Connected components of open semigroups in semi-simple Lie groups*

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Abstract

This paper studies connected components of open subsemigroups of non-compact semi-simple Lie groups through the control sets in the flag manifolds and their coverings. A method for computing the number of components we call recurrent, which includes the semigroup components, is developed and it is proved that the union of this set of components is a subsemigroup. The idea of mid-reversibility comes up to show that an open semigroup has just one semigroup component if the identity belongs to its closure. A necessary and sufficient condition for mid-reversibility is proved showing that e.g. in a complex group any open semigroup is mid-reversible.

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1 Introduction

Let G be a connected non-compact semi-simple Lie group with finite center. The purpose of this paper is to study the connected components of an open subsemigroup $S \subset G$. Our approach is through the action of S on the flag manifolds of G and the corresponding control sets (for results related to this method we refer to [8], [9], [10], [11]). Thus the main efforts are dedicated towards the description of the connected components in terms of the control sets. Having this in mind we divide the connected components of S into two classes. The first one comprises those components $\Gamma \subset S$ such that some power Γ^k, $k \geq 2$, meets Γ (and hence is contained in Γ). We say that such a component has finite index or is recurrent. The other class contains the infinite index or transient components. Among the recurrent components there are those which are themselves subsemigroups (semigroups components, for short).

In this paper we get sharper results for the recurrent components. This is due to the fact that the properties of the semigroup S captured by its action on the flag manifolds are usually related to high powers of the elements of S.

We better describe our results by summarizing the contents of the paper: Sections 2, 3 and 4 are preparatory. In section 2 we discuss generalities about connected components of semigroups, and prove a lemma used throughout the paper, ensuring that in a connected nilpotent Lie group an open subsemigroup has just one semigroup component. In Section 3 we set notations and recall some of the above mentioned results about semigroups in semi-simple Lie groups and their control sets on flag manifolds. This includes a discussion about the parabolic type of a semigroup and the introduction of the open subsets sets C^+ and C^- of the maximal flag manifold, which we call the attractor and the repeller sets of S, respectively. These sets play a central role in the study of recurrent components of S. In Section 4 the invariant control sets in the covering G/P_0 of the flag manifold are determined. These control sets are used afterwards in the description of the recurrent components.

In Section 5 we relate the Jordan decomposition of a $g \in S$ with the parabolic type of S, by showing how the latter influences the semi-simple and unipotent components of g. This result has independent interest (and improves Corollary 4.4 of [11]). From the knowledge of the Jordan decompositions we prove that any $g \in S$ has a unique fixed point in the attractor subset of the flag manifold determined by the parabolic type of S (see Theorem

5.9). This result is used in Section 6 to show that a connected component of S, say Υ, leaves invariant a unique connected component of C^+, denoted by $K^+(\Upsilon)$, and analogously Υ^{-1} leaves invariant a unique component $K^-(\Upsilon)$ of C^-.

The uniqueness of the invariant component opens the way to study the semigroup components in Section 7. In Theorem 7.1 we prove that for a pair of connected components K_1 of C^+ and K_2 of C^- there exists a unique semigroup component, say $\Gamma(K_1, K_2)$, which leaves invariant K_1 and whose inverse leaves invariant K_2. This is one of the main results of the paper. It gives the number of semigroup components in terms of the control sets. In proving this theorem some results of independent interest are obtained, like the fact that the attractor and repeller sets of $\Gamma(K_1, K_2)$ are K_1 and K_2, respectively, and that the parabolic type of a semigroup component of S coincides with the parabolic type of S.

The recurrent components which are not semigroups are studied in Section 8. They are described by the components of the invariant control sets in the covering G/P_0 mentioned above. As a consequence we prove that the set of recurrent components is a subsemigroup of the semigroup of components, or equivalently, the union of recurrent components is a subsemigroup of S.

In Section 9 we relate the connected components with the concept of mid-reversibility introduced by Ruppert [6] (S is mid-reversible in G if $G = SS^{-1}S$). This algebraic property of a semigroup has deep links to connectivity properties as already appears in [6]. In fact, Theorem 3.9 of [6] shows that an open subsemigroup of a group G containing the identity in its closure is connected in case every open subsemigroup of G is mid-reversible. In our semi-simple context we give a necessary and sufficient condition for an open semigroup in a semi-simple Lie group to be mid-reversible (see Theorem 9.4). This condition depends on the connected components of the centralizer of a split-torus (the M-group). In particular, the condition of Ruppert that every open subsemigroup is mid-reversible holds if M is connected (which happens for instance if G is a complex Lie group). For these groups Theorem 3.9 of [6] applies. In case M is not connected we show anyway that a semigroup S with $1 \in \mathrm{cl}S$ has exactly one semigroup component. Still in this subject we prove that compression semigroups are mid-reversible and for in general we give a rough upper bound for the number of factors S, S^{-1} required to generate the group.

Finally, in Section 10 we provide some examples and counter-examples related to the results of the paper. In particular we illustrate how to prove connectedness of a semigroup by showing that the compression semigroup

S_W of a pointed and generating cone $W \subset \mathbb{R}^n$ is connected.

2 Semigroup of connected components

In this section we discuss some concepts and general facts about connected components of semigroups which are used throughout the paper. Let G be a semitopological group and $S \subset G$ a subsemigroup.

Given $x \in S$ we denote by K_x the connected component of S containing x. The subset K_xK_y is connected and contains xy thus $K_xK_y \subset K_{xy}$. This provides the set of connected components of S the structure of a semigroup, where the product K_xK_y of two components K_x and K_y is the component containing xy. In the sequel we often write $K_xK_y = K_{xy}$, meaning the product in the semigroup of connected components, instead of product of sets (when equality may not be true).

A connected component K_x is a subsemigroup, (that is $K_xK_x \subset K_x$) if and only if it is an idempotent in the semigroup of components, which is equivalent to $K_{x^2} = K_x$, that is, $x^2 \in K_x$. Such a connected component is called *semigroup component*. In dealing with semigroups in Lie groups it is convenient to have the following terminology for components meeting one-parameter semigroups.

Definition 2.1 *A connected component K of a semigroup $S \subset G$ is said to be an* exit *component provided there exists a one-parameter subgroup $\phi : \mathbb{R} \to G$, and $T_0 \in \mathbb{R}$, such that the interval $\phi(T_0, +\infty) \subset K$.*

An exit component has the form $K = K_{\phi(t)}$, $t > T_0$. Since $\phi(t)\phi(t) = \phi(2t)$, it is clear that an exit component is a subsemigroup. Exit components are easily built with the aid of the following simple fact.

Lemma 2.2 *Let $\Gamma \subset \mathbb{R}_+$ be a semigroup of positive reals, and suppose that for $0 < t_0 \in \Gamma$ there exists $\varepsilon > 0$ with $(t_0 - \varepsilon, t_0 + \varepsilon) \subset \Gamma$. Then, $(nt_0, +\infty) \subset \Gamma$ if $n > \dfrac{t_0 - \varepsilon}{2\varepsilon}$.*

Proof: Given an integer $n > 0$, the interval $(nt_0 - n\varepsilon, nt_0 + n\varepsilon)$ is contained in Γ. Hence $(nt_0, +\infty) \subset \Gamma$ if $kt_0 + k\varepsilon > (k+1)t_0 - (k+1)\varepsilon$ for all $k \geq n$. But this inequality holds if $n > \dfrac{t_0 - \varepsilon}{2\varepsilon}$. ∎

This lemma implies immediately the following statement about existence of exit components (cf. [6], Proposition 3.1).

Proposition 2.3 *Let ϕ be a one-parameter subgroup of the topological group G. Let $S \subset G$ be a subsemigroup and suppose that $\phi(s) \in \mathrm{int}S$ for some $s > 0$. Then there exists $T_0 > 0$ such that $\phi(T_0, +\infty) \subset \mathrm{int}S$. Clearly $\phi(T_0, +\infty)$ is entirely contained in an exit component.*

Therefore, any open semigroup meeting a one-parameter group contains an exit component, and hence a semigroup component.

Now, let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X \in \mathfrak{g}$ is such that the one-parameter semigroup $\exp tX$, $t \geq 0$, meets the open semigroup S we denote by $E(X, S)$, or simply $E(X)$, the exit component of S containing $\exp tX$ for large $t > 0$. Also, if $h \in S$ and $X = \log h$ is well defined we put $E(h) = E(X)$.

At this point we recall that a Lie group G is said to have finite index if for every $x \in G$ some power x^k belongs to the image of the exponential map (cf. Dokovic and Hofmann [2]). Any open subsemigroup of a group with finite index meets a one-parameter group and thus has exit components. Furthermore, any semigroup component is an exit component, so that in these groups both concepts are equivelent. Note also that in a finite index group G a connected component Υ of an open semigroup $S \subset G$ must have some power meeting a semigroup component.

In particular, open semigroups in a connected nilpotent Lie group has semigroup (exit) components. Next we show the uniquenes of such components, a fact which is used extensively in the sequel, applied to abelian groups. For the proof we use the concept of reversibility. Recall that a subsemigroup T of a group G is said to be left (respectively right) reversible if TT^{-1} (respectively $T^{-1}T$) is a group, which must be G if T has non-empty interior and G is connected.

Proposition 2.4 *Let G be a connected nilpotent Lie group and $S \subset G$ an open subsemigroup. Then S contains exactly one semigroup (exit) component K.*

Proof: The nilpotent Lie group G is exponential, so that a connected component $K \subset S$ is exit if and only if it is a subsemigroup. For the uniqueness we use the fact that any open semigroup in a nilpotent Lie group is (right and left) reversible (see [6], Proposition 1.5). Thus suppose that $K_1, K_2 \subset S$ are semigroup components of S. Take $y \in K_2$. By right reversibility of K_1, there exists $x \in K_1$ such that $xy \in K_1$. Hence, $K_1 K_2 \subset K_1$. But by left reversibility of K_2, for any $x \in K_1$ there exists $z \in K_2$ such that $xz \in K_2$.

Therefore, $K_1 K_2 \subset K_2$, that is, $K_1 = K_2$. ∎

Remark: Notice that this proof uses both right and left reversibility of open subsemigroups of nilpotent Lie groups. Actually one sided reversibility is not enough. In fact, after looking at the connected components of semigroups in semi-simple Lie groups we can find easily an example of a subsemigroup S of an exponential solvable Lie group G such that every component of S is right reversible and nevertheless S has more than one semigroup component (see the remark at the end of Section 7, below). ∎

3 Semigroups in semi-simple groups

The purpose of this section is to establish notations and background results to be used afterwards. Let G be a connected noncompact Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. The flag manifolds of G are labelled by subsets of the set of simple (restricted) roots of $\mathfrak{g}$. Precisely, choose an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let Π be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Denote by Π^+ and Σ the set of positive and simple roots, respectively, which correspond to the nilpotent component $\mathfrak{n}$, that is,

$$\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ stands for the α-root space. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and put $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for the corresponding minimal parabolic subalgebra. By definition, the maximal flag manifold $\mathbb{B}$ of G is the set of subalgebras $\mathrm{Ad}\,(G)\,\mathfrak{p}$, where Ad stands for the adjoint representation of G in $\mathfrak{g}$. There is an identification of $\mathbb{B}$ with G/P where P is the normalizer of $\mathfrak{p}$ in G. Furthermore, $P = MAN$, $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$ and M is the centralizer of A in $K = \exp \mathfrak{k}$.

Given a subset $\Theta \subset \Sigma$, denote by $\mathfrak{p}_\Theta$ the corresponding parabolic subalgebra, namely,

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p},$$

where $\mathfrak{n}^-(\Theta)$ is the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle \Theta \rangle$. Here $\langle \Theta \rangle$ is the set of positive roots generated by Θ. The set of parabolic subalgebras conjugate to $\mathfrak{p}_\Theta$ identifies with the homogenous space G/P_Θ, where P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \{g \in G : \mathrm{Ad}\,(g)\,\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

This construction yields the flag manifold $\mathbb{B}_\Theta = G/P_\Theta$, $\Theta \subset \Sigma$.

Let

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\}$$

be the Weyl chamber associated to Σ. We say that $X \in \mathfrak{g}$ is split-regular in case $X = \mathrm{Ad}(g)(H)$ for some $g \in G$, $H \in \mathfrak{a}^+$. Analogously, $x \in G$ is said to be split-regular in case $x = ghg^{-1}$ with $h \in A^+ = \exp \mathfrak{a}^+$, that is, $x = \exp X$, with X split-regular in $\mathfrak{g}$.

Let $\mathfrak{n}^- = \sum_{\alpha\in\Pi} \mathfrak{g}_{-\alpha}$ be the nilpotent subalgebra opposed to $\mathfrak{n}$. Put $N^- = \exp \mathfrak{n}^-$. Then in any flag manifold $\mathbb{B}_\Theta$, the orbit $\mathrm{Ad}(N^-)\,\mathfrak{p}_\Theta$ (called open Bruhat cell) is open and dense. Furthermore, if $h \in A^+$ then $\lim h^k y = \mathfrak{p}_\Theta$ for any $y \in \mathrm{Ad}(N^-)\,\mathfrak{p}_\Theta$. In other words, $\mathfrak{p}_\Theta$ is an attractor in $\mathbb{B}_\Theta$ for any $h \in A^+$, with $\mathrm{Ad}(N^-)\,\mathfrak{p}_\Theta$ the corresponding stable manifold. Similarly, for $x \in G$ the element $g = xhx^{-1}$ is split-regular. Its attractor in $\mathbb{B}_\Theta$ is $\mathrm{Ad}(g)\,\mathfrak{p}_\Theta$ with open and dense stable manifold $\mathrm{Ad}\left(xN^-x^{-1}\right)$. In the sequel we denote the attractor fixed point of g in $\mathbb{B}_\Theta$ by $\mathrm{att}_\Theta(g)$, while the corresponding stable manifold is denoted by $\mathrm{st}_\Theta(h)$. Analogous remarks hold for the repeller $\mathrm{rpp}_\Theta(g)$ in $\mathbb{B}_\Theta$. In case $\mathbb{B}_\Theta = \mathbb{B}$ is the maximal flag manifold we suppress the indices and write simply $\mathrm{att}(g)$ and $\mathrm{rpp}(g)$.

Given two subsets $\Theta_1 \subset \Theta_2 \subset \Sigma$, the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $G/P_{\Theta_1} \to G/P_{\Theta_2}$, $gP_{\Theta_1} \mapsto gP_{\Theta_2}$. Alternatively, the fibration assigns to the parabolic subalgebra $\mathfrak{q} \in \mathbb{B}_{\Theta_1}$ the unique parabolic subalgebra in $\mathbb{B}_{\Theta_2}$ containing $\mathfrak{q}$. In particular, $\mathbb{B} = \mathbb{B}_\emptyset$ projects onto every flag manifold $\mathbb{B}_\Theta$.

From the structure of the parabolic subgroup P_Θ the fiber P_Θ/P of $\mathbb{B} \to \mathbb{B}_\Theta$ is obtained. We follow closely the notation of Warner [12], Section 1.2. Denote by $\mathfrak{a}_\Theta$ the annihilator of Θ in $\mathfrak{a}$:

$$\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta\}.$$

Let L_Θ stand for the centralizer of $\mathfrak{a}_\Theta$ in G and put $M_\Theta(K) = L_\Theta \cap K$ for the centralizer of $\mathfrak{a}_\Theta$ in K. The Lie algebra $\mathfrak{l}_\Theta$ of L_Θ is reductive and decomposes as $\mathfrak{l}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta$ with $\mathfrak{m}_\Theta$ semi-simple. Let M_Θ^0 be the connected subgroup whose Lie algebra is $\mathfrak{m}_\Theta$ and put $M_\Theta = M_\Theta(K)\,M_\Theta^0$. It follows that the identity component of M_Θ is M_Θ^0. The Bruhat-Moore Theorem (see [12], Theorem 1.2.4.8), provides the following decompositions:

1. $P_\Theta = M_\Theta A_\Theta N_\Theta$, where $A_\Theta = \exp \mathfrak{a}_\Theta$ and N_Θ is the unipotent radical of P_Θ, that is, $N_\Theta = \exp \mathfrak{n}_\Theta$, with $\mathfrak{n}_\Theta$ the nilradical of $\mathfrak{p}_\Theta$.

2. $P_\Theta = M_\Theta(K)\,AN$.

This second decomposition ensures that the fiber P_Θ/P is equal to the coset space $M_\Theta(K)/M$. It turns out that $M_\Theta(K)/M = M_\Theta/(M_\Theta \cap P)$. This last coset space is the maximal flag manifold of M_Θ, since $M_\Theta \cap P$ is a minimal parabolic subgroup of M_Θ.

We discuss now semigroups in G. The following facts can be proved for any semigroup S with $\mathrm{int}S \neq \emptyset$, provided G has finite center. Consider the action of S in the flag manifolds of G. It was proved in [11], Theorem 6.2, that S is not transitive in $\mathbb{B}_\Theta$ unless $S = G$. Moreover, there exists just one closed invariant subset $C_\Theta(S) \subset \mathbb{B}_\Theta$ such that Sx is dense in $C_\Theta(S)$ for all $x \in C_\Theta(S)$. This subset is called the invariant control set of S in $C_\Theta(S)$ (abbreviated S-i.c.s.). Since S is not transitive, $C_\Theta(S) \neq B_\Theta$.

The fact that Sx is dense in $C_\Theta(S)$ for all $x \in C_\Theta(S)$ implies the existence of an open subset $C_\Theta^+(S) \subset C_\Theta(S)$ such that for all $x, y \in C_\Theta^+$ there exists $g \in S$ with $gx = y$. Furthermore, $C_\Theta^+(S)$ is dense in $C_\Theta(S)$. In view of Proposition 3.1 below we call $C_\Theta^+(S)$ the *attractor set* of S in $\mathbb{B}_\Theta$. Replacing S by S^{-1} we get a subset $C_\Theta^-(S)$ which we call the *repeller set* of S in $\mathbb{B}_\Theta$. In case $\mathbb{B}_\Theta = \mathbb{B}$ is the maximal flag manifold, we suppress de subscripts and write simply $C^\pm(S)$ for $C_\Theta^\pm(S)$. Also, if the semigroup is understood we write simply $C_\Theta^\pm$ instead of $C_\Theta^\pm(S)$. Usually $C_\Theta^\pm$ will be associated to a given semigroup S while $C_\Theta^\pm(T)$ is extensively used for subsemigroups $T \subset S$.

For later reference we note that $C_\Theta^\pm(S)$ is connected in case S is connected, because S is transitive on $C_\Theta^\pm(S)$ and the evaluation map $g \mapsto gx$ is continuous.

Proposition 3.1 *The attractor set C_Θ^+ is given* $\mathrm{att}(h)$ *with h running through the split-regular elements in* $\mathrm{int}S$. *Analogously the repeller set C_Θ^- is formed by* $\mathrm{rpp}(h)$, $h \in \mathrm{int}S$.

The semigroups in G are distinguished according to the geometry of their invariant control sets. This geometry is described by the following statements, proved in [11].

Proposition 3.2 *There exists $\Theta \subset \Sigma$ such that $\pi_\Theta^{-1}(C_\Theta) \subset \mathbb{B}$ is the invariant control set in the maximal flag manifold. Among the subsets Θ satisfying this property there exists a maximal one, in the sense that it contains the others.*

We denote the maximal subset by $\Theta(S)$ and say that it is the *parabolic type* of S. Alternatively, we say also that the parabolic type of S is the corresponding flag manifold $\mathbb{B}(S) = \mathbb{B}_{\Theta(S)}$ (see [8], [10], [11], for further discussions about the parabolic type of a semigroup).

When $\Theta = \Theta(S)$, the invariant control set $C_{\Theta(S)}$ has the following nice properties:

Proposition 3.3 *The set $R(S)$ of split-regular elements in* $\mathrm{int}S$ *is not empty, and if $h \in R(S)$ then $\mathrm{att}_\Theta(h) \in C_\Theta^+$ for any Θ and $.C_{\Theta(S)} \subset \mathrm{st}_{\Theta(S)}(h)$.*

We conclude this section by proving the following reversibility properties inside identity component of minimal parabolic subgroups.

Lemma 3.4 *Let T be an open semigroup and take $x \in C^+(T)$. Denote by P the isotropy subgroup at x and let P_0 be its identity component. Then $T \cap P_0 \neq \emptyset$ is left reversible.*

Proof: By definition of C^+, $T \cap P$ has non-empty interior in P. Since we are assuming that G has finite center, the number of connected components of P is finite, hence $T \cap P_0$ also has non-empty interior. Now, left reversibility follows from [6], Lemma 4.6. In fact, $T \cap P_0$ contains a split-regular $h = \exp H$, which belongs to a Weyl chamber positive for P. This means that the eigenvalues of $\mathrm{ad}(H)$ in the Lie algebra of P are ≥ 0. ∎

Using the same argument for the inverse semigroup we get right reversibility inside the isotropy subgroups at repeller points.

Lemma 3.5 *Let T be an open semigroup and take $x \in C^-(T)$. Denote by P the isotropy subgroup at x and let P_0 be its identity component. Then $T \cap P_0 \neq \emptyset$ right reversible.*

4 Control sets on G/P_0

Let $P = MAN$ be a minimal parabolic subgroup and put $P_0 = M_0AN$ for its identity component. Given an open semigroup S let C^+ be its attracting set on $\mathbb{B} = G/P$. Without loss of generality we can assume that P is the isotropy subgroup at $x \in C^+$. In this case $S \cap P$ is a nonempty open semigroup meeting P_0 (if G has finite center). In order to have a notation for the components of P meeting S we put

$$M(S, P) = \{tP_0 \in P/P_0 : S \cap tP_0 \neq \emptyset\}.$$

Clearly $M(S,P)$ a subsemigroup of $P/P_0 = M/M_0$. Notice that $M(S,P)$ is actually a group in case G has finite center, because in this case M/M_0 is finite.

The group $M(S,P)$ is also described in terms of control sets in G/P_0. Let

$$\pi : G/P_0 \longrightarrow \mathbb{B} = G/P$$

be the canonical fibration with typical fiber $P/P_0 = M/M_0$. This is simultaneously a covering and a principal bundle. The group M/M_0 acts on the right on G/P_0, and this action commutes with left action of G. Since we are assuming that G has finite center M/M_0 is finite and G/P_0 is compact. Thus any open semigroup $S \subset G$ has invariant control sets in G/P_0, in general not a unique one. As before we assume that P is the isotropy at $x \in C^+$. In this case P_0 is the isotropy subgroup at any y in the fiber $\pi^{-1}\{x\}$ over x.

Now, let $D \subset G/P_0$ be an invariant control set for S, and put D^+ for its set of transitivity. By general facts about control sets on fiber bundles $\pi(D) = C$, $\pi(D^+) = C^+$ and any point of $\pi^{-1}(C)$ belongs to an invariant control set. Furthermore, since the left action of G commutes with the right action of M (or rather M/M_0), it follows that for any $m \in M$, Dm is also an S-i.c.s. This implies that $Dm = D$ or $Dm \cap D = \emptyset$, and the invariant control sets of S on G/P_0 have the form Dm, $m \in M/M_0$. We define

$$M(S,D) = \{m \in M/M_0 : Dm = D\}.$$

It is easy to check that $M(S,D)$ is a subgroup of M/M_0. The following proposition establishes the relation between $M(S,P)$ and $M(S,D)$.

Proposition 4.1 *Let P be the isotropy subgroup at a given $x \in C^+$ and fix $y \in D \cap \pi^{-1}\{x\}$. Let $m \mapsto ym$ be the bijection between M/M_0 and the fiber through y. Then $M(S,D) = yM(S,P)$.*

Proof: Let $m \in M$ be such that the component mP_0 belongs to $M(S,D)$. Since $y \in D$, $ym \in D$, so that there exists $g \in S$ such that $gy = ym$. Clearly, g leaves invariant the fiber over x, so that $g \in P$. Moreover, $gy = ym$ implies that $g = mt$ for some $t \in P_0$. Hence, $g \in mP_0$, showing that $S \cap mP_0 \neq \emptyset$, that is $mM_0 \in M(S,P)$.

Conversely, suppose that $g \in S \cap mP_0$. Then $gy = ym \in D$, hence the coset mM_0 belongs to $M(S,D)$. ∎

Corollary 4.2 *The number of invariant control sets for S on G/P_0 is the order of $(M/M_0)/M(S,D)$.*

We get a more detailed information about $M(S,D)$ with the aid of the parabolic type $\Theta(S)$ of the semigroup S. To do this we discuss first the restriction to open cells of the bundles over a flag $\mathbb{B}_\Theta$. Fix an open Bruhat cell $\sigma \subset \mathbb{B}_\Theta$. Recall that the restriction of the bundle $\pi_\Theta : \mathbb{B} \to \mathbb{B}_\Theta$ to σ is trivial, meaning that $\pi_\Theta^{-1}(\sigma)$ is diffeomorphic to $\sigma \times F$, where $F = P_\Theta/P$ is the fiber of $\mathbb{B} \to \mathbb{B}_\Theta$. Analogously, the restriction of $G/P_0 \to \mathbb{B}_\Theta$ to σ gives the product $\sigma \times F_0$, where $F_0 = P_\Theta/P_0$. The decomposition of the fiber F_0 into connected components reads

$$P_\Theta/P_0 = \left(P_\Theta^0/P_0\right) \times \left(P_\Theta/P_\Theta^0\right)$$

where P_Θ^0 is the identity component of P_Θ. The first factor P_Θ^0/P_0 is equal to P_Θ/P, since any connected component of P_Θ contains a component of P (see [12], Lemma 1.2.4.5). The second component is writen in terms of the M-group as follows: Write $M(\Theta) = M \cap P_\Theta^0$. Then the set of components of P_Θ is $M/M(\Theta)$ (that is, M meets every component of P_Θ and $M(\Theta)$ is contained in the identity component of P_Θ; see [12], Lemma 1.2.4.5). Therefore, the restriction of $G/P_0 \to \mathbb{B}_\Theta$ over σ is diffeomorphic to the product $\sigma \times (P_\Theta/P) \times (M/M(\Theta))$.

Now we carry this decomposition to invariant control sets in G/P_0 by taking $\Theta = \Theta(S)$. Recall that the invariant control set $C \subset \mathbb{B}$ is given by $C = \pi_{\Theta(S)}^{-1}\left(C_{\Theta(S)}\right)$, and there exists an open cell $\sigma \subset \mathbb{B}_{\Theta(S)}$ with $C_{\Theta(S)} \subset \sigma$. Hence the restriction of $G/P_0 \to \mathbb{B}_{\Theta(S)}$ above $C_{\Theta(S)}$ is diffeomorphic to

$$C_{\Theta(S)} \times (P_\Theta/P) \times (M/M(\Theta))$$

while C is diffeomorphic to $C_{\Theta(S)} \times (P_\Theta/P)$. Notice that the projection of $C_{\Theta(S)} \times (P_\Theta/P) \times (M/M(\Theta))$ onto the first two components is just the restriction of $G/P_0 \to \mathbb{B}$. Therefore, the slices $C_{\Theta(S)} \times (P_\Theta/P) \times \{a\}$, $a \in M/M(\Theta)$, are the leaves above $C = C_{\Theta(S)} \times (P_\Theta/P)$, and each one is contained in an invariant control set in G/P_0. Also, let $D \subset C_{\Theta(S)} \times (P_\Theta/P) \times (M/M(\Theta))$. Then by definition of $M(D,S)$, it follows that

$$D = C_{\Theta(S)} \times (P_\Theta/P) \times (M(S,D)/M(\Theta)).$$

For later reference we summarize this description of the invariant control sets in the following proposition.

Proposition 4.3 *Keep the above notations with $\Theta(S)$ the parabolic type of S. Suppose, without loss of generality, that the standard parabolic subgroup $P \subset P_{\Theta(S)}$ is the isotropy of $x \in C^{+}$. Then any slice $C_{\Theta(S)} \times (P_{\Theta}/P) \times \{a\}$, $a \in M/M(\Theta)$, is entirely contained in a control set. Furthermore, an invariant control set $D \subset G/P_0$ is diffeomorphic to $C_{\Theta(S)} \times (P_{\Theta}/P) \times M_D$, where $M_D = M(S,D)/M(\Theta)$.*

5 Jordan decompositions

As before we let $S \subset G$ be an open semigroup in the connected semi-simple Lie group G, with finite center with parabolic type $\Theta(S)$. In this section we relate the Jordan decompositions of elements in S with $\Theta(S)$. As a result we get Theorem 5.9, ensuring that a $g \in S$ has exactly one fixed point in $C^{+}_{\Theta(S)}$.

Recall that a $g \in G$ is said to be unipotent or semi-simple if $\mathrm{Ad}(g)$ is unipotent or semi-simple, respectivley. The Jordan decomposition of $g \in G$ writes $g = g_s g_u = g_u g_s$ uniquely with $g_s, g_u \in G$ semi-simple and unipotent, respectively (see [12], Proposition 1.4.3.3). In real groups the semi-simple component g_s can be decomposed further into compact and radial parts.

Proposition 5.1 *Given $g \in G$, there are an Iwasawa decomposition $G = KAN$ and $u \in K$, $h = \exp H \in A$ such that $g_s = uh$, so that $g = uhg_u$, with g_u unipotent. Furthermore, the components u, h, g_u commute with each other.*

Proof: See [4], Theorem IX, 7.2. ∎

Now, for $g \in S$ we shall relate the decomposition $g = uhn$ with the parabolic type of S. Since h belongs to A there exists a Weyl chamber A^{+} such that $h \in \mathrm{cl}A^{+}$. Denote by Σ the simple system of roots associated to A^{+}, and put

$$\Theta(h) = \{\alpha \in \Sigma : \alpha(\log h) = 0\}. \qquad (1)$$

This subset defines the standard parabolic subgroup $P_{\Theta(h)}$, whose reductive Levi component $L_{\Theta(h)}$ is the centralizer of h in G (see [12]).

The next few results are intended to prove that $\Theta(h)$ is contained in the parabolic type $\Theta(S)$ of S.

Lemma 5.2 *As above decompose $g \in S$ as $g = uhg_u$. Then for some integer $k > 0$, $h^k \in S$.*

Proof: Since S is open and u belongs to a torus, we can find v of finite order so that $vhg_u \in S$. Thus, for some integer l, $h^l g_u^l \in S$. On the other hand, g_u^l is a unipotent element in the reductive Lie group $L_{\Theta(h)}$. Thus for any neighborhood U of g_u^l in $L_{\Theta(h)}$ there exists $z \in U$ with finite order (see [7], Lemma 4.1). If U is small enough, $h^l z \in S$. Taking into account that $z \in L_{\Theta(h)}$ commutes with h^l, we conclude that $\left(h^l\right)^s \in S$, if s is the order of z, showing the lemma. ∎

Since h^k belongs to the closure of A^+, the lemma implies that $S \cap A^+ \neq \emptyset$. Therefore, if we denote by $L^0_{\Theta(h)}$ the identity component of $L_{\Theta(h)}$, we have $S \cap L^0_{\Theta(h)} \neq \emptyset$. This intersection is in fact, quite large:

Lemma 5.3 *Denote by $M_{\Theta(h)}$ the semi-simple component of $L^0_{\Theta(h)}$, and consider the projection $p : L^0_{\Theta(h)} \to M_{\Theta(h)}$ modulo the center. Then $p\left(S \cap L^0_{\Theta(h)}\right) = M_{\Theta(h)}$.*

Proof: Clearly, $p\left(S \cap L^0_{\Theta(h)}\right)$ is an open semigroup in $M_{\Theta(h)}$. By the previous lemma $h^k \in S \cap L^0_{\Theta(h)}$, and since $p\left(h^k\right)$ is the identity in $M_{\Theta(h)}$, the result follows. ∎

Corollary 5.4 *For every $x \in M_{\Theta(h)}$ there exists $a \in A$ such that $xa \in S$.*

Proof: The center of $L^0_{\Theta(h)}$ has the form $Z_K Z_{\mathfrak{p}}$ with $Z_K \subset K$, compact and $Z_{\mathfrak{p}} \subset A$. Given $x \in M_{\Theta(h)}$ the lemma shows the existence of $a \in Z_K Z_{\mathfrak{p}}$ with $ax = xa \in S$. Since Z_K is compact we can argue as in the proof of Lemma 5.2, and get $ax \in S$, with $a \in Z_{\mathfrak{p}} \subset A$. ∎

Corollary 5.5 *Given a flag manifold $\mathbb{B}_\Theta$ denote by b^Θ the attractor of A^+ in $\mathbb{B}_\Theta$ and by C^+_Θ the attractor set of S, also in $\mathbb{B}_\Theta$. Then the orbit $M_{\Theta(h)} b^\Theta \subset C^+_\Theta$.*

Proof: Take $x \in M_{\Theta(h)}$, and let $a \in A$ be such that $xa \in S$. Then $xb^\Theta = xab^\Theta$. But $b^\Theta_0 \in C^+_\Theta$, and since $xa \in S$, it follows that $xb^\Theta \subset C^+_0$. ∎

Now it is easy to prove that the orbit $M_{\Theta(h)} b^\Theta$ is entirely contained in the attractor set C^+_Θ.

Lemma 5.6 *Keep the previous notations with b^{Θ} the attractor of A^{+} in $\mathbb{B}_{\Theta}$. Then, $M_{\Theta(h)}b^{\Theta}$ is contained in the open Bruhat cell determined by A^{+} if and only if $\Theta(h) \subset \Theta$.*

Proof: Suppose that $\Theta(h) \subset \Theta$. Then $M_{\Theta(h)} \subset P_{\Theta}$, the isotropy at b^{Θ}. Hence $M_{\Theta(h)}b^{\Theta} = b^{\Theta}$.

For the converse denote by $\mathcal{W}_{\Theta}$ the subgroup of the Weyl group generated by the reflections with respect to the roots in Θ. Suppose that some $\alpha \in \Theta(h)$ is not in Θ, and let r_{α} be the reflection with respect to α. Then $r_{\alpha} \notin \mathcal{W}_{\Theta}$, so that if w_{α} is a representative of r_{α} in the normalizer M^{*} of A then $w_{\alpha}b^{\Theta} \neq b^{\Theta}$. However, $w_{\alpha} \in M_{\Theta(h)}$ so that $M_{\Theta(h)}b^{\Theta}$ is not contained in the open cell. ∎

Corollary 5.7 *For $g \in S$ write $g = uhg_{u}$, and define $\Theta(h)$ as in (1). Then $\Theta(h) \subset \Theta(S)$.*

Proof: Follows immediately from the previous lemma and the definition of the parabolic type of S, after taking into account that $S \cap A^{+} \neq \emptyset$. ∎

Corollary 5.8 $gb^{\Theta(S)} = b^{\Theta(S)}$.

Proof: $L_{\Theta(h)} \subset P_{\Theta(h)} \subset P_{\Theta(S)}$. ∎

Finally we arrive that any element of the open semigroup S leaves fixed just one point of the attractor set of S in the flag manifold corresponding to the parabolic type.

Theorem 5.9 *Let S be an open semigroup. Then any $g \in S$ has a unique fixed point, say $\mathrm{fix}_{\Theta(S)}(g)$, in $C^{+}_{\Theta(S)}$.*

Proof: It remains to check that $b^{\Theta(S)}$ is the only fixed point in $C^{+}_{\Theta(S)}$. Write $g = g_{s}g_{u}$, $g_{s} = uh$. It is standard that a g fixed point is also fixed under g_{s} and g_{u} [proof: $\mathrm{Ad}(g_{s})$ and $\mathrm{Ad}(g_{u})$ are polynomial functions of $\mathrm{Ad}(g)$. Thus any subspace invariant under $\mathrm{Ad}(g)$ is also invariant under $\mathrm{Ad}(g_{s})$ and $\mathrm{Ad}(g_{u})$. The claim then follows by the remark that any flag manifold can be realized as an orbit in a certain Grassmannian of subspaces of $\mathfrak{g}$.] Now, $b^{\Theta(S)}$ is the only fixed point under g_{s} in the open cell $\sigma \subset \mathbb{B}(S)$ determined by A^{+}, since u leaves σ invariant and $h^{k}x \to b^{\Theta(S)}$ for all $x \in \sigma$, so that a $x \in \sigma$, $x \neq b^{\Theta(S)}$ is not a fixed point. Since $C^{+}_{\Theta(S)} \subset \sigma$ the result follows. ∎

We note that the same result holds with S^{-1} in place of S, taking care to consider flag manifold $\mathbb{B}_{\Theta^{*}(S)}$ dual to $\mathbb{B}_{\Theta(S)}$ in the sense of [8].

Remark: It becomes clear from the proof above that $g^{n}x \to \mathrm{fix}_{\Theta(S)}(g)$ for every x in the open cell σ determined by g (or A^{+}). This open cell contains the invariant control set $C_{\Theta(S)}$ so that $g^{n}x \to \mathrm{fix}_{\Theta(S)}(g)$ for every $g \in S$ and $x \in C_{\Theta(S)}$. ∎

Remark: With some extra effort one can use the previous results (specially Corollary 5.7) to prove that the the map $g \in S \mapsto \mathrm{fix}_{\Theta(S)}(g) \in \mathbb{B}_{\Theta(S)}$ is continous in S. We do not give the details here since continuity of $\mathrm{fix}_{\Theta}(\cdot)$ is not needed in the sequel. ∎

6 Invariance of connected components

This section starts the study of the conected components. We prove here that a component of S leaves invariant a unique component of the attractor set C^{+}. Recall that C^{+} is given by $\pi^{-1}_{\Theta(S)}\left(C^{+}_{\Theta(S)}\right)$ where $\pi_{\Theta(S)} : \mathbb{B} \to \mathbb{B}_{\Theta(S)}$ is the canonical projection. Hence, the connected components of C^{+} have the form $\pi^{-1}(K)$ where K is a connected component of $C^{+}_{\Theta(S)}$. Analogously, the connected components of C^{-} have the form $\pi^{-1}_{\Theta(S^{-1})}\left(C^{+}_{\Theta(S^{-1})}\right)$, with obvious notation.

Lemma 6.1 *Let $\Upsilon \subset S$ be a connected component, and take $g \in \Upsilon$. Then there exists a semi-simple element $\widetilde{g} \in \Upsilon$ with $\mathrm{fix}_{\Theta(S)}(\widetilde{g}) = \mathrm{fix}_{\Theta(S)}(g)$.*

Proof: We keep in mind the notations and results of the previous section. With the given choice of A^{+} we have that the isotropy at $\mathrm{fix}_{\Theta(S)}(g)$ is $P_{\Theta(S)}$ and $g \in L_{\Theta(S)}$. Now, the set of semi-simple elements in the reductive group $L_{\Theta(S)}$ is dense. Since $\Upsilon \cap L_{\Theta(S)} \neq \emptyset$ is open in $L_{\Theta(S)}$, there exists a semi-simple $\widetilde{g} \in \Upsilon \cap L_{\Theta(S)}$. Clearly $\mathrm{fix}_{\Theta(S)}(g)$ is fixed under $\widetilde{g}$, hence by Theorem 5.9, $\mathrm{fix}_{\Theta(S)}(g)$ is the unique $\widetilde{g}$-fixed point in $C^{+}_{\Theta(S)}$, that is $\mathrm{fix}_{\Theta(S)}(\widetilde{g}) = \mathrm{fix}_{\Theta(S)}(g)$. ∎

Notice that a connected component $\Upsilon \subset S$ maps components of C^{+} into components, because the evaluation map $g \mapsto gx$ is continuous for any x and

C^+ is invariant. Furthermore, if $g\kappa_1 \subset \kappa_2$ for some $g \in \Upsilon$ and components κ_1 and κ_2, then $\Upsilon\kappa_1 \subset \kappa_2$. Analogous remarks, hold for Υ^{-1} and components of the repeller set C^-.

Our objective now is to prove that Υ leaves invariant exactly one connected component of C^+. For this we recall the well known construction of the flag manifold as an adjoint orbit. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition and fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Given $H_0 \in \mathrm{cl}\mathfrak{a}^+$ its adjoint orbit under $K = \exp \mathfrak{k}$ identifies with the flag manifold $\mathbb{B}_{\Theta(H_0)} = G/P_{\Theta(H_0)}$ where $\Theta(H_0)$ is the set of simple roots annihilating H_0. This embedding permits to define for $H \in \mathfrak{s}$ the height function $f_H(\cdot) = \langle H, \cdot \rangle$. Now, there exists in $\mathbb{B}_{\Theta(H_0)}$ a K-invariant Riemannian metric, say $(\cdot,\cdot)_{H_0}$, such that for any $H \in \mathfrak{s}$ the gradient of f_H with respect to $(\cdot,\cdot)_{H_0}$ is precisely the vector field $\widetilde{H}$ on $\mathbb{B}_{\Theta(H_0)}$ induced by H (see Borel [1] and Duistermaat, Kolk, Varadarajan [3]).

Proposition 6.2 *Let $\Upsilon \subset S$ be a connected component. Then there exists a unique connected component κ_1 of C^+ which is invariant under Υ. Also, there exists a unique component κ_2 of C^- invariant under Υ^{-1}.*

Proof: It is enough to prove the result for Υ and C^+. By Theorem 5.9 a $g \in S$ has a unique fixed point in $C^+_{\Theta(S)}$. Take $g \in \Upsilon$, let x be its fixed point in $C^+_{\Theta(S)}$, and denote by $\widetilde{\kappa}_1$ the connected component of $C^+_{\Theta(S)}$ containing x. Then $g\widetilde{\kappa}_1 \subset \widetilde{\kappa}_1$ and thus $\Upsilon\widetilde{\kappa}_1 \subset \widetilde{\kappa}_1$. It follows that $\Upsilon\kappa_1 \subset \kappa_1$ if $\kappa_1 = \pi^{-1}\widetilde{\kappa}_1$.

For the uniqueness it is enough to show that $\widetilde{\kappa}_1$ is the only Υ-invariant component of $C^+_{\Theta(S)}$. Take g as above with $x = \mathrm{fix}_{\Theta(S)}(g) \in \widetilde{\kappa}_1$. By Lemma 6.1 we can assume that g is semi-simple, and write $g = uh = hu$, with u eliptic and $h = \exp H$ hyperbolic, that is, there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, such that $u \in \exp \mathfrak{k}$ and $h \in \mathrm{cl}\exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ is a Weyl chamber in $\mathfrak{s}$. Thus the vector field $\widetilde{H}$ induced by H on $\mathbb{B}_{\Theta(S)}$ is the gradient of the height function f_H, with respect to a Borel metric on $\mathbb{B}_{\Theta(S)}$, so that $f_H(hz) \geq f_H(z)$ for any $z \in \mathbb{B}_{\Theta(S)}$ with strict inequality if z is not a singularity of $\widetilde{H}$. Note that f_H attains its maximum at x (because x is an attractor of h), and f_H is constant along the orbits of u in $\mathbb{B}_{\Theta(S)}$ (because $\mathrm{Ad}(u)H = H$). Also, if $\sigma \subset \mathbb{B}_{\Theta(S)}$ stands for the open cell corresponding to $\mathfrak{a}^+$ then $\widetilde{H}$ has no singularity in σ (this is due again to the fact that x is an isolated fixed point of h so that the roots outside $\langle \Theta(S) \rangle$ do not vanish on H).

Now, let $\widetilde{\kappa} \neq \widetilde{\kappa}_1$ be a connected component of $C^+_{\Theta(S)}$. Since some power of h belongs to S (see Lemma 5.2) $C^+_{\Theta(S)}$ (and hence $\widetilde{\kappa}$) is contained in $\mathrm{st}_{\Theta(S)}(h)$. Hence, $f_H(gz) = f_H(hz) > f_H(z)$ for all $z \in \mathrm{cl}\widetilde{\kappa}$. Thus if the maximum of f_H on $\mathrm{cl}\widetilde{\kappa}$ is attained in $z \in \mathrm{cl}\widetilde{\kappa}$ then $hz \notin \mathrm{cl}\widetilde{\kappa}$, showing that $\mathrm{cl}\widetilde{\kappa}$ is not invariant under h, so that $\widetilde{\kappa}$ cannot be invariant under Υ.

So far we have proved existence and uniqueness of invariant component of $C^+_{\Theta(S)}$. However, the components of C^+ have the form $\kappa = \pi^{-1}(\widetilde{\kappa})$, where $\pi : \mathbb{B} \to \mathbb{B}_{\Theta(S)}$ is the canonical projection. Thus the result follows for components of C^+. ∎

Since the control sets in $\mathbb{B}_\Theta$ are obtained by projeting control sets in $\mathbb{B}$, this result implies immediately existence and uniqueness of invariant components in arbitrary flag manifolds.

Corollary 6.3 *A connected component $\Upsilon \subset S$ leaves invariant a unique component of C^+_Θ, while there exists just one component of C^-_Θ invariant under Υ^{-1}.*

Notation: We denote by $K^+(\Upsilon)$ the component of C^+ invariant under Υ and by $K^-(\Upsilon)$ the Υ^{-1}-invariant component of C^-. Analogously $K^\pm_\Theta(\Upsilon)$ are the invariant components of $C^\pm_\Theta$. ∎

Specializing these facts to the parabolic type $\mathbb{B}_{\Theta(S)}$ we have.

Corollary 6.4 *For a connected component $\Upsilon \subset S$ the set $\{\mathrm{fix}_{\Theta(S)}(g) : g \in \Upsilon\}$ is contained in $K^+_{\Theta(S)}(\Upsilon)$.*

Proof: A fixed point $\mathrm{fix}_{\Theta(S)}(g)$, $g \in \Upsilon$, belongs to a Υ-invariant component. ∎

Still another consequence of the existence and uniqueness of invariant components obtains:

Corollary 6.5 *Given a semigroup component $\Upsilon \subset S$ and an integer $k > 0$, $K^\pm_\Theta(\Upsilon^k) = K^\pm_\Theta(\Upsilon)$.*

Proof: First consider the K^+ case. If $\Theta = \Theta(S)$ the result follows from the previous corollary because $\mathrm{fix}_{\Theta(S)}(g^k) = \mathrm{fix}_{\Theta(S)}(g)$. This implies the result in the maximal flag manifold $\mathbb{B}$ because the components of C^+ have the form $\pi^{-1}(\kappa)$, with $\kappa \subset C^{+}_{\Theta(S)}$ a connected component. From $\mathbb{B}$ the result is carried out to $\mathbb{B}_{\Theta}$ by projecting components. Finally, the K^{-} case is obtained by taking S^{-1}. ∎

For a semigroup component Γ the invariant component $K^{+}(\Gamma)$ actually satisfies $\Gamma C^{+} \subset K^{+}(\Gamma)$. In fact, the attractor $\mathrm{att}(h)$ of any split-regular $h \in \Gamma$ belongs to $K^{+}(\Gamma)$. Therefore, for any connected component $K \subset C^{+}$, there exists a large enough integer $k > 0$ such that $h^{k}K \cap K^{+}(\Gamma) \neq \emptyset$, implying that $h^{k}K \subset K^{+}(\Gamma)$, and thus $\Gamma K \subset K^{+}(\Gamma)$. Analogously, $\Gamma^{-1}C^{-} \subset K^{-}(\Gamma)$. The same reasoning yields that the attractor and repeller sets of Γ satisfy $C^{\pm}(\Gamma) \subset K^{\pm}(\Gamma)$.

For other components we have the following useful property of the sets $K^{\pm}(\cdot)$.

Lemma 6.6 *For $g \in S$ and a semigroup component Γ, denote by $\Gamma(g)$ the component containing Γg and by $(g)\Gamma$ the component containing $g\Gamma$. Then $K^{+}(\Gamma(g)) = K^{+}(\Gamma)$ and $K^{-}((g)\Gamma) = K^{-}(\Gamma)$.*

Proof: For the first equality note that $gK^{+}(\Gamma) \subset C^{+}$ so that $\Gamma g(K^{+}(\Gamma)) \subset \Gamma C^{+} \subset K^{+}(\Gamma)$. The second equality follows analogously. ∎

Remark: The proof of uniqueness of the invariant component could be done in a different route, exploiting the continuity of the map $g \in S \mapsto \mathrm{fix}_{\Theta(S)}(g) \in C_{\Theta(S)}$ (see the remark at the end of Section 5). In fact, since the map is continuous the set of fixed points $\mathrm{fix}_{\Theta(S)}(g)$ with g running through Υ is contained in a connected component of $C^{+}_{\Theta(S)}$, which must be the only Υ-invariant component. ∎

7 Semigroup components

This section is devoted to the proof of the following characterization of the semigroup components of S. Denote by $\#(C^{\pm})$ the number (possibly infinite) of connected components of $C^{\pm}$ and by $\#_{s}(S)$ the number of semigroup components of S.

Theorem 7.1 *Given a pair of connected components K_1 of C^{+} and K_2 of C^{-} there exists a unique semigroup component Γ such that $C^{+}(\Gamma) \subset K_1$*

and $C^-(\Gamma) \subset K_2$*. Furthermore,* Γ *satisfies* $C^+(\Gamma) = K^+(\Gamma) = K_1$ *and* $C^-(\Gamma) = K^-(\Gamma) = K_2$*. Therefore,* $\#_s(S) = \#(C^+) \cdot \#(C^-)$*.*

Notation: Given components $K_1 \subset C^+$ and $K_2 \subset C^-$ we denote by $\Gamma(K_1, K_2)$ the unique semigroup component whose attractor and repeller sets are K_1 and K_2, respectively. ∎

Corollary 7.2 *Given an arbitrary connected component* $\Upsilon \subset S$ *the set* $\bigcup_{k\geq 1} \Upsilon^k$ *meets a unique semigroup component. We denote this semigroup component by* $\Gamma(\Upsilon)$*.*

Proof: Existence follows immediately from Lemma 5.2 (and its proof). On the other hand, by Corollary 6.4 $K^\pm(\Upsilon^k) = K^\pm(\Upsilon)$ so that if Υ^k is contained in a semigroup component Γ then $C^\pm(\Gamma) = K^\pm(\Upsilon)$. Hence uniqueness follows from the theorem. ∎

We separate the proof of Theorem 7.1 in two steps corresponding to existence and uniqueness of Γ.

7.1 Existence

Given the connected components $K_1 \subset C^+$ and $K_2 \subset C^-$ in order to prove the existence of a semigroup component Γ with $C^+(\Gamma) \subset K_1$ and $C^-(\Gamma) \subset K_2$ it is enough exhibit a split-regular $h \in S$ whose attractor belongs to K_1 and repeller to K_2. In fact, some positive power h^k belongs to a semigroup component satisfying the required conditions.

To start with let us take split-regular elements $h_1, h_2 \in S$ such that $\mathrm{att}(h_1) \in K_1$ and $\mathrm{rpp}(h_2) \in K_2$. Denote by Γ_1 and Γ_2 the exit components of h_1 and h_2, respectively. Then $C^+(\Gamma_1) \subset K_1$ whereas $C^-(\Gamma_2) \subset K_2$. Now, the basic idea is provided by the following lemma combined with Proposition 6.2 (and its corollary).

Lemma 7.3 *Let* Γ_1 *and* Γ_2 *be semigroup components with* $C^+(\Gamma_1) \subset K_1$ *and* $C^-(\Gamma_2) \subset K_2$*. Then* K_1 *is invariant under* $\Gamma_1\Gamma_2$ *while* K_2 *invariant under* $(\Gamma_1\Gamma_2)^{-1} = \Gamma_2^{-1}\Gamma_1^{-1}$*.*

Proof: For the invariance of K_1 it is enough to show that there exists $a \in \Gamma_1\Gamma_2$ with $aK_1 \cap K_1 \neq \emptyset$. For this take $g \in \Gamma_2$ and a split-regular

$h \in \Gamma_1$. The attractor $\text{att}(h) \in K_1$, so that $h^n x \in K_1$ for x in a dense subset and n large enough. Since gK_1 is open, there exists $z \in K_1$, and an integer $n > 0$ such that $h^n gz \in K_1$. Hence $a = h^n g$ is the required element in $\Gamma_1\Gamma_2$. The proof for K_2 is analogous. ∎

Now the existence proof can be performed. Put $\Upsilon = \Gamma_1\Gamma_2$. By the previous lemma $K^+(\Upsilon) = K_1$ and $K^+(\Upsilon) = K_2$. Since our group G has finite index there is an integer $k > 0$ such that Υ^k is contained in a semigroup component, say Γ. However, we have by Corollary 6.5, that $K^\pm(\Gamma) = K^\pm(\Upsilon)$. Therefore, Γ is the required semigroup component, concluding the proof.

7.2 Uniqueness

For the proof of uniqueness of semigroup component we fix advance connected components $K_1 \subset C^+$ and $K_2 \subset C^-$, and assume that there exists a semigroup component, say Γ_0, with $C^+(\Gamma_0) \subset K_1$ and $C^-(\Gamma_0) \subset K_2$.

Lemma 7.4 $K_1 = C^+(\Gamma)$ *with* Γ *running through the semigroup components such that* $C^+(\Gamma) \subset K_1$. *An analogous result holds for* K_2 *and* $C^-(\Gamma)$.

Proof: Given $x \in K_1$ there exists a split-regular $h \in S$ with $x = \text{att}(h)$. Denote by $E(h)$ the exit component of h, so that for some $k > 0$, $h^k \in E(h)$. Then the attractor set $C^+(E(h)) \subset K_1$, since $h^k x = x$ and hence K_1 is the unique component left invariant by $E(h)$. But $x \in C^+(E(h)) \subset K_1$, proving the statement for K_1. The proof for K_2 is analogous. ∎

This lemma shows that in the eventuality that $C^+(\Gamma_0)$ differs from K_1 there must exist another semigroup component Γ_1 such that $C^+(\Gamma_0) \cap C^+(\Gamma_1) \neq \emptyset$. In the next two lemmas we look at this possibility.

Lemma 7.5 *The semigroup components* Γ_1 *and* Γ_2 *satisfy* $\Gamma_1\Gamma_2 \subset \Gamma_2$ *if* $C^+(\Gamma_1) \cap C^+(\Gamma_2) \neq \emptyset$. *Analogously,* $\Gamma_1\Gamma_2 \subset \Gamma_1$ *if* $C^-(\Gamma_1) \cap C^-(\Gamma_2) \neq \emptyset$.

Proof: Take $x \in C^+(\Gamma_1) \cap C^+(\Gamma_2)$ and denote by P the istoropy at x. By assumption the semigroups $T_1 = \Gamma_1 \cap P_0$ and $T_2 = \Gamma_2 \cap P_0$ are open in P_0 and non-empty. By Lemma 3.4, T_2 is left reversible in P_0. Thus for every $x \in T_1$ there exists $y \in T_2$ such that $xy \in T_2$. It follows that $\Gamma_1\Gamma_2 \subset \Gamma_2$, as claimed.

Now applying this fact to the inverse semigroups, we get $\Gamma_2^{-1}\Gamma_1^{-1} \subset \Gamma_1^{-1}$, which is equivalent to the last statement. ∎

Lemma 7.6 *Let Γ_1 and Γ_2 be semigroup components, and suppose that $C^+(\Gamma_1) \cap C^+(\Gamma_2) \neq \emptyset$. Then $C^+(\Gamma_1) = C^+(\Gamma_2)$.*

Proof: Given $x \in C^+(\Gamma_1)$ there exists a split regular $h \in \Gamma_1$ such that $x = \mathrm{att}(h)$. Thus $h^n y \to x$ for y in a dense subset. In particular, for all $g \in \Gamma_2$, $gC^+(\Gamma_2)$ is open, so that there exists $y \in C^+(\Gamma_2)$ such that $h^n gy \to x$, $n \to +\infty$. By the previous lemma $h^n g \in \Gamma_2$, for all n. Hence $h^n gy \in C^+(\Gamma_2)$, and since the invariant control sets are closed, it follows that $x \in C(\Gamma_2)$.

It remains to check that x is actually in $C^+(\Gamma_2)$. Take $z \in C(\Gamma_1)_0 \cap C(\Gamma_2)_0$. Then there are $a \in \Gamma_2$ and $b \in \Gamma_1$ such that $ax = z$ (because $x \in C(\Gamma_2)$ and $z \in C^+(\Gamma_2)$) and $bz = x$ (because $x, y \in C^+(\Gamma_1)$). Therefore, $bax = x$. But, $ba \in \Gamma_2$, hence $x \in C^+(\Gamma_2)$, as desired. Thus $C^+(\Gamma_1) \subset C^+(\Gamma_2)$. The reverse inclusion follows by symmetry. ∎

It is now an easy consequence of the previous lemmas that the attractor set of a semigroup component is a connected component of the attractor set of S.

Proposition 7.7 *Let Γ a semigroup component, and denote by K the connected component of C^+ which contains $C^+(\Gamma)$. Then $C^+(\Gamma) = K$.*

Proof: By Lemma 7.4, K is a union of attractor sets of semigroup components. However Lemma 7.6 shows that two of these attractor sets are equal or disjoint. Since these sets are open and K is connected, the result follows. ∎

Applying this proposition to the semigroup S^{-1}, we get the corresponding result for the repeller sets.

Proposition 7.8 *Let Γ a semigroup component, and denote by K the connected component of C^- which contains $C^-(\Gamma)$. Then $C^-(\Gamma) = K$.*

Now let Γ_0 and Γ_1 be semigroup components whose attractor and repeller sets are contained in the same component of C^+ and C^-, respectively. By Propositions 7.7 and 7.8, $C^\pm(\Gamma_0) = C^\pm(\Gamma_1)$. On the other hand, Lemma

7.5 ensures that $\Gamma_0\Gamma_1 \subset \Gamma_0$ and $\Gamma_0\Gamma_1 \subset \Gamma_1$. Hence $\Gamma_0 = \Gamma_1$. Concluding the uniqueness part of the proof of Theorem 7.1.

We conclude this section stating the following consequence of our proof, which might be interesting in itself.

Proposition 7.9 *Let $\Gamma \subset S$ be a semigroup component. Then Γ has the same parabolic type as S.*

Proof: By Proposition 7.7 the Γ-i.c.s. on $\mathbb{B}$ is the closure of a connected component $K \subset C^+$. Such a component has the form $\pi^{-1}_{\Theta(S)}(K_1)$, with K_1 a component of $C^+_{\Theta(S)}$. This shows that the parabolic type $\Theta(\Gamma)$ of Γ contains $\Theta(S)$. Since $\Gamma \subset S$ they must be equal. ∎

Remark: These results about semigroup components show that in general semigroups in connected solvable groups can have more than one semigroup component. To see this take $S \subset G$ an open semigroup in such a way that C^- contains two different connected components, say K_1 and K_2. Fixing $x \in C^+$ let P be the isotropy at x and K the component of C^+ containing x. Then $S \cap P_0$ has at least two semigroup components, namely $\Gamma(K, K_1) \cap P_0$ and $\Gamma(K, K_2) \cap P_0$. ∎

8 Recurrent components

Definition 8.1 *A connected component $\Upsilon \subset S$ is said to be* recurrent *(or to have* finite index*) if $\Upsilon^k \cap \Upsilon \neq \emptyset$ for some integer $k > 1$. The smallest $k \geq 2$ satisfying this condition is the* index *of Υ. Otherwise the component is* transient *or has* infinite index.

Alternatively, Υ has finite index if and only if $\Upsilon^k \subset \Upsilon$ for some integer $k > 1$, which means that Υ has finite index in the semigroup of components. Of course, a component is a semigroup if and only if it has index 2. Furthermore, if k is the index of Υ then $\Upsilon^n \subset \Upsilon$ for any multiple $n = lk$, $l \geq 1$.

In this section we present a description of the finite index components in a manner similar to the semigroup components. Now we look at the control sets on G/P_0 instead of $\mathbb{B}$. First we prove an useful property of recurrent components, which holds in general.

Lemma 8.2 *Given a recurrent component Υ, suppose there exists a semigroup component Γ containing a power Υ^s. Then $\Gamma\Upsilon \subset \Upsilon$ and $\Upsilon\Gamma \subset \Upsilon$.*

Proof: Let $n > s$ be such that $\Upsilon^n \subset \Upsilon$. Then any $g \in \Upsilon^{n-s}$ satisfies $g\Gamma, \Gamma g \subset \Upsilon$. We can change g by gh, $h \in \Gamma$, without affecting the inclusion $g\Gamma \subset \Upsilon$. But, $gh \in \Upsilon$, so that $\Upsilon\Gamma \subset \Upsilon$. Analogously, $hg \in \Upsilon$ so that $\Gamma\Upsilon \subset \Upsilon$. ∎

In semi-simple groups the following kind of converse to this lemma also holds.

Proposition 8.3 *Let the notations be as in the prefious section and take a component $\Upsilon \subset S$. Suppose that t Γ is a recurrent component such that $K^{\pm}(\Upsilon) = K^{\pm}(\Gamma)$ and Υ contains $g\Gamma$ or Γg for some $g \in S$. Then Υ is recurrent.*

Proof: Assume first that Γ is a semigroup component and $g\Gamma \subset \Upsilon$. Denote by Υ_g the component of S containing g, so that $\Upsilon_g\Gamma \subset \Upsilon$. Since Υ_g is open it contains semi-simple elements, so that we can assume without loss of generality that g is semi-simple. Write $g = mh = hm$, with h split and m compact. Using again that Υ_g is open we can assume that m has finite index, say k. Since $h = \exp H$, $H \in \mathfrak{g}$, there exists a power h^s which is contained in a semigroup component. This component must be Γ, because $\mathrm{fix}(g)$ and $\mathrm{fix}\left(g^{-1}\right)$ are fixed points of h. Hence, for large $s = lk+1$, $h^s \in \Gamma$ and $mh^s \in \Upsilon$. Therefore, $(mh^s)^{k+1} = mh^{ks} \in \Upsilon$, showing that $\Upsilon^{k+1} \subset \Upsilon$, that is, Υ has finite index. The proof in case $\Gamma g \subset \Upsilon$ is the same.

Now, if Γ is recurrent then $K^{\pm}(\Gamma) = K^{\pm}(\Gamma_1)$, for a semigroup component Γ_1, so that by the previous lemma there exists $g_1 \in S$ with $g_1\Gamma_1 \subset \Gamma$. Hence, $gg_1\Gamma_1 \subset \Upsilon$, and the result follows by the first part of the proof. ∎

In order to proceed we recall that by Corollary 7.2, there exists a unique semigroup component $\Gamma(\Upsilon)$ which contains some power of Υ. It is defined by the conditions $K^{\pm}(\Gamma(\Upsilon)) = K^{\pm}(\Upsilon)$. Hence Lemma 8.2 applies to our context. Consider the projection $\pi : G/P_0 \to \mathbb{B}$. The pre-images $\pi^{-1}(K^{\pm}(\Upsilon))$ are also invariant under Υ, so the components of $\pi^{-1}(K^{+}(\Upsilon))$ are mapped into each other by Υ. To see the behavior of these mappings we look first at the semigroup components.

Lemma 8.4 *Let* Γ *be a semigroup component, and* κ *a connected component of* $\pi^{-1}(K^{+}(\Gamma))$*. Then* $\Gamma y = \kappa$ *for every* $y \in \kappa$*. Analogously, the components of* $\pi^{-1}(K^{-}(\Gamma))$ *are orbits of* Γ^{-1}*.*

Proof: Since Γ is a semigroup it contains a split-regular element h. Denote by A^{+} the chamber containing h, and let P be the corresponding parabolic subgroup. If P is the isotropy subgroup at $x \in \mathbb{B}$, then $x \in K^{+}(\Gamma)$, and P_0 is the isotropy subgroup of any y in the fiber $\pi^{-1}\{x\}$. Thus $h \in \Gamma$ fixes every point over x, showing that $\Gamma y \subset \kappa$ for every $y \in \kappa$. This inclusion is an equality because Γ acts transitively on $K^{+}(\Gamma)$. ∎

Combining this lemma with Lemma 8.2 we see that the recurrent components map components in G/P_0 onto components.

Proposition 8.5 *Let* Υ *be a recurrent component and* κ *a connected component of* $\pi^{-1}(K^{+}(\Upsilon))$*. Then* Υy *is a connected component of* $\pi^{-1}(K^{+}(\Upsilon))$ *for every* $y \in \kappa$*. Analogous result holds for* $\pi^{-1}(K^{-}(\Upsilon))$ *and* Υ^{-1}*.*

Proof: By Lemma 8.2, there exists $g \in S$ with $\Gamma(\Upsilon) g \subset \Upsilon$. Then, for every $y \in \kappa$, $\Gamma(\Upsilon) gy \subset \Upsilon y$, so that Υy equals the component containing gy. ∎

Now we can prove existence and uniqueness of a recurrent component mapping a connected component of a control set in G/P_0 onto another component.

Lemma 8.6 *Let* Γ *be a semigroup component, and* κ_1*,* κ_2 *connected components of* $\pi^{-1}(K^{+}(\Gamma))$*. Then there exists at most one finite index component* Υ *with* $\Gamma(\Upsilon) = \Gamma$ *and* $\Upsilon\kappa_1 = \kappa_2$*.*

Proof: Let Υ and Υ_1 be recurrent components with $\Gamma(\Upsilon) = \Gamma(\Upsilon_1)$, and suppose that Υ and Υ_1 map κ_1 into κ_2. Take $g \in \Upsilon$ and $y \in \kappa_1$. By the previous corollary there exists $g_1 \in \Upsilon_1$ such that $gy = g_1 y$. Then $g^{-1}g_1 \in P_0$, the isotropy at y. Now, $\Gamma \cap P_0$ is left reversible in P_0, so that $g^{-1}g_1\Gamma \cap \Gamma \neq \emptyset$, that is, $g\Gamma \cap g_1\Gamma \neq \emptyset$. However, by Lemma 8.2, $g\Gamma \subset \Upsilon$ and $g_1\Gamma \subset \Upsilon_1$. Hence, $\Upsilon \cap \Upsilon_1 \neq \emptyset$, showing the uniqueness of the component. ∎

Lemma 8.7 *Let* Γ *be a semigroup component, and* κ_1*,* κ_2 *connected components of* $\pi^{-1}(K^{+}(\Gamma))$*. Suppose that* κ_1 *and* κ_2 *are contained in the same* *invariant control set in G/P_0. Then there exists a recurrent component Υ such that $\Upsilon\kappa_1 = \kappa_2$.*

Proof: By assumption there exists $g \in S$ with $g\kappa_1 \subset \kappa_2$. Denote by Υ the component of S containing $g\Gamma$. We claim that $K^{\pm}(\Upsilon) = K^{\pm}(\Gamma)$. In fact, the inclusion $g\kappa_1 \subset \kappa_2$ implies that $gK^{+}(\Gamma) \subset K^{+}(\Gamma)$, so that $g\Gamma$ leaves invariant $K^{+}(\Gamma)$, hence $K^{+}(\Upsilon) = K^{+}(\Gamma)$. On the other hand $K^{-}(\Upsilon) = K^{-}(\Gamma)$ by Lemma 6.6. Therefore, by Proposition 8.3, Υ is recurrent. Now, Lemma 8.4 ensures that Υ maps κ_1 into κ_2. ∎

Finally we prove that the product of recurrent components is recurrent, so that this set of components is a subsemigroup.

Proposition 8.8 *The set of recurrent components is a subsemigroup of the semigroup of components of S, that is, the union of the recurrent components is a subsemigroup of S.*

Proof: First we check that the product $\Upsilon = \Gamma(K_1, K_2)\Gamma(L_1, L_2)$ of semigroup components is recurrent (cf. the notation following Theorem 7.1). Note that by Lemma 7.3, $K_1 = K^{+}(\Upsilon)$ while $L_2 = K^{-}(\Upsilon)$. Now, by Lemma 7.5, $\Gamma(K_1, L_2)\Gamma(K_1, K_2) \subset \Gamma(K_1, K_2)$, so that $\Gamma(K_1, L_2)\Upsilon \subset \Upsilon$. Hence, by Proposition 8.3, Υ is recurrent.

Now if Υ is recurrent and Γ is a semigroup component, then Υ contains $g\Gamma_1$ with Γ_1 the semigroup component such that $K^{\pm}(\Upsilon_1) = K^{\pm}(\Gamma_1)$, so $\Upsilon\Gamma$ contains $g\Gamma_1\Gamma$. By the first part of the proof $\Gamma_1\Gamma$ is recurrent, hence by Proposition 8.3, $\Upsilon\Gamma$ is recurrent, because $K^{\pm}(\Upsilon\Gamma) = K^{\pm}(\Gamma_1\Gamma)$. The same way one proves that the product of arbitrary recurrent components is recurrent. ∎

9 Mid-reversibility

A subsemigroup $S \subset G$ is said to be *mid-reversible* if $SgS \cap S \neq \emptyset$, for all $g \in G$, or $G = SS^{-1}S$ or $G = S^{-1}SS^{-1}$. Accordingly the *mid-reversor* of S is defined to be the set

$$\mathrm{Mid}(S) = \{g \in G : SgS \cap S \neq \emptyset\}$$

and S is mid-reversible if $\mathrm{Mid}(S) = G$. These concepts were introduced in [6] where they are related to the connected components of S when S is open. In this section we pursue further on these relations for semigroups in semi-simple Lie groups.

Let us take in advance a minimal parabolic subgroup P in such a say that the corresponding origin $b_0 \in \mathbb{B} = G/P$ belongs to $C^+(S)$. As we shall see mid-reversibility and questions involving the way S generates G are related to the intersection of S with the connected components of P.

Lemma 9.1 $G = S^{-1}P$.

Proof: It is known that for every $x \in \mathbb{B}$, $C(S) \subset Sx$ (this is a consequence of the uniqueness of the S-i.c.s., see e.g. [7]). Hence $\mathbb{B} = S^{-1}b_0$, and since P is the isotropy subgroup at B_0 the lemma follows. ∎

This simple fact yields the following sufficient condition for S to be mid-reversible.

Proposition 9.2 *Suppose that $S \cap P$ meets every connected component of P. Then S is mid-reversible.*

Proof: We show that $P \subset SS^{-1}$. Take $g \in P$. By assumption there exists $t \in S \cap P$ in the connected component of g. Then $t^{-1}g \in P_0$. By Lemma 3.4, $S \cap P_0$ is left reversible, so that there exists $s \in S$ with $t^{-1}gs \in S \cap P_0$. Hence $g \in SS^{-1}$. Using the previous lemma we get $G = S^{-1}SS^{-1}$, concluding the proof. ∎

Corollary 9.3 *Every open semigroup in G is mid-reversible if the minimal parabolic subgroups are connected.*

Examples of G with connected P (or equivalently, connected M) are the groups with real rank one with $\dim \mathbb{B} > 1$ and the complex groups. For these groups a result of Ruppert (see [6], Theorem 3.9) shows that subsemigroups containing 1 in the closure are connected. Of course this phenomena does not occur in general. For example the interior of the semigroup $\pm \mathrm{Sl}(2, \mathbb{R})^+$ contains the identity in its closure but is not connected. However, we shall prove below that in general an open semigroup has just one semigroup component if the identity is a cluster point.

Remark: The criterion of Theorem 9.4 works also for semigroups with nonempty interior, since in case $\mathrm{int}T \neq \emptyset$ it is easy to prove that T is mid-reversible if and only if $\mathrm{int}T$ is mid-reversible. ∎

We observe that Proposition 9.2 (and hence its corollary) does not require that the group G has finite center. So the condition that $S \cap P$ meets every component of P is sufficient for mid-reversibility even in groups with infinite center. We prove next the converse to Proposition 9.2, using explicitly that G has finite center. Note that by the discussion in Section 4 uniqueness of invariant control set in G/P_0 holds if and only if $S \cap P$ meets every component of P if P is the isotropy subgroup at $x \in C^+$.

Theorem 9.4 *An open semigroup $S \subset G$ is mid-reversible if and only if S has exactly one invariant control set on G/P_0.*

Proof: If there is uniqueness of invariant control set then S meets every component of P so it is mid-reversible by Proposition 9.2.

For the converse we assume that some component of P, say mP_0, does not meet S and prove that $SmS \cap S = \emptyset$. In fact, suppose to the contrary that $tms \in S$ with $t, s \in S$. Arguing as in the first part of the proof we see that there exists $t_1 \in S$ such that $p = t_1 t \in P$. We can write $p = m_1 p_1$, with $p_1 \in P_0$, $m_1 \in M(S, P)$. Then $pm = m_1 m q$, where $q = m^{-1} p_1 m \in P_0$. Note that $m_1 m \notin M(S, P)$, since $m \notin M(S, P)$.

Now, recall the notation rounding Proposition 4.3, and identify $\pi^{-1}(C)$, $\pi : G/P_0 \to \mathbb{B}$, with $C_{\Theta(S)} \times (P_\Theta/P) \times (M(S, D)/M(\Theta))$. By Proposition 4.3 the set $D = C_{\Theta(S)} \times (P_\Theta/P) \times M_D$ is an invariant control set for S. Since $q \in P_0$ any slice $C_{\Theta(S)} \times \{a\} \times \{b\}$, $a \in \left(P_{\Theta(S)}\right)_0 / P_0$, $b \in \left(P_{\Theta(S)} / \left(P_{\Theta(S)}\right)_0\right)$ is invariant under q. Also, $s \in S$, so that $sD \subset D$. Hence, $qsD \subset D$. On the other hand, $m_1 m D \cap D = \emptyset$, because $m_1 m \notin M(S, P)$. Hence we arrive at the contradiction that $(tms) D \cap D = \emptyset$ with $tms \in S$ and D an invariant control set. ∎

As an application of these results we can prove that compression semigroups are always mid-reversible.

Proposition 9.5 *Let $T \subset G$ be a semigroup with non-empty interior, and denote by C its invariant control set on $\mathbb{B}$. Suppose that*

$$T = \mathrm{comp} C = \{g \in G : gC \subset C\}.$$

Then T has a unique i.c.s on G/P_0, namely $\pi^{-1}(C)$, where $\pi : G/P_0 \to \mathbb{B}$ is the standard projection. Therefore, T is mid-reversible.

Proof: Fix $b \in C^{+}$ and take a split-regular $h \in \mathrm{int}T$ such that $b = \mathrm{att}\,(h)$. Suppose that $h \in A^{+}$, corresponding to the decomposition $P = MAN$, where P is the isotropy at b. It is enough to show that for any $m \in M$ there exists an integer $k > 0$ with $mh^{k} \in T$.

Let $\Theta\,(T)$ be the parabolic type of T and put $C_{\Theta(T)}$ for the T-i.c.s on $\mathbb{B}_{\Theta(T)}$. Since $C = \pi^{-1}_{\Theta(T)}\left(C_{\Theta(T)}\right)$, $g \in T$ if and only if $gC_{\Theta(T)} \subset C_{\Theta(T)}$. Put $b_{\Theta} = \pi_{\Theta(T)}b$. Then $h^{n}C_{\Theta(T)} \to \{b_{\Theta}\}$ as $n \to +\infty$, and b_{Θ} is fixed under M. The latter ensures given $m \in M$ there exists a neighborhood $U \subset C^{+}_{\Theta}$ of b_{Θ} such $m\,(U) \subset C^{+}_{\Theta}$. Now, take k large enough so that $h^{k}C_{\Theta} \subset U$. Then $mh^{k}C_{\Theta} \subset mU \subset C_{\Theta}$, so that $mh^{k} \in \mathrm{int}T$, concluding the proof. ∎

A slight change of the proof of Theorem 9.4 yields the following information about the mid-reversor of S.

Proposition 9.6 *Let A^{+} be a Weyl chamber and denote by P and N^{-} the subgroups determined by A^{+}. Suppose that the attractor set C^{+} of S is contained in the open cell $\sigma_{A^{+}}$ determined by A^{+}. Then, the subset $M\,(S,P)\,N^{-}P_{0}$ is contained in the mid-reversor of the open semigroup S.*

Proof: As in Proposition 4.3, write a control set D on G/P_{0} as

$$D = C_{\Theta(S)} \times \left(P_{\Theta(S)}/P\right) \times M_{D}$$

with $C_{\Theta(S)}$ contained in the open Bruhat cell $\sigma_{\Theta(S)}$ in $\mathbb{B}_{\Theta(S)}$ determined by P. Take $g \in M\,(S,P)\,N^{-}P_{0}$. If y is an element in the fiber over x then gy is contained in $\sigma_{\Theta(S)} \times \left(P^{0}_{\Theta(S)}/P_{0}\right) \times M\,(S,P)$. Now a split-regular $h \in P_{0}$ leaves invariant the slices $\sigma_{\Theta(S)} \times \{a\} \times \{b\}$. In particular, if $h \in S \cap P_{0}$ hgy belongs to the same slice as gy, so that if h is large enough $hgy \in D$. Thus there exists $g_{1} \in S$ such that $g_{1}g \in M\,(S,P)\,P_{0}$. Therefore the same argument as in the first part of the proof above shows that $g \in \mathrm{Mid}\,(S)$, as desired. ∎

We turn now to the relation between mid-reversibiltity and the connected components. First note that the set $N^{-}P_{0}$ is a neighborhood of $1 \in G$, since the product map $N^{-} \times P_{0} \to G$ has full rank at the identity. From this remark and the above proposition we get the following fact which is a slight change ot [6], Theorem 3.9 (i).

Lemma 9.7 *Suppose $1 \in \mathrm{cl}S$, and let A^{+} be a Weyl chamber with $S \cap A^{+} \neq \emptyset$. Take N^{-} and P_{0} subgroups corresponding to A^{+} and put $U =$*

N^-P_0. Then U meets exactly one component Γ of S, and this component is a semigroup containing 1 in its closure.

Proof: By assumption there exists $g \in S \cap U$. Let Γ be the component containing g and select an neighborhood of the identity V such that $VgV \subset \Gamma$. Put S_V for the interior of the semigroup generated by $S \cap V$. Clearly $S_V \subset S$ hence its invariant control set is contained in C. Therefore by Proposition 9.6 U is contained in the mid-reversor of S_V. Now, the proof that Γ is a semigroup and the unique component meeting U follows verbatim the proof of Theorem 3.9 (i) of [6]. We sketch it: The choice of V implies that $S_V\Gamma \subset \Gamma$ and $S_V g S_V \subset \Gamma$. Thus

$$\emptyset \neq S_V g S_V \cap S_V \subset \Gamma \cap S_V.$$

But for $h \in \Gamma \cap S_V$ it holds $h\Gamma \subset S\Gamma \subset \Gamma$, so that Γ is a semigroup. Furthermore, let Υ be a component meeting V. Repeating the above arguments we get $\Upsilon g \Upsilon \cap \Upsilon \neq \emptyset$ and $\Upsilon g \Upsilon \subset \Gamma$. Hence $S \cap V \subset \Gamma$, so that $1 \in \mathrm{cl} S$, and since $g \in U$ was arbitrary, the uniqueness follows, concluding the proof. ∎

Theorem 9.8 *Let S be an open semigrouup in the connected semi-simple Lie group G with finite center. If $1 \in \mathrm{cl}S$ then any component of S has finite index. Also, there exists exactly one semigroup component, which is the only component containing 1 in its closure.*

Proof: For the uniqueness of the semigroup component it is enough to check that the sets $C^{\pm}$ are connected. Let Γ be the unique (semigroup) component containing 1 in its closure, as ensured by the above lemma. If κ is a connected component of C^+ then $\Gamma\kappa \subset \kappa$ because $1 \in \mathrm{cl}S$. Since by Lemma 6.2, Γ leaves invariant just one component of C^+ it follows that this set is connected. Analogously, C^- is connected showing that Γ is the only semigroup component of S. Using again the fact that $1 \in \mathrm{cl}\Gamma$ we see that $\Gamma\Upsilon \subset \Upsilon$ for any connected component Υ. Therefore, Proposition 8.3 ensures that the components are recurrent. ∎

In concluding we shall exploit a step further our method of proving mid-reversibility to get a (rough) estimate of the number of factors S, S^{-1} required to produce G. The idea is that if P is the isotropy subgroup at $b_0 \in C^+$ then $P_0 \subset SS^{-1}$, so that if $x \in G/P_0$ is fixed under P_0 and for a certain product $A = S^{i_1} \cdots S^{i_k}$, $Ax = G/P_0$ then $G = ASS^{-1}$. Now, let

$x_1, \ldots, x_k$ be the fixed point set of P_0 in G/P_0. By making a right choice of N^-, we get open orbits $N^- x_i$, $i = 1, \ldots, k$, whose union contain the invariant control sets in G/P_0. Accordingly, we choose $h \in S \cap P_0$ such that $h^n y \to x_i$ for every $y \in N^- x_i$. Write $C_i = \pi^{-1}(C^+) \cap N^- x_i$ then the sets $h^{-n} C_i$, $n \geq 1$, cover $N x_i$, so that $N^- x_i \subset S^{-1} x_i$.

Lemma 9.9 *If two slices satisfy* $I_{ij} = \mathrm{cl}(N^- x_i) \cap \mathrm{cl}(N^- x_j) \neq \emptyset$ *then* $(C_j)_0 \subset SS^{-1} x_i$.

Proof: The closed set I_{ij} contains h-fixed points, since I_{ij} is h-invariant and a limit $\lim h^n z$ is fixed under h. If $y \in I_{ij}$ is an h-fixed point then there exists an effective control set, say D, with $y \in D_0$, so that $D_0 \cap N^- x_i \neq \emptyset \neq D_0 \cap N^- x_j$. Now, $D_0 \cap N^- x_i \neq \emptyset$ implies that $D_0 \cap S^{-1} x_i \neq \emptyset$, and hence $D_0 \subset S^{-1} x_i$, because S^{-1} is transitive on D_0. On the other hand, $h^n z \to x_j$ for any $z \in N^- x_j$. Therefore, $(C_j)_0$ meets $SD_0 \subset SS^{-1} x_i$ implying that $(C_j)_0 \subset SS^{-1} x_i$, as claimed. ∎

Thus by applying SS^{-1} we cover different sets $(C_j)_0$. We can do this successively and get the following upper bound for the number of factors. Note that the order of the fixed point set of P_0 in G/P_0 is the number of connected components of P.

Proposition 9.10 *Let* $k = |P/P_0|$. *Then* $G = S^{-1}\left(SS^{-1}\right)^k$.

Proof: With the notations as above we prove by induction on $l = 1, \ldots, k-1$ that $\left(SS^{-1}\right)^l x_1$ contains $l+1$ different sets $(C_i)_0$. For $l = 1$ the statement follows from the above lemma. Thus suppose that $\left(SS^{-1}\right)^{l-1} x_1$ contains the sets $\left(C_{i_j}\right)_0$, $j = 1, \ldots, l$. Since G/P_0 is connected there exists a slice $N^- x_i$, $i \neq i_j$, $j = 1, \ldots, l$, such that $\mathrm{cl}(N^- x_i)$ meets some $\mathrm{cl}\left(N^- x_{i_j}\right)$. Applying again the lemma, it follows that $(C_{x_i})_0 \subset \left(SS^{-1}\right)^l x_1$.

Therefore, $\bigcup_{i=1}^k (C_{x_i})_0 \subset \left(SS^{-1}\right)^{k-1}$, so that $G/P_0 \subset S^{-1}\left(SS^{-1}\right)^{k-1}$. As mentioned above, this implies that $G = S^{-1}\left(SS^{-1}\right)^k$, concluding the proof. ∎

10 Examples

In this section we provide some examples and counter-examples related to the results of the paper.

10.1 Transient components

The following is an example of a transient component of an open semigroup in $G = \mathrm{Sl}(2,\mathbb{R})$. For positive reals $r < s$ put

$$U_{r,s} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R}) : r < a < s;\ b,c,d \geq 0 \right\}.$$

Since the entries are positive, it follows that $r^{n+1} < aa'$ if $x \in U_{r,s}$ and $x' \in U_{r,x}^{n}$, where

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad x^{'} = \begin{pmatrix} a^{'} & b^{'} \\ c^{'} & d^{'} \end{pmatrix}.$$

Hence, $U_{r,s} \cap U_{r,s}^{n} = \emptyset$ for all $n \geq 2$ if $1 < r < s < r^2$. Under this condition the connected compoenents of $U_{r,s}$ are transient components in the semigroup $S_{r,s}$ generated by $U_{r,s}$. Actually, it is not hard to check that $U_{r,s}$ is itself connected, so that it is a transient component in $S_{r,s}$ if $1 < r < s < r^2$.

10.2 Fixed points

For the results Section 5 to hold the condition that S is an open semigroup is essential. This condition appears explicitly for instance in the proof of Lemma 5.2 and subsequently.

Actually, even for semigroups with non-empty interior the uniqueness of fixed points stated in Theorem 5.9 holds only for the interior points. Here is an example of a semigroup $S \subset \mathrm{Sl}(3,\mathbb{R})$ with non-empty interior such that boundary elements of S can have infinity fixed points in the attractor set $C_{\Theta(S)}^{+}$.

Given a basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ define the flag $b_0 = (V_1 \subset V_2)$, $V_1 = \mathrm{span}\{e_1\}$, $V_2 = \mathrm{span}\{e_1, e_2\}$, and write lower triangular matrices as

$$(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}.$$

Consider the diagonal matrix

$$H = \mathrm{diag}\{2, -1, -1\}.$$

Then $\exp(tH)\,(a,b,c)\exp(-tH) = \left(e^{-3t}a, b, e^{-3t}c\right)$, so that the semigroup $\exp(tH)$, $t \geq 0$, leaves invariant subsets of the form

$$R_{\alpha,\beta,\gamma} = \{(a,b,c)\, b_0 : |a| \leq \alpha,\ |b| \leq \beta,\ |c| \leq \gamma\}.$$

Thus if we let S be the compression semigroup of $R_{\alpha,\beta,\gamma}$, $\alpha, \beta\gamma > 0$, then $\exp(tH) \in S$ if $t \geq 0$. By general facts about compression semigroup the flag type of S is the full flag manifold, $R_{\alpha,\beta,\gamma}$ is the invariant control set of S and $C^+ = \mathrm{int} R_{\alpha,\beta,\gamma}$. The points $(0, b, 0)\, b_0 \in R_{\alpha,\beta,\gamma}$ are fixed under $\exp(tH)$, $t \geq 0.$, and belong to C^+. Of course, $\exp(tH)$, $t \geq 0$, are boundary elements of S.

10.3 Components of i.c.s.

The connected components of the semigroups were studied via the components of the attractor and repeller sets $C^{\pm}$. The attractor set C^+ is dense in the invariant control set C, so that it is natural to ask about the relation between the components of C^+ and C. In general C^+ can have much more components than C. Examples of semigroups with C connected, but C^+ not connected are easily given as compression semigroups of closed connected sets, having non-connected interior. For instance, consider the situation in $\mathrm{Sl}(3, \mathbb{R})$ of the previous example. In the open cell $N^- b_0$ let C be the union of two tangent balls, i.e., C is prescribed by

$$a^2 + b^2 + (c+1)^2 \leq 1 \text{ or } a^2 + b^2 + (c-1)^2 \leq 1.$$

The compression semigroup of C has at least two semigroup components although C is connected.

10.4 Compression semigroup of a cone

Let $W \subset \mathbb{R}^n$ be a pointed generating cone, and consider the compression semigroup

$$S_W = \{g \in \mathrm{Sl}(n, \mathbb{R}) : gW \subset W\}.$$

In Ribeiro and San Martin [5] it was proved that S_W is connected. We shall use our results above to give an alternative proof of this fact. It is well know (and easy to see) that $S = \mathrm{int} S_W$ is dense in S_W, so that it is enough to check that $\mathrm{int} S_W$ is connected. Of course, $1 \in \mathrm{cl} S$ so that by 9.8 it has only recurrent components. Now, the parabolic type Θ_W of S_W is given by the condition that $\mathrm{Sl}(n, \mathbb{R}) / P_{\Theta_W}$ is the projective space $\mathbb{P}^{n-1}$. Hence its covering $\mathrm{Sl}(n, \mathbb{R}) / P^0_{\Theta_W}$ is the sphere $\mathbb{S}^{n-1}$. Also, the invariant control set of S in $\mathbb{P}^{n-1}$ is the set of lines contained in W, which is connected. This set splits into two components in $\mathbb{S}^{n-1}$, namely the set of rays starting at the origin and contained in W. Of course the later components are invariant under W. Therefore by the characterization of the finite index components it follows that S has just one component, meaning that S_W is connected.

10.5 Product of semigroup components

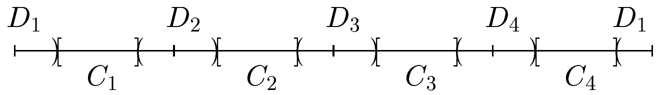
In view of Theorem 8.8 it is natural to ask if the product of two semigroup components is still a semigroup component. The example given here shows that in general this is not true: In $\mathbb{R}^2$ consider the pointed cones $W_1 = \{(x, y) : 0 \leq y \leq x\}$ and $W_2 = \{(x, y) : -y \leq x \leq 0\}$, form the compression semigroup in $\mathrm{Sl}(2, \mathbb{R})$, $T = \mathrm{comp}(\pm W_1 \cup \pm W_2)$ and put $S = \mathrm{int} T$. In the projective line $\mathbb{P}^1$ the attractor set C^+ of S has two components, namely $K_1 = \mathrm{int}[\pm W_1]$ and $K_2 = \mathrm{int}[\pm W_2]$. The repeller set C^- also has two connected components $L_1 = \mathrm{int}[\pm U_1]$ and $L_2 = \mathrm{int}[\pm U_2]$, where $U_1 = \{(x, y) : 0 \leq x \leq y\}$ and $U_2 = \{(x, y) : -y \leq x \leq 0\}$. We claim that the product $\Gamma(K_2, L_2)\, \Gamma(K_1, L_1)$ is not a semigroup component. In fact, take $g \in \Gamma(K_1, L_1)$ such that $gK_2 \subset K_1$ and $h \in \Gamma(K_2, L_2)$ with $hK_1 \subset K_2$. The matrix g has real eigenvalues $\lambda_1 > 1 > \lambda_2$ whose principal eigenspace is contained in $\pm W_1$ and the secondary one is in $\pm L_1$. Analogously, the principal eigenspace of h is contained in $\pm W_2$ and the secondary in $\pm L_2$. Taking into account that a matrix leaves invariant a half-space bounded by an eigenspace we see that for any $x \in W_2$, $gx \in -W_1$, and $hgx \in -W_2$. Hence hg does not leave invariant a connected component of a control set in the double covering $\mathbb{S}^1 \to \mathbb{P}^1$. By Lemma 8.4, $\Gamma(K_2, L_2)\, \Gamma(K_1, L_1)$ is not a semigroup component.

10.6 Number of factors of S generating G

Let G be a two fold covering of $\mathrm{Sl}(2, \mathbb{R})$ and $S \subset G$ the semigroup generated by the exponential of the Lie wedge $\mathfrak{sl}^+(2, \mathbb{R})$ formed by the matrices

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad b, c \geq 0.$$

The picture below depicts the control sets in in the four fold covering G/P_0 of the projective line $\mathbb{P}^1 = G/P$. In the picture the C's represent the invariant control sets and the D's the open control sets. Also, the points marked inside the D's are delimiters of open intervals which are the sheets given by open N-orbits (cf. Proposition 9.10).



Now, given $x \in C_1$, $S^{-1}x$ is contained in $C_1 \cup D_1 \cup D_2$, since $S^{-1}x$ is connected and does not meet C_2 and C_4 because these are S-invariant. By similar reasons we see that $SS^{-1}x$ is contained in $C_1 \cup C_2 \cup C_4 \cup D_1 \cup D_2$ and $S^{-1}SS^{-1}x$ does not meet C_3, so that $SS^{-1}S \neq G$. It is clear that if we take higher coverings of $\mathrm{Sl}(2, \mathbb{R})$ we can apply this method to find examples semigroups such that the number of factors needed to generate the group is as large as we please.

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