Graded identities for the algebra of $n \times n$ upper triangular matrices over an infinite field

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Abstract

We consider the algebra $U_n(K)$ of $n \times n$ upper triangular matrices over an infinite field K equipped with its usual $\mathbb{Z}_n$-grading. We describe a basis of the ideal of the graded polynomial identities for this algebra, and compute some of the numerical invariants of this ideal. An extended version of this research will be published elsewhere.

2000 AMS Mathematics Subject Classification: 16P90, 16R10, 16R20, 16R50, 16W50

Introduction

The interest into graded polynomial identities was inspired by their importance for the structure theory of PI algebras (see for example [16] or [5]).

*Partially supported by CNPq and by FAEP, UNICAMP

†Partially supported by MURST of Italy

Shortly afterwards they became an object of independent studies. This in turn was determined by the various applications that the graded identities can find in PI theory. We mention some of the most important results concerning graded polynomial identities. Thus for example, in [3] and in [10] it was proved that if G is a finite abelian group and A is a G-graded algebra, then A is PI if and only if its 0-component is PI. In [3, 6, 7, 8, 9, 12, 13], several numerical characteristics of T-ideals were transferred to the graded case and lots of their properties were deduced. In [11], bases of the 2-graded identities for several important algebras were described. As a corollary it was obtained a rather elementary proof of the fact that the algebras $M_{11}(G)$ and $G \otimes G$ satisfy the same polynomial identities when the base field is of characteristic 0. Here G stands for the infinite dimensional Grassmann algebra, and $M_{11}(G)$ is the algebra of the 2×2 matrices over G whose main diagonal elements are even (central) elements of G, and whose other diagonal elements are odd elements of G. In [19] a basis of the n-graded identities of the algebra of $n \times n$ matrices $M_n(K)$ was found when K is a field of characteristic 0, and in [20] this was extended to the $\mathbb{Z}$-gradings of $M_n(K)$. Furthermore, in [17] the results of [11] were extended to algebras over an infinite field of characteristic $p \neq 2$ (and as a consequence a really elementary proof of the coincidence of the T-ideals of $M_{11}(G)$ and $G \otimes G$ was obtained). In [2] it was established that the main result of [19] holds for algebras over an infinite field.

It is worth mentioning that the study of graded identities was one of the key ingredients in Kemer's methods for developing the structure theory of PI algebras and in particular, for resolving positively the Specht problem in characteristic 0. Graded identities, along with other kinds of "weaker" identities play essential role in study of the polynomial identities satisfied by concrete algebras. Some applications of graded identities satisfied by concrete algebras can be found in [11, 13, 17].

The polynomial identities satisfied by the algebra $U_n(K)$ of the $n \times n$ upper triangular matrices are of particular interest. It is well known that for every field K and every n they are finitely based (as T-ideal), see for example [21]. Thus, when the field K is infinite, the T-ideal of $U_n(K)$ is generated by the identity $[x_1, x_2][x_3, x_4] \ldots [x_{2n-1}, x_{2n}]$ where $[a, b] = ab - ba$ is the usual commutator. It is also well known that the identities of $U_n(K)$ are closely related to the problem of the description of the subvarieties of the variety of associative algebras generated by $M_2(K)$. Since the latter is still open when $\mathrm{char} K \neq 0$ it is important to obtain further information about the identities in $U_n(K)$. A detailed description of the 2-graded identities satisfied by the algebra $U_2(K)$ when $\mathrm{char}K = 0$ is given in [18].

In this paper we describe a basis of the n-graded polynomial identities for the algebra $U_n(K)$ over an infinite field K and as an application we compute the asymptotics for the sequence of its graded codimensions.

These results generalize the ones in [18]. We hope they can be useful in describing the behaviour of the subvarieties of the variety generated by the matrix algebra of order 2 over an infinite field of characteristic not 2.

1 Preliminaries

We fix an infinite field K, and consider all algebras (graded and ungraded), vector spaces etc. over K. Denote by $U_n(K)$ the algebra of $n \times n$ upper triangular matrices over K i.e., $n \times n$ matrices with zero entries below the main diagonal. The algebra $U_n(K)$ has a natural $\mathbb{Z}_n$-grading $U_n(K) = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$ with

$$V_i = \{a_{1,i+1}e_{1,i+1} + a_{2,i+2}e_{2,i+2} + \cdots + a_{n-i,n}e_{n-i,n} \mid a_{r,s} \in K\}, \quad 0 \le i \le n-1$$

where the e_{ij} are the elementary matrices having 1 as (i,j)-th entry and 0 elsewhere. In particular V_0 consists of all diagonal matrices. Clearly all V_i are subspaces of $U_n(K)$ and $V_iV_j \subseteq V_{i+j}$ where $i+j$ is taken modulo n. Observe that if $i+j \ge n$ then $V_iV_j = 0$.

Let X be a countable infinite set, and let $K(X)$ be the free associative algebra freely generated by X over K. Suppose that $X = X_0 \cup X_1 \cup \ldots \cup X_{n-1}$ where $X_i \cap X_j = \emptyset$ if $i \ne j$, all X_i being infinite, and $X_i = \{x_{i1}, x_{i2}, \ldots\}$. The weight of the variable x_{ij} is $w(x_{ij}) = i$, and if $m = x_{i_1j_1}x_{i_2j_2}\ldots x_{i_kj_k}$ is a monomial then its weight is defined as $w(m) = i_1 + i_2 + \cdots + i_k$, the last sum is taken modulo n. Define $K(X)_i$ as the span of all monomials of weight i, $0 \le i \le n-1$, then $K(X) = K(X)_0 \oplus K(X)_1 \oplus \cdots \oplus K(X)_{n-1}$ is a $\mathbb{Z}_n$-graded algebra. We shall use the term n-graded instead of $\mathbb{Z}_n$-graded. This algebra is the free n-graded algebra in the sense that given an n-graded algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{n-1}$, every map $\varphi\colon X \to A$ such that $\varphi(X_i) \subseteq A_i$ can be extended uniquely to a graded homomorphism $\Phi\colon K(X) \to A$.

Let $f = f(x_{i_1j_1}, x_{i_2j_2}, \ldots, x_{i_mj_m}) \in K(X)$ be a polynomial, and let A be an n-graded algebra. Then f is an n-graded identity for the algebra A if $f(a_{i_1j_1}, a_{i_2j_2}, \ldots, a_{i_mj_m}) = 0$ for every $a_{i_kj_k} \in A_{i_k}$. In other words f becomes zero when we substitute its variables for homogeneous elements of A having the same weight as the respective variables. Obviously the set $Id^{gr}(A)$ of all n-graded identities is an ideal of $K(X)$, and this ideal is stable under all graded endomorphisms of A. We call such ideals T_n-ideals. The class of all n-graded algebras satisfying the graded identities of $Id^{gr}(A)$ is the variety $\mathrm{var}\,(A)$ of n-graded algebras determined by A (and by $Id^{gr}(A)$).

The quotient algebra $K(X)/Id^{gr}(A)$ is called the relatively free n-graded algebra. We shall use the same symbols x_{ij} instead of $x_{ij}+Id^{gr}(A)$ for the generators of the relatively free graded algebra. Then if $B \in \mathrm{var}\,(A)$, every homogeneous map $\varphi: X \to B$ (i.e. a map that preserves the grading) can be extended uniquely to a homomorphism of graded algebras $\Phi: K(X)/Id^{gr}(A) \to B$.

Since our field is infinite then every ideal of n-graded identities is generated by its multihomogeneous elements. Note that when the base field is of positive characteristic it may occur that the multilinear identities (graded identities) of an algebra do not generate the T-ideal (the T_n-ideal, respectively) of the given algebra.

2 The graded identities of $U_n(K)$

Throughout this section $A = U_n(K)$ is the algebra of $n \times n$ upper triangular matrices with its natural $\mathbb{Z}_n$-grading $U_n(K) = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$.

Now if $S \subseteq K(X)$ is any set of polynomials, we denote by $\langle S \rangle_{T_n}$ the T_n-ideal of $K(X)$ generated by S. We start with the following lemma.

Lemma 1 *The algebra $U_n(K)$ satisfies the graded identities*

$$x_{01}x_{02} - x_{02}x_{01} \equiv 0 \quad (1)$$

$$x_{i_1 1}x_{i_2 2} \equiv 0 \quad (2)$$

whenever $i_1 + i_2 \geq n$.

Proof. The identity $x_{01}x_{02} - x_{02}x_{01} \equiv 0$ holds in $U_n(K)$ since two diagonal matrices commute. Moreover since $i + j \geq n$ implies $V_iV_j = 0$, the graded identities $x_{i_1 1}x_{i_2 2} \equiv 0$ with $i_1 + i_2 \geq n$, hold for $U_n(K)$ as well. ♢

Denote by I the ideal of n-graded identities generated by the identities (1) and (2), and set $J = Id^{gr}(U_n(K)$.

We wish to show that $I = J$. We start with an obvious observation.

Remark 1 *For any* $i_1 + i_2 + \cdots + i_k \geq n$, $x_{i_1 1} x_{i_2 2} \ldots x_{i_k k} \equiv 0$ *is a graded identity of* $U_n(K)$.

Proof. It follows from (2) by the invariance of T_n-ideals under endomorphisms preserving the grading. ♢

Consider the set of monomials of $K\langle X\rangle$ of the type

$$u = w_0 x_{k_1 i_1} w_1 \ldots w_{t-1} x_{k_t i_t} w_t \tag{3}$$

where $k_1 + \cdots + k_t < n$ and $w_0, \ldots, w_t$ are (possibly empty) monomials in the variables x_{0i} of homogeneous degree 0 and in each w_i these variables are written in increasing order from left to right.

The next theorem gives the multilinear structure of the relatively free n-graded algebra in the variety determined by $U_n(K)$ and as a consequence, a basis of the n-graded identities of $U_n(K)$.

Theorem 2 *If* K *is an infinite field, the monomials (3) are a basis of* $K\langle X\rangle$ *modulo* $Id^{gr}(U_n(K)$. *Moreover* $Id^{gr}(U_n(K)) = \langle [x_{01}, x_{02}], x_{i1}x_{j2} \mid i + j \geq n\rangle_{T_n}$.

Proof. We first claim that the monomials (3) are linearly independent modulo $Id^{gr}(U_n(K))$.

In fact suppose that $f = \sum \alpha_i u_i \in Id^{gr}(U_n(K))$ for some $\alpha_i \in K$ where the monomials u_i are of the type (3). Since K is an infinite field, graded ideals are multihomogeneous. Hence we may assume that all u_i are multihomogeneous of the same degree sequence, i.e., the same variables appear in each monomial u_i. Fix one monomial with nonzero coefficient say u_1 and let $u_1 = w_0 x_{k_1 i_1} w_1 \ldots w_{t-1} x_{k_t i_t} w_t$. We now evaluate f on $U_n(K)$ as follows: evaluate each variable appearing in w_0, into e_{11}, the variable $x_{k_1 i_1}$ into e_{1,k_1+1}, each variable appearing in w_1 into e_{k_1+1,k_1+1}, the variable $x_{k_2 i_2}$ into e_{k_1+1,k_1+k_2+1}, $\ldots$, the variable $x_{k_t i_t}$ into $e_{k_1+\cdots+k_{m-1}+1,k_1+\cdots+k_m+1}$ and each variable in w_t into e_{jj} where $j = k_1 + \cdots + k_m + 1$. It follows that u_1 evaluates into e_{1j} and all other monomials of f evaluate to zero. Thus $f \not\equiv 0$ on $U_n(K)$, a contradiction.

Next we claim that, modulo $\langle [x_{01}, x_{02}], x_{i1}x_{j2} | i + j \geq n\rangle_{T_n}$, every element of $K\langle X\rangle$ can be written as a linear combination of monomials of type (3). In fact, this is clear since T_n-ideals are invariant under homogeneous substitutions.

It follows that the monomials (3) are a basis of $K\langle X\rangle$ modulo $Id^{gr}(U_n(K)$ and that $Id^{gr}(U_n(K)) = \langle [x_{01}, x_{02}], x_{i1}x_{j2}, i + j \geq n\rangle_{T_n}$. ♢

3 Generic matrices and graded identities

We now introduce a model for $K(X)/Id^{gr}(U_n)$, the relatively free n-graded algebra. This model is based on a modification of the construction of the ring of generic matrices.

Let y_{ijk}, i, j, $k \geq 0$, be commuting variables and consider the polynomial algebra $K[y_{ijk}]$ in these variables. The tensor product $U_n(K) \otimes_K K[y_{ijk}]$ is canonically isomorphic to the algebra $U_n(K[y_{ijk}])$. Let

$$Y_{ik} = y_{1,i+1,k}e_{1,i+1} + y_{2,i+2,k}e_{2,i+2} + \cdots + y_{n-i,n,k}e_{n-i,n}$$

with $0 \leq i \leq n-1, k = 1, 2, \ldots$, and set G_n the subalgebra of $U_n(K[y_{ijk}])$ generated by the matrices Y_{ik}, $0 \leq i \leq n$, $k = 1, 2, \ldots$ The algebra G_n is n-graded in a natural way, precisely the same as $U_n(K)$. Furthermore it satisfies all graded identities of $U_n(K)$.

Lemma 3 *The algebra G_n is isomorphic as a graded algebra to the relatively free algebra $K(X)/Id^{gr}(U_n(K)$ of the variety of n-graded algebras var$(U_n(K))$.*

Proof. The map $x_{ij} \to Y_{ij}$ defines an epimorphism $\Phi\colon K(X) \to G_n$. Furthermore Φ is actually an isomorphism. The reasoning is exactly the same as in the case of the generic matrix algebra since our field K is infinite. Obviously Φ preserves the grading and hence is a graded isomorphism. $\diamondsuit$

We shall work in the algebra G_n instead of the relatively free algebra. We use some ideas from [19] and from [2].

Lemma 4 *Let $m_s = m_s(x_{i_1j_1}, x_{i_2j_2}, \ldots, x_{i_kj_k})$, $s = 1, 2$, be two monomials in $K(X)$. Suppose that the matrices*

$$M_1 = m_1(Y_{i_1j_1}, Y_{i_2j_2}, \ldots, Y_{i_kj_k}) \quad \text{and} \quad M_2 = m_2(Y_{i_1j_1}, Y_{i_2j_2}, \ldots, Y_{i_kj_k})$$

in G_n have the same nonzero entries in the same positions in their first rows. (In other words, $M_1 - M_2$ has zeros in its first row.) Then $M_1 = M_2$.

Proof. We know that the (unique) nonzero entry in the first row occurs in position $(1, r+1)$, and the same entry occurs in M_2, at the same position $(1, r+1)$, for some r, $0 \leq r \leq n-1$. One computes these entries as follows. For M_1 we have that the $(1, r+1)$-st entry equals $y_{a_1,b_1,c_1}y_{a_2,b_2,c_2} \cdots y_{a_k,b_k,c_k}$

where $a_1 = 1$, $b_1 = i_1 + 1$, $c_1 = j_1$, and $c_t = j_t$ for all t; thus one obtains the recurrence formula

$$a_{t+1} = b_t, b_{t+1} = a_{t+1} + i_{t+1}, c_{t+1} = j_{t+1}, 1 \leq t \leq k-1,$$

where $a_1 = 1$, $b_1 = i_1+1$, $c_1 = j_1$. Note that $b_{t+1} = b_t + i_{t+1}$. Furthermore, in order to obtain a nonzero entry one needs $b_k \leq n$. But this is always the case if $b_k = i_1 + i_2 + \cdots + i_k + 1 = r + 1 \leq n$. Now observe that these recurrencies determine uniquely the monomial M_1, and hence $M_1 = M_2$. $\diamondsuit$

Corollary 5 *Under the assumptions and in the notation of the preceding Lemma, we have* $m_1 - m_2 \in Id^{gr}(U_n(K))$.

Proof. The proof is straightforward since $M_1 - M_2 = 0$ in G_n which is the relatively free n-graded algebra. $\diamondsuit$

Using the properties of the relatively free graded algebra we give an alternative proof of Theorem 2.

Proof of Theorem 2. Let I be the T_n-ideal generated by the polynomials (1) and (2), and let $J = Id^{gr}(U_n(K))$. Since $I \subseteq J$, it suffices to prove that $J \subseteq I$. Suppose on the contrary that there exists a multihomogeneous polynomial $f \in J$ and $f \notin I$. We work in the relatively free n-graded algebra $K\langle X\rangle/I = G_n$ and choose $f \in G_n$ of minimal degree, and among these f, choose one that is expressed in the form $f = \alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_s m_s$ for m_t being distinct monomials in G_n, all $\alpha_t \neq 0$, $\alpha_t \in K$, and s the least possible. Hence $s \geq 1$.

Suppose $m_t = m_t(x_{i_1 j_1}, x_{i_2 j_2}, \ldots, x_{i_k j_k})$. Then

$$m_1(Y_{i_1 j_1}, Y_{i_2 j_2}, \ldots, Y_{i_k j_k}) = \sum_{z=2}^{r} \beta_z m_z(Y_{i_1 j_1}, Y_{i_2 j_2}, \ldots, Y_{i_k j_k})$$

where $\beta_z = -\alpha_z/\alpha_1 \neq 0$, $z = 2, 3, \ldots, r$.

On the other hand we have that $m_1(Y_{i_1 j_1}, Y_{i_2 j_2}, \ldots, Y_{i_k j_k}) \neq 0$ and in the first row of this matrix there will be some nonzero entry. This nonzero entry appears on the right-hand side as well. Say it comes from the monomial m_2. But then the monomials m_1 and m_2 have the same nonzero entry in the same position on their first rows, and hence $m_1 - m_2 \in J$, according to Corollary 5. But then $m_1 = m_2$, and we reduce f to $t-1$ monomials which contradicts the choice of t. Hence $t = 0$ and $I = J$.

We now prove that the monomials (3) are linearly independent modulo $Id^{gr}(U_n(K))$. Suppose that the polynomial $\sum_{i=1}^{t} \alpha_i m_i \in Id^{gr}(U_n(K))$ where m_i are distinct monomials of the type (3), for $0 \neq \alpha_i \in K$. Since K is infinite we may assume that the monomials m_i are all multihomogeneous and of the same multidegree. One can express m_1 in the following way: $m_1 = -\sum_{i=2}^{t} \beta_i m_i$. Now substitute the variables for the respective generic graded matrices; the first row of m_1 will contain some nonzero entry since otherwise m_1 would belong to $Id^{gr}(U_n(K))$. The same nonzero entry must appear in some of the monomials on the right-hand side, say in m_2. Hence $m_1 - m_2 \in Id^{gr}(U_n(K))$, and we reduce our linear combination to $t-1$ terms. Finally we obtain single monomial, but it cannot be a graded identity for $U_n(K)$ since it is of weight $< n$, and it will contain in its first row some nonzero entry when evaluated on G_n. $\diamondsuit$

4 Applications

Here we apply the results of the previous section and describe some of the numerical invariants of the T_n-ideal of $U_n(K)$. We start with one of the various kinds of codimensions one may define for graded identities.

Suppose that $g_1 h_1 g_2 h_2 \ldots g_k h_k$ is a multilinear monomial that is nonzero in the relatively free n-graded algebra $K(X)/Id^{gr}(U_n(K))$. Let g_i depend on the 0-variables x_{0j}, and h_i depend on some x_{zj}, $1 \leq z \leq n-1$. Suppose further that the number of variables x_{0j} is s, and of the x_{zj} is $m-s$. If g_u depends on t_u variables, $1 \leq u \leq k$, we fix these variables and we obtain $(m-s)!$ monomials of the desired type, permuting only the variables x_{zj}. Now, dividing the variables x_{0j} into groups, t_i in the i-th group, we will obtain a multiple of the multinomial coefficient

$$\frac{s!(m-s)!}{(t_1!t_2!\ldots t_k!)} = (m-s)!\binom{s}{t_1!t_2!\ldots t_k!}.$$

Now summing up over all such divisions of the x_{0j} into k groups (some of them may be empty) one gets that the m-th graded codimension in the fixed variables equals $(m-s)!(m-s+1)^s$. Of course we impose one further condition namely that the weight of the monomial be less than n. Hence we proved the following proposition

Proposition 6 *Let m be a fixed positive integer and let x_{0j_1}, x_{0j_2}, $\ldots$, x_{0j_s} be fixed. Suppose further that x_{z_1,r_1}, x_{z_2,r_2}, $\ldots$, x_{z_g,r_g}, $g = m-s$, are fixed,*

$1 \leq z_i \leq n-1$ *for all* i *and* $z_1 + z_2 + \cdots + z_g \leq n-1$. *Then the span of all multilinear monomials on these variables in* $K(X)/Id^{gr}(U_n(K))$ *is of dimension* $(m-s)!(m-s+1)^s$. ♢

One can consider another kind of graded codimensions, see for example [3]. We recall the definition of these codimensions that represent a direct generalization of the ungraded case. Let m be fixed positive integer, and consider the (graded) variables x_{ij}, $0 \leq i \leq n$, $1 \leq j \leq m$, in $K(X)$. We consider the multilinear monomials in these variables that are of the form

$$x_{i_i,j_1} x_{i_2,j_2} \dots x_{i_m,j_m} \tag{4}$$

where $\{j_1, j_2, \dots, j_m\} = \{1, 2, \dots, m\}$ and $0 \leq i_t \leq n-1$.

In other words, we do not admit repetitions of the second indices in the variables. These monomials can be obtained by the usual multilinear monomials $x_{j_1} x_{j_2} \dots x_{j_m}$ by assigning all possible weights on the variables. It is straightforward that the span P_m^n of these monomials in $K(X)$ is of dimension $c_m^{(n)} = n^m m!$. Denote by $P_m^n(A)$ the quotient $P_m^n/(P_m^n \cap Id^{gr}(A))$ where A is an n-graded algebra, and set $c_m^{(n)}(A) = \dim_K P_m^n(A)$. Next we compute the codimensions $c_m = c_m^{(n)}(U_n(K))$.

Theorem 7 *The codimensions* c_m *equal*

$$c_m = \sum_{q=0}^{M} \binom{m}{q} \binom{n-1}{q} q!(q+1)^{m-q}$$

where $M = \min\{m, n-1\}$.

Proof. Let $f = g_1 y_1 g_2 y_2 \dots g_k y_k g_{k+1} \in P_m^n(U_n(K))$ be a nonzero multilinear monomial in the variables x_{ij}, $0 \leq i \leq n-1$, $1 \leq j \leq m$ where for $i = 1$, …, $k+1$, the $g_i^{\prime}s$ are monomials (possibly empty) in the variables x_{0j} only, and for $r = 1, \dots, k$, $y_r = x_{a_r, b_r}$, with $1 \leq a_r \leq n-1$, $1 \leq b_r \leq m$. Since $f \in P_m^n(U_n(K))$ we have to impose $a_1 + a_2 + \cdots + a_q \leq n-1$.

Denote by $A_{n-1}(q)$ the number of q-tuples of positive integers $a_1, a_2, \dots, a_q$ such that $a_1 + a_2 + \cdots + a_q \leq n-1$, and as $B_{n-1}(q)$ the number of such q-tuples with $a_1 + a_2 + \cdots + a_q = n-1$. Then obviously $A_{n-1}(q) = B_{n-1}(q) + B_{n-2}(q) + \cdots + B_q(q)$. On the other hand, $B_r(q) = \binom{r-1}{q-1}$ is the number of *compositions* of r into q parts, see for example [1, p. 54]. Hence, using some elementary transformations on binomial coefficients, we obtain that

$$A_{n-1}(q) = \sum_{r=q}^{n-1} B_r(q) = \sum_{r=q}^{n-1} \binom{q+r-1}{q-1} = \binom{n-1}{q} = \frac{(n-1)!}{(q!(n-1-q)!)}.$$

Now, for the indices b_r of $y_r = x_{a_r,b_r}$ we have $m!/(m-q)! = q!\binom{m}{q}$ choices (no repetition!). Therefore there exist

$$q!\binom{m}{q}\binom{n-1}{q}$$

nonzero monomials h in G_n with the properties described above.

Now let us consider the variables x_{0j}. We have $m-q$ choices for the index j (again no repetitions!), and we split the set of all such x_{0j} into $k+1$ subsets (some of them possibly empty). If we denote by $C(k,q)$ the number of nonzero monomials f of type (4) that can be represented as $f = g_1y_1g_2y_2\dots g_ky_kg_{k+1}$, g_r are monomials (possibly empty) in x_{0j}, and $y_r = x_{a_r,b_r}$ with $1 \le a_r \le n-1$, then we have

$$c_m = \sum_{q=0}^{n-1}\sum_{k=0}^{q}(C(k,q) - C(k-1,q))$$

where

$$C(k,q) = (k+1)^{m-q}q!\binom{m}{q}\binom{n-1}{q}.$$

Therefore

$$\begin{aligned} c_m &= \sum_{q=1}^{n-1}\sum_{k=0}^{q} q!\binom{m}{q}\binom{n-1}{q}((k+1)^{m-q} - k^{m-q}) \\ &= \sum_{q=0}^{m}\binom{m}{q}\binom{n-1}{q}q!(q+1)^{m-q}. \end{aligned}$$

Note that the binomial coefficient $\binom{n-1}{q}$ equals 0 whenever $q > n-1$, and this last observation yields the formula for c_m from the theorem. $\diamondsuit$

As a consequence of Theorem 7 we can now compute the precise asymptotics of the sequence of graded codimensions $c_m(U_n(K))$.

Recall that if $f(x)$ and $g(x)$ are two functions of a natural argument then $f(x)$ and $g(x)$ are asymptotically equal, and we write $f(x) \simeq g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Corollary 8 *For all m,*

$$c_m(U_n(K)) \simeq \frac{1}{n^{n-1}} m^{n-1} n^m.$$

Proof. For $M = \min\{m, n-1\}$ we have

$$c_m(U_n(K)) = \sum_{q=0}^{M} \binom{m}{q} \binom{n-1}{q} q!(q+1)^{m-q} \simeq \binom{m}{n-1} (n-1)! n^{m-n+1}$$

which in turn equals

$$\frac{m(m-1)\cdots(m-n+2)}{(n-1)!}(n-1)! n^{m-n+1} \simeq m^{n-1} n^{m-n+1} = \frac{1}{n^{n-1}} m^{n-1} n^m.$$

$\diamondsuit$

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