Metric Clifford Algebra[*]

V. V. Fernández[1][†], A. M. Moya[1][‡] and W. A. Rodrigues Jr.[1,2][§]

[1] Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP CP 6065
13083-970 Campinas-SP, Brazil

[2] Department of Mathematical Sciences, University of Liverpool
Liverpool, L69 3BX, UK

10/30/2001

Abstract

In this paper we introduce the concept of metric Clifford algebra $\mathcal{C}\ell(V, g)$ for a n-dimensional real vector space V endowed with a metric extensor g whose signature is (p, q), with $p+q = n$. The metric Clifford product on $\mathcal{C}\ell(V, g)$ appears as a well-defined *deformation* (induced by g) of an euclidean Clifford product on $\mathcal{C}\ell(V)$. Associated with the metric extensor g, there is a gauge metric extensor h which codifies all the geometric information just contained in g. The precise form of such h is here determined. Moreover, we present and give a proof of the so-called *golden formula,* which is important in many applications that naturally appear in ours studies of multivector functions, and differential geometry and theoretical physics.

Contents

1

1   Introduction

This is paper IV of a series of seven. Here, we introduce the concept of a metric Clifford algebra for a n-dimensional real vector space V endowed with a metric extensor g of an arbitrary signature (p, q), with $p + q = n$. The novelty, regarding previous presentations of the subject as, e.g., in ([1]-[3]), is that a metric Clifford product appears as a well-defined *deformation* of an euclidean Clifford product. More important, we show that associated to any metric extensor g there is a gauge metric extensor h (defined 'modulus' a gauge) such that $g = h^{\dagger} \circ \eta \circ h$, where η is a standard orthogonal metric extensor over V with the same signature as g. This theorem for the decomposition of g (which is somewhat analogous to Silvester's theorem) will play a fundamental role in the intrinsic formulation of the differential geometry on smooth manifolds. The paper ends with the proof of the so-called *golden formula*, which will show worth to deserve that name. We introduce also the concepts of standard and metric Hodge (star) operators, and find a formula connecting them.

2 Metric Clifford Algebra of Multivectors

2.1 Metric Scalar Product

Let us consider $\bigwedge V$ endowed with the eulidean scalar product associated to any fixed basis for V denoted by $\{b_k\}$, as in previous papers of this series ([4]-[5]), i.e., the b-scalar product on $\bigwedge V$.

Now, take a $(1,1)$-extensor over V, say g, adjoint symmetric ($g = g^\dagger$) and non-degenerate ($\det[g] \neq 0$). It will be called a *metric extensor* over V.

We can define another scalar product of multivectors $X, Y \in \bigwedge V$ by

$$X \underset{g}{\cdot} Y = \underline{g}(X) \cdot Y, \tag{1}$$

where $\underline{g}$ is the *extended* of g. It will be called a *metric scalar product* generated by g. Or, *g-scalar product*, for short.

As we can see, this scalar product is a well-defined scalar product on $\bigwedge V$. It is symmetric, satisfies the distributive laws, has the mixed associativity property and is non-degenerate, i.e., $X \underset{g}{\cdot} Y = 0$ for all X, implies $Y = 0$.

All the properties just mentioned above are immediate consequences of the corresponding ones for the b-scalar product. But, a g-scalar product is not necessarily *positive definite*.

We present now some of the most important properties of the g-scalar product of multivectors.

g1 For any $\alpha, \beta \in \mathbb{R}$

$$\alpha \underset{g}{\cdot} \beta = \alpha\beta \text{ (real product).} \tag{2}$$

g2 For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \underset{g}{\cdot} Y_k = 0, \text{ if } j \neq k. \tag{3}$$

g3 For any simple k-vectors $v_1 \wedge \ldots v_k \in \bigwedge^k V$ and $w_1 \wedge \ldots w_k \in \bigwedge^k V$

$$(v_1 \wedge \ldots v_k) \underset{g}{\cdot} (w_1 \wedge \ldots w_k) = \det \begin{bmatrix} v_1 \underset{g}{\cdot} w_1 & \ldots & v_1 \underset{g}{\cdot} w_k \\ \ldots & \ldots & \ldots \\ v_k \underset{g}{\cdot} w_1 & \ldots & v_k \underset{g}{\cdot} w_k \end{bmatrix}, \tag{4}$$

3

where (as before) by $\det \begin{bmatrix} v_p \underset{g}{\cdot} w_q \end{bmatrix}$ we denote the classical $k \times k$ determinant.

Eqs.(2-4) follow without difficulties from the corresponding properties of the b-scalar product, by taking into account that $\underline{g}(\alpha) = \alpha$ with $\alpha \in \mathbb{R}$, the grade-preserving property of the outermorphisms (i.e., of the extension operator [5]), and that $\underline{g}(v_1 \wedge \ldots v_k) = \underline{g}(v_1) \wedge \ldots \underline{g}(v_k)$ with $v_1, \ldots, v_k \in V$.

g4 For any $X, Y \in \bigwedge V$

$$\widehat{X} \underset{g}{\cdot} Y = X \underset{g}{\cdot} \widehat{Y}, \tag{5}$$

$$\widetilde{X} \underset{g}{\cdot} Y = X \underset{g}{\cdot} \widetilde{Y}. \tag{6}$$

The proof is immediate and left to the reader. Hint: take into account that $\widehat{\underline{g}(X)} = \underline{g}(\widehat{X})$ and $\widetilde{\underline{g}(X)} = \underline{g}(\widetilde{X})$ with $X \in \bigwedge V$.

2.2 Metric Reciprocal Bases

Let $(\{e_k\}, \{e^k\})$ be an arbitrary pair of b-reciprocal bases of V, i,e., $e_k \cdot e^l = \delta_k^l$.

Theorem 1 *Take an invertible $(1,1)$-extensor over V, say f. We can construct two bases for V, say $\{E_k\}$ and $\{E^k\}$, by the following formulas*

$$E_k = f(e_k), \tag{7}$$
$$E^k = g^{-1} \circ f^*(e^k) \text{ for each } k = 1, \ldots, n. \tag{8}$$

These bases satisfy the metric scalar product conditions

$$E_k \underset{g}{\cdot} E^l = \delta_k^l. \tag{9}$$

Reciprocally, given two arbitrary bases $\{E_k\}$ and $\{E^k\}$ which satisfy eq.(9), there exists an unique invertible $(1,1)$-extensor f such that the eqs.(7) and (8) hold.

Proof. Since $\{e_k\}$ and $\{e^k\}$ are bases for V and, f and g are invertible $(1,1)$-extensors over V, it follows that the n vectors $E_1, \ldots, E_n \in V$ and the n vectors $E^1, \ldots, E^n \in V$ must also determine two well-defined bases for V.

Now, a straightforward calculation gives

$$E_k \underset{g}{\cdot} E^l = g \circ f(e_k) \cdot g^{-1} \circ f^*(e^l) = e_k \cdot f^\dagger \circ g \circ g^{-1} \circ f^*(e^l) = e_k \cdot e^l = \delta_k^l,$$

4

and the first statement follows.

Now, $\{e_k\}$ and $\{e^k\}$ are bases for V, and $\{E_k\}$ and $\{E^k\}$ are supposed to be also bases for V. Then, there must exist exactly two invertible $(1,1)$-extensors over V, say f_1 and f_2, such that

$$\begin{aligned} E_k &= f_1(e_k), \\ E^k &= f_2(e^k) \text{ for each } k = 1, \ldots, n. \end{aligned}$$

It is easy to check that f_1 and f_2 are given by

$$\begin{aligned} f_1(v) &= (e^s \cdot v)E_s, \\ f_2(v) &= (e_s \cdot v)E^s. \end{aligned}$$

But, using eq.(9) we have

$$f_1(e_k) \underset{g}{\cdot} f_2(e^l) = \delta_k^l \Rightarrow e_k \cdot f_1^\dagger \circ g \circ f_2(e^l) = \delta_k^l \Rightarrow f_1^\dagger \circ g \circ f_2(e^l) = e^l,$$

for each $l = 1, \ldots, n$. Thus, $f_1^\dagger \circ g \circ f_2 = i_V$.

Then, choosing $f_1 = f$ and $f_2 = g^{-1} \circ f^*$, the second statement follows. ∎

Two bases $\{E_k\}$ and $\{E^k\}$ satisfying $E_k \underset{g}{\cdot} E^l = \delta_k^l$ are said to be a pair of *metric reciprocal bases*, and we say that $\{E^k\}$ is the *metric reciprocal basis* of $\{E_k\}$.

We end this section presenting two interesting and useful formulas for the expansion of multivectors in terms of a g-scalar product.

Proposition 2 *Let $(\{E_k\}, \{E^k\})$ be any pair of metric reciprocal bases for V, i.e., $E_k \underset{g}{\cdot} E^l = \delta_k^l$. We have the following two expansion formulas. For all $X \in \bigwedge V$*

$$X = X \underset{g}{\cdot} 1 + \sum_{k=1}^n \frac{1}{k!} X \underset{g}{\cdot} (E^{j_1} \wedge \ldots E^{j_k})(E_{j_1} \wedge \ldots E_{j_k}) \tag{10}$$

and

$$X = X \underset{g}{\cdot} 1 + \sum_{k=1}^n \frac{1}{k!} X \underset{g}{\cdot} (E_{j_1} \wedge \ldots E_{j_k})(E^{j_1} \wedge \ldots E^{j_k}). \tag{11}$$

The proof is left to the reader. (Hint: use eq.(1), eq.(7), eq.(8) and some of the properties of extension operator, and take also into account the expansion formula for multivectors in the euclidean Clifford algebra as defined in [4] (paper I of this series) .

2.3 Metric Interior Algebras

We define now the *metric left* and *right contracted products* of multivectors $X, Y \in \bigwedge V$, denoted respectively by $\underset{g}{\lrcorner}$ and $\underset{g}{\llcorner}$,

$$X \underset{g}{\lrcorner} Y = \underline{g}(X) \lrcorner Y, \tag{12}$$

$$X \underset{g}{\llcorner} Y = X \llcorner \underline{g}(Y). \tag{13}$$

When no confusion arises we call $\underset{g}{\lrcorner}$ and $\underset{g}{\llcorner}$ the *g-contracted products*, for short.

These g-contracted products $\underset{g}{\lrcorner}$ and $\underset{g}{\llcorner}$ are internal laws on $\bigwedge V$. Both of them satisfy the distributive laws (on the left and on the right) but they are not associative products.

The vector space $\bigwedge V$ endowed with the g-contracted product either $\underset{g}{\lrcorner}$ or $\underset{g}{\llcorner}$ is a non-associative algebra. They are called *metric interior algebras of multivectors*. Or, *g-interior algebras*, for short.

We present now some of the basic properties of the metric interior algebras.

mi1 For any $\alpha, \beta \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha \underset{g}{\lrcorner} \beta = \alpha \underset{g}{\llcorner} \beta = \alpha\beta \text{ (real product)}, \tag{14}$$

$$\alpha \underset{g}{\lrcorner} X = X \underset{g}{\llcorner} \alpha = \alpha X \text{ (multiplication by scalars)}. \tag{15}$$

mi2 For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$ with $j \leq k$

$$X_j \underset{g}{\lrcorner} Y_k = (-1)^{j(k-j)} Y_k \underset{g}{\llcorner} X_j. \tag{16}$$

mi3 For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \underset{g}{\lrcorner} Y_k = 0, \text{ if } j > k, \tag{17}$$

$$X_j \underset{g}{\llcorner} Y_k = 0, \text{ if } j < k. \tag{18}$$

mi4 For any $X_k, Y_k \in \bigwedge^k V$

$$X_k \underset{g}{\lrcorner} Y_k = X_k \underset{g}{\llcorner} Y_k = \widetilde{X_k} \underset{g}{\cdot} Y_k = X_k \underset{g}{\cdot} \widetilde{Y_k}. \tag{19}$$

mi5 For any $v \in V$ and $X, Y \in \bigwedge V$

$$v \underset{g}{\lrcorner} (X \wedge Y) = (v \underset{g}{\lrcorner} X) \wedge Y + \overline{X} \wedge (v \underset{g}{\lrcorner} Y). \tag{20}$$

All these properties easily follow from the corresponding properties of the euclidean interior algebras, once we take into account the properties of extension operator [4].

Proposition 3 *For all $X, Y, Z \in \bigwedge V$ it holds*

$$(X \underset{g}{\lrcorner} Y) \underset{g}{\cdot} Z = Y \underset{g}{\cdot} (\widetilde{X} \wedge Z), \tag{21}$$

$$(X \underset{g}{\llcorner} Y) \underset{g}{\cdot} Z = X \underset{g}{\cdot} (Z \wedge \widetilde{Y}). \tag{22}$$

These properties are completely equivalent to the definitions of the right and left contracted products given in eqs.(12-13), and can be proved without difficulties by using the properties of extension operator [4].

Proposition 4 *For all $X, Y, Z \in \Lambda V$ it holds*

$$X \underset{g}{\lrcorner} (Y \underset{g}{\lrcorner} Z) = (X \wedge Y) \underset{g}{\lrcorner} Z, \tag{23}$$

$$(X \underset{g}{\llcorner} Y) \underset{g}{\llcorner} Z = X \underset{g}{\llcorner} (Y \wedge Z). \tag{24}$$

Proof. We prove only the first statement. Take $X, Y, Z \in \bigwedge V$. Using the multivector identity $A \lrcorner (B \lrcorner C) = (A \wedge B) \lrcorner C$ and a property of extension operator, we have

$$
\begin{aligned}
X \underset{g}{\lrcorner} (Y \underset{g}{\lrcorner} Z) &= \underline{g}(X) \lrcorner (\underline{g}(Y) \lrcorner Z) = (\underline{g}(X) \wedge \underline{g}(Y)) \lrcorner Z \\
&= \underline{g}(X \wedge Y) \lrcorner Z = (X \wedge Y) \underset{g}{\lrcorner} Z. \blacksquare
\end{aligned}
$$

2.4   Metric Clifford Algebra

We define a *metric Clifford product* of $X, Y \in \bigwedge V$ associated to g by the following axioms:

A1 For all $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$

$\alpha \underset{g}{} X = X\alpha$ equals multiplication of multivector X by scalar α.

7

A2 For all $v \in V$ and $X \in \bigwedge V$

$$v \underset{g}{} X = v \underset{g}{\lrcorner} X + v \wedge X \text{ and } X \underset{g}{} v = X \underset{g}{\llcorner} v + X \wedge v.$$

A3 For all $X, Y, Z \in \bigwedge V$

$$X \underset{g}{} (Y \underset{g}{} Z) = (X \underset{g}{} Y) \underset{g}{} Z.$$

This metric Clifford product is an internal law on $\bigwedge V$. It is associative (by the axiom A3) and satisfies the distributive laws (on the left and on the right) which follow from the corresponding distributive laws of the euclidean contracted and exterior products [4].

$\bigwedge V$ endowed with this metric Clifford product is an associative algebra. It will be called a metric *Clifford algebra of multivectors* generated by g, or simply, *g-Clifford algebra*. It will be denoted by $\mathcal{C}\ell(V, g)$.

We present now some of the most basic properties which hold in $\mathcal{C}\ell(V, g)$.

clg1 For any $v \in V$ and $X \in \bigwedge V$

$$v \underset{g}{\lrcorner} X \;=\; \frac{1}{2}(v \underset{g}{} X - \widehat{X} \underset{g}{} v) \text{ and } X \underset{g}{\llcorner} v = \frac{1}{2}(X \underset{g}{} v - v \underset{g}{} \widehat{X}), \qquad (25)$$

$$v \wedge X \;=\; \frac{1}{2}(v \underset{g}{} X + \widehat{X} \underset{g}{} v) \text{ and } X \wedge v = \frac{1}{2}(X \underset{g}{} v + v \underset{g}{} \widehat{X}). \qquad (26)$$

clg2 For any $X, Y \in \bigwedge V$

$$X \underset{g}{\cdot} Y = \left\langle \widetilde{X} \underset{g}{} Y \right\rangle_0 = \left\langle X \underset{g}{} \widetilde{Y} \right\rangle_0. \qquad (27)$$

clg3 For $X, Y, Z \in \bigwedge V$

$$(X \underset{g}{} Y) \underset{g}{\cdot} Z \;=\; Y \underset{g}{\cdot} (\widetilde{X} \underset{g}{} Z) = X \underset{g}{\cdot} (Z \underset{g}{} \widetilde{Y}), \qquad (28)$$

$$X \underset{g}{\cdot} (Y \underset{g}{} Z) \;=\; (\widetilde{Y} \underset{g}{} X) \underset{g}{\cdot} Z = (X \underset{g}{} \widetilde{Z}) \underset{g}{\cdot} Y. \qquad (29)$$

clg4 For any $X, Y \in \bigwedge V$

$$\widehat{X \underset{g}{} Y} \;=\; \widehat{X} \underset{g}{} \widehat{Y}, \qquad (30)$$

$$\widetilde{X \underset{g}{} Y} \;=\; \widetilde{Y} \underset{g}{} \widetilde{X}. \qquad (31)$$

8

clg5 Let $I \in \bigwedge^n V$, then for any $v \in V$ and $X \in \bigwedge V$

$$I \underset{g}{\lrcorner} (v \wedge X) = (-1)^{n-1} v \underset{g}{\lrcorner} (I \underset{g}{\lrcorner} X). \tag{32}$$

Eq.(32) will be called the metric duality identity, or g-duality identity, for short.

3 Eigenvalues and Eigenvectors

Let t be a $(1,1)$-extensor over V. A scalar $\lambda \in \mathbb{R}$ and a non-zero vector $v \in V$ are said to be an eigenvalue and an eigenvector of t, respectively, if and only if

$$t(v) = \lambda v. \tag{33}$$

We say that λ and v are naturally to be associated to each other. This means that, if $\lambda \in \mathbb{R}$ is an eigenvalue of t, then there is some non-zero $v \in V$ (the associated eigenvector of t) such that eq.(33) holds, and if a non-zero $v \in V$ is an eigenvector of t, then there is some $\lambda \in \mathbb{R}$ (the associated eigenvalue of t) such that eq.(33) is satisfied.

A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of t if and only if it satisfies the following algebraic equation of degree n

$$\det[\lambda i_V - t] = 0, \tag{34}$$

where $i_V \in ext_1^1(V)$ is the known identity $(1,1)$-extensor over V.

Theorem 5 *For any adjoint symmetric $(1,1)$-extensor s, i.e., $s = s^\dagger$, there exists a set of n eigenvectors of s which is a b-orthonormal basis for V. This means that there are exactly n linearly independent non-zero vectors $v_1, \ldots, v_n \in V$ and n scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that*

$$s(v_k) = \lambda_k v_k, \text{ for each } k = 1, \ldots, n$$

and $\{v_k\}$ is a basis for V which satisfies $v_j \cdot v_k = \delta_{jk}$.

Corollary 6 *All eigenvalues of a metric extensor g (i.e., $g \in ext_1^1(V)$, $g = g^\dagger$ and $\det[g] \neq 0$) are non-zero real numbers.*

9

Proof. We first calculate $\det[g]$ by using $\{v_k\}$, taking into account the eigenvalue equation of g and recalling that, in this case, the euclidean reciprocal vectors v^k are equal to the vectors v_k,

$$
\begin{aligned}
\det[g] &= (g(v_1) \wedge \ldots g(v_n)) \cdot (v^1 \wedge \ldots v^n) \\
&= (\lambda_1 v_1 \wedge \ldots \lambda_n v_n) \cdot (v_1 \wedge \ldots v_1) \\
&= \lambda_1 \ldots \lambda_n (v_1 \wedge \ldots v_1) \cdot (v_1 \wedge \ldots v_1) \\
\det[g] &= \lambda_1 \ldots \lambda_n.
\end{aligned}
\tag{35}
$$

Since $\det[g] \neq 0$, all $\lambda_1, \ldots, \lambda_n$ must be non-zero real numbers. ∎

The integer number $s = p - q$, where p, q are non negative integer numbers, respectively the numbers of positive and negative eigenvalues of t and $p + q = n$, is called the signature of t. We already have used (and will continue to do so) the usual convention of physicists and denote the signature of g by the pair (p, q).

4 Gauge Metric Extensor

Lemma 7 *Any b-orthogonal symmetric $(1,1)$-extensor over V, say σ, (i.e., $\sigma = \sigma^*$ and $\sigma = \sigma^\dagger$) can only have eigenvalues ± 1.*

Proof. If $\lambda \in R$ is an eigenvalue of σ, there is an non-zero $v \in V$, the associated eigenvector of σ, such that $\sigma(v) = \lambda v$. And, the orthogonality and symmetry of σ yield $\sigma^2 = i_V$.

Thus, we have that $v = \lambda^2 v$. Since $v \neq 0$, it follows that $1 - \lambda^2 = 0$, i.e., $\lambda = \pm 1$. ∎

Lemma 8 *Let $\{b_k\}$ be the fiducial basis for V (i.e., $b_j \underset{b}{\cdot} b_k = \delta_{jk}$). We can construct a fiducial b-orthogonal metric extensor over V, say $\eta \in ext_1^1(V)$, (i.e., $\eta = \eta^*$ and $\eta = \eta^\dagger$, $\det[\eta] \neq 0$) with signature (p, q) and which b-orthonormal basis of V, made of the eigenvectors of η, is exactly $\{b_k\}$.*

Such a $(1,1)$-extensor over V is given by

$$
\eta(v) = \sum_{j=1}^{p} (v \cdot b_j) b_j - \sum_{j=p+1}^{p+q} (v \cdot b_j) b_j.
\tag{36}
$$

Proof. We first shall prove that η has p eigenvalues $+1$ with associated eigenvectors $b_1, \ldots, b_p$ and q eigenvalues -1 with associated eigenvectors $b_{p+1}, \ldots, b_{p+q}$.

Take b_k with $k = 1, \ldots, p$ we have

$$\eta(b_k) = \sum_{j=1}^{p} (b_k \cdot b_j) b_j - \sum_{j=p+1}^{p+q} (b_k \cdot b_j) b_j = \sum_{j=1}^{p} \delta_{kj} b_j - \sum_{j=p+1}^{p+q} 0 b_j = b_k,$$

and, for b_k with $k = p+1, \ldots, p+q$, it yields

$$\eta(b_k) = \sum_{j=1}^{p} (b_k \cdot b_j) b_j - \sum_{j=p+1}^{p+q} (b_k \cdot b_j) b_j = \sum_{j=1}^{p} 0 b_j - \sum_{j=p+1}^{p+q} \delta_{kj} b_j = -b_k.$$

Then, we have

$$\eta(b_k) = \begin{cases} b_k, & k = 1, \ldots, p \\ -b_k, & k = p+1, \ldots, p+q \end{cases}. \tag{37}$$

Now, we shall prove that η as defined above is a metric extensor over V, i.e., $\eta = \eta^\dagger$ and $\det[\eta] \neq 0$.

Take $v, w \in V$ then

$$\begin{aligned} \eta^\dagger(v) \cdot w &= v \cdot \eta(w) = v \cdot \left(\sum_{j=1}^{p} (w \cdot b_j) b_j - \sum_{j=p+1}^{p+q} (w \cdot b_j) b_j \right) \\ &= \sum_{j=1}^{p} (v \cdot b_j)(w \cdot b_j) - \sum_{j=p+1}^{p+q} (v \cdot b_j)(w \cdot b_j) \\ &= \left(\sum_{j=1}^{p} (v \cdot b_j) b_j - \sum_{j=p+1}^{p+q} (v \cdot b_j) b_j \right) \cdot w = \eta(v) \cdot w, \end{aligned}$$

i.e., $\eta^\dagger = \eta$.

We calculate the determinant of η by using the fundamental formula with $\{b_k\}$, i.e., $\det[\eta] = \eta(b_1) \wedge \ldots \eta(b_n) \cdot (b^1 \wedge \ldots b^n)$. Recall that, in this case, the b-reciprocal basis vectors b^k coincide with b_k for $k = 1, \ldots, n$.

$$\begin{aligned} &\det[\eta] \\ =\ & (\eta(b_1) \wedge \ldots \eta(b_p) \wedge \eta(b_{p+1}) \wedge \ldots \eta(b_{p+q})) \cdot (b_1 \wedge \ldots b_p \wedge b_{p+1} \wedge \ldots b_{p+q}) \\ =\ & (b_1 \wedge \ldots b_p \wedge (-1)^q b_{p+1} \wedge \ldots b_{p+q}) \cdot (b_1 \wedge \ldots b_p \wedge b_{p+1} \wedge \ldots b_{p+q}), \end{aligned}$$

11

i.e., $\det[\eta] = (-1)^q$.

Next, we shall prove that $\eta^2 = i_V$, i.e., $\eta^{-1} = \eta$.

Take $v \in V$ then

$$
\begin{aligned}
\eta \circ \eta(v) &= \sum_{k=1}^{n} (v \cdot b_k) \eta \circ \eta(b_k) \\
&= \sum_{k=1}^{n} (v \cdot b_k) \begin{cases} \eta(b_k) & k = 1, \ldots, p \\ -\eta(b_k) & k = p+1, \ldots, p+q \end{cases} \\
&= \sum_{k=1}^{n} (v \cdot b_k) b_k = v,
\end{aligned}
$$

i.e., $\eta^2 = i_V$. ∎

This lemma allows us to construct, associated to the fiducial basis $\{b_k\}$, a fiducial b-orthogonal metric extensor $\eta \in ext_1^1(V)$ with signature (p, q). Such a $(1, 1)$-extensor over V has p eigenvalues $+1$ and q eigenvalues -1, and their corresponding associated eigenvectors are the vectors of $\{b_k\}$.

Theorem 9 *For any metric extensor $g \in ext_1^1(V)$ whose signature is (p, q), there exists an invertible extensor $h \in ext_1^1(V)$ such that*

$$
g = h^\dagger \circ \eta \circ h, \tag{38}
$$

where $\eta \in ext_1^1(V)$ is just the fiducial b-orthogonal metric extensor with signature (p, q), as considered in eq.(36).

Such a $(1, 1)$-extensor over V is given by

$$
h(a) = \sum_{j=1}^{n} \sqrt{|\lambda_j|}(a \cdot v_j) b_j, \tag{39}
$$

where $\lambda_1, \ldots \lambda_n \in \mathbb{R}$ are the eigenvalues of g and $v_1, \ldots, v_n \in V$ are the corresponding associated eigenvectors of g.

Proof. First we need calculate the adjoint extensor of h.

Take $a, b \in V$ then

$$
\begin{aligned}
h^\dagger(a) \cdot b &= a \cdot h(b) = a \cdot \left(\sum_{j=1}^{n} \sqrt{|\lambda_j|}(b \cdot v_j) b_j \right) \\
&= \left(\sum_{j=1}^{n} \sqrt{|\lambda_j|}(a \cdot b_j) v_j \right) \cdot b,
\end{aligned}
$$

12

i.e., $h^\dagger(a) = \sum\limits_{j=1}^{n} \sqrt{|\lambda_j|}(a \cdot b_j)v_j$.

Now, let $a \in V$. A straightforward calculation yields

$$
\begin{aligned}
h^\dagger \circ \eta \circ h(a) &= \sum_{j=1}^{n}\sum_{k=1}^{n} \sqrt{|\lambda_j \lambda_k|}\eta(b_j) \cdot b_k(a \cdot v_j)v_k \\
&= \sum_{j=1}^{p}\sum_{k=1}^{p} \sqrt{|\lambda_j \lambda_k|}\eta(b_j) \cdot b_k(a \cdot v_j)v_k \\
&\quad + \sum_{j=1}^{p}\sum_{k=p+1}^{p+q} \sqrt{|\lambda_j \lambda_k|}\eta(b_j) \cdot b_k(a \cdot v_j)v_k \\
&\quad + \sum_{j=p+1}^{p+q}\sum_{k=1}^{p} \sqrt{|\lambda_j \lambda_k|}\eta(b_j) \cdot b_k(a \cdot v_j)v_k \\
&\quad + \sum_{j=p+1}^{p+q}\sum_{k=p+1}^{p+q} \sqrt{|\lambda_j \lambda_k|}\eta(b_j) \cdot b_k(a \cdot v_j)v_k,
\end{aligned}
$$

and, by taking into account eq.(37) we have

$$
\begin{aligned}
h^\dagger \circ \eta \circ h(a) &= \sum_{j=1}^{p}\sum_{k=1}^{p} \sqrt{|\lambda_j \lambda_k|}\delta_{jk}(a \cdot v_j)v_k + 0 \\
&\quad + 0 - \sum_{j=p+1}^{p+q}\sum_{k=p+1}^{p+q} \sqrt{|\lambda_j \lambda_k|}\delta_{jk}(a \cdot v_j)v_k \\
&= \sum_{j=1}^{p} |\lambda_j|(a \cdot v_j)v_j - \sum_{j=p+1}^{p+q} |\lambda_j|(a \cdot v_j)v_j \\
&= \sum_{j=1}^{p} \lambda_j(a \cdot v_j)v_j + \sum_{j=p+1}^{p+q} \lambda_j(a \cdot v_j)v_j = \sum_{j=1}^{n} \lambda_j(a \cdot v_j)v_j.
\end{aligned}
$$

On the last step we have used that the signature of g is (p, q), i.e., g has p positive eigenvalues and q negative eigenvalues.

And, by using the eigenvalues equation of g, i.e., $g(v_j) = \lambda_j v_j$ for each $j = 1, \ldots, n$, we have

$$
h^\dagger \circ \eta \circ h(a) = \sum_{j=1}^{n} (a \cdot v_j)g(v_j) = g(\sum_{j=1}^{n} (a \cdot v_j)v_j) = g(a),
$$

13

i.e., $h^\dagger \circ \eta \circ h = g$.

Finally, since

$$\det[g] = \det[h^\dagger \circ \eta \circ h] = \det[h^\dagger]\det[\eta]\det[h] = \det[\eta]\det{}^2[h] = (-1)^q \det{}^2[h],$$

and $\det[g] \neq 0$, then $\det[h] \neq 0$, and so h is an invertible $(1,1)$-extensor. ∎

It should be noted that h satisfying eq.(38) is not *unique*. If there is some $h \in ext_1^1(V)$ which satisfies eq.(38), then $h' \equiv \Lambda \circ h$, where Λ is a η-orthogonal $(1,1)$-extensor over V (i.e., $\Lambda^\dagger \circ \eta \circ \Lambda = \eta)^1$, also satisfies eq.(38).

Indeed, we have $h'^\dagger \circ \eta \circ h' = (\Lambda \circ h)^\dagger \circ \eta \circ \Lambda \circ h = h^\dagger \circ \Lambda^\dagger \circ \eta \circ \Lambda \circ h = h^\dagger \circ \eta \circ h = g$.

In general, an invertible extensor $h \in ext_1^1(V)$ which satisfies eq.(38) will be said to be *a gauge metric extensor for the metric extensor g*.

4.1   Gauge Metric Bases

Let $\{e_k\}$ and $\{e^k\}$ be two b-reciprocal bases to each other for the vector space V, i.e., $e_k \cdot e^l = \delta_k^l$. Since h, a gauge metric extensor over V, is non-degenerate, i.e., $\det[h] \neq 0$, it follows that the n vectors $h(e_1), \ldots, h(e_n) \in V$ and the n vectors $h^*(e^1), \ldots, h^*(e^n) \in V$ will be also well-defined bases for V.

As the reader can easily prove $(\{h(e_k)\}, \{h^*(e^k)\})$ is also a pair of b-reciprocal bases of V, i.e.,

$$h(e_k) \cdot h^*(e^l) = \delta_k^l. \tag{40}$$

Two other remarkable properties of these bases are:

$$h(e_j) \underset{\eta}{\cdot} h(e_k) = g(e_j) \cdot e_k \equiv g_{jk}, \tag{41}$$

$$h^*(e^j) \underset{\eta^{-1}}{\cdot} h^*(e^k) = g^{-1}(e^j) \cdot e^k \equiv g^{jk}. \tag{42}$$

These bases $\{h(e_k)\}$ and $\{h^*(e^k)\}$ will be said to be a pair of *gauge metric bases* for V.

Theorem 10 *Given n non-zero real numbers $\rho_1, \ldots, \rho_n$ and a b-orthogonal $(1,1)$-extensor l over V (i.e., $l = l^*$), we can construct an invertible $(1,1)$-extensor h over V using the following formula*

$$h(v) = \sum_{j=1}^{n} \rho_j (l(v) \cdot b_j) b_j. \tag{43}$$

---

[1]As the reader can prove without difficulties, a η-orthogonal $(1,1)$-extensor Λ preserves the η-scalar products, i.e., for all $v, w \in V : \Lambda(v) \underset{\eta}{\cdot} \Lambda(w) = v \underset{\eta}{\cdot} w$.

Then, the $(1,1)$-extensor g over V defined by

$$g = h^\dagger \circ \eta \circ h, \tag{44}$$

where $\eta \in ext_1^1(V)$ is just the fiducial b-orthogonal metric extensor over V with signature (p, q), as considered in eq.(36).

The p positive real numbers $\rho_1^2, \ldots, \rho_p^2$ are the eigenvalues of g with the associated eigenvectors $l^\dagger(b_1), \ldots, l^\dagger(b_p)$ of g, and the q negative real numbers $-\rho_{p+1}^2, \ldots, -\rho_{p+q}^2$ are the eigenvalues of g with the associated eigenvectors $l^\dagger(b_{p+1}), \ldots, l^\dagger(b_{p+q})$ of g.

The set of n non-zero vectors $\{l^\dagger(b_1), \ldots, l^\dagger(b_p), l^\dagger(b_{p+1}), \ldots, l^\dagger(b_{p+q})\}$ is a b-orthonormal basis of V, made of the eigenvectors of g.

The signature of g is also (p, q).

Proof. We first must check that g is symmetric and non-degenerate. Using eq.(44) we have

$$g^\dagger = h^\dagger \circ \eta^\dagger \circ (h^\dagger)^\dagger = h^\dagger \circ \eta \circ h = g,$$

thus, $g = g^\dagger$, i.e., g is symmetric.

We now calculate $\det[g]$,

$$\det[g] = \det[h^\dagger] \det[\eta] \det[h] = \det[h] \det[\eta] \det[h] = (-1)^q \det{}^2[h].$$

But, it is possible to calculate $\det[h]$ by using a trick. We shall evaluate $\det[h \circ l^\dagger]$ in two different ways.

First, using eq.(43) and the b-orthogonality of l, i.e., $l^{-1} = l^\dagger$, we have

$$h \circ l^\dagger(v) = \sum_{j=1}^{n} \rho_j (a \cdot b_j) b_j.$$

Sencondly, using the fundamental formula for the determinant of a $(1, 1)$-extensor (see [5]) we have,

$$
\begin{aligned}
\det[h \circ l^\dagger] &= (h \circ l^\dagger(b_1) \wedge \ldots h \circ l^\dagger(b_n)) \cdot (b^1 \wedge \ldots b^n) \\
&= (\rho_1 b_1 \wedge \ldots \rho_n b_n) \cdot (b_1 \wedge \ldots b_n) \\
&= \rho_1 \ldots \rho_n.
\end{aligned}
$$

Now, taking into account that $\det[l] = \pm 1$ and property (d1) of the determinant (see [5]) we get,

$$\det[h \circ l^\dagger] = \det[h] \det[l^\dagger] = \det[h] \det[l] = \pm \det[h].$$

Thus, we have $\det[h] = \pm\rho_1 \ldots \rho_n$. And, therefore $\det[g] = (-1)^q \rho_1^2 \ldots \rho_n^2$. Since $\rho_1, \ldots, \rho_n$ are non-zero real numbers, $\det[g] \neq 0$, i.e., g is non-degenerate.

The proof of the first statement is then complete.

In order to prove the second statement, related to the eigenvalues and eigenvectors of g, we shall use the following equations: $h \circ l^\dagger(b_k) = \rho_k b_k$ (just used above), $h^\dagger(b_k) = \rho_k l^\dagger(b_k)$ (obtained from eq.(43)) and the eigenvalue equation of η, i.e., $\eta(b_k) = \begin{cases} b_k, & k = 1, \ldots, p \\ -b_k, & k = p+1, \ldots, p+q \end{cases}$.

We have

$$
\begin{aligned}
g \circ l^\dagger(b_k) &= h^\dagger \circ \eta \circ h \circ l^\dagger(b_k) = h^\dagger \circ \eta(\rho_k b_k) = \rho_k h^\dagger \circ \eta(b_k) \\
&= \begin{cases} \rho_k h^\dagger(b_k), & k = 1, \ldots, p \\ -\rho_k h^\dagger(b_k), & k = p+1, \ldots, p+q \end{cases} \\
g \circ l^\dagger(e_k) &= \begin{cases} \rho_k^2 l^\dagger(b_k), & k = 1, \ldots, p \\ -\rho_k^2 l^\dagger(b_k), & k = p+1, \ldots, p+q \end{cases}.
\end{aligned}
$$

This establishes the second statement.

From the euclidean b-orthogonality of l it follows easily that the eigenvectors of g, are b-orthonormal. It is also obvious that the signature of g is also (p, q). Thus, the third and fourth statement are proved. ∎

5   The Golden Formula

Proposition 11 *Let h be any gauge operator for g, i.e., $g = h^\dagger \circ \eta \circ h$, and let $\underset{g}{*}$ mean either $\wedge$ (exterior product), $\underset{g}{\cdot}$ (g-scalar product), $\underset{g}{\lrcorner} \underset{g}{\llcorner}$ (g-contracted products) or $\underset{g}{}$ (g-Clifford product), and analogously for $\underset{\eta}{*}$. The g-metric products $\underset{g}{*}$ and the η-metric products $\underset{\eta}{*}$ are related by the following remarkable formula. For all $X, Y \in \bigwedge V$*

$$\underline{h}(X \underset{g}{*} Y) = \underline{h}(X) \underset{\eta}{*} \underline{h}(Y), \tag{45}$$

where $\underline{h}$ denotes the extended of h. Eq.(45) will be called the golden formula

Proof.

By recalling the fundamental properties for the outermorphism of an operator: $\underline{t}(X \wedge Y) = \underline{t}(X) \wedge \underline{t}(Y)$ and $\underline{t}(\alpha) = \alpha$, we have that the multivector identity above holds for the exterior product, i.e.,

$$X \wedge Y = \underline{h}^{-1}[\underline{h}(X) \wedge \underline{h}(Y)] \tag{46}$$

and for the g-scalar product and the η-scalar product, i.e.,

$$X \underset{g}{\cdot} Y = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{\cdot} \underline{h}(Y)]. \tag{47}$$

By using the multivector identities for an invertible operator: $\underline{t}^{\dagger}(X) \lrcorner Y = \underline{t}^{-1}[X \lrcorner \underline{t}(Y)]$ and $X \llcorner \underline{t}^{\dagger}(Y) = \underline{t}^{-1}[\underline{t}(X) \llcorner Y]$, and the gauge equation $g = h^{\dagger} \circ \eta \circ h$ we can easily prove that the multivector identity above holds for the g-contracted product and the η-contracted product, i.e.,

$$X \underset{g}{\lrcorner} Y = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{\lrcorner} \underline{h}(Y)] \tag{48}$$

$$X \underset{g}{\llcorner} Y = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{\llcorner} \underline{h}(Y)]. \tag{49}$$

To prove eq.(48) see that we can write

$$X \underset{g}{\lrcorner} Y = \underline{h^{\dagger} \circ \eta \circ h}(X) \lrcorner Y = \underline{h}^{-1}[\underline{\eta \circ h}(X) \lrcorner \underline{h}(Y)] = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{\lrcorner} \underline{h}(Y)],$$

where the definitions of $\underset{g}{\lrcorner}$ and $\underset{\eta}{\lrcorner}$ have been used. The proof of eq.(49) is completely analogous, the definitions of $\underset{g}{\llcorner}$ and $\underset{\eta}{\llcorner}$ should be used.

In order to prove that the multivector identity above holds for the g-Clifford product and the η-Clifford product, i.e.,

$$X \underset{g}{} Y = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{} \underline{h}(Y)], \tag{50}$$

we first must prove four particular cases of it.

Take $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$. By using the axioms of the g and η Clifford products: $\alpha \underset{g}{} X = X \underset{g}{} \alpha = \alpha X$ and $\alpha \underset{\eta}{} X = X \underset{\eta}{} \alpha = \alpha X$, we can write

$$\alpha \underset{g}{} X = \alpha X = \underline{h}^{-1}[\alpha \underline{h}(X)] = \underline{h}^{-1}[\alpha \underset{\eta}{} \underline{h}(X)],$$

i.e.,

$$\alpha \underset{g}{} X = \underline{h}^{-1}[\underline{h}(\alpha) \underset{\eta}{} \underline{h}(X)]. \tag{51}$$

Analogously, we have

$$X \underset{g}{} \alpha = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{} \underline{h}(\alpha)]. \tag{52}$$

Take $v \in V$ and $X \in \bigwedge V$. By using the axioms of the g and η Clifford products: $v \underset{g}{} X = v \underset{g}{\lrcorner} X + v \wedge X$ and $v \underset{\eta}{} X = v \underset{\eta}{\lrcorner} X + v \wedge X$, and eqs.(48) and (46) we can write

$$v \underset{g}{} X = v \underset{g}{\lrcorner} X + v \wedge X = \underline{h}^{-1}[h(v) \underset{\eta}{\lrcorner} \underline{h}(X)] + \underline{h}^{-1}[h(v) \wedge \underline{h}(X)],$$

i.e.,

$$v \underset{g}{} X = \underline{h}^{-1}[h(v) \underset{\eta}{} \underline{h}(X)]. \tag{53}$$

From the axioms of the g and η Clifford products: $X \underset{g}{} v = X \underset{g}{\llcorner} v + X \wedge v$ and $X \underset{\eta}{} v = X \underset{\eta}{\llcorner} v + X \wedge v$, and eqs.(49) and (46) we get

$$X \underset{g}{} v = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{} h(v)]. \tag{54}$$

Take $v_1, v_2, \ldots, v_k \in V$. By using $k-1$ times eq.(53) we have indeed

$$
\begin{aligned}
v_1 \underset{g}{} v_2 \cdots \underset{g}{} v_k &= \underline{h}^{-1}[h(v_1) \underset{\eta}{} \underline{h}(v_2 \cdots \underset{g}{} v_k)] \\
&= \underline{h}^{-1}[h(v_1) \underset{\eta}{} h(v_2) \cdots \underset{\eta}{} h(v_k)], \\
v_1 \underset{g}{} v_2 \cdots \underset{g}{} v_k &= \underline{h}^{-1}[h(v_1) \underset{\eta}{} h(v_2) \cdots \underset{\eta}{} h(v_k)]. \tag{55}
\end{aligned}
$$

Take $v_1, v_2, \ldots, v_k \in V$ and $X \in \bigwedge V$. By using $k-1$ times eq.(53) and eq.(55) we have indeed

$$
\begin{aligned}
(v_1 \underset{g}{} v_2 \cdots \underset{g}{} v_k) \underset{g}{} X &= \underline{h}^{-1}[h(v_1) \underset{\eta}{} \underline{h}(\underset{g}{} v_2 \cdots \underset{g}{} v_k \underset{g}{} X)] \\
&= \underline{h}^{-1}[h(v_1) \underset{\eta}{} h(v_2) \cdots \underset{\eta}{} h(v_k) \underset{\eta}{} \underline{h}(X)], \\
(v_1 \underset{g}{} v_2 \cdots \underset{g}{} v_k) \underset{g}{} X &= \underline{h}^{-1}[\underline{h}(v_1 \underset{g}{} v_2 \cdots \underset{g}{} v_k) \underset{\eta}{} \underline{h}(X)]. \tag{56}
\end{aligned}
$$

We now can prove the general case of eq.(50). We shall use an expansion formula for multivectors: $X = X^0 + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} e_{j_1} \cdots \underset{g}{} e_{j_k}$, where $\{e_j\}$ is a basis of V, eq.(51) and eq.(56). We can write

$$X \underset{g}{} Y = X^0 \underset{g}{} Y + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \ldots j_k} (e_{j_1} \cdots \underset{g}{} e_{j_k}) \underset{g}{} Y$$

18

$$
\begin{aligned}
&= \ \underline{h}^{-1}[\underline{h}(X^0) \underset{\eta}{} \underline{h}(Y)] + \underline{h}^{-1}[\sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \dots j_k} \underline{h}(e_{j_1} \underset{g}{\cdots} e_{j_k}) \underset{\eta}{} \underline{h}(Y)] \\
&= \ \underline{h}^{-1}[\underline{h}(X^0 + \sum_{k=1}^{n} \frac{1}{k!} X^{j_1 \dots j_k} e_{j_1} \underset{g}{\cdots} e_{j_k}) \underset{\eta}{} \underline{h}(Y)],
\end{aligned}
$$

$$
X \underset{g}{} Y = \underline{h}^{-1}[\underline{h}(X) \underset{\eta}{} \underline{h}(Y)].
$$

Hence, eq.(46), eq.(47), eqs.(48) and (49), and eq.(50) have set the golden formula. ∎

6   Metric Adjoint Operators

Let g be a metric operator on V, i.e., $g \in ext_1^1(V)$ such that $g = g^\dagger$ and $\det[g] \neq 0$. To each $t \in 1\text{-}ext(\overset{\diamond}{\underset{1}{\bigwedge}} V; \overset{\diamond}{\underset{2}{\bigwedge}} V)$. We define the metric adjoint operator $t^{\dagger(g)} \in 1\text{-}ext(\overset{\diamond}{\underset{2}{\bigwedge}} V; \overset{\diamond}{\underset{1}{\bigwedge}} V)$ by

$$
t^{\dagger(g)} = \underline{g}^{-1} \circ t^\dagger \circ \underline{g}. \tag{57}
$$

As we can easily see, $t^{\dagger(g)}$ is the unique extensor from $\overset{\diamond}{\underset{2}{\bigwedge}} V$ to $\overset{\diamond}{\underset{1}{\bigwedge}} V$ which satisfies the following property: for any $X \in \overset{\diamond}{\underset{1}{\bigwedge}} V$ and $Y \in \overset{\diamond}{\underset{2}{\bigwedge}} V$

$$
X \underset{g}{\cdot} t^{\dagger(g)}(Y) = t(X) \underset{g}{\cdot} Y. \tag{58}
$$

This is the 'metric version' of the fundamental property given by the formula $t^\dagger(X) \cdot Y = X \cdot t(Y)$ in the paper II of this series.

Finally, we notice the very important formula that

$$
\det[t^{\dagger(g)}] = \det[t^\dagger] = \det[t]. \tag{59}
$$

7   Standard Hodge Extensor

Let $(\{e_j\}, \{e^j\})$ be a pair of b-reciprocal bases to each other for V, i.e., $e_j \underset{b}{\cdot} e^k = \delta_j^k$. Associated to them we define a non-zero pseudoscalar

$$
\tau = \sqrt{e_\wedge \cdot e_\wedge} e^\wedge \tag{60}
$$

19

where $e_\wedge \equiv e_1 \wedge \ldots \wedge e_n \in \bigwedge^n V$ and $e^\wedge \equiv e^1 \wedge \ldots \wedge e^n \in \bigwedge^n V$. Note that $e_\wedge \cdot e_\wedge > 0$, since the scalar product is positive definite. It will be called a *standard volume pseudoscalar for V*. It has the fundamental property

$$\tau \cdot \tau = \tau \lrcorner \widetilde{\tau} = \tau \widetilde{\tau} = 1, \tag{61}$$

it follows from the equation $e_\wedge \cdot e^\wedge = 1$.

From eq.(61), we can get an expansion formula for pseudoscalars

$$I = (I \cdot \tau)\tau. \tag{62}$$

The extensor $\star \in ext(V)$ which is defined by $\star : \bigwedge V \to \bigwedge V$ such that

$$\star X = \widetilde{X} \lrcorner \tau = \widetilde{X}\tau, \tag{63}$$

will be called a *standard Hodge extensor on V*.

It should be noticed that if $X \in \bigwedge^p V$, then $\star X \in \bigwedge^{n-p} V$. It means that $\star$ can be also defined as a $(p, n-p)$-extensor over V.

The extensor over V, $\star^{-1} : \bigwedge V \to \bigwedge V$ such that

$$\star^{-1} X = \tau \llcorner \widetilde{X} = \tau \widetilde{X} \tag{64}$$

is the *inverse* extensor of $\star$.

Indeed, take $X \in \bigwedge V$. Eq.(61) gives $\star^{-1} \circ \star X = \tau \widetilde{\tau} X = X$, and $\star \circ \star^{-1} X = X \widetilde{\tau} \tau = X$, i.e., $\star^{-1} \circ \star = \star \circ \star^{-1} = i_{\bigwedge V}$, where $i_{\bigwedge V} \in ext(V)$ is the so-called *identity function* for $\bigwedge V$.

Let us take $X, Y \in \bigwedge V$. Using the multivector identity $(XA) \cdot Y = X \cdot (Y\widetilde{A})$ and eq.(61) we get

$$(\star X) \cdot (\star Y) = X \cdot Y. \tag{65}$$

It means that the standard Hodge extensor preserves the euclidean scalar product.

Let us take $X, Y \in \bigwedge^p V$. By using eq.(62) together with the multivector identity $(X \wedge Y) \cdot Z = Y \cdot (\widetilde{X} \lrcorner Z)$, and eq.(65) we get

$$X \wedge (\star Y) = (X \cdot Y)\tau. \tag{66}$$

This identity is completely equivalent to the definition of standard Hodge extensor given by eq.(63).

Take $X \in \bigwedge^p V$ and $Y \in \bigwedge^{n-p} V$. By using the multivector identity $(X \lrcorner Y) \cdot Z = Y \cdot (\widetilde{X} \wedge Z)$ and eq.(62) we get

$$((\star X) \cdot Y)\tau = X \wedge Y. \tag{67}$$

8 Metric Hodge Extensor

Let g be a metric extensor over V of signature (p, q), i.e., $g \in ext_1^1(V)$ such that $g = g^\dagger$ and $\det[g] \neq 0$. It has p positive and q negative eigenvalues. Associated to a pair $(\{e_j\}, \{e^j\})$ of b-reciprocal bases we can define another non-zero pseudoscalar

$$\tau_g = \sqrt{\left| e_\wedge \underset{g}{\cdot} e_\wedge \right|} e^\wedge = \sqrt{|\det[g]|}\tau. \tag{68}$$

It will be called a *metric volume pseudoscalar* for V. It has the fundamental property

$$\tau_g \underset{g^{-1}}{\cdot} \tau_g = \tau_g \underset{g^{-1}}{\lrcorner} \tilde{\tau}_g = \tau_g \underset{g^{-1}}{\cdot} \tilde{\tau}_g = (-1)^q. \tag{69}$$

It follows from eq.(61) by taking into account the definition of determinant of a linear operator on V, and recalling that $sgn(\det[g]) = (-1)^q$.

An expansion formula for the pseudoscalars can be obtained from eq.(69), i.e.,

$$I = (-1)^q (I \underset{g^{-1}}{\cdot} \tau_g)\tau_g. \tag{70}$$

The extensor $\underset{g}{\star} \in ext(V)$ which is defined by $\underset{g}{\star} : \bigwedge V \to \bigwedge V$ such that

$$\underset{g}{\star} X = \tilde{X} \underset{g^{-1}}{\lrcorner} \tau_g = \tilde{X} \underset{g^{-1}}{\cdot} \tau_g, \tag{71}$$

will be called a *metric Hodge extensor* on V. It should be noticed that the definition of $\underset{g}{\star}$ needs the use of both the g and g^{-1} metric Clifford algebras, a non trivial fact.

It is clear that if $X \in \bigwedge^p V$, then $\underset{g}{\star} X \in \bigwedge^{n-p} V$.

The extensor over V, $\underset{g}{\star}^{-1} : \bigwedge V \to \bigwedge V$ such that

$$\underset{g}{\star}^{-1} X = (-1)^q \tau_g \underset{g^{-1}}{\llcorner} \tilde{X} = (-1)^q \tau_g \underset{g^{-1}}{\cdot} \tilde{X}, \tag{72}$$

is the *inverse* extensor of $\underset{g}{\star}$.

Indeed, take $X \in \Lambda V$. By using eq.(69), we verify that $\underset{g}{\star}^{-1} \circ \underset{g}{\star} X = (-1)^q \tau_g \underset{g^{-1}}{\cdot} \tilde{\tau}_g \underset{g^{-1}}{\cdot} X = X$, and $\underset{g}{\star} \circ \underset{g}{\star}^{-1} X = (-1)^q X \underset{g^{-1}}{\cdot} \tilde{\tau}_g \underset{g^{-1}}{\cdot} \tau_g = X$, i.e., $\underset{g}{\star}^{-1} \circ \underset{g}{\star} = \underset{g}{\star} \circ \underset{g}{\star}^{-1} = i_{\bigwedge V}$.

Take $X, Y \in \bigwedge V$. The identity $(X \underset{g^{-1}}{\lrcorner} A) \underset{g^{-1}}{\cdot} Y = X \underset{g^{-1}}{\cdot} (Y \underset{g^{-1}}{\lrcorner} \widetilde{A})$ and eq.(69) yield

$$(\underset{g}{\star} X) \underset{g^{-1}}{\cdot} (\underset{g}{\star} Y) = (-1)^q X \underset{g^{-1}}{\cdot} Y. \tag{73}$$

Take $X, Y \in \bigwedge^p V$. Eq.(70), the identity $(X \wedge Y) \underset{g^{-1}}{\cdot} Z = Y \underset{g^{-1}}{\cdot} (\widetilde{X} \underset{g^{-1}}{\lrcorner} Z)$ and eq.(73) allow us to get

$$X \wedge (\underset{g}{\star} Y) = (X \underset{g^{-1}}{\cdot} Y)\tau_g. \tag{74}$$

This remarkable property is completely equivalent to the definition of the metric Hodge extensor.

Take $X \in \bigwedge^p V$ and $Y \in \bigwedge^{n-p} V$. The use of identity $(X \underset{g^{-1}}{\lrcorner} Y) \underset{g^{-1}}{\cdot} Z = Y \underset{g^{-1}}{\cdot} (\widetilde{X} \wedge Z)$ and eq.(70) yield

$$((\underset{g}{\star} X) \underset{g^{-1}}{\cdot} Y)\tau_g = (-1)^q X \wedge Y. \tag{75}$$

It might as well be asked what is the relationship between the standard and metric Hodge extensors as defined above.

Take $X \in \bigwedge V$. By using eq.(68), the multivector identity for an invertible $(1,1)$-extensor $\underline{t}^{-1}(X) \lrcorner Y = \underline{t}^\dagger(X \lrcorner \underline{t}^*(Y))$ and the definition of determinant of a $(1,1)$-extensor, we have

$$
\begin{aligned}
\underset{g}{\star} X &= \underline{g}^{-1}(\widetilde{X}) \lrcorner \sqrt{|\det[g]|}\,\tau = \sqrt{|\det[g]|}\,\underline{g}(\widetilde{X} \lrcorner \underline{g}^{-1}(\tau)) \\
&= \frac{\sqrt{|\det[g]|}}{\det[g]}\underline{g}(\widetilde{X} \lrcorner \tau) = \frac{sgn(\det[g])}{\sqrt{|\det[g]|}}\underline{g} \circ \star(X),
\end{aligned}
$$

ie.,

$$\underset{g}{\star} = \frac{(-1)^q}{\sqrt{|\det[g]|}}\underline{g} \circ \star. \tag{76}$$

Eq.(76) is the formula which relates[2] $\underset{g}{\star}$ with $\star$.

We recall that for any metric operator $g \in ext_1^1(V)$ there exists a non-degenerate operator $h \in ext_1^1(V)$ such that

$$g = h^\dagger \circ \eta \circ h, \tag{77}$$

---

[2]It is a very important formula and good use of it will be done in our theory of the gravitational field to be presented in another series of papers.

where $\eta \in ext_1^1(V)$ is an orthogonal metric operator with the same signature as g. Such a h is called a gauge operator for g.

The g and g^{-1} metric contracted products $\underset{g}{\lrcorner}$ and $\underset{g^{-1}}{\lrcorner}$ are related to the η-metric contracted product $\underset{\eta}{\lrcorner}$ (recall that $\eta = \eta^{-1}$) by the following formulas

$$\underline{h}(X \underset{g}{\lrcorner} Y) = \underline{h}(X) \underset{\eta}{\lrcorner} \underline{h}(Y), \tag{78}$$

$$\underline{h}^*(X \underset{g^{-1}}{\lrcorner} Y) = \underline{h}^*(X) \underset{\eta}{\lrcorner} \underline{h}^*(Y). \tag{79}$$

We can also get a noticeable formula which relates a g-metric Hodge extensor with a η-metric Hodge extensor.

Now, take $X \in \bigwedge V$. By using eq.(79), eq.(68), the definition of determinant of a $(1,1)$-extensor, eq.(77) and the equation $\tau_\eta = \tau$, we have

$$\underset{g}{\star} X = \underline{h}^\dagger(\underline{h}^*(\widetilde{X}) \underset{\eta}{\lrcorner} \underline{h}^*(\tau)) = \sqrt{|\det[g]|}\underline{h}^\dagger(\underline{h}^*(\widetilde{X}) \underset{\eta}{\lrcorner} \det[h^*]\tau)$$

$$= |\det[h]| \det[h^*]\underline{h}^\dagger(\widetilde{\underline{h}^*(X)} \underset{\eta}{\lrcorner} \tau) = sgn(\det[h])\underline{h}^\dagger \circ \underset{\eta}{\star} \circ \underline{h}^*(X),$$

i.e.,

$$\underset{g}{\star} = sgn(\det[h])\underline{h}^\dagger \circ \underset{\eta}{\star} \circ \underline{h}^*. \tag{80}$$

Eq. (80) is the formula which relates $\underset{g}{\star}$ with $\underset{\eta}{\star}$.

9   Conclusions

We showed that any metric Clifford product on $\mathcal{C}\ell(V, g)$ can be considered as *deformation* of the euclidean Clifford product on $\mathcal{C}\ell(V)$, induced by the metric extensor g. We also proved that any metric extensor g is decomposable in terms of a *gauge metric extensor h* and a *fiducial b-orthogonal extensor η* which has the same signature as g. Although h is not unique, since two h's satisfying that property differ only by a composition with a general transformation Λ which is a η-orthogonal $(1,1)$-extensor. For the case that V is 4-dimensional and η is a Lorentzian metric extensor (i.e., with signature $(1,3)$), Λ is just a general Lorentz transformation. The paper contains a proof of the non trivial golden formula, which as the future papers will show, really deserves its name. Indeed, the formula is a key in our theory of the intrinsic formulation of differential geometry on arbitrary manifolds that we

will present in future papers, and also find applications in some some problems of Theoretical Physics as, e.g., in geometric theories of gravitation and Lagrangian formulation of the theory of multivector and extensor fields.

References

[1] Lounesto, P., *Clifford Algebras and Spinors*, London Math. Soc., Lecture Notes Series **239**, Cambridge University Press, Cambridge, 1997.

[2] Porteous, I. R, *Topological Geometry*, Van Nostrand Reinhold, London, 1969, 2nd edition, Cambridge University Press, Cambridge, 1981.

[3] Porteous, I. R., *Clifford Algebras and the Classical Groups*, Cambridge Studies in Advanced Mathematics vol.**50**, Cambridge University Press, Cambridge, 1995.

[4] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Euclidean Clifford Algebra (paper I in a series of seven), in publication in *Adv. Appl. Clifford Algebras* **11**(S3), (2001)

[5] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Extensors (paper II in a series of seven), in publication in *Adv. Appl. Clifford Algebras* **11**(S3), (2001)