VI and VII of the present series of papers, where we obtain important results concerning to the theory of multivector functions of a multivector variable, and to the theory of multivector functionals.

Before ending, we quote that the concept of multivector functions (of real variable or multivector variable) has been first introduced in [2], and used together with the notion of multivector functionals by some authors, in order to study problems ranging from linear algebra to applications to physical sciences and engineering (e.g., [3][4]). We believe that our approach is a real contribution to those presentations of these subjects.

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