Metric Tensor Vs. Metric Extensor*

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Abstract

In this paper we give a comparison between the formulation of the concept of metric for a real vector space of finite dimension in terms of *tensors* and *extensors*. A nice property of metric extensors is that they have inverses which are also themselves metric extensors. This property is not shared by metric tensors because tensors do *not* have inverses. We relate the definition of determinant of a metric extensor with the classical determinant of the corresponding matrix associated to the metric tensor in a given vector basis. Previous identifications of these concepts are equivocated. The use of metric extensor permits sophisticated calculations without the introduction of matrix representations.

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1 Introduction

This is the third paper of a series of seven. Here, we explore the concept of metric on a n-dimensional real vector space V by using the concepts of tensors and extensors. We show that for each metric tensor G, there is an unique metric extensor g, and vice versa. The metric extensor is a fundamental tool for future developments that we have in mind, and which are going to be presented in following papers of this series and subsequent series of papers. Besides this fact, it is worth to emphasize here that the concept of metric extensor has a prior status in the foundations of linear algebra relative to its corresponding metric tensor. This is because a metric extensor g has an inverse g^{-1} which is of course itself a metric extensor. As a simple application of this concept, we use g and g^{-1}, to recall some results involving the well-known metric isomorphism between V and V^* induced by G.

2 Standard Isomorphism

Let $\{b_k\}$ be an arbitrary, but fixed, basis for V and $\{\beta^k\}$ its corresponding dual basis for V^*, i.e., $\beta^k(b_j) = \delta^k_j$. There exists a linear isomorphism between V and V^* realized by the linear mappings $V \ni v \mapsto \underset{b}{\iota}(v) \in V^*$ and $V^* \ni \omega \mapsto \underset{\beta}{\iota}^{-1}(\omega) \in V$ such that

$$\underset{b}{\iota}(v) = \sum_{k=1}^{n} \beta^k(v)\beta^k, \tag{1}$$

$$\underset{\beta}{\iota}^{-1}(\omega) = \sum_{k=1}^{n} \omega(b_k)b_k. \tag{2}$$

A suggested by the notations, $\underset{\beta}{\iota}^{-1}$ is inverse mapping of $\underset{b}{\iota}$.

Proof. The linearity property for both $\underset{b}{\iota}$ and $\underset{\beta}{\iota}^{-1}$ holds. We must prove that $\underset{\beta}{\iota}^{-1} \circ \underset{b}{\iota} = i_V$ and $\underset{b}{\iota} \circ \underset{\beta}{\iota}^{-1} = i_{V^*}$, where i_V and i_{V^*} are the identity mappings in V and V^*, respectively.

First take $v \in V$, using the linearity property of $\underset{\beta}{\iota}^{-1}$, eq.(2), the duality condition $\beta^k(b_j) = \delta_j^k$ and the elementary expansion for vectors, i.e., $v = \beta^k(v)b_k$, we have

$$\underset{\beta}{\iota}^{-1} \circ \underset{b}{\iota}(v) = \sum_{k=1}^{n} \beta^k(v)\iota^{-1}(\beta^k) = \sum_{k=1}^{n} \beta^k(v) \sum_{s=1}^{n} \delta_s^k b_s = \beta^k(v)b_k = v,$$

hence, $\underset{\beta}{\iota}^{-1} \circ \underset{b}{\iota} = i_V$.

Now take $\omega \in V^*$, using the linearity property of $\underset{b}{\iota}$, eq.(1), the duality condition $\beta^k(b_j) = \delta_j^k$ and the elementary expansion for forms, i.e., $\omega = \omega(b_k)\beta^k$, we have

$$\underset{b}{\iota} \circ \underset{\beta}{\iota}^{-1}(\omega) = \sum_{k=1}^{n} \omega(b_k)\iota(b_k) = \sum_{k=1}^{n} \omega(b_k) \sum_{s=1}^{n} \delta_k^s \beta^s = \omega(b_k)\beta^k = \omega,$$

hence, $\underset{b}{\iota} \circ \underset{\beta}{\iota}^{-1} = i_{V^*}$. ■

This linear isomorphism between V and V^* induced by the pair of bases $(\{b_k\}, \{\beta^k\})$ will be called a *standard* isomorphism.

As introduced above this linear isomorphism is the unique satisfying the following two conditions: $\underset{b}{\iota}(b_k) = \beta^k$ and $\underset{\beta}{\iota}^{-1}(\beta^k) = b_k$.

2.1 b-Scalar Products

We recall from [1] (paper I on this series) that we can define a b-scalar product of vectors v, $w \in V$ by

$$v \underset{b}{\cdot} w = \underset{b}{\iota}(v)(w) = \sum_{k=1}^{n} \beta^k(v)\beta^k(w). \tag{3}$$

Also, a b-scalar product of forms ω and σ can be defined by

$$\omega \underset{\beta}{\cdot} \sigma = \sigma(\underset{\beta}{\iota}^{-1}(\omega)) = \sum_{k=1}^{n} \omega(b_k)\sigma(b_k). \tag{4}$$

The products defined by eq.(3) and eq.(4) are well-defined scalar products in V and V^*, associated with the arbitrary pair of bases $(\{b_k\},\{\beta^k\})$, and also these scalar products are positive definite, e.g., for the b-scalar product of vectors: $v \underset{b}{\cdot} v \geq 0$ for all v, and $v \underset{b}{\cdot} v = 0$ if and only if $v = 0$.

The vector basis $\{b_k\}$ and the form basis $\{\beta^k\}$ are both orthonormal in each of spaces V and V^*, respectively, i.e.,

$$b_j \underset{b}{\cdot} b_k = \delta_{jk}, \tag{5}$$

$$\beta^j \underset{b}{\cdot} \beta^k = \delta^{jk}. \tag{6}$$

Proof. The use of eq.(3) and the duality condition of $(\{b_k\},\{\beta^k\})$ gives

$$b_j \underset{b}{\cdot} b_k = \sum_{p=1}^{n} \beta^p(b_j)\beta^p(b_k) = \sum_{p=1}^{n} \delta_j^p \delta_k^p = \delta_{jk},$$

by using eq.(4) and once again the duality condition of $(\{b_k\},\{\beta^k\})$ we get

$$\beta^j \underset{b}{\cdot} \beta^k = \sum_{q=1}^{n} \beta^j(b_q)\beta^k(b_q) = \sum_{q=1}^{n} \delta_q^j \delta_q^k = \delta^{jk}. \blacksquare$$

In what follows we will use the simplified notation $(\cdot)$ for the b-scalar product $(\underset{b}{\cdot})$.

2.2 b-Reciprocal Bases

Let $(\{e_k\},\{\varepsilon^k\})$ be any bases for V and V^*, the second one being dual basis of the first one, i.e., $\varepsilon^k(e_j) = \delta_j^k$. Associated to it we can introduce another pair of bases $(\{e^k\},\{\varepsilon_k\})$, the first one for V and the second one for V^*, given by

$$e^k = \iota^{-1}(\varepsilon^k) = \sum_{s=1}^{n} \varepsilon^k(b_s) b_s, \tag{7}$$

$$\varepsilon_k = \iota(e_k) = \sum_{s=1}^{n} \beta^s(e_k)\beta^s. \tag{8}$$

Since ι and ι^{-1} are linear mappings between V and V^*, the n vectors $e^1,\ldots,e^n \in V$ and the n forms $\varepsilon_1,\ldots,\varepsilon_n \in V^*$ are linearly independent, hence they are a vector basis for V and a form basis for V^*, respectively.

i. The pairs of bases $(\{e_k\},\{e^k\})$ and $(\{\varepsilon^k\},\{\varepsilon_k\})$ satisfy the remarkable b-scalar product conditions

$$\begin{aligned} e_k\cdot e^l &= \delta_k^l, && (9)\\ \varepsilon^k\cdot\varepsilon_l &= \delta_l^k. && (10)\end{aligned}$$

Proof. By straightforward calculation, employing eq.(3) and eq.(7), and the duality condition of $(\{e_k\},\{\varepsilon^k\})$ we have

$$\begin{aligned} e_k\cdot e^l &= e^l\cdot e_k\\ &= \iota(e^l)(e_k)=\iota\circ\iota^{-1}(\varepsilon^l)(e_k)=\varepsilon^l(e_k)=\delta_k^l,\end{aligned}$$

and employing eq.(4) and eq.(8), and once again the duality condition of $(\{e_k\},\{\varepsilon^k\})$ we get

$$\varepsilon_k\cdot\varepsilon^l=\varepsilon^l(\iota^{-1}(\varepsilon_k))=\varepsilon^l(\iota^{-1}\circ\iota(e_k))=\varepsilon^l(e_k)=\delta_k^l. \blacksquare$$

Due to the property expressed by eq.(9) the vector bases $\{e_k\}$ and $\{e^k\}$ are called a *b-reciprocal bases* of V. The form bases $\{\varepsilon^k\}$ and $\{\varepsilon_k\}$ because the property given by eq.(10) are called a *b-reciprocal bases* of V^*.

ii. $\{\varepsilon_k\}$ is dual basis of $\{e^k\}$, i.e.,

$$\varepsilon_k(e^j)=\delta_k^j. \tag{11}$$

Proof. The equations (8), (3) and (9) yield

$$\varepsilon_k(e^j)=\iota(e_k)(e^j)=e_k\cdot e^j=\delta_k^j,$$

Also, the equations (7), (4) and (10) yield

$$\varepsilon_k(e^j)=\varepsilon_k(\iota^{-1}(\varepsilon^j))=\varepsilon^j\cdot\varepsilon_k=\delta_k^j. \blacksquare$$

3 2-Tensors vs. $(1,1)$-Extensors

For any 2-tensor $T\in T_2(V)$, there exists an unique $(1,1)$-extensor $t\in ext_1^1(V)$, such that for all vectors $v,w\in V$

$$T(v,w)=t(v)\cdot w. \tag{12}$$

Proof. We will prove that $t(v)=T(v,e_k)e^k$ (or, also $t(v)=T(v,e^k)e_k$), where $(\{e_k\},\{e^k\})$ is an arbitrary pair of b-reciprocal bases for V, satisfies eq.(12). Indeed, by using an expansion formula for vectors [1] and the linearity property of tensors, we have

$$t(v) \cdot w = T(v, e_k)e^k \cdot w = T(v, e^k \cdot w e_k) = T(v, w).$$

Now, t is unique. Indeed suppose that there is another $t' \in ext_1^1(V)$ for which $T(v, w) = t'(v) \cdot w$ for arbitrary v and w. Then, using the expansion formula for vectors it follows that

$$t'(v) = t'(v) \cdot e_k e^k = T(v, e_k)e^k = t(v).$$

Thus, the *existence* and *uniqueness* of such a $(1, 1)$-extensor satisfying eq.(12) is established. ■

The above theorem means that there exists an unique linear isomorphism between the vector spaces $T_2(V)$ and $ext_1^1(V)$ such that the jk-th covariant components of T with respect to $\{e_k\}$ equals the jk-matrix element of t with respect to $\{e_k\}$, i.e., $T(e_j, e_k) = t(e_j) \cdot e_k$.

T is *symmetric* (or, *skew-symmetric*) if and only if t is *adjoint symmetric* (or, *adjoint skew-symmetric*), i.e.,

$$T(v, w) = T(w, v), \text{ for all } v, w \in V \Leftrightarrow t = t^\dagger, \tag{13}$$

or,

$$T(v, w) = -T(w, v), \text{ for all } v, w \in V \Leftrightarrow t = -t^\dagger. \tag{14}$$

Let T_{jk} be the jk-entries of the $n \times n$ real matrix associated to T with respect to a basis $\{e_k\}$, i.e., $T_{jk} = T(e_j, e_k)$. The classical determinant $\det[T_{jk}]$ and $\det[t]$ are related by the remarkable formula[1]

$$\det[T_{jk}] = \det[t](e_1 \wedge \ldots \wedge e_n) \cdot (e_1 \wedge \ldots \wedge e_n). \tag{15}$$

Proof. By definition of classical determinant of $n \times n$ real matrix. Using eq.(12), the formula $(v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k) = \epsilon^{s_1 \ldots s_k} v_1 \cdot w_{s_1} \ldots v_k \cdot w_{s_k}$ and the property $\underline{t}(v_1 \wedge \ldots \wedge v_k) = t(v_1) \wedge \ldots t(v_k)$, where $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are vectors. We have

$$\begin{aligned}
|T_{jk}| &= \epsilon^{s_1 \ldots s_n} T_{1s_1} \ldots T_{ns_n} = \epsilon^{s_1 \ldots s_n} T(e_1, e_{s_1}) \ldots T(e_n, e_{s_n}) \\
&= \epsilon^{s_1 \ldots s_n} t(e_1) \cdot e_{s_1} \ldots t(e_n) \cdot e_{s_n} = (t(e_1) \wedge \ldots t(e_n)) \cdot (e_1 \wedge \ldots e_n) \\
&= \underline{t}(e_1 \wedge \ldots e_n) \cdot (e_1 \wedge \ldots e_n),
\end{aligned}$$

[1] Note that some previous papers that appeared in the literature and which use the concept of extensor [3] have made a wrong identification between the classical determinant of the matrix formed with the elements T_{jk}, and $\det[t]$.

hence, by definition of $\det[t]$, i.e., $\underline{t}(I) = \det[t] I$ for all non-null pseudoscalar I, the required result follows. ■

The $n \times n$ real matrix T_{jk} has inverse (i.e., there exists an unique $n \times n$ real matrix whose jk-entries, say T^{jk}, satisfying $T^{js}T_{sk} = \delta_k^j$ and $T_{js}T^{sk} = \delta_j^k$) if and only if t has inverse, say t^{-1} (i.e., t^{-1} is the unique $(1,1)$-extensor that satisfies $t^{-1} \circ t = t \circ t^{-1} = i_V$). The real numbers T^{jk} are equal to the jk-matrix elements of t^{-1}, i.e.,

$$T^{jk} = t^{-1}(e^j) \cdot e^k. \tag{16}$$

Proof. The first statement is an immediate consequence of eq.(15). In order to prove the second statement we shall use the expansion formula for $v \in V : v = v \cdot e^s e_s$, to get

$$\begin{aligned} T^{js}T_{sk} &= t^{-1}(e^j) \cdot e^s t(e_s) \cdot e_k = t(t^{-1}(e^j) \cdot e^s e_s) \cdot e_k \\ &= t \circ t^{-1}(e^j) \cdot e_k = e^j \cdot e_k, \end{aligned}$$

hence, by the b-reciprocity condition of $(\{e_k\}, \{e^k\})$, i.e., $e_k \cdot e^l = \delta_k^l$, we have that $T^{js}T_{sk} = \delta_k^j$. Analogously it can be proved that $T_{js}T^{sk} = \delta_j^k$. ■

4 Metric Isomorphism

We recall some definitions of the theory of ordinary linear algebra.

Let G be a (covariant) metric tensor over V, i.e., $G \in T_2(V)$ such that: G is symmetric $(G(v,w) = G(w,v))$ and G is non-degenerate ($\det[G_{jk}] \neq 0$, $G_{jk} = G(e_j, e_k)$ where $\{e_k\}$ is any basis for V).

According to equations (12), (13) and (15) there exists a $(1,1)$-extensor over V, say g, symmetric ($g = g^\dagger$) and non-degenerate ($\det[g] \neq 0$), such that for all $v, w \in V : G(v,w) = g(v) \cdot w$. By our previous nomenclature $g \in ext_1^1(V)$ is called a *metric extensor over V*.

This algebraic object *codifies* all the information contained into the classical concept of metric which appears in ordinary linear algebra.

The well-defined 2-tensor over V^*, $G^* \in T^2(V)$ such that $G^*(\omega, \sigma) = G^{jk}\omega(e_j)\sigma(e_k)$, where G^{jk} are the jk-entries of the inverse matrix of $G_{jk} = G(e_j, e_k)$, is said to be the (contravariant) metric tensor over V^*.

There exists a linear isomorphism between V and V^* realized by the linear mappings $V \ni v \mapsto \underset{G}{\iota}(v) \in V^*$and $V^* \ni \omega \mapsto \underset{G}{\iota}^{-1}(\omega) \in V$ such that

$$\underset{G}{\iota}(v)(w) = G(v,w), \tag{17}$$

$$\sigma(\underset{G}{\iota}^{-1}(\omega)) = G^*(\omega, \sigma). \tag{18}$$

As the notations point out $\underset{G}{\iota}^{-1}$, is the inverse mapping of $\underset{G}{\iota}$.

Proof. The linearity property for both $\underset{G}{\iota}$ and $\underset{G}{\iota}^{-1}$ holds. We must prove that $\underset{G}{\iota}^{-1} \circ \underset{G}{\iota} = i_V$ and $\underset{G}{\iota} \circ \underset{G}{\iota}^{-1} = i_{V^*}$, where i_V and i_{V^*} are the identity functions in V and V^*, respectively.

Let $(\{e_k\}, \{\varepsilon^k\})$ be an arbitrary pair of dual basis (of V and V^*), i.e., $\varepsilon^k(e_j) = \delta^k_j$. Recall the elementary expansions for vectors and forms, i.e., $v = \varepsilon^k(v)e_k$ and $\omega = \omega(e_k)\varepsilon^k$.

Take $v \in V$ and $\omega \in V^*$, $\underset{G}{\iota}(v)$ and $\underset{G}{\iota}^{-1}(\omega)$. Some well-known formulas involving the matrix elements of G and G^* can be written as follows

$$\begin{aligned}
\underset{G}{\iota}(v) &= \underset{G}{\iota}(v)(e_k)\varepsilon^k = G(v, e_k)\varepsilon^k \\
&= G(\varepsilon^j(v)e_j, e_k)\varepsilon^k = G(e_j, e_k)\varepsilon^j(v)\varepsilon^k, \\
\underset{G}{\iota}(v) &= G_{jk}\varepsilon^j(v)\varepsilon^k. \qquad (19)
\end{aligned}$$

Also,

$$\begin{aligned}
\underset{G}{\iota}^{-1}(\omega) &= \varepsilon^k(\underset{G}{\iota}^{-1}(\omega))e_k = G^*(\omega, \varepsilon^k)e_k \\
&= G^*(\omega(e_j)\varepsilon^j, \varepsilon^k)e_k = G^*(\varepsilon^j, \varepsilon^k)\omega(e_j)e_k \\
\underset{G}{\iota}^{-1}(\omega) &= G^{jk}\omega(e_j)e_k. \qquad (20)
\end{aligned}$$

Next, take $v \in V$. Using eq.(19), eq.(17) and the formula for expansion of vectors we have

$$\begin{aligned}
\underset{G}{\iota}^{-1} \circ \underset{G}{\iota}(v) &= G^{jk}\, \underset{G}{\iota}(v)(e_j)e_k = G^{jk}G(v, e_j)e_k = G^{jk}G(e_s, e_j)\varepsilon^s(v)e_k \\
&= G^{jk}G_{sj}\varepsilon^s(v)e_k = \delta^k_s\varepsilon^s(v)e_k = \varepsilon^k(v)e_k = v,
\end{aligned}$$

hence, $\underset{G}{\iota}^{-1} \circ \underset{G}{\iota} = i_V$.

By the same way, taking an arbitrary $\omega \in V^*$ and employing eq.(20), eq.(18) and the elementary expansion for forms, we finally get $\underset{G}{\iota} \circ \underset{G}{\iota}^{-1} = i_{V^*}$. ■

The linear isomorphism between V and V^* showed above, which is induced by the (covariant) metric tensor G is usually called a *metric isomorphism*.

The inverse mappings $\underset{G}{\iota}$ and $\underset{G}{\iota}^{-1}$ can be written in suggestive forms involving the $(1,1)$-extensors g and g^{-1}, and the inverse mappings ι and ι^{-1} of the standard isomorphism, i.e.,

$$\underset{G}{\iota} = \iota \circ g, \tag{21}$$

$$\underset{G}{\iota}^{-1} = g^{-1} \circ \iota^{-1}. \tag{22}$$

Proof. First take $v \in V$, using equations (17), (12) and (7) we have

$$\begin{aligned} \underset{G}{\iota}(v) &= \underset{G}{\iota}(v)(e_k)\varepsilon^k = G(v, e_k)\varepsilon^k = g(v) \cdot e_k \varepsilon^k \\ &= g(v) \cdot e_k \iota(e^k) = \iota(g(v) \cdot e_k e^k) \\ &= \iota \circ g(v), \end{aligned}$$

hence, $\underset{G}{\iota} = \iota \circ g$.

Now, using the property: 'inverse of composition equals composition of inverse into reversed order', we finally get

$$\underset{G}{\iota}^{-1} = (\iota \circ g)^{-1} = g^{-1} \circ \iota^{-1}. \blacksquare$$

5 Conclusions

We investigated the relationship between metric tensors and metric extensors associated to a n-dimensional real vector space, and translated some well known results of tensor theory using extensors. The results obtained, specially eq.(15) that relates the classical determinant of the matrix whose entries are the components of a 2-tensor in a given vector basis with the determinant of the corresponding extensor, is important for many calculations which will appear in the next papers reporting ours studies on the theory of multivector functions, and some problems of differential geometry and theoretical physics. We emphasize moreover that even in elementary linear algebra it is an advantage to use metric extensors instead of metric tensors because a metric extensor has an inverse[2] which is itself a metric extensor. Also, with the concept of metric extensor sophisticated calculations can be done without the introduction of matrix representations.

[2]Of course, a metric tensor has no inverse, see [1].

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References

[1] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr, Euclidean Clifford Algebra (paper I of a series of seven), in publication in *Adv. Appl. Clifford Algebras* **11**(S3), (2001).

[2] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Extensors (paper II of a series of seven), in publication in *Adv. Appl. Clifford Algebras* **11**(S3), (2001).

[3] Lasenby, A., Doran, C., and Gull, S., Gravity, Gauge Theories and Geometrical Algebras, *Phil. Trans. R. Soc.* **356**, 4877-582 (1998).