# Extensors*

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**Abstract**

In this paper we introduce a class of mathematical objects called *extensors* and develop some aspects of their theory with considerable detail. We give special names to several particular but important cases of extensors. The *extension, adjoint* and *generalization* operators are introduced and their properties studied. For the so-called $(1,1)$-extensors we define the concept of *determinant*, and their properties are investigated. Some preliminary applications of the theory of extensors are presented in order to show the power of the new concept in action. An useful formula for the inversion of $(1,1)$-extensors is obtained.

## Contents



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# 1 Introduction

In this second paper of a series of seven we introduce the so-called *extensors,* and develop some aspects of their theory with considerable detail. We give special names and study in detail some important cases of extensors which will appear frequently in future developments of ours theories. The *extension, adjoint* and *generalization* operators are introduced and their main properties are determined. In particular, the generalization operator plays a crucial role in our theory of covariant derivative operators on smooth manifolds to be presented in a forthcoming series of papers. We define moreover the concept of determinant of a $(1, 1)$-extensor (a scalar which is an characteristic invariant associated to the extensor), and obtain their basic properties which are very similar, but not identical, to the classical determinat of a real square matrix. We present some preliminary applications of the concept of extensor which will be useful later. In particular, we present the concept of the *changing basis extensor*, and derive also an interesting formula for the inversion of $(1,1)$-extensors. Some related, but not equivalent material, appears in ( [1],[3]).

## 2 General $k$-Extensors

Let $V$ be a real vector space of finite dimension, and let $V^*$ be its dual space. Denote by $\bigwedge^p V$ the space of $p$-vectors over $V$. Recall that if $\dim V = n$, then $\dim \bigwedge^p V = \binom{n}{p}$.

As defined in paper I [4] a multivector over $V$ is simply a formal sum of scalar, vector,..., pseudovector and pseudoscalar. The space of these objects has been denoted by $\bigwedge V$, i.e., $\bigwedge V = \mathbb{R} + V + \cdots + \bigwedge^{n-1} V + \bigwedge^n V$. Recall that if $\dim V = n$ then $\dim \bigwedge V = 2^n$.

Let $\overset{\diamond}{\underset{1}{\bigwedge}} V, \ldots, \overset{\diamond}{\underset{k}{\bigwedge}} V$ to $\overset{\diamond}{\bigwedge} V$ be $k+1$ subspaces of $\bigwedge V$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V$, and $\overset{\diamond}{\bigwedge} V$ be either any sum of homogeneous subspaces of $\bigwedge V$ or the trivial subspace $\{0\}$. A multilinear mapping from the cartesian product $\overset{\diamond}{\underset{1}{\bigwedge}} V \times \cdots \times \overset{\diamond}{\underset{k}{\bigwedge}} V$ to $\overset{\diamond}{\bigwedge} V$ will be called a *general $k$-extensor* over $V$, i.e., $t : \overset{\diamond}{\underset{1}{\bigwedge}} V \times \cdots \times \overset{\diamond}{\underset{k}{\bigwedge}} V \to \overset{\diamond}{\bigwedge} V$ such that for any $\alpha_j, \alpha'_j \in \mathbb{R}$ and $X_j, X'_j \in \overset{\diamond}{\underset{j}{\bigwedge}} V$

$$t(\ldots, \alpha_j X_j + \alpha'_j X'_j, \ldots) = \alpha_j t(\ldots, X_j, \ldots) + \alpha'_j t(\ldots, X'_j, \ldots), \qquad (1)$$

for each $j$ with $1 \leq j \leq k$.

It should be noticed that the linear operators on $V$, $\bigwedge^p V$ or $\bigwedge V$ which appear in ordinary linear algebra are particular cases of 1-extensors over $V$. A covariant $k$-tensor over $V$ is just being a $k$-extensor over $V$.

In this way, the concept of general $k$-extensor generalizes and unifies both of the well-known concepts of linear operator and covariant $k$-tensor. These mathematical objects are of the same nature!

The set of general $k$-extensors over $V$, denoted by the suggestive notation $k\text{-}ext(\overset{\diamond}{\underset{1}{\bigwedge}} V, \cdots, \overset{\diamond}{\underset{k}{\bigwedge}} V, \overset{\diamond}{\bigwedge} V)$, has a natural structure of real vector space. Its dimension is given by noticeable formula

$$\dim k\text{-}ext(\overset{\diamond}{\bigwedge_1} V, \ldots, \overset{\diamond}{\bigwedge_k} V; \overset{\diamond}{\bigwedge} V) = \dim \overset{\diamond}{\bigwedge_1} V \cdots \dim \overset{\diamond}{\bigwedge_k} V \dim \overset{\diamond}{\bigwedge} V. \tag{2}$$

We shall need to consider only some particular cases of these general $k$-extensors over $V$. So, special names and notations will be given for them.

We will equip $V$ with an arbitrary (but fixed, once and for all) euclidean metric $G_E$. And as usual we will denote the scalar product of multivectors $X, Y \in \bigwedge V$ with respect to the euclidean metric structure $(V, G_E)$, namely $X \underset{G_E}{\cdot} Y$, by the more simple notation $X \cdot Y$.

Let $\{e_j\}$ be any basis for $V$, and $\{e^j\}$ be its euclidean reciprocal basis for $V$, i.e., $e_j \cdot e^k = \delta_j^k$.

## 2.1 $(p, q)$-Extensors

Let $p$ and $q$ be two integer numbers with $0 \leq p, q \leq n$. A linear mapping which sends $p$-vectors to $q$-vectors will be called a $(p, q)$-*extensor* over $V$. The space of them, namely $1\text{-}ext(\bigwedge^p V; \bigwedge^q V)$, will be denoted by $ext_p^q(V)$ for short. By using eq.(2) we get

$$\dim ext_p^q(V) = \binom{n}{p}\binom{n}{q}. \tag{3}$$

For instance, we see that the $(1,1)$-extensors over $V$ are just the linear operators on $V$.

The set of $\binom{n}{p}\binom{n}{q}$ extensors belonging to $ext_p^q(V)$, namely $\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}$, defined by

$$\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}(X) = (e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot X e^{k_1} \wedge \ldots \wedge e^{k_q}, \tag{4}$$

for all $X \in \bigwedge^p V$, is a $(p, q)$-extensor basis for $ext_p^q(V)$.

This set of extensors are indeed linearly independent, and for each $t \in ext_p^q(V)$ there exist $\binom{n}{p}\binom{n}{q}$ real numbers, say $t_{j_1 \ldots j_p; k_1 \ldots k_q}$, given by

$$t_{j_1 \ldots j_p; k_1 \ldots k_q} = t(e_{j_1} \wedge \ldots \wedge e_{j_p}) \cdot (e_{k_1} \wedge \ldots \wedge e_{k_q}) \tag{5}$$

such that

$$t = \frac{1}{p!}\frac{1}{q!} t_{j_1 \ldots j_p; k_1 \ldots k_q} \varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}. \tag{6}$$

Such $t_{j_1 \ldots j_p; k_1 \ldots k_q}$ will be called the $j_1 \ldots j_p; k_1 \ldots k_q$-th *covariant components* of $t$ with respect to the $(p,q)$-*extensor basis* $\{\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}\}$.

Of course, there are still other $(p,q)$-extensor bases for $ext_p^q(V)$ besides the ones given by eq.(4) which can be constructed with the vector bases $\{e_j\}$ and $\{e^j\}$ of $V$. Indeed, there are $2^{p+q}$ of such $(p,q)$-extensor bases for $ext_p^q(V)$. For instance, if we use the basis $(p,q)$-extensors $\varepsilon_{j_1 \ldots j_p; k_1 \ldots k_q}$ and the real numbers $t^{j_1 \ldots j_p; k_1 \ldots k_q}$ defined by

$$\varepsilon_{j_1 \ldots j_p; k_1 \ldots k_q}(X) = (e_{j_1} \wedge \ldots \wedge e_{j_p}) \cdot X e_{k_1} \wedge \ldots \wedge e_{k_q}, \tag{7}$$

$$t^{j_1 \ldots j_p; k_1 \ldots k_q} = t(e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot (e^{k_1} \wedge \ldots \wedge e^{k_q}), \tag{8}$$

we get an expansion formula for $t \in ext_p^q(V)$ analogous to that given by eq.(6), i.e.,

$$t = \frac{1}{p!} \frac{1}{q!} t^{j_1 \ldots j_p; k_1 \ldots k_q} \varepsilon_{j_1 \ldots j_p; k_1 \ldots k_q}. \tag{9}$$

Such $t^{j_1 \ldots j_p; k_1 \ldots k_q}$ are called the $j_1 \ldots j_p; k_1 \ldots k_q$-th *contravariant components* of $t$ with respect to the $(p,q)$-*extensor basis* $\{\varepsilon_{j_1 \ldots j_p; k_1 \ldots k_q}\}$.

## 2.2 Extensors

A linear mapping which sends multivectors to multivectors will be simply called an *extensor* over $V$. They are the linear operators on $\bigwedge V$. For the space of extensors over $V$, namely 1-$ext(\bigwedge V; \bigwedge V)$, we will use the short notation $ext(V)$. By using eq.(2) we get

$$\dim ext(V) = 2^n 2^n. \tag{10}$$

For instance, we will see that the so-called Hodge star operator is just being a well-defined extensor over $V$ which can be also thought as $(p, n-p)$-extensor over $V$. The so-called extended of $t \in ext_1^1(V)$ is just being an extensor over $V$, i.e., $\underline{t} \in ext(V)$.

There are $2^n 2^n$ extensors over $V$, namely $\varepsilon^{J;K}$, given by[1]

$$\varepsilon^{J;K}(X) = (e^J \cdot X) e^K, \tag{11}$$

for all $X \in \bigwedge V$, which set extensor bases for $ext(V)$.

[1] $J$ and $K$ are colective indices. Recall, for example, that: $e_J = 1, e_{j_1}, e_{j_1} \wedge e_{j_2}, \ldots (e^J = 1, e^{j_1}, e^{j_1} \wedge e^{j_2}, \ldots)$ and $\nu(J) = 0, 1, 2, \ldots$ for $J = \emptyset, j_1, j_1 j_2, \ldots$, where all index $j_1, j_2, \ldots$ runs from 1 to $n$.

In fact they are lineary independent, and for each $t \in ext(V)$ there exist $2^n 2^n$ real numbers, say $t_{J;K}$, given by

$$t_{J;K} = t(e_J) \cdot e_K \tag{12}$$

such that

$$t = \sum_J \sum_K \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t_{J;K} \varepsilon^{J;K}. \tag{13}$$

Such $t_{J;K}$ will be called the $J;K$-th *covariant components* of $t$ with respect to the *extensor basis* $\{\varepsilon^{J;K}\}$.

We notice that exactly $(2^{n+1}-1)^2$ extensor bases for $ext(V)$ can be constructed from the vector bases $\{e_j\}$ and $\{e^j\}$ of $V$. For instance, whenever the basis extensors $\varepsilon_{J;K}$ and the real numbers $t^{J;K}$ defined by

$$\varepsilon_{J;K}(X) = (e_J \cdot X) e_K, \tag{14}$$
$$t^{J;K} = t(e^J) \cdot e^K \tag{15}$$

are being used, an expansion formula for $t \in ext(V)$ analogous to that given by eq.(13) can be obtained, i.e.,

$$t = \sum_J \sum_K \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t^{J;K} \varepsilon_{J;K}. \tag{16}$$

Such $t^{J;K}$ are called the $J;K$-th *contravariant components* of $t$ with respect to the *extensor basis* $\{\varepsilon_{J;K}\}$.

## 2.3 Elementary $k$-Extensors

A multilinear mapping which takes $k$-uple of vectors into $q$-vectors will be called an *elementary $k$-extensor* over $V$of degree $q$. The space of them, namely $k$-$ext(V, \ldots, V; \bigwedge^q V)$, will be denoted by $k$-$ext^q(V)$. It is easy to verify by using eq.(2) that

$$\dim k\text{-}ext^q(V) = n^k \binom{n}{q}. \tag{17}$$

It should be noticed that a elementary $k$-extensor over $V$ of degree 0 is just being a *covariant $k$-tensor* over $V$, i.e., $k$-$ext^0(V) \equiv T_k(V)$. It is easily realized that 1-$ext^q(V) \equiv ext_1^q(V)$.

The elementary $k$-extensors of degrees $0, 1, 2, \ldots$ etc. are sometimes said to be *scalar*, *vector, bivector*, ... etc. *elementary $k$-extensors.*

There are $n^k \binom{n}{q}$ elementary $k$-extensors of degree $q$ belonging to $k\text{-}ext^q(V)$, namely $\varepsilon^{j_1,\ldots,j_k;k_1\ldots k_q}$, defined by

$$\varepsilon^{j_1,\ldots,j_k;k_1\ldots k_q}(v_1, \ldots, v_k) = (v_1 \cdot e^{j_1}) \ldots (v_k \cdot e^{j_k}) e^{k_1} \wedge \ldots \wedge e^{k_q}, \tag{18}$$

for all $(v_1, \ldots, v_k) \in \underbrace{V \times \cdots \times V}_{k \text{ copies}}$, which set elementary $k$-extensor of degree $q$ bases for $k\text{-}ext^q(V)$.

In fact they are linearly independent, and for all $t \in k\text{-}ext^q(V)$ there are $n^k \binom{n}{q}$ real numbers, say $t_{j_1,\ldots,j_k;k_1\ldots k_q}$, given by

$$t_{j_1,\ldots,j_k;k_1\ldots k_q} = t(e_{j_1}, \ldots, e_{j_k}) \cdot (e_{k_1} \wedge \ldots \wedge e_{k_q}) \tag{19}$$

such that

$$t = \frac{1}{q!} t_{j_1,\ldots,j_k;k_1\ldots k_q} \varepsilon^{j_1,\ldots,j_k;k_1\ldots k_q}. \tag{20}$$

Such $t_{j_1,\ldots,j_k;k_1\ldots k_q}$ will be called the $j_1, \ldots, j_k; k_1 \ldots k_q$-th *covariant components* of $t$ with respect to the *bases* $\{\varepsilon^{j_1,\ldots,j_k;k_1\ldots k_q}\}$.

We notice that exactly $2^{k+q}$ elementary $k$-extensors of degree $q$ bases for $k\text{-}ext^q(V)$ can be constructed from the vector bases $\{e_j\}$ and $\{e^j\}$ of $V$. For instance, we might define the basis elementary $k$-extensor of degree $q$ $\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}$ and the real numbers $t^{j_1,\ldots,j_k;k_1\ldots k_q}$ by the following equations

$$\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}(v_1, \ldots, v_k) = (v_1 \cdot e_{j_1}) \ldots (v_k \cdot e_{j_k}) e_{k_1} \wedge \ldots \wedge e_{k_q}, \tag{21}$$

$$t^{j_1,\ldots,j_k;k_1\ldots k_q} = t(e^{j_1}, \ldots, e^{j_k}) \cdot (e^{k_1} \wedge \ldots \wedge e^{k_q}). \tag{22}$$

Then, we might have other expansion formula for $t \in k\text{-}ext^q(V)$ than that given by eq.(20), i.e.,

$$t = \frac{1}{q!} t^{j_1,\ldots,j_k;k_1\ldots k_q} \varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}. \tag{23}$$

Such $t^{j_1,\ldots,j_k;k_1\ldots k_q}$ are called the $j_1, \ldots, j_k; k_1 \ldots k_q$-th *contravariant components* of $t$ with respect to the *basis* $\{\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}\}$.

Note that $1\text{-}ext^p(V) \equiv ext_1^p(V)$.

A completely skew-symmetric $k$-extensor over $V$ of degree $p$ $(k \geq 2)$, i.e., $\Theta \in k\text{-}ext^p(V)$ such that for any $v_i, v_j \in V$ with $1 \leq i < j \leq k$,

$$\Theta(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\Theta(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) \tag{24}$$

will be called a *$k$-exform over $V$ of degree $p$.*

The vector space of $k$-exforms over $V$ of degree $p$ will be denoted by $k\text{-}exf^p(V)$.

It is also convenient to define a 0-exform of degree $p$ to be a $p$-vector (i.e., $0\text{-}exf^p(V) = \bigwedge^p V$) and an 1-exform of degree $p$ to be a 1-extensor of degree $p$ (i.e., $1\text{-}exf^p(V) = 1\text{-}ext^p(V)$).

If $\dim V = n$, then $\dim k\text{-}exf^p(V) = \binom{n}{k}\binom{n}{p}$.

The $k$-exforms of degree $0, 1, 2, \ldots$ etc. are said to be 'scalar $k$-exforms', vector $k$-exforms', 'bivector $k$-exforms',... etc.

Note that a scalar $k$-exform is just a $k$-form, i.e., $k\text{-}exf^0(V) = \bigwedge^k V$. In this way, we see that the concept of $k$-exform generalizes the concept of $k$-form.

## 3 Projectors

Let $\overset{\diamond}{\bigwedge} V$ be either any sum of homogeneous subspaces[2] of $\bigwedge V$ or the trivial subspace $\{0\}$. Associated to $\bigwedge^\diamond V$, a noticeable extensor from $\bigwedge V$ to $\overset{\diamond}{\bigwedge} V$, namely $\langle\,\rangle_{\overset{\diamond}{\bigwedge} V}$, can defined by

$$\langle X\rangle_{\overset{\diamond}{\bigwedge} V} = \begin{cases} \langle X\rangle_{p_1} + \cdots + \langle X\rangle_{p_\nu}, & \text{if } \overset{\diamond}{\bigwedge} V = \bigwedge^{p_1} V + \cdots + \bigwedge^{p_\nu} V \\ 0, & \text{if } \overset{\diamond}{\bigwedge} V = \{0\} \end{cases}. \quad (25)$$

Such $\langle\,\rangle_{\overset{\diamond}{\bigwedge} V} \in 1\text{-}ext(\bigwedge V; \overset{\diamond}{\bigwedge} V)$ will be called the *$\bigwedge^\diamond V$-projector extensor.*

We notice that if $\overset{\diamond}{\bigwedge} V$ is any homogeneous subspace of $\bigwedge V$, i.e., $\overset{\diamond}{\bigwedge} V = \bigwedge^p V$, then the projector extensor is reduced to the so-called *$p$-part operator*, i.e., $\langle\,\rangle_{\overset{\diamond}{\bigwedge} V} = \langle\,\rangle_p$.

We now summarize the fundamental properties for the $\overset{\diamond}{\bigwedge} V$-projector extensors.

Let $\overset{\diamond}{\underset{1}{\bigwedge}} V$ and $\overset{\diamond}{\underset{2}{\bigwedge}} V$ be two subspaces of $\bigwedge V$. If each of them is either any

[2]Note that for such a subspace $\overset{\diamond}{\bigwedge} V$ there are $\nu$ integers $p_1, \ldots, p_\nu$ ($0 \leq p_1 < \cdots < p_\nu \leq n$) such that $\overset{\diamond}{\bigwedge} V = \bigwedge^{p_1} V + \cdots + \bigwedge^{p_\nu} V$.

sum of homogeneous subspaces of $\bigwedge V$ or the trivial subspace $\{0\}$, then

$$\left\langle \langle X\rangle_{\overset{\diamond}{\bigwedge_1} V} \right\rangle_{\overset{\diamond}{\bigwedge_2} V} = \langle X\rangle_{\overset{\diamond}{\bigwedge_1} V \cap \overset{\diamond}{\bigwedge_2} V} \tag{26}$$

$$\langle X\rangle_{\overset{\diamond}{\bigwedge_1} V} + \langle X\rangle_{\overset{\diamond}{\bigwedge_2} V} = \langle X\rangle_{\overset{\diamond}{\bigwedge_1} V \cup \overset{\diamond}{\bigwedge_2} V}. \tag{27}$$

Let $\overset{\diamond}{\bigwedge} V$ be either any sum of homogeneous subspaces of $\bigwedge V$ or the trivial subspace $\{0\}$. It holds

$$\langle X\rangle_{\overset{\diamond}{\bigwedge} V} \cdot Y = X \cdot \langle Y\rangle_{\overset{\diamond}{\bigwedge} V}. \tag{28}$$

In this sense we might say that the concept of $\overset{\diamond}{\bigwedge} V$-projector extensor is just a well-done generalization of the concept of $p$-part operator.

# 4 The Extension Operator

Let $\{e_j\}$ be any basis for $V$, and $\{\varepsilon^j\}$ be its dual basis for $V^*$. As we know, $\{\varepsilon^j\}$ is the unique 1-form basis associated to the vector basis $\{e_j\}$ such that $\varepsilon^j(e_i) = \delta_i^j$.

The linear mapping $ext_1^1(V) \ni t \mapsto \underline{t} \in ext(V)$ such that for any $X \in \bigwedge V$ : if $X = X_0 + \sum\limits_{k=1}^{n} X_k$, then

$$\underline{t}(X) = X_0 + \sum_{k=1}^{n} \frac{1}{k!} X_k(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \tag{29}$$

will be called the *extension operator*. We call $\underline{t}$ the *extended* of $t$. It is the well-known *outermorphism* of $t$ in the ordinary linear algebra.

The extension operator is well-defined since it does not depend on the choice of $\{e_j\}$.

We summarize now the basic properties satisfied by the extension operator.

**e1** The extension operator is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^p V, \text{ then } \underline{t}(X) \in \bigwedge^p V. \tag{30}$$

It is an obvious result which follows from eq.(29).

**e2** For any $\alpha \in \mathbb{R}$, $v \in V$ and $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$ it holds

$$\begin{aligned} \underline{t}(\alpha) &= \alpha, & (31)\\ \underline{t}(v) &= t(v), & (32)\\ \underline{t}(v_1 \wedge \ldots \wedge v_k) &= t(v_1) \wedge \ldots \wedge t(v_k). & (33) \end{aligned}$$

**Proof.**

The first statement trivially follows from eq.(29). The second one can easily be deduced from eq.(29) by recalling the elementary expansion formula for vectors and the linearity of extensors. In order to prove the third statement we will use the remarkable formulas: $v_1 \wedge \ldots \wedge v_k(\omega^1, \ldots, \omega^k) = \epsilon^{i_1 \ldots i_k} \omega^1(v_{i_1}) \ldots \omega^k(v_{i_k})$ and $w_{i_1} \wedge \ldots \wedge w_{i_k} = \epsilon_{i_1 \ldots i_k} w_1 \wedge \ldots \wedge w_k$, where $v_1, \ldots, v_k \in V$, $w_1, \ldots, w_k \in V$ and $\omega^1, \ldots, \omega^k \in V^*$, and the combinatorial formula $\epsilon^{i_1 \ldots i_k} \epsilon_{i_1 \ldots i_k} = k!$. By recalling the elementary expansion formula for vectors and the linearity of extensors we have that

$$\begin{aligned} \underline{t}(v_1 \wedge \ldots \wedge v_k) &= \frac{1}{k!} v_1 \wedge \ldots \wedge v_k(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \\ &= \frac{1}{k!} \epsilon^{i_1 \ldots i_k} \varepsilon^{j_1}(v_{i_1}) \ldots \varepsilon^{j_k}(v_{i_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \\ &= \frac{1}{k!} \epsilon^{i_1 \ldots i_k} t(v_{i_1}) \wedge \ldots \wedge t(v_{i_k}), \\ &= t(v_1) \wedge \ldots \wedge t(v_k). \blacksquare \end{aligned}$$

**e3** For any $X, Y \in \bigwedge V$ it holds

$$\underline{t}(X \wedge Y) = \underline{t}(X) \wedge \underline{t}(Y). \tag{34}$$

It is an immediate result which follows from eq.(33).

We emphasize that the three fundamental properties as given by eq.(31), eq.(32) and eq.(34) together are completely equivalent to the *extension procedure* as defined by eq.(29).

We present next some important properties of the extension operator.

**e4** Let us take $s, t \in ext_1^1(V)$. Then, the following result holds

$$\underline{s \circ t} = \underline{s} \circ \underline{t}. \tag{35}$$

**Proof.**

It is enough to present the proofs for scalars and simple $k$-vectors.

For $\alpha \in \mathbb{R}$, by using eq.(31), we get

$$\underline{s \circ t}(\alpha) = \alpha = \underline{s}(\alpha) = \underline{s}(\underline{t}(\alpha)) = \underline{s} \circ \underline{t}(\alpha).$$

For a simple $k$-vector $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$, by using eq.(33), we get

$$\begin{aligned} \underline{s \circ t}(v_1 \wedge \ldots \wedge v_k) &= s \circ t(v_1) \wedge \ldots \wedge s \circ t(v_k) = s(t(v_1)) \wedge \ldots \wedge s(t(v_k)) \\ &= \underline{s}(t(v_1) \wedge \ldots \wedge t(v_k)) = \underline{s}(\underline{t}(v_1 \wedge \ldots \wedge v_k)), \\ &= \underline{s} \circ \underline{t}(v_1 \wedge \ldots \wedge v_k). \end{aligned}$$

Next we can easily generalize to multivectors by linearity of extensors. It yields

$$\underline{s \circ t}(X) = \underline{s} \circ \underline{t}(X). \blacksquare$$

**e5** Let us take $t \in ext_1^1(V)$ with inverse $t^{-1} \in ext_1^1(V)$, i.e., $t^{-1} \circ t = t \circ t^{-1} = i_V$. Then, $\underline{(t^{-1})} \in ext(V)$ is the inverse of $\underline{t} \in ext(V)$, i.e.,

$$(\underline{t})^{-1} = \underline{(t^{-1})}. \tag{36}$$

Indeed, by using eq.(35) and the obvious property $\underline{i}_V = i_{\bigwedge V}$, we have that

$$t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow \underline{(t^{-1})} \circ \underline{t} = \underline{t} \circ \underline{(t^{-1})} = i_{\bigwedge V}.$$

It means that the *inverse of the extended* of $t$ equals the *extended of the inverse* of $t$.

In accordance with the corollary above we might use a more simple notation $\underline{t}^{-1}$ to denote both of $(\underline{t})^{-1}$ and $\underline{(t^{-1})}$.

**e6** Let $\{e_j\}$ be any basis for $V$, and $\{e^j\}$ its euclidean reciprocal basis for $V$, i.e., $e_j \cdot e^k = \delta_j^k$. There are two interesting and useful formulas for calculating the extended of $t \in ext_1^1(V)$, i.e.,

$$\begin{aligned} \underline{t}(X) &= 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!}(e^{j_1} \wedge \ldots \wedge e^{j_k}) \cdot X t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \qquad (37) \\ &= 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!}(e_{j_1} \wedge \ldots \wedge e_{j_k}) \cdot X t(e^{j_1}) \wedge \ldots \wedge t(e^{j_k}). \qquad (38) \end{aligned}$$

# 5 Standard Adjoint Operator

Let as above $\overset{\diamond}{\underset{1}{\bigwedge}} V$ and $\overset{\diamond}{\underset{2}{\bigwedge}} V$ be two subspaces of $\bigwedge V$ such that each of them is either any sum of homogeneous subspaces of $\bigwedge V$. Let $\{e_j\}$ and $\{e^j\}$ be two euclidean reciprocal bases to each other for $V$, i.e., $e_j \cdot e^k = \delta_j^k$.

The *standard adjoint operator* is the linear mapping 1-$ext(\overset{\diamond}{\underset{1}{\bigwedge}} V; \overset{\diamond}{\underset{2}{\bigwedge}} V) \ni t \to t^\dagger \in$ 1-$ext(\overset{\diamond}{\underset{2}{\bigwedge}} V; \overset{\diamond}{\underset{1}{\bigwedge}} V)$ such that for any $Y \in \overset{\diamond}{\underset{2}{\bigwedge}} V$ :

$$
\begin{aligned}
t^\dagger(Y) &= t(\langle 1\rangle_{\overset{\diamond}{\underset{1}{\bigwedge}} V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t(\langle e^{j_1} \wedge \ldots e^{j_k}\rangle_{\overset{\diamond}{\underset{1}{\bigwedge}} V}) \cdot Y e_{j_1} \wedge \ldots e_{j_k} &(39)\\
&= t(\langle 1\rangle_{\overset{\diamond}{\underset{1}{\bigwedge}} V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t(\langle e_{j_1} \wedge \ldots e_{j_k}\rangle_{\overset{\diamond}{\underset{1}{\bigwedge}} V}) \cdot Y e^{j_1} \wedge \ldots e^{j_k}, &(40)
\end{aligned}
$$

or into a more compact notation by employing the *colective index* $J$,

$$
\begin{aligned}
t^\dagger(Y) &= \sum_J \frac{1}{\nu(J)!} t(\langle e^J\rangle_{\overset{\diamond}{\bigwedge_1} V}) \cdot Y e_J &(41)\\
&= \sum_J \frac{1}{\nu(J)!} t(\langle e_J\rangle_{\overset{\diamond}{\bigwedge_1} V}) \cdot Y e^J, &(42)
\end{aligned}
$$

We call $t^\dagger$ the *standard adjoint extensor* of $t$. It should be noticed the use of the $\overset{\diamond}{\underset{1}{\bigwedge}} V$-projector extensor.

The standard adjoint operator is well-defined since the sums appearing in each of places above do not depend on the choice of $\{e_j\}$.

Let us take $X \in \overset{\diamond}{\underset{1}{\bigwedge}} V$ and $Y \in \overset{\diamond}{\underset{2}{\bigwedge}} V$. A straightforward calculation yields

$$
\begin{aligned}
X \cdot t^\dagger(Y) &= \sum_J \frac{1}{\nu(J)!} t(\langle e^J\rangle_{\overset{\diamond}{\bigwedge_1} V}) \cdot Y (X \cdot e_J)\\
&= t(\sum_J \frac{1}{\nu(J)!} \langle (X \cdot e_J) e^J\rangle_{\overset{\diamond}{\bigwedge_1} V}) \cdot Y\\
&= t(\langle X\rangle_{\overset{\diamond}{\bigwedge_1} V}) \cdot Y,
\end{aligned}
$$

i.e.,

$$
X \cdot t^\dagger(Y) = t(X) \cdot Y. \qquad (43)
$$

It is a generalization of the well-known property which holds for linear operators.

Let us take $t \in 1\text{-}ext(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V)$ and $u \in 1\text{-}ext(\bigwedge_2^\diamond V; \bigwedge_3^\diamond V)$. We can note that $u \circ t \in 1\text{-}ext(\bigwedge_1^\diamond V; \bigwedge_3^\diamond V)$ and $t^\dagger \circ u^\dagger \in 1\text{-}ext(\bigwedge_3^\diamond V; \bigwedge_1^\diamond V)$. Then, let us take $X \in \bigwedge_1^\diamond V$ and $Z \in \bigwedge_3^\diamond V$, by using eq.(43) we have that

$$X \cdot (u \circ t)^\dagger(Z) = (u \circ t)(X) \cdot Z = t(X) \cdot u^\dagger(Z) = X \cdot (t^\dagger \circ u^\dagger)(Z).$$

Hence, we get

$$(u \circ t)^\dagger = t^\dagger \circ u^\dagger. \tag{44}$$

Let us take $t \in 1\text{-}ext(\bigwedge^\diamond V; \bigwedge^\diamond V)$ with inverse $t^{-1} \in 1\text{-}ext(\bigwedge^\diamond V; \bigwedge^\diamond V)$, i.e., $t^{-1} \circ t = t \circ t^{-1} = i_{\bigwedge^\diamond V}$, where $i_{\bigwedge^\diamond V} \in 1\text{-}ext(\bigwedge^\diamond V; \bigwedge^\diamond V)$ is the so-called identity function for $\bigwedge^\diamond V$. By using eq.(44) and the obvious property $i_{\bigwedge^\diamond V} = i^\dagger_{\bigwedge^\diamond V}$, we have that

$$t^{-1} \circ t = t \circ t^{-1} = i_{\bigwedge^\diamond V} \Rightarrow t^\dagger \circ (t^{-1})^\dagger = (t^{-1})^\dagger \circ t^\dagger = i_{\bigwedge^\diamond V},$$

hence,

$$(t^\dagger)^{-1} = (t^{-1})^\dagger, \tag{45}$$

i.e., the inverse of the adjoint of $t$ equals the adjoint of the inverse of $t$. In accordance with the above corollary it is possible to use a more simple symbol, say $t^*$, to denote both of $(t^\dagger)^{-1}$ and $(t^{-1})^\dagger$.

Let us take $t \in ext_1^1(V)$. We note that $\underline{t} \in ext(V)$ and $\underline{(t^\dagger)} \in ext(V)$. A straightforward calculation by using eqs.(37) and (38) yields

$$\begin{aligned}
\underline{(t^\dagger)}(Y) &= 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!}(e^{j_1} \wedge \ldots e^{j_k}) \cdot Y t^\dagger(e_{j_1}) \wedge \ldots t^\dagger(e_{j_k}) \\
&= 1 \cdot Y + \\
&\quad \sum_{k=1}^{n} \frac{1}{k!}(e^{j_1} \wedge \ldots e^{j_k}) \cdot Y t^\dagger(e_{j_1}) \cdot e_{p_1} e^{p_1} \wedge \ldots t^\dagger(e_{j_k}) \cdot e_{p_k} e^{p_k} \\
&= 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!}(e_{j_1} \cdot t(e_{p_1}) e^{j_1} \wedge \ldots e_{j_k} \cdot t(e_{p_k}) e^{j_k}) \cdot Y e^{p_1} \wedge \ldots e^{p_k} \\
&= \underline{t}(1) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} \underline{t}(e_{p_1} \wedge \ldots e_{p_k}) \cdot Y e^{p_1} \wedge \ldots e^{p_k} \\
&= (\underline{t})^\dagger(Y).
\end{aligned}$$

Hence, we get

$$\underline{(t^{\dagger})} = (\underline{t})^{\dagger}. \tag{46}$$

This means that the extension operator commutes with the adjoint operator. In accordance with the property above we may use a more simple notation $\underline{t}^{\dagger}$ to denote without ambiguities both of $\underline{(t^{\dagger})}$ and $(\underline{t})^{\dagger}$.

In many applications the adjoint operator is used for the cases where $\overset{\diamond}{\underset{1}{\bigwedge}} V = \bigwedge^p V$ and $\overset{\diamond}{\underset{2}{\bigwedge}} V = \bigwedge^q V$ are homogenous subspaces of $\bigwedge V$. In this particular case the adjoint operator is simply a linear mapping acting on these vector spaces of extensors, namely $ext_q^p(V) \ni t \mapsto t^{\dagger} \in ext_p^q(V)$. We have from eq.(41) and eq.(42) the simple formulas

$$\begin{aligned} t^{\dagger}(Y) &= \frac{1}{p!}\underline{t}(e_{j_1} \wedge \ldots e_{j_p}) \cdot Y(e^{j_1} \wedge \ldots e^{j_p}) \quad (47)\\ &= \frac{1}{p!}\underline{t}(e^{j_1} \wedge \ldots e^{j_p}) \cdot Y(e_{j_1} \wedge \ldots e_{j_p}), \quad (48)\end{aligned}$$

for all $Y \in \bigwedge^q V$, where Einstein's convention has been used.

# 6 The Generalization Operator

Let $\{e_k\}$ be any basis for $V$, and $\{e^k\}$ be its euclidean reciprocal basis for $V$, as we know, $e_k \cdot e^l = \delta_k^l$.

The linear mapping $ext_1^1(V) \ni t \mapsto \underset{\sim}{t} \in ext(V)$ such that for any $X \in \bigwedge V$

$$\underset{\sim}{t}(X) = t(e^k) \wedge (e_k \lrcorner X) = t(e_k) \wedge (e^k \lrcorner X) \tag{49}$$

will be called the *generalization operator*. We call $\underset{\sim}{t}$ the *generalized of* $t$.

The generalization operator is well-defined since it does not depend on the choice of $\{e_k\}$.

We present now some important properties which are satisfied by the generalization operator.

**g1** The generalization operator is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k V, \text{ then } \underset{\sim}{t}(X) \in \bigwedge^k V. \tag{50}$$

**g2** The grade involution $\hat{}\in ext(V)$, reversion $\tilde{}\in ext(V)$, and conjugation $\bar{} \in ext(V)$ commute with the generalization operator, i.e.,

$$\underset{\sim}{t}(\widehat{X}) = \widehat{\underset{\sim}{t}(X)}, \tag{51}$$

$$\underset{\sim}{t}(\widetilde{X}) = \widetilde{\underset{\sim}{t}(X)}, \tag{52}$$

$$\underset{\sim}{t}(\overline{X}) = \overline{\underset{\sim}{t}(X)}. \tag{53}$$

They are immediate consequences of the grade-preserving property.

**g3** For any $\alpha \in \mathbb{R}$, $v \in V$ and $X, Y \in \bigwedge V$ it holds

$$\underset{\sim}{t}(\alpha) = 0, \tag{54}$$

$$\underset{\sim}{t}(v) = t(v), \tag{55}$$

$$\underset{\sim}{t}(X \wedge Y) = \underset{\sim}{t}(X) \wedge Y + X \wedge \underset{\sim}{t}(Y). \tag{56}$$

The proof of eq.(54) and eq.(55) are left to the reader. Hint: $v \lrcorner \alpha = 0$ and $v \lrcorner w = v \cdot w$. Now, the identities: $a \lrcorner (X \wedge Y) = (a \lrcorner X) \wedge Y + \widehat{X} \wedge (a \lrcorner Y)$ and $a \wedge X = \widehat{X} \wedge a$, with $a \in V$ and $X, Y \in \bigwedge V$, allow us to prove the property given by eq.(56).

We can prove that the basic properties given by eq.(54), eq.(55) and eq.(56) together are completely equivalent to the *generalization procedure* as defined by eq.(49).

**g4** The generalization operator commutes with the adjoint operator, i.e.,

$$(\underset{\sim}{t})^\dagger = (\underset{\sim}{t^\dagger}), \tag{57}$$

or put it on another way, the *adjoint of the generalized* of $t$ is just the *generalized of the adjoint* of $t$.

**Proof.** A straightforward calculation, by using eq.(43) and the multivector identities: $X \cdot (a \wedge Y) = (a \lrcorner X) \wedge Y$ and $X \cdot (a \lrcorner Y) = (a \wedge X) \cdot Y$, with $a \in V$ and $X, Y \in \bigwedge V$, gives

$$\begin{aligned}
(\underset{\sim}{t})^\dagger(X) \cdot Y &= X \cdot \underset{\sim}{t}(Y) \\
&= (e_j \wedge (t(e^j) \lrcorner X)) \cdot Y = (e_j \wedge (t(e^j) \cdot e^k e_k \lrcorner X)) \cdot Y \\
&= (e^j \cdot t^\dagger(e^k) e_j \wedge (e_k \lrcorner X)) \cdot Y = (t^\dagger(e^k) \wedge (e_k \lrcorner X)) \cdot Y \\
&= (\underset{\sim}{t^\dagger})(X) \cdot Y.
\end{aligned}$$

Hence, by non-degeneracy of the euclidean scalar product, the required result follows. ■

In accordance with the above property we might use a more simple symbol $\underset{\sim}{t}^\dagger$ to mean $(\underset{\sim}{t})^\dagger$ and $(\underset{\sim}{t^\dagger})$ .

**g5** The symmetric (skew-symmetric) part of the generalized of $t$ is just the generalized of the symmetric (skew-symmetric) part of $t$, i.e.,

$$(\underset{\sim}{t})_\pm = (\underset{\sim}{t_\pm}). \tag{58}$$

This property follows immediately from eq.(57).

We see also that it is possible to use the more simple notation $\underset{\sim\pm}{t}$ to mean $(\underset{\sim}{t})_\pm$ and $(\underset{\sim}{t_\pm})$.

**g6** The skew-symmetric part of the generalized of $t$ can be factorized by the noticeable formula[3]

$$\underset{\sim-}{t}(X) = \frac{1}{2}biv[t] \times X, \tag{59}$$

where $biv[t] \equiv t(e^k) \wedge e_k$ is an *characteristic invariant* of $t$, the so-called *bivector of* $t$.

**Proof.** By using eq.(58), the well-known identity $t_-(a) = \frac{1}{2}biv[t] \times a$ and the remarkable multivector identity $B \times X = (B \times e^k) \wedge (e_k \lrcorner X)$, with $B \in \bigwedge^2 V$ and $X \in \bigwedge V$, we have that

$$\underset{\sim-}{t}(X) = t_-(e^k) \wedge (e_k \lrcorner X) = (\frac{1}{2}biv[t] \times e^k) \wedge (e_k \lrcorner X) = \frac{1}{2}biv[t] \times X. \ \blacksquare$$

**g7** A noticeable formula holds for the skew-symmetric part of the generalized of $t$. For all $X, Y \in \bigwedge V$

$$\underset{\sim-}{t}(X * Y) = \underset{\sim-}{t}(X) * Y + X * \underset{\sim-}{t}(Y), \tag{60}$$

where $*$ is any product either $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (*Clifford product*).

In order to prove this property we should use eq.(59) and the noticeable multivector identity $B \times (X * Y) = (B \times X) * Y + X * (B \times Y)$, with $B \in \bigwedge^2 V$

[3] Recall that $X \times Y \equiv \frac{1}{2}(XY - YX)$.

and $X, Y \in \bigwedge V$. By taking into account eq.(54) we can see that the following property for the euclidean scalar product of multivectors holds

$$\underset{\sim_-}{t}(X) \cdot Y + X \cdot \underset{\sim_-}{t}(Y) = 0. \tag{61}$$

It is consistent with the well-known property: *the adjoint of a skew-symmetric extensor equals minus the extensor!*

# 7 Determinant

We now define a characteristic scalar associated to any $(1,1)$-extensor $t$. It is the unique real number, denoted by $\det[t]$, such that

$$\underline{t}(I) = \det[t]I, \tag{62}$$

for all non-zero pseudoscalar $I$. It will be called the *determinant of* $t$.

It is a well-defined scalar invariant since it does not depend on the choice of $I$.

We present now some of the most important properties satisfied by the determinant.

**d1** Let $t$ and $u$ be two $(1,1)$–extensors. It holds

$$\det[u \circ t] = \det[u] \det[t]. \tag{63}$$

Take a non-zero pseudoscalar $I \in \bigwedge^n V$. Then, by using eq.(35) and eq.(62) we can write that

$$\begin{aligned} \det[u \circ t]I &= \underline{u \circ t}(I) = \underline{u} \circ \underline{t}(I) = \underline{u}(\underline{t}(I)) \\ &= \underline{u}(\det[t]I) = \det[t]\underline{u}(I), \\ &= \det[t] \det[u]I. \end{aligned}$$

**d2** Let us take $t \in ext_1^1(V)$ with inverse $t^{-1} \in ext_1^1(V)$. It holds

$$\det[t^{-1}] = (\det[t])^{-1}. \tag{64}$$

Indeed, by using eq.(63) and the obvious property $\det[i_V] = 1$, we have that

$$t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow \det[t^{-1}] \det[t] = \det[t] \det[t^{-1}] = 1.$$

It means that the *determinant of the inverse equals the inverse of the determinant.*

Due to the above corollary it is often convenient to use the short notation $\det^{-1}[t]$ for both of $\det[t^{-1}]$ and $(\det[t])^{-1}$.

**d3** Let us take $t \in ext_1^1(V)$. It holds

$$\det[t^\dagger] = \det[t]. \tag{65}$$

Indeed, take a non-zero $I \in \bigwedge^n V$. Then, by using eq.(62), eq.(43) and eq.(46), we have that

$$\det[t^\dagger]I \cdot I = \underline{t}^\dagger(I) \cdot I = I \cdot \underline{t}(I) = I \cdot \det[t]I = \det[t]I \cdot I,$$

from where the expected result follows.

Let $\{e_j\}$ be any basis for $V$, and $\{e^j\}$ be its euclidean reciprocal basis for $V$, i.e., $e_j \cdot e^k = \delta_j^k$. There are two interesting and useful formulas for calculating $\det[t]$, i.e.,

$$\begin{aligned} \det[t] &= \underline{t}(e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n), & (66) \\ &= \underline{t}(e^1 \wedge \ldots \wedge e^n) \cdot (e_1 \wedge \ldots \wedge e_n). & (67) \end{aligned}$$

They follow from eq.(62) by using $(e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n) = 1$ which is an immediate consequence of the formula for the euclidean scalar product of simple $k$-vectors and the reciprocity property of $\{e_k\}$ and $\{e^k\}$.

Each of eq.(66) and eq.(67) is completely equivalent to the definition of determinant given by eq.(62).

# 8 Some Applications

**Theorem 1** *Let* $(\{b_k\}, \{b^k\})$ *and* $(\{e_k\}, \{e^k\})$ *be two pairs of euclidean bases for* $V$*. There exists an unique invertible* $(1,1)$*-extensor* $f$ *such that*

$$\begin{aligned} e_k &= f(b_k), & (68) \\ e^k &= f^*(b^k) \text{ for each } k = 1, \ldots, n. & (69) \end{aligned}$$

*And, reciprocally given an arbitrary invertible* $(1,1)$*-extensor, say* $f$*, it is possible construct from a pair of reciprocal bases, say* $(\{b_k\}, \{b^k\})$*, with the above formulas another pair of reciprocal bases, say* $(\{e_k\}, \{e^k\})$*.*

**Proof.** Since each one of the sets $\{e_k\}$ and $\{e^k\}$ is a basis of $V$, there must be exactly two invertible $(1,1)$-extensors over $V$, say $f_1$ and $f_2$, such that

$$\begin{aligned} e_k &= f_1(b_k), \\ e^k &= f_2(b^k) \text{ for each } k = 1, \ldots, n. \end{aligned}$$

It is not difficult to see that $f_1$ and $f_2$ are given by

$$\begin{aligned} f_1(v) &= \sum_{j=1}^{n} (b_j \cdot v) e_j, \\ f_2(v) &= \sum_{j=1}^{n} (b^j \cdot v) e^j. \end{aligned}$$

Due to the reciprocity property of $(\{e_k\}, \{e^k\})$ we have

$$f_1(b_k) \cdot f_2(b^l) = \delta_k^l \Rightarrow b_k \cdot f_1^\dagger \circ f_2(b^l) = \delta_k^l \Rightarrow f_1^\dagger \circ f_2(b^l) = b^l,$$

for each $l = 1, \ldots, n$. Thus, $f_1^\dagger \circ f_2 = i_V$.

Now, let us choose $f_1 = f$. Then, it must be $f_2 = f^*$ (recall that $f^* = (f^\dagger)^{-1} = (f^{-1})^\dagger$), and so the first statement follows.

To prove the second statement we must check that $\{e_k\}$ and $\{e^k\}$ given by eqs.(68) and (69) satisfy the reciprocity property. Indeed, by using eq.(43), we can write

$$e_k \cdot e^l = f(b_k) \cdot f^*(b^l) = b_k \cdot f^\dagger \circ f^*(b^l) = b_k \cdot b^l = \delta_k^l. \blacksquare$$

## 8.1 Orthonormal Bases

It should be noted that if $f$ is an *orthogonal* $(1,1)$-*extensor* (i.e., $f^\dagger = f^{-1}$, or equivalently $f = f^*$ (the adjoint of $f$ being taken with respect to the euclidean scalar product), then $\{e_k\}$, as defined in eq.(68), is an *orthonormal basis* for $V$, i.e., $e_j \cdot e_k = \delta_{jk}$, if and only if $\{b_k\}$ is an *orthonormal basis* for $V$, i.e., $b_j \cdot b_k = \delta_{jk}$. Indeed, $e_j \cdot e_k = f(b_j) \cdot f(b_k) = f^\dagger \circ f(b_j) \cdot b_k = b_j \cdot b_k = \delta_{jk}$.

## 8.2 Changing Basis Extensor

Theorem 1 implies that for two arbitrary pairs of reciprocal bases of $V$, say $(\{e_k\}, \{e^k\})$ and $(\{e'_k\}, \{e^{k\prime}\})$, there must be an unique invertible $(1,1)$-extensor over $V$, say $\varepsilon$, such that

$$\varepsilon(e_k) = e'_k, \tag{70}$$

$$\varepsilon^*(e^k) \;=\; e^{k\prime}. \tag{71}$$

Indeed, there are exactly two invertible $(1,1)$-extensors, say $f$ and $f'$, such that $e_k = f(b_k)$, $e^k = f^*(b^k)$ and $e'_k = f'(b_k)$, $e^{k\prime} = f'^*(b^k)$ for each $k = 1,\ldots,n$. From these equations we get $e'_k = f' \circ f^{-1}(e_k)$ and $e^{k\prime} = f'^* \circ f^\dagger(e^k)$. It means that there is an unique invertible $(1,1)$-extensor which satisfies eqs.(70) and (71). Such one is given by $\varepsilon = f' \circ f^{-1}$.

Such $\varepsilon \in ext_1^1(V)$ will be called the *changing basis extensor relative to* $(\{e_k\},\{e^k\})$ *and* $(\{e'_k\},\{e^{k\prime}\})$ (in this order!).

The changing basis extensor $\varepsilon$, as is not difficult to see, can be defined equivalently by

$$\varepsilon(v) = (e^s \cdot v)e'_s. \tag{72}$$

Also, we can easily see that $\varepsilon^{-1}$, can be alternatively defined by

$$\varepsilon^{-1}(v) = (e^{s\prime} \cdot v)e_s, \tag{73}$$

and, by using eq.(43), a straightforward calculation yields

$$\varepsilon^\dagger(v) \;=\; (e'_s \cdot v)e^s, \tag{74}$$

$$\varepsilon^*(v) \;=\; (e_s \cdot v)e^{s\prime}. \tag{75}$$

As we know, the vector bases $\{e_k\}$ and $\{e'_k\}$ induce the $k$-vector bases $\{e_{j_1} \wedge \ldots \wedge e_{j_k}\}$ and $\{e'_{j_1} \wedge \ldots \wedge e'_{j_k}\}$. From eq.(70) by using eq.(33) it follows that

$$\underline{\varepsilon}(e_{j_1} \wedge \ldots \wedge e_{j_k}) = e'_{j_1} \wedge \ldots \wedge e'_{j_k}. \tag{76}$$

Analogously for the vector bases $\{e^j\}$ and $\{e^{j\prime}\}$ it holds

$$\underline{\varepsilon}^*(e^{j_1} \wedge \ldots \wedge e^{j_k}) = e^{j_1\prime} \wedge \ldots \wedge e^{j_k\prime}. \tag{77}$$

## 8.3 Inversion of a Non-singular $(1,1)$-Extensor

We will end this section presenting an useful formula for the inversion of a non-singular $(1,1)$–extensor.

Let us take $t \in ext_1^1(V)$. If $t$ is non-singular, i.e., $\det[t] \neq 0$, then there exists its inverse $t^{-1} \in ext_1^1(V)$ which is given by

$$t^{-1}(v) = \det{}^{-1}[t]\underline{t}^\dagger(vI)I^{-1}, \tag{78}$$

where $I \in \bigwedge^n V$ is any non-zero pseudoscalar.

**Proof.** We must prove that $t^{-1}$ given by the formula above satisfies both of conditions $t^{-1}\circ t = i_V$ and $t\circ t^{-1} = i_V$.

Let $I \in \bigwedge^n V$ be a non-zero pseudoscalar. Take $v \in V$, by using the extensor identities[4] $\underline{t}^\dagger(t(v)I)I^{-1} = t(\underline{t}^\dagger(vI)I^{-1}) = \det[t]v$, we have that

$$t^{-1}\circ t(v) = t^{-1}(t(v)) = \det{}^{-1}[t]\underline{t}^\dagger(t(v)I)I^{-1} = \det{}^{-1}[t]\det[t]v = i_V(v).$$

And

$$t\circ t^{-1}(v) = t(t^{-1}(v)) = \det{}^{-1}[t]t(\underline{t}^\dagger(vI)I^{-1}) = \det{}^{-1}[t]\det[t]v = i_V(v). \blacksquare$$

# 9 Conclusions

We introduced and developed some aspects of the theory of extensors, and made preliminary applications of it. The concept of extensor when used together with the euclidean Clifford algebra (as introduced in [4], paper I of this series) permits an intrinsic formulation of the key concepts of linear algebra theory, and plays a crucial role in our study of more sophisticated concepts which are developed in subsequent papers of the present series. And also in some forthcoming new series of papers.

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