# Euclidean Clifford Algebra*

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**Abstract**

Let $V$ be a $n$-dimensional real vector space. In this paper we introduce the concept of *euclidean* Clifford algebra $\mathcal{C}\ell(V,G_E)$ for a given euclidean structure on $V$, i.e., a pair $(V,G_E)$ where $G_E$ is a euclidean metric for $V$ (also called an euclidean scalar product). Our construction of $\mathcal{C}\ell(V,G_E)$ has been designed to produce a powerful computational tool. We start introducing the concept of *multivectors* over $V$. These objects are elements of a linear space over the real field, denoted by $\bigwedge V$. We introduce moreover, the concepts of exterior and euclidean scalar product of multivectors. This permits the introduction of two *contraction operators* on $\bigwedge V$, and the concept of euclidean *interior* algebras. Equipped with these notions an euclidean Clifford product is easily introduced. We worked out with considerable details several important identities and useful formulas, to help the reader to develope a skill on the subject, preparing himself for the reading of the following papers in this series.

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# Contents



# 1 Introduction

This is the first paper of a series of seven. We introduce $\mathcal{C}\ell(V, G_E)$, an *euclidean* Clifford algebra of multivectors associated to an euclidean structure on a $n$-dimensional real vector space $V$. By an euclidean structure we mean a pair $(V, G_E)$ where $G_E$ is an euclidean metric on $V$. Our construction of $\mathcal{C}\ell(V, G_E)$ has been designed in order to produce a powerful computational tool. It starts by introducing the concept of multivectors over $V$. These objects are elements of a linear space over the real field, denoted by $\bigwedge V$. We introduce in $\bigwedge V$, the concepts of *exterior product* and *euclidean scalar product* of multivectors. This permits the introduction of two *contraction operators* on $\bigwedge V$, and the concept of euclidean *interior* algebras. Equipped with these notions an euclidean Clifford product is introduced. We worked out with considerable details several important identities and useful formulas, to help the reader to develope a skill on the subject, preparing himself for the reading of the following papers in this series[1] (and also for the ones in two forthcoming series of papers). We have the following resume concerning to the content of the other papers of the present series.

[1] The papers in this series will be denoted when quoted within or in another paper in the series by I, II, III,...

In paper II we introduce the fresh concept of *general extensors.* The theory of these objects is developed and the properties of some particular extensors that appear frequently in our theory of multivector functions and multivector functionals (the subject of papers V,VI and VII) are worked in details.

In paper III we study the relationship between the concepts of a *metric tensor* $G$ (of arbitrary signature $s = p - q$ ( or $(p, q)$ as physicists say), with $p + q = n$) for a $n$-dimensional vector space $V$ and a *metric extensor* $g$ for that space (with the same signature), showing the advantage of using the later object even in elementary linear algebra theory. It is worthwhile to emphasize here that our introduction of the concept of metric extensor plays a crucial role in our definition of *metric* Clifford algebras, which are introduced in paper IV. There, a *metric* Clifford product is introduced by a well-defined *deformation* (induced by $g$) of a given euclidean Clifford product. We introduce also the concept of *gauge metric extensor* $h$ associated to a metric extensor $g$, and present and prove the so-called *golden formula.* The gauge extensors[2] appear naturally in our theory of the differential geometry on manifolds.

In papers V and VI we present a theory of *multivector functions* of a real variable, and a theory of multivector functions of a $p$-vector variable. The notions of limit, continuity and differentiability are carefully studied. Particular emphasis is given to relate the basic concept of *directional derivative* with other types of derivatives such as the *generalized* curl, divergence and gradient.

In paper VII we develop a theory of *multivector functionals*, a key concept for the developments that we have in mind, and that has not been properly studied until now.

In two future series of papers the material developed in the present series will be used as the expression and calculational tool of several different mathematical and physical theories. We quote here our theory of possible different kinds of *covariant derivatives operators* for *multivector* and *extensor fields* on arbitrary metric manifolds which explicitly shows how these different possible covariant derivative operators are related through the concept of gauge extensors. In particular, the concept of *deformed* derivative operators will be seen to play a key rule in our formulation of families of mathematically possible geometric theories of gravitation (using Clifford al-

[2]More precisely, in differential geometry the key objects are gauge extensor *fields*.

gebras methods). We believe that our presentation of these theories have a new flavor in relation to old presentations of possible theories of gravitation (of the Riemann-Cartan-Weyl types). Also, it will be seen that the concept of multivector functionals (developed in paper VII) is a necessary one for a presentation of a rigorous formulation of a Lagrangian theory for multivector and extensor fields having support on an arbitrary manifold (or subsets of it) representing spacetime.[3] [4]

Before starting our enterprise we recall that Clifford Algebras and their applications in Mathematical Physics are now respectable subjects of research whose wealth can be appreciate by an examination of the topics presented in the last five international conference on this subject[5] ([1],[6]). We have no intention to present here even a small history of the subject, and we do not claim even to have given any reasonable list of references on papers and books on the subject. Only a few points and references will be recalled here[6]. Clifford algebras[7] has been applied since a long time ago for presentations of Maxwell theory, see e.g., ([7]-[10]), of Dirac theory, see e.g., ([11]-[19]) and the theory of the gravitational field, see e.g., ([9],[21],[22]). Hestenes in 1966 [7] wrote a small book on the subject which has been source of inspiration for many scientists and eventually spread unnecessary misconceptions on the subject of Clifford algebras and their applications in physics. In 1984 Hestenes and Sobczyk published a book[8], emphasizing that Clifford algebras

[3]The definition of a spacetime will be given at the appropriate place.

[4]Our approach immediately suggests possible improvements of the Einstein's gravitational theory as well as other interpretations, where the gravitational field, contrary to what happens in Einstein's theory is understood as a physical field in the sense of Faraday.

[5]The subject has even a journal, *Advances in Applied Clifford Algebras* , edited by J. Keller and in publication since 1991. Keller is an enthusiastic of the applications of Clifford algebras in theoretical physics and contributed with several beautiful papers to the subject. His ideas are nicely described in his recent book [19].

[6]We antecipately apologize to the author of any important contribution on the subject that has not been quoted in our brief account.

[7]Formulations of Maxwell and Dirac theories which use only Clifford algebras (more properly speaking Clifford bundles), and do not use the concept of *extensor* are incomplete [20]. These theories, e.g., cannot capture the essential mathematical nature of the physical concepts of energy-momentum and angular momentum associated with physical fields (in the sense of Faraday), since they must be mathematically represented by extensor fields. In particular, approaches to Dirac theory which do not use the concept of extensors are incomplete, to say the less. On this issue, see ( [14]-[18]).

[8]This book is essentially based on Sobczyk Ph.D. thesis presented at the Department of Mathematics of the University of Arizona. We are grateful to Professor P. Lounesto

leads naturally to a *geometric calculus* [23]. Some of the ideas that we will explore in papers I-VII have their inspiration on that book. However, it must be emphasized that our approach differs substantially from the one of those authors in many aspects as the reader can verify. In particular, the theory of multivector functionals and their derivatives with the full generality that the subject deserves appears in this series for the first time. Our exposition of the *differential geometry* on manifolds (the subject of a new series of papers), with a general theory of connections using the concept of extensor fields is (we believe) a fresh approach to the subject. Our theory is *not* based on the concept of vector manifolds used in [23] (which presents some problems [24]) and can be applied rigorously to general manifolds of arbitrary topology[9]. Our development of a Lagrangian theory of multivector fields [10] improves preliminary presentations[11], since now we give a formulation valid for multivector fields and extensor fields over arbitrary Lorentzian manifolds equipped with a general connection (*not* necessarily metric compatible)[12]. We emphasize also that our presentation of Einstein's gravitational theory (in a future series of papers) using the multivector-extensor calculus on manifolds will demonstrates that preliminary attempts [22] towards a theory of the gravitational field based on these concepts is paved with some serious mathematical (and also physical) misconceptions (see also [25] in this respect) which invalidate them. We think that our presentation of the basic working ideas about euclidean and metric Clifford algebras is reasonable self complete for our purposes. However, there is still more concerning Clifford algebra theory that has not been developed or even quoted in this series of papers. These results are important for many applications ranging from pure and applied mathematics to engineering and recent physical theories (see, e.g., [30] [31]).

For readers that are newcomers to the subject we recommend the books

(Helsinki) for this important information.

[9]A preliminary presentation of the general theory of connection using multivector-extensor calculus on Minkowski manifolds appears in [25].

[10]In the paper dealing with the Lagrangian formalism for fields we make use of the concept of a spinor field that (roughly speaking) can be said to be an equivalence class of a sum of non homogenous multivector fields. A tentative definition of these objects appear in [26], which unfortunately contains many misprints and some important errors. These are correct in ( [31],[32]).

[11]See [27] for a list of references on the subject.

[12]It is also possible to present a theory of *spinor* fields, where these objects are (loosely speaking) represented by certain equivalence classes of multivector fieds on an arbitrary manifold. A rigorous presentation of that theory is given elsewhere ([28],[29]).

by Lounesto [32] and Porteous ([33],[34]) for complementary points of view and material relative to the developments that follows.

# 2 The Euclidean Clifford Algebra of Multivectors

Let $V$ be a vector space over $\mathbb{R}$ with finite dimension, i.e., $\dim V = n$, where $n \in N$, and let $V^*$ be the dual vector space to $V$. Recall that $\dim V = \dim V^* = n$.

Let $k$ be an integer number with $0 \leq k \leq n$. The vector spaces of $k$-vectors and $k$-forms over $V$ as usual will be denoted by $\bigwedge^k V$ and $\bigwedge^k V^*$, respectively.[13]

As well known, a 0-vector can be identified with a real number, i.e., $\bigwedge^0 V = \mathbb{R}$, an 1-vector is the name of objects living on $V$, i.e., $\bigwedge^1 V = V$, and a $k$-vector with $2 \leq k \leq n$ is precisely a skew-symmetric contravariant $k$-tensor over $V$. A 0-form is also a real number, i.e., $\bigwedge^0 V^* = \mathbb{R}$. A 1-form is a form (or covector) belonging to $V^*$, i.e., $\bigwedge^1 V^* = V^*$, and a $k$-form with $2 \leq k \leq n$ is exactly a skew-symmetric covariant $k$-tensor over $V$. Recall that $\dim \bigwedge^k V = \dim \bigwedge^k V^* = \binom{n}{k}$.

The 0-vectors, 2-vectors,..., $(n-1)$-vectors and $n$-vectors are sometimes called scalars, bivectors,..., pseudovectors and pseudoscalars, respectively. The 0-forms, 2-forms,..., $(n-1)$-forms and $n$-forms are named as scalars, biforms,..., pseudoforms and pseudoscalars, respectively.

## 2.1 Multivectors

A formal sum of $k$-vectors over $V$ with $k$ running from 0 to $n$,

$$X = X_0 + X_1 + \cdots + X_n, \tag{1}$$

is called a *multivector* over $V$.

The set of multivectors over $V$ has a natural structure of vector space over $\mathbb{R}$ and is usually denoted by $\bigwedge V = \mathbb{R} + V + \cdots + \bigwedge^n V$. We have that $\dim \bigwedge V = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

[13] If the reader is not familiar with exterior algebra he must consult texts on the subject. See, e.g., ([35],[36][37]). However, care must be taken when reading different books which use *different* definitions for the *exterior* product and still use all the same symbol for that different products. About this issue, see comments on Appendix A.

### 2.1.1 $k$-Part, Grade Involution and Reversion Operators

Let $k$ be an integer number with $0 \leq k \leq n$. The linear mapping $\bigwedge V \ni X \mapsto \langle X \rangle_k \in \bigwedge^k V$ such that for any $j$ with $0 \leq j \leq n$ : if $j \neq k$, then $\langle X \rangle_k = 0$, i.e.,

$$\langle X \rangle_k = X_k, \tag{2}$$

for each $k = 0, 1, \cdots, n$, is called the $k$-part operator. $\langle X \rangle_k$ is read as the $k$-part of $X$.

It is evident that any multivector can be written as sum of their $k$-parts

$$X = \sum_{k=0}^{n} \langle X \rangle_k \,. \tag{3}$$

There are several important automorphisms (or antiautomorphisms) on $\bigwedge V$. For what follows, we shall need to introduce some automorphisms that are involutions on $\bigwedge V$. We have:

**i** The linear mapping $\bigwedge V \ni X \mapsto \hat{X} \in \bigwedge V$ such that

$$\hat{X} = \sum_{k=0}^{n} (-1)^k \langle X \rangle_k \,, \tag{4}$$

is called the *main automorphim* operator or *grade involution* operator. $\hat{X}$ is called the grade involution of $X$.

**ii** The linear mapping $\bigwedge V \ni X \mapsto \widetilde{X} \in \bigwedge V$ such that

$$\widetilde{X} = \sum_{k=0}^{n} (-1)^{\frac{1}{2}k(k-1)} \langle X \rangle_k \,, \tag{5}$$

is an antiautomorphism called the *reversion* operator. $\widetilde{X}$ is called the reverse of $X$.

Since the main automorphisms and reversion operators are involutions on the vector space of multivectors, we have that $\widehat{\hat{X}} = X$ and $\widetilde{\widetilde{X}} = X$. Both involutions commute with the $k$-part operator, i.e., $\widehat{\langle X \rangle_k} = \left\langle \hat{X} \right\rangle_k$ and $\widetilde{\langle X \rangle_k} = \left\langle \widetilde{X} \right\rangle_k$, for each $k = 0, 1, \ldots, n$.

The composition of the main automorphism with the reversion operator (in any order) is called the *conjugate* operator. The conjugate of $X$ will be denoted by $\bar{X}$. We have

$$\bar{X} = \widetilde{\hat{X}} = \widehat{\widetilde{X}}. \tag{6}$$

## 2.2 Exterior Algebra

We define the exterior product[14] of $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ by

$$X_p \wedge Y_q = \frac{(p+q)!}{p!q!}\mathcal{A}(X_p \otimes Y_q), \tag{7}$$

where $X_p \otimes Y_q$ is the tensor product of $X_p$ by $Y_q$ (see Appendix A) and $\mathcal{A}$ is the *antisymmetrization operator*, i.e., a linear mapping $\mathcal{A} : T^k V \to \bigwedge^k V$ such that

(**i**) for all $\alpha \in \mathbb{R} : \mathcal{A}\alpha = \alpha$,
(**ii**) for all $v \in V : \mathcal{A}v = v$,
(**iii**) for all $t \in T^k V$, with $k \geq 2$,

$$\mathcal{A}t(\omega^1, \dots, \omega^k) = \frac{1}{k!}\epsilon_{i_1 \dots i_k} t(\omega^{i_1}, \dots, \omega^{i_k}), \tag{8}$$

where $\epsilon_{i_1 \dots i_k}$ is the permutation symbol of order $k$,

$$\epsilon_{i_1 \dots i_k} = \begin{cases} 1, & \text{if } i_1 \dots i_k \text{ is a even permutation of } 1 \dots k \\ -1, & \text{if } i_1 \dots i_k \text{ is odd permutation of } 1 \dots k \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Eq.(7), with $p \geq 1$ and $q \geq 1$ means that for $\omega^1, \dots, \omega^p, \omega^{p+1}, \dots, \omega^{p+q} \in V^*$,

$$\begin{aligned} & X_p \wedge Y_q(\omega^1, \dots, \omega^p, \omega^{p+1}, \dots, \omega^{p+q}) \\ = \; & \frac{1}{p!q!}\epsilon_{i_1 \dots i_p i_{p+1} \dots i_{p+q}} X_p(\omega^{i_1}, \dots, \omega^{i_p}) Y_q(\omega^{i_{p+1}}, \dots, \omega^{i_{p+q}}). \end{aligned} \tag{10}$$

From eq.(7) by using a well-known property of the antisymmetrization operator, namely: $\mathcal{A}(\mathcal{A}t \otimes u) = \mathcal{A}(t \otimes \mathcal{A}u) = \mathcal{A}(t \otimes u)$, a noticeable formula for expressing simple $k$-vectors in terms of tensor products of $k$ vectors can be easily deduced. It is[15],

$$v_1 \wedge \dots \wedge v_k = \epsilon^{j_1 \dots j_k} v_{j_1} \otimes \dots \otimes v_{j_k}. \tag{11}$$

[14] There are several different definitions of the exterior product in the literature differing by factors and all using the same symbol.. This may lead to confusion if care is not taken. See Appendix A for some details.

[15] Recall that $\epsilon^{j_1 \dots j_k} \equiv \epsilon_{j_1 \dots j_k}$.

If $\omega^1, \ldots, \omega^k \in V^*$, then

$$v_1 \wedge \ldots \wedge v_k(\omega^1, \ldots, \omega^k) = \epsilon^{j_1 \ldots j_k} \omega^1(v_{j_1}) \ldots \omega^k(v_{j_k}). \tag{12}$$

Now, we define the exterior product of *multivectors* $X$ and $Y$ as being the mutivector with components $\langle X \wedge Y \rangle_k$ such that

$$\langle X \wedge Y \rangle_k = \sum_{j=0}^{k} \langle X \rangle_j \wedge \langle Y \rangle_{k-j}, \tag{13}$$

for each $k = 0, 1, \ldots, n$. Note that on the right side there appears the exterior product of $j$-vectors and $(k-j)$-vectors with $0 \leq j \leq n$.

This exterior product is an *internal* composition law on $\bigwedge V$. It is associative and satisfies the usual distributive laws (on the left and on the right).

The vector space $\bigwedge V$ endowed with this exterior product $\wedge$ is an associative algebra called the *exterior algebra* of multivectors.

We recall now for future use some important properties of the exterior algebra of multivectors:

**ei** For any $\alpha, \beta \in \mathbb{R}$, $X \in \bigwedge V$

$$\alpha \wedge \beta = \beta \wedge \alpha = \alpha\beta \text{ (real product)}, \tag{14}$$
$$\alpha \wedge X = X \wedge \alpha = \alpha X \text{ (multiplication by scalars)}. \tag{15}$$

**eii** For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \wedge Y_k = (-1)^{jk} Y_k \wedge X_j. \tag{16}$$

**eiii** For any $X, Y \in \bigwedge V$

$$\widehat{X \wedge Y} = \hat{X} \wedge \hat{Y}, \tag{17}$$
$$\widetilde{X \wedge Y} = \widetilde{Y} \wedge \widetilde{X}. \tag{18}$$

## 2.3 Euclidean Scalar Product

Let $G_E$ be an *euclidean metric* for $V$, i.e., a mapping $G_E : V \times V \to \mathbb{R}$ which is a symmetric, non-degenerate and positive definite bilinear form over $V$,

$$G_E(v, w) = G_E(w, v) \ \forall v, w \in V \tag{19}$$
$$\text{If } G_E(v, w) = 0 \ \forall w \in V, \text{ then } v = 0 \tag{20}$$
$$G_E(v, v) \geqslant 0 \ \forall v \in V \text{ and if } G_E(v, v) = 0, \text{ then } v = 0. \tag{21}$$

It is usual to write

$$G_E(v,w) \equiv v \cdot w \tag{22}$$

where $v \cdot w$ is said to be the *scalar product* of the vectors $v, w \in V$.

This practice forgets that any scalar product is relative to a given $G_E$, it is a fact which will be important for the developments that follows, the correct notation should be $v \underset{G_E}{\cdot} w$. Nevertheless, when no confusion arises we will follow the standard practice.

The pair $(V, G_E)$ is called an *euclidean structure* for $V$. Sometimes, an euclidean structure is also called an *euclidean space.* It is very important to realize that there are an infinite of euclidean structures for a real vector space $V$. Two euclidean structures $(V, G_E)$ and $(V, G'_E)$ are equal if and only if $G_E = G'_E$.

Let $\mathfrak{B}$ be the set of all basis of $V$. It means that a generic element of $\mathfrak{B}$ is an *ordered* set of linearly independent vectors of $V$, say $(e_1, e_2, ..., e_n)$, which will be denoted simply by $\{e_k\}$ in what follows.

Now, given an euclidean structure for $V$, we can immediately select a subset $\mathfrak{B}_O$ of $\mathfrak{B}$ whose elements are of the orthonormal bases according to the euclidean structure. This means that if $\{f_k\} \in \mathfrak{B}_O$, then

$$G_E(f_i, f_j) \equiv f_i \cdot f_j = \delta_{ij}, \tag{23}$$

where $\delta_{ij} = \begin{cases} 1, & i = j = 1, 2, ..., n \\ 0, & i \neq j \end{cases}$. It is trivial to realize that any two basis $\{f_k\}, \{f'_k\} \in \mathfrak{B}_O$ are related by a linear orthogonal transformation, i.e., $f'_k = L_k{}^i f_i$, where the matrix $\mathbf{L}$ whose entries are the real numbers $L_k{}^i$ is orthogonal, i.e., $\mathbf{L}^t\mathbf{L} = \mathbf{L}\mathbf{L}^\mathbf{t} = \mathbf{1}$.

Once an euclidean structure $(V, G_E)$ has been set we can equip $\bigwedge^p V$ with an *euclidean scalar product of p-vectors.* $\bigwedge V$ can be endowed with an *euclidean scalar product of multivectors.*

Let $\{e_k\}$ be any basis of $V$, and $\{\varepsilon^k\}$ be the dual basis of $\{e_k\}$. As we know, $\{\varepsilon^k\}$ is the unique basis of $V^*$ such that $\varepsilon^k(e_j) = \delta^k_j$. Associated to $(V, G_E)$ we define the scalar product of $p$-vectors $X_p, Y_p \in \bigwedge^p V$, namely $X_p \cdot Y_p \in \mathbb{R}$, by the following axioms:

**Ax-i** For all $\alpha, \beta \in \mathbb{R}$,

$$\alpha \cdot \beta = \alpha\beta \text{ (real product)}. \tag{24}$$

**Ax-ii** For all $X_p, Y_p \in \bigwedge^p V$ with $p \geq 1$,

$$X_p \cdot Y_p = (\frac{1}{p!})^2 X_p(\varepsilon^I) Y_p(\varepsilon^J) \det\left[G_E(e_I, e_J)\right], \tag{25}$$

where we use (conveniently) the short notations

$$X_p(\varepsilon^I) \equiv X_p(\varepsilon^{i_1}, \dots, \varepsilon^{i_p}) = \underset{p}{X}^{i_1 \dots i_p}, \tag{26}$$

$$Y_p(\varepsilon^J) \equiv Y_p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p}) = \underset{p}{Y}^{j_1 \dots j_p}. \tag{27}$$

$X_p(\varepsilon^{i_1}, \dots, \varepsilon^{i_p})$ and $Y_p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p})$ are the components of $X_p$ and $Y_p$ with respect to the $p$-vector basis $\{e_{j_1} \wedge \dots \wedge e_{j_p}\}$ and $1 \leq j_1 < \cdots j_p \leq n$, i.e.,

$$X_p = \frac{1}{p!} X_p(\varepsilon^{i_1}, \dots, \varepsilon^{i_p}) e_{i_1} \wedge \dots e_{i_p} \text{ and } Y_p = \frac{1}{p!} Y_p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p}) e_{j_1} \wedge \dots e_{j_p}. \tag{28}$$

Also,

$$\det\left[G_E(e_I, e_J)\right] \equiv \det \begin{bmatrix} G_E(e_{i_1}, e_{j_1}) & \dots & G_E(e_{i_1}, e_{j_k}) \\ \dots & \dots & \dots \\ G_E(e_{i_k}, e_{j_1}) & \dots & G_E(e_{i_k}, e_{j_k}) \end{bmatrix}. \tag{29}$$

Note that in eq.(25) the Einstein *convention* for sums over the indices $I \equiv i_1, \dots, i_p = 1, \dots, n$ and $J \equiv j_1, \dots, j_p = 1, \dots, n$ was used.

It is not difficult to realize that the scalar product defined by the axioms **i-ii** does not depend on the bases $\{e_k\}$ and $\{\varepsilon^k\}$ for calculating it.

It is a well-defined *euclidean scalar product on* $\bigwedge^p V$, since it is symmetric, satisfies the distributive laws, has the mixed associative property and is non-degenerate, i.e., if $X_p \cdot Y_p = 0$ fot all $Y_p$, then $X_p = 0$. It is also satisfying the strong property of being *positive definite*, i.e., $X_p \cdot X_p \geq 0$ for all $X_p$ and if $X_p \cdot X_p = 0$, then $X_p = 0$.

So the scalar product on $\bigwedge^p V$ as defined by eqs.(24) and (25) will be called the euclidean scalar product of $p$-vectors associated to $(V, G_E)$.

Now, associated to $(V, G_E)$ we define the scalar product of multivectors $X, Y \in \bigwedge V$, namely $X \cdot Y \in \mathbb{R}$, by

$$X \cdot Y = \sum_{k=0}^{n} \langle X \rangle_k \cdot \langle Y \rangle_k \,. \tag{30}$$

Note that on the right side there appears the scalar product of $k$-vectors with $0 \leq k \leq n$, as defined by eqs.(24) and (25).

By using eqs.(24) and (25) we can easily note that eq.(30) can be written as

$$X \cdot Y = \langle X\rangle_0 \langle Y\rangle_0 + \sum_{k=1}^{n} (\frac{1}{k!})^2 \langle X\rangle_k (\varepsilon^I) \langle Y\rangle_k (\varepsilon^J) \det[G_E(e_I, e_J)]\,. \tag{31}$$

Recall that in eq.(31) the Einstein *convention* for sums over the indices $I \equiv i_1, \ldots, i_k = 1, \ldots, n$ and $J \equiv j_1, \ldots, j_k = 1, \ldots, n$ was used.

It is very important here to notice that the scalar product as defined by eq.(30) is a well-defined *euclidean scalar product on* $\bigwedge V$. It is symmetric, satisfies the distributive laws, has the mixed associative property and is non-degenerate, i.e., if $X \cdot Y = 0$ for all $Y$, then $X = 0$. In addition, it has also the strong property of being *positive definite,* i.e., $X \cdot X \geq 0$ for all $X$ and if $X \cdot X = 0$, then $X = 0$.

So the scalar product on $\bigwedge V$ as defined by eq.(30) will be called the euclidean scalar product of multivectors associated to $(V, G_E)$.

Now, note that if we take any orthonormal basis $\{f_k\}$ with respect to $(V, G_E)$, i.e., $f_j \cdot f_k = \delta_{jk}$, whose dual basis is $\{\varphi^k\}$, i.e., $\varphi^k(f_j) = \delta_j^k$, we have that $\det[G_E(f_I, , f_J)] = \epsilon_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \epsilon_{j_1 \ldots j_k}^{i_1 \ldots i_k}$. Then, by taking into account[16] that $\dfrac{1}{k!}\epsilon_{i_1 \ldots i_k}^{j_1 \ldots j_k} \langle X\rangle_k (\varphi^{i_1}, \ldots, \varphi^{i_k}) = \langle X\rangle_k (\varphi^{j_1}, \ldots, \varphi^{j_k})$, we can easily see that eq.(31) can be written as

$$X \cdot Y = \langle X\rangle_0 \langle Y\rangle_0 + \sum_{k=1}^{n} \frac{1}{k!} \sum_{j_1 \ldots j_k = 1}^{n} \langle X\rangle_k (\varphi^{j_1}, \ldots, \varphi^{j_k}) \langle Y\rangle_k (\varphi^{j_1}, \ldots, \varphi^{j_k}). \tag{32}$$

It should be noted that eq.(32) in the particular case of vectors is reduced to

$$v \cdot w = \sum_{j=1}^{n} \varphi^j(v)\varphi^j(w). \tag{33}$$

We summarize now the basic properties of the euclidean scalar product of multivectors.

[16]Recall that $\epsilon_{i_1 \ldots i_k}^{j_1 \ldots j_k}$ is the so-called generalized permutation symbol of order $k$,

$$\epsilon_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \det \begin{bmatrix} \delta_{i_1}^{j_1} & \ldots & \delta_{i_1}^{j_k} \\ \ldots & \ldots & \ldots \\ \delta_{i_k}^{j_1} & \ldots & \delta_{i_k}^{j_k} \end{bmatrix}, \text{ with } i_1, \ldots, i_k = 1, \ldots, n \text{ and } j_1, \ldots, j_k = 1, \ldots, n.$$

**esi** For any $\alpha, \beta \in \mathbb{R}$ :

$$\alpha \cdot \beta = \alpha\beta \text{ (real product).} \tag{34}$$

**esii** For any $v, w \in V$ :

$$v \cdot w = G_E(v, w). \tag{35}$$

It shows that eq.(30) contains the scalar product of vectors.

**esiii** For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$ :

$$X_j \cdot Y_k = 0, \text{ if } j \neq k. \tag{36}$$

The properties given by eq.(34), eq.(35) and eq.(36) follow directly from the definition given by eq.(30).

**esiv** For any simple $k$-vectors $v_1 \wedge \ldots v_k \in \bigwedge^k V$ and $w_1 \wedge \ldots w_k \in \bigwedge^k V$ :

$$(v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k) = \det \begin{bmatrix} v_1 \cdot w_1 & \ldots & v_1 \cdot w_k \\ \ldots & \ldots & \ldots \\ v_k \cdot w_1 & \ldots & v_k \cdot w_k \end{bmatrix}. \tag{37}$$

**Proof.**

We will use eq.(32). Then, by using eq.(12) and eq.(33), and recalling the $k \times k$ determinant formula, $\det [a_{pq}] = \dfrac{1}{k!}\epsilon^{p_1 \ldots p_k}\epsilon^{q_1 \ldots q_k} a_{p_1 q_1} \ldots a_{p_k q_k}$, we have

$$\begin{aligned}
& (v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k) \\
= \;& \frac{1}{k!} \sum_{j_1 \ldots j_k = 1}^{n} (v_1 \wedge \ldots v_k)(\varphi^{j_1}, \ldots, \varphi^{j_k})(w_1 \wedge \ldots w_k)(\varphi^{j_1}, \ldots, \varphi^{j_k}) \\
= \;& \frac{1}{k!} \sum_{j_1 \ldots j_k = 1}^{n} \epsilon^{p_1 \ldots p_k}\epsilon^{q_1 \ldots q_k} \varphi^{j_1}(v_{p_1}) \ldots \varphi^{j_k}(v_{p_k}) \varphi^{j_1}(w_{q_1}) \ldots \varphi^{j_k}(w_{q_k}) \\
= \;& \frac{1}{k!}\epsilon^{p_1 \ldots p_k}\epsilon^{q_1 \ldots q_k} \sum_{j_1 = 1}^{n} \varphi^{j_1}(v_{p_1})\varphi^{j_1}(w_{q_1}) \ldots \sum_{j_k = 1}^{n} \varphi^{j_k}(v_{p_k})\varphi^{j_k}(w_{q_k}) \\
= \;& \frac{1}{k!}\epsilon^{p_1 \ldots p_k}\epsilon^{q_1 \ldots q_k} (v_{p_1} \cdot w_{q_1}) \ldots (v_{p_k} \cdot w_{q_k}), \\
= \;& \det [v_p \cdot w_q] \,. \blacksquare
\end{aligned}$$

**Proposition 1** *Let* $(\{e_k\},\{e^k\})$ *be any pair of euclidean reciprocal bases of* $V$, *i.e.,* $e_k\cdot e^l\equiv G_E(e_k,e^l)=\delta_k^l$. *For all* $X\in\bigwedge V$ *we have the following two expansion formulas*

$$X = X\cdot 1+\sum_{k=1}^{n}\frac{1}{k!}X\cdot(e^{j_1}\wedge\ldots e^{j_k})(e_{j_1}\wedge\ldots e_{j_k}) \tag{38}$$

$$X = X\cdot 1+\sum_{k=1}^{n}\frac{1}{k!}X\cdot(e_{j_1}\wedge\ldots e_{j_k})(e^{j_1}\wedge\ldots e^{j_k}). \tag{39}$$

**Proof.** We give here the proof for vectors and $p$-vectors, with $p\geq 2$.

For $v\in V$, since $\{e_k\}$ and $\{e^k\}$ are bases of $V$, there are unique real numbers $v^i$ and $v_i$ with $i=1,\ldots,n$ such that

$$v=v^ie_i=v_ie^i.$$

Let us calculate $v\cdot e^j$ and $v\cdot e_j$. Then, by taking into account the reciprocity condition of $(\{e_k\},\{e^k\})$, we get

$$v=(v\cdot e^j)e_j=(v\cdot e_j)e^j.$$

It is standard practice to call $v\cdot e^j$ and $v\cdot e_j$ respectively the contravariant and covariant $j$-th components of $v$.

For $X\in\bigwedge^p V$, there are unique real numbers $X^{i_1\ldots i_p}$ and $X_{i_1\ldots i_p}$ with $i_1,\ldots,i_p=1,\ldots,n$ such that

$$X=\frac{1}{p!}X^{i_1\ldots i_p}e_{i_1}\wedge\ldots e_{i_p}=\frac{1}{p!}X_{i_1\ldots i_p}e^{i_1}\wedge\ldots e^{i_p}. \tag{40}$$

Then, by taking for example the scalar products $X\cdot(e^{j_1}\wedge\ldots e^{j_p})$. By using eq.(37), the reciprocity condition of $(\{e_k\},\{e^k\})$ and the combinatorial formula $\frac{1}{p!}\epsilon^{j_1\ldots j_p}_{i_1\ldots i_p}X^{i_1\ldots i_p}=X^{j_1\ldots j_p}$, we have

$$\begin{aligned}
X\cdot(e^{j_1}\wedge\ldots e^{j_p}) &= \frac{1}{p!}X^{i_1\ldots i_p}(e_{i_1}\wedge\ldots e_{i_p})\cdot(e^{j_1}\wedge\ldots e^{j_p})\\
&= \frac{1}{p!}X^{i_1\ldots i_p}\det\begin{bmatrix} e_{i_1}\cdot e^{j_1} & \ldots & e_{i_1}\cdot e^{j_p}\\ \ldots & \ldots & \ldots\\ e_{i_p}\cdot e^{j_1} & \ldots & e_{i_p}\cdot e^{j_p}\end{bmatrix}\\
&= \frac{1}{p!}\epsilon^{j_1\ldots j_p}_{i_1\ldots i_p}X^{i_1\ldots i_p}=X^{j_1\ldots j_p},
\end{aligned}$$

i.e., $X^{j_1 \ldots j_p} = X \cdot (e^{j_1} \wedge \ldots e^{j_p})$. Analogously, we can prove that $X_{j_1 \ldots j_p} = X \cdot (e_{j_1} \wedge \ldots e_{j_p})$.

Then, we get

$$X = \frac{1}{p!} X \cdot (e^{j_1} \wedge \ldots e^{j_p}) e_{j_1} \wedge \ldots e_{j_p} = \frac{1}{p!} X \cdot (e_{j_1} \wedge \ldots e_{j_p}) e^{j_1} \wedge \ldots e^{j_p}.$$

Hence, eqs.(38) and (39) follows from the statement above and essentially from eq.(36). ■

## 2.4 $b$-Metric

Let $\{b_k\}$ be any but fixed basis of $V$, and let $\{\beta^k\}$ be a basis of $V^*$dual to $\{b_k\}$, i.e., $\beta^k(b_j) = \delta^k_j$. Associated to $\{b_k\}$ we can introduce an euclidean metric on $V$, say $\underset{b}{G_E}$, defined by

$$\underset{b}{G_E}\,(v, w) = \delta_{jk} \beta^j(v) \beta^k(w), \tag{41}$$

i.e., $\underset{b}{G_E} = \delta_{jk} \beta^j \otimes \beta^k$.

It is a well defined euclidean metric on $V$, since $\underset{b}{G_E} \in T_2(V)$ is symmetric non-degenerate and positive definite, as it is easy to verify. Such $\underset{b}{G_E}$ will be called a *fiducial metric on $V$ induced by $\{b_k\}$*, or for short, a *$b$-metric.* The euclidean structure $(V, \underset{b}{G_E})$ will be called a *fiducial metric structure for $V$ induced by $\{b_k\}$*, or for short, a *$b$-metric structure.* The pair $(V, \{b_k\})$ could be called a *fiducial structure for $V$ associated to $\{b_k\}$*, or for short, a *$b$-structure.*

On another way of thinking we are equipping $V$ with a positive definite scalar product of vectors naturally induced by $\{b_k\}$. We write

$$v \underset{b}{\cdot} w \equiv \underset{b}{G_E}\,(v, w). \tag{42}$$

We present now two remarkable properties of a $b$-metric structure.

**i** The basis $\{b_k\}$ is orthonormal with respect to $(V, \underset{b}{G_E})$, i.e.,

$$b_j \underset{b}{\cdot} b_k = \delta_{jk}. \tag{43}$$

**ii** The scalar product of multivectors associated to $(V, \underset{b}{G_E})$ is given by the noticeable formula

$$X \underset{b}{\cdot} Y = \langle X\rangle_0 \langle Y\rangle_0 + \sum_{k=1}^{n} \frac{1}{k!} \sum_{j_1 \ldots j_k = 1}^{n} \langle X\rangle_k (\beta^{j_1}, \ldots, \beta^{j_k}) \langle Y\rangle_k (\beta^{j_1}, \ldots, \beta^{j_k}). \tag{44}$$

We know that all $b$-metric structure is a well-defined euclidean structure. However, it might as well be asked if any euclidean structure $(V, G_E)$ is some $b$-metric structure $(V, \underset{b}{G_E})$. The answer is YES.

Given an euclidean metric $G_E$, by the Gram-Schmidt procedure, there is an orthonormal basis $\{b_k\}$ with respect to $(V, G_E)$, i.e., $b_j \cdot b_k \equiv G_E(b_j, b_k) = \delta_{jk}$, such that the $b$-metric $\underset{b}{G_E}$ induced by $\{b_k\}$ coincides with $G_E$. Indeed, if $\{\beta^k\}$ is the dual basis of $\{b_k\}$, then

$$\begin{aligned} \underset{b}{G_E}(v, w) &= \delta_{jk}\beta^j(v)\beta^k(w) = G_E(b_j, b_k)\beta^j(v)\beta^k(w) \\ &= G_E(\beta^j(v)b_j, \beta^k(w)b_k) = G_E(v, w), \end{aligned}$$

i.e., $\underset{b}{G_E} = G_E$

### 2.4.1 $b$-Reciprocal Bases

Let $\{e_k\}$ be any basis of $V$, and $\{\varepsilon^k\}$ be its dual basis of $V^*$, i.e., $\varepsilon^k(e_j) = \delta_j^k$. Let us take a $b$-metric structure $(V, \underset{b}{G_E})$. Associated to $\{e_k\}$, it is possible to define another basis for $V$, say $\{e^k\}$, given by

$$e^k = \sum_{j=1}^{n} \varepsilon^k(b_j) b_j. \tag{45}$$

Since the set of the $n$ forms $\varepsilon^1, \ldots, \varepsilon^n \in V^*$, is a basis for $V^*$, they are linearly independent covectors. It follows that the $n$ vectors $e^1, \ldots, e^n \in V$ are also linearly independent and constitutes a well-defined basis for $V$.

**Proposition 2** *The bases $\{e_k\}$ and $\{e^k\}$ satisfy the following b-scalar product conditions*

$$e_k \underset{b}{\cdot} e^l = \delta_k^l. \tag{46}$$

**Proof.** Using eqs.(45) and (43), and the duality condition of $(\{e_k\},\{\varepsilon^k\})$ we have

$$e_k \underset{b}{\cdot} e^l = \sum_{j=1}^{n} \varepsilon^l(b_j)(e_k \underset{b}{\cdot} b_j) = \varepsilon^l(\sum_{j=1}^{n}(e_k \underset{b}{\cdot} b_j)b_j) = \varepsilon^l(e_k) = \delta_k^l. \blacksquare$$

It is noticeable that $\{e^k\}$ given by eq.(45) is the unique basis of $V$ which satisfies eq.(46). Such a basis $\{e^k\}$ will be called the *b-reciprocal basis* of $\{e_k\}$. In what follows we say that $\{e_k\}$ and $\{e^k\}$ are $b$-reciprocal bases to each other.

In particular, the $b$-reciprocal basis of $\{b_k\}$ is itself, i.e.,

$$b^k = b_k \text{ for each } k = 1,\ldots,n, \tag{47}$$

It follows directly from eq.(45) and the duality condition of $(\{b_k\},\{\beta^k\})$.

## 2.5 Euclidean Interior Algebras

Let us take an euclidean structure $(V, G_E)$. We can define two kind of *contracted products* for multivectors, namely $\lrcorner$ and $\llcorner$. If $X, Y \in \bigwedge V$ then $X \lrcorner Y \in \bigwedge V$ and $X \llcorner Y \in \bigwedge V$ such that

$$\begin{aligned} (X \lrcorner Y) \cdot Z &= Y \cdot (\widetilde{X} \wedge Z) &(48)\\ (X \llcorner Y) \cdot Z &= X \cdot (Z \wedge \widetilde{Y}), &(49)\end{aligned}$$

for all $Z \in \bigwedge V$.

These contracted products $\lrcorner$ and $\llcorner$ are internal laws on $\bigwedge V$. Both contracted products satisfy distributive laws (on the left and on the right) but they are *not* associative.

The vector space $\bigwedge V$ endowed with each of these contracted products (either $\lrcorner$ or $\llcorner$) is a non-associative algebra. They are called the *euclidean interior algebras of multivectors.*

We present now some of the most important properties of the contracted products.

**eip-i** For any $\alpha, \beta \in \mathbb{R}$ and $X \in \bigwedge V$ :

$$\begin{aligned} \alpha \lrcorner \beta &= \alpha \llcorner \beta = \alpha\beta \text{ (real product)}, &(50)\\ \alpha \lrcorner X &= X \llcorner \alpha = \alpha X \text{ (multiplication by scalars)}. &(51)\end{aligned}$$

**eip-ii** For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$ $(j \leq k)$ :

$$X_j \lrcorner Y_k = (-1)^{j(k-j)} Y_k \llcorner X_j. \tag{52}$$

**eip-iii** For any $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$ :

$$\begin{aligned} X_j \lrcorner Y_k &= 0, \text{ if } j > k, & (53) \\ X_j \llcorner Y_k &= 0, \text{ if } j < k. & (54) \end{aligned}$$

**eip-iv** For any $X_k, Y_k \in \bigwedge^k V$ :

$$X_k \lrcorner Y_k = X_k \llcorner Y_k = \widetilde{X_k} \cdot Y_k = X_k \cdot \widetilde{Y_k}. \tag{55}$$

**eip-v** For any $v \in V$ and $X, Y \in \bigwedge V$ :

$$v \lrcorner (X \wedge Y) = (v \lrcorner X) \wedge Y + \widehat{X} \wedge (v \lrcorner Y). \tag{56}$$

## 2.6 Euclidean Clifford Algebra

We define now an *euclidean Clifford product* of multivectors $X$ and $Y$ relative to a given euclidean structure $(V, G_E)$, denoted by juxtaposition, by the following axioms:

**Ax-i** For all $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$ : $\alpha X = X\alpha$ equals the multiplication of multivector $X$ by scalar $\alpha$.

**Ax-ii** For all $v \in V$ and $X \in \bigwedge V$ : $vX = v \lrcorner X + v \wedge X$ and $Xv = X \llcorner v + X \wedge v$.

**Ax-iii** For all $X, Y, Z \in \bigwedge V$ : $X(YZ) = (XY)Z$.

The Clifford product is an internal law on $\bigwedge V$. It is *associative* (by the axiom (Ax-iii)) and satisfies distributive laws (on the left and on the right). The distributive laws follow from the corresponding distributive laws of the contracted and exterior products.

The vector space of multivectors over $V$ endowed with the Clifford product is an associative algebra. It will be called *euclidean Clifford algebra of multivectors* and denoted by $\mathcal{C}\ell(V, G_E)$.

Some important formulas which hold in $\mathcal{C}\ell(V, G_E)$ are the following.

**eca-i** For any $v \in V$ and $X \in \bigwedge V$ :

$$v \lrcorner X = \frac{1}{2}(vX - \widehat{X}v) \quad (57)$$
$$\text{and } X \llcorner v = \frac{1}{2}(Xv - v\widehat{X}).$$
$$v \wedge X = \frac{1}{2}(vX + \widehat{X}v) \quad (58)$$
$$\text{and } X \wedge v = \frac{1}{2}(Xv + v\widehat{X}).$$

**eca-ii** For any $X, Y \in \bigwedge V$ :

$$X \cdot Y = \left\langle \widetilde{X}Y \right\rangle_0 = \left\langle X\widetilde{Y} \right\rangle_0 . \quad (59)$$

**eca-iii** For any $X, Y, Z \in \bigwedge V$ :

$$(XY) \cdot Z = Y \cdot (\widetilde{X}Z) = X \cdot (Z\widetilde{Y}), \quad (60)$$
$$X \cdot (YZ) = (\widetilde{Y}X) \cdot Z = (X\widetilde{Z}) \cdot Y. \quad (61)$$

**eca-iv** For any $X, Y \in \bigwedge V$ :

$$\widehat{XY} = \widehat{X}\,\widehat{Y}, \quad (62)$$
$$\widetilde{XY} = \widetilde{Y}\widetilde{X}. \quad (63)$$

**eca-v** Let $I \in \bigwedge^n V$. Then, for any $v \in V$ and $X \in \bigwedge V$ :

$$I(v \wedge X) = (-1)^{n-1} v \lrcorner (IX). \quad (64)$$

Eq.(64) is sometimes called the *duality identity* and since it appears in several contexts in what follows we prove it.

**Proof.** By using eq.(58), $Iv = (-1)^{n-1} vI$ and $\widehat{I} = (-1)^n I$ where $v \in V$ and $I \in \bigwedge^n V$ and, eqs.(62) and (57) we have

$$\begin{aligned} I(v \wedge X) &= \frac{1}{2}(IvX + I\widehat{X}v) = \frac{1}{2}((-1)^{n-1} vIX + (-1)^n \widehat{I}\,\widehat{X}v) \\ &= (-1)^{n-1}\frac{1}{2}(vIX - \widehat{IX}v) = (-1)^{n-1} v \lrcorner (IX). \blacksquare \end{aligned}$$

**eca-vi** For any $X, Y, Z \in \bigwedge V$ :

$$X \lrcorner (Y \lrcorner Z) = (X \wedge Y) \lrcorner Z, \quad (65)$$
$$(X \llcorner Y) \llcorner Z = X \llcorner (Y \wedge Z). \quad (66)$$

**Proof.** We prove only eq.(65). The proof of eq.(66) is analogous and left to the reader.

Let $W \in \bigwedge V$. By using eq.(48) and eq.(18) we have

$$\begin{aligned}(X \lrcorner (Y \lrcorner Z)) \cdot W &= (Y \lrcorner Z) \cdot (\widetilde{X} \wedge W) = Z \cdot ((\widetilde{Y} \wedge \widetilde{X}) \wedge W) \\ &= Z \cdot ((\widetilde{X \wedge Y}) \wedge W) = ((X \wedge Y) \lrcorner Z) \cdot W.\end{aligned}$$

Hence, by the non-degeneracy of the euclidean scalar product, the first statement follows. ■

To end, we call the readers attention to the fact that all Clifford algebra associated to all possible euclidean structure $(V, G_E)$ over the same vector space $V$ are equivalent each to other, i.e., define the same abstract Clifford algebra. Indeed all euclidean structures for $V$ are isomorphic to the euclidean structure $(\mathbb{R}^n, \bullet)$, where $\bullet$ is the canonical scalar product on $\mathbb{R}^n$. The Clifford algebra associated to the euclidean structure $(\mathbb{R}^n, \bullet)$ is conveniently denoted ([33][34]) by $\mathbb{R}_n$.

# 3 Conclusions

The euclidean Clifford algebra[17] $\mathcal{C}\ell(V, G_E)$ introduced above will serve as our basic calculational tool for the development of the theories of multivector functions and multivector functionals that we develop in this series of papers, and also for many applications that will be reported elsewhere. When $\mathcal{C}\ell(V, G_E)$ is used together with the concept of extensor (to be introduced in paper II) we obtain a powerful formalism which permits among other things an intrinsic presentation (i.e., without the use of matrices) of the principal results of classical linear algebra theory. Also, endowed $V$ with an arbitrary *metric extensor* $g$ (of signature $s = p - q$ or $(p, q)$ as physicists like to say, with $p + q = n$) we can construct a metric Clifford algebra $\mathcal{C}\ell(V, g)$ as a well-defined *deformation* of the euclidean Clifford algebra $\mathcal{C}\ell(V, G_E)$, see paper IV.

**Appendix A**

In the literature we can find several *different definitions* (differing by numerical factors $(p!q!$ , $(p+q)!$, $(p+q)!/\ p!q!$,) for the exterior product $X_p \wedge Y_q$ in terms of *some antisymmetrization* of the tensor product $X_p \otimes Y_q$.

[17]The classification of all euclidean algebras for arbitrary finite dimensional space and their matrix representations can be found, e.g., in [30].

Before continuing we recall that the tensor product of $t \in T^pV$ and $u \in T^qV$, namely $t \otimes u \in T^{p+q}V$, is defined by

(**ti**) for all $\alpha, \beta \in T^0V \equiv \mathbb{R} : \alpha \otimes \beta = \beta \otimes \alpha = \alpha\beta$ (real product),

(**tii**) for all $\alpha \in \mathbb{R}, u \in T^qV : \alpha \otimes u = u \otimes \alpha = \alpha u$ (scalar multiplication of $u$ by $\alpha$), and

(**tiii**) for all $t \in T^pV, u \in T^qV$ $(p, q \geq 1)$ and $\omega^1, \ldots, \omega^p, \omega^{p+1}, \ldots, \omega^{p+q} \in V^*$,

$$t \otimes u(\omega^1, \ldots, \omega^p, \omega^{p+1}, \ldots, \omega^{p+q}) = t(\omega^1, \ldots, \omega^p)u(\omega^{p+1}, \ldots, \omega^{p+q}) \quad \text{(A.1)}$$

Now, the exterior algebra $\bigwedge V$ is defined in the *modern approach* to algebraic structures as the quotient $\bigwedge V = TV/I$, where $TV = \sum_{p=0}^{\infty} T^pV$ is the *tensor algebra* and $I$ is the *bilateral ideal* generated by elements of the form $x \otimes x$. In this case, it is necessary to define the product of $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ by

$$X_p \overset{qa}{\wedge} Y_q = \mathcal{A}(X_p \otimes Y_q), \quad \text{(A.2)}$$

instead of eq.(7). This observation means that when reading books with chapters on the theory of the exterior algebras or scientific papers, it is necessary to take care and to be sure about which product has been defined, for otherwise great confusion may arise. In particular for *not* distinguishing $\wedge$ as defined in eq.(7) from $\overset{qa}{\wedge}$ as defined by eq.(A.2) the following error appears frequently. Let $X_p \in \bigwedge^p V$, let $\{e_i\}$ be any basis of $V$ and $\{\varepsilon^i\}$ its corresponding dual basis of $V^*$, and consider the $p$ 1-forms $\omega^1, \ldots, \omega^p \in V^*$. Then, using the elementary expansions $\omega^1(e_{j_1})\varepsilon^{j_1}, \ldots$ etc., we have

$$\begin{aligned} X(\omega^1, \ldots, \omega^p) &= X(\omega^1(e_{j_1})\varepsilon^{j_1}, \ldots, \omega^p(e_{j_p})\varepsilon^{j_p}) \\ &= \omega^1(e_{j_1}) \ldots \omega^p(e_{j_p})X(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) \\ &= X^{j_1 \ldots j_p} e_{j_1} \otimes \ldots \otimes e_{j_p}(\omega^1, \ldots, \omega^p), \end{aligned}$$

i.e.,

$$X = X^{j_1 \ldots j_p} e_{j_1} \otimes \ldots \otimes e_{j_p}. \quad \text{(A.3)}$$

The real numbers $X(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) = X^{j_1 \ldots j_p}$ are called the $j_1 \ldots j_p$-th (contravariant) components of $X$ relative to the basis $\{e_{j_1} \otimes \ldots \otimes e_{j_p}\}$ of $T^pV$.

Now, since $X \in \bigwedge^p V$ is a completly antisymmetric tensor it must satisfy

$$\mathcal{A}X = X, \quad \text{(A.4)}$$

and using the definition of the operator $\mathcal{A}$ (see eq.(8)) we get the identity

$$X^{j_1 \ldots j_p} = \frac{1}{p!} \epsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p} X^{i_1 \ldots i_p}, \tag{A.5}$$

and of course the components $X^{j_1 \ldots j_p}$ are antisymmetric in all indices.

Using eq.(A.5) in eq.(A.3) we obtain,

$$X = \frac{1}{p!} \epsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p} X^{i_1 \ldots i_p} e_{j_1} \otimes \ldots \otimes e_{j_p}. \tag{A.6}$$

Now, if we use the definition of the exterior product given by eq.(7), more exactly an particular case of eq.(11), the well-known combinatorial formula: $e_{i_1} \wedge \ldots \wedge e_{i_p} = \epsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p} e_{j_1} \otimes \ldots \otimes e_{j_p}$, we see that eq.(A.6) can be written as

$$X = \frac{1}{p!} X^{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}. \tag{A.7}$$

Eq.(A.7) is the expansion that has been used in this paper and in all the others of this series.

Now, if we use the definition of the exterior product as given by eq.(A.2), then by repeting the above calculations we get that $X$ can be witten as

$$X = X^{i_1 \ldots i_p} e_{i_1} \overset{qa}{\wedge} \ldots \overset{qa}{\wedge} e_{i_p}. \tag{A.8}$$

To write $X = \frac{1}{p!} X^{i_1 \ldots i_p} e_{i_1} \overset{qa}{\wedge} \ldots \overset{qa}{\wedge} e_{i_p}$ instead of eq.(A.8) is clearly wrong if it is supposed that the meaning of $X(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p})$ is that $X(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) = X^{j_1 \ldots j_p}$ as in eq.(A.3). This confusion appears, e.g., in [36].

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