Poincaré-Hopf Inequalities

M.A. BERTOLIM*
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo São Carlos SP Brazil
bertolim@icmc.sc.usp.br

M.P. MELLO
Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas Campinas SP Brazil
margarid@ime.unicamp.br

K.A. de REZENDE†
Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas Campinas SP Brazil
ketty@ime.unicamp.br

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Abstract

In this article the main theorem establishes the necessity and sufficiency of the Poincaré-Hopf inequalities in order for the Morse inequalities to hold under the hypothesis that the flow and the reverse flow satisfy the Conley index duality condition on components of the chain recurrent set. The convex hull of the collection of all Betti number vectors which satisfy the Morse inequalities for a pre-assigned index data determines a Morse polytope defined on the nonnegative orthant. Using results from network flow theory, a scheme is provided for constructing all possible Betti number vectors which satisfy the Morse inequalities for a pre-assigned index data. Geometrical properties of this polytope are described.

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1 Introduction

For a continuous flow ϕ_t on a closed manifold M^n with finite component chain recurrent set R where each component R_k is an isolated invariant set it is possible to compute the dimensions of the Conley homology indices $(h_0, \dots, h_n)_k$ for each R_k. In [2] it is proven that, given ϕ_t on M with $R = \cup R_k$ and $h_j = \sum_k (h_j)_k$ the following generalized Morse inequalities hold.

$$
\begin{array}{rcll}
\gamma_n - \gamma_{n-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & = & h_n - h_{n-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(n)} \\
\gamma_{n-1} - \gamma_{n-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_{n-1} - h_{n-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(n} - 1\text{)} \\
\vdots & & \vdots & \\
\gamma_j - \gamma_{j-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_j - h_{j-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(j)} \\
\gamma_{j-1} - \gamma_{j-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 & \leq & h_{j-1} - h_{j-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(j} - 1\text{)} \\
\vdots & & \vdots & \\
\gamma_2 - \gamma_1 + \gamma_0 & \leq & h_2 - h_1 + h_0 & (2) \\
\gamma_1 - \gamma_0 & \leq & h_1 - h_0 & (1) \\
\gamma_0 & \leq & h_0 & (0)
\end{array}
\qquad (1)
$$

The *Conley index duality condition* on the indices $(h_j)_k$ of the components R_k of the chain recurrent set of the flow ϕ_t implies that $\overline{h}_j = h_{n-j}$ where $(\overline{h}_j)_k$ are the indices of the components $\overline{R}_k$ for the reverse flow ϕ_{-t}. This property is easily verified for Morse-Smale flows. We refer the reader to [2] for more details about index pairs, isolating blocks and Conley homology index. Note that the Morse inequalities (1) hold for the dual indices $(\overline{h}_j)_k$ and we will refer to it as the dual Morse inequalities. This corresponds to passing the bar only over the right hand side of the inequalities since the left hand side remains unaltered.

In [1] the Poincaré-Hopf inequalities in all generality are introduced for flows on isolating blocks N and their Lyapunov graph L_N in order to ensure the continuation of L_N to a Morse type Lyapunov graph. These inequalities involve the Betti numbers of the exiting and entering boundaries of N. We refer to these inequalities as the Poincaré-Hopf inequalities for isolating blocks which will be presented in Section 2.

For the purpose of this article, we consider a particular case of the Poincaré-Hopf inequalities for isolating blocks which we will refer to as the Poincaré-Hopf inequalities for closed manifolds (2), (3) and (4). The Poincaré-Hopf inequalities give bounds on the numbers h_j with respect to alternating sums of h_s with $s < j$ and their duals h_{n-s}. In the case of Morse flows these inequalities provide bounds on the number of singularities c_j of Morse index j with respect to alternating sums of c_s

with $s < j$ and their duals c_{n-s}.

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
n = 2i+1 \left\{ \; -h_i \leq (h_{i+2} - h_{i-1}) - (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \leq h_{i+1} \right. \\
\\
n = 2i \left\{ \; -h_i \leq (h_{i+1} - h_{i-1}) - (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i} - h_0) \leq h_i \right.
\end{array}
\right. \quad \text{(i)} \\
\\
\\
\left\{
\begin{array}{l}
-h_j \leq (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \leq h_{n-j} \text{ (j)} \\
\qquad\qquad \vdots \\
-h_2 \leq (h_{n-1} - h_1) - (h_n - h_0) \leq h_{n-2} \text{ (2)} \\
\\
\left\{
\begin{array}{l}
h_1 \geq h_0 - 1 \\
h_{n-1} \geq h_n - 1
\end{array}
\right. \quad (1)
\end{array}
\right.
\end{array}
\right.
\tag{2}
$$

In the case $n = 2i+1$ we have

$$\sum_{j=0}^{2i+1} (-1)^j h_j = 0 \tag{3}$$

and in the case $n = 2i = 2 \mod 4$ we have

$$h_i - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) \text{ be even.} \tag{4}$$

A nonnegative integral vector $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$ satisfying $\gamma_{n-k} = \gamma_k$, for $k = 0, \ldots, n$, $\gamma_0 = \gamma_n = 1$ and $\gamma_{n/2}$ even if n is even, is called a Betti number vector.

Our main theorem asserts that under the hypothesis that the flow satisfies the Conley index duality condition on components of the chain recurrent set, the Poincaré-Hopf inequalities (2) hold if and only if the Morse inequalities (1) hold for some Betti number vector $(\gamma_0, \ldots, \gamma_n)$.

We can state the main theorem in terms of a collection of nonnegative numbers $(h_0, h_1 \ldots, h_n)$.

Theorem 1.1 *A set of nonnegative numbers $(h_0, h_1 \ldots, h_n)$ satisfies the Poincaré-Hopf inequalities in (2) if and only if it satisfies the Morse inequalities (1) for some Betti number vector $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$.*

An important role in our proof is played by a more elaborate classification of singularities. Given a nondegenerate singularity, a classical approach is to associate to it, its Morse index j. More generally, one can associate to it the dimensions of the Conley homology indices, $h_j = 1$ and $h_k = 0$ for all $k \neq j$. In [3] singularities are classified not only by their index, but also by the effect caused on the Betti numbers of the entering and exiting boundaries, N^+ and N^-, of the flow defined on an isolating block N containing the singularity. In other words, a singularity of Morse index j can increase (resp. decrease) the j-th (resp. $j-1$-th) Betti number of N^+ with respect to the j-th (resp. $j-1$-th) Betti number of N^-. We refer to this singularity as h_j^d (resp. h_j^c) , where the d stands for disconnecting (where the c stands for connecting). In the case $n = 2i = 0 \mod 4$, a singularity of index i is β-i, if all Betti numbers are kept constant.

The crucial step in the proof of the main theorem is to define a linear system, henceforth called h^{cd}-system which can be characterized by the dimensions of the Conley homology indices $(h_0, \ldots, h_n)$ and whose unknowns are precisely $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$. Nonnegative integer solutions to this system correspond to different ways one can choose h_j^c and h_j^d for a pre-assigned index data $(h_0, \ldots, h_n)$. In [1] it was shown that the h^{cd}-system has a nonnegative solution if and only if the Poincaré-Hopf inequalities (2), (3) and (4) are satisfied. We also show in [1] that this h^{cd}-system constitutes a network-flow problem, with possible additional constraints. The nature of the network involved allows a complete characterization of all possible solutions of the h^{cd}-system by means of one particular solution of the system (an extreme point, or vertex, of the polytope associated with the h^{cd}-system) and the simple circulations of the network.

The novelty here is that the h_j^c and h_j^d will be used to define a Betti number vector which satisfies the Morse inequalities for a pre-assigned index data $(h_0, \ldots, h_n)$. By considering all possible combinations of circulations, we may construct the convex hull of all Betti number vectors which satisfy the Morse inequalities for a pre-assigned index data. The convex hull of the collection of Betti number vectors that satisfy the Morse inequalities constitutes a polytope. The Morse polytope $\mathcal{P}(h_0, \ldots, h_n)$ (or simply $\mathcal{P}$, if we consider a generic fixed $(h_0, \ldots, h_n)$) is the intersection of this convex hull with the nonnegative orthant.

This article is divided in the following sections. Section 2 will summarize the Poincaré-Hopf inequalities for isolating blocks obtained in [1]. In Section 3 the main equivalence results are established, that is, the h^{cd}-system has nonnegative integral solutions if and only if there exist nonnegative integral Betti number vectors that satisfy the Morse inequalities. Lastly, Section 4 describes the Morse polytope $\mathcal{P}$ and presents additional geometric properties thereof.

2 Poincaré-Hopf Inequalities for isolating blocks

The Poincaré-Hopf inequalities for an isolated invariant set Λ in an isolating block N with entering set for the flow N^+ and exiting set for the flow N^-, are obtained by analysis of long exact sequences of (N, N^+) and (N, N^-). This analysis can be found in [1].

Note that (N, N^-) is an index pair for Λ and (N, N^+) is an index pair for the isolated invariant set of the reverse flow, Λ'.

Consider the long exact sequences for the pairs (N, N^-) and (N, N^+):

$$\begin{aligned}
&0 \to H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N,N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}} \\
&\to H_{n-1}(N,N^-) \xrightarrow{\partial_{n-1}} H_{n-2}(N^-) \xrightarrow{i_{n-2}} H_{n-2}(N) \xrightarrow{p_{n-2}} H_{n-2}(N,N^-) \xrightarrow{\partial_{n-2}} \dots \\
&\xrightarrow{\partial_4} H_3(N^-) \xrightarrow{i_3} H_3(N) \xrightarrow{p_3} H_3(N,N^-) \xrightarrow{\partial_3} H_2(N^-) \xrightarrow{i_2} H_2(N) \xrightarrow{p_2} H_2(N,N^-) \xrightarrow{\partial_2} \\
&\to H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N,N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N,N^-) \to 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
&0 \to H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N,N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}} \\
&\to H_{n-1}(N,N^+) \xrightarrow{\partial'_{n-1}} H_{n-2}(N^+) \xrightarrow{i'_{n-2}} H_{n-2}(N) \xrightarrow{p'_{n-2}} H_{n-2}(N,N^+) \xrightarrow{\partial'_{n-2}} \dots \\
&\xrightarrow{\partial'_4} H_3(N^+) \xrightarrow{i'_3} H_3(N) \xrightarrow{p'_3} H_3(N,N^+) \xrightarrow{\partial'_3} H_2(N^+) \xrightarrow{i'_2} H_2(N) \xrightarrow{p'_2} H_2(N,N^+) \xrightarrow{\partial'_2} \\
&H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N,N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N,N^+) \to 0
\end{aligned} \tag{6}$$

We will assume throughout our analysis that the Conley duality condition on the indices holds. That is, the isolated invariant sets Λ and Λ' have the property that $\operatorname{rank} H_j(N, N^-) = h_j$ and $\operatorname{rank} H_j(N, N^+) = \overline{h}_j = h_{n-j}$. Denote $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$ and $\operatorname{rank}(H_j(N^{\pm})) = B_j^{\pm}$.

By simultaneously analyzing the following pairs of maps

$$\{[(p_i, \partial'_i), (p'_i, \partial_i)], \dots [(p_2, \partial'_2), (p'_2, \partial_2)]\}$$

and analyzing p_1 and p'_1 we obtain the Poincaré-Hopf inequalities in all its generality.

$$
\left\{
\begin{array}{l}
n=2i+1\left\{
\begin{array}{l}
h_i \geq -(B^+_{i-1}-B^-_{i-1})+(B^+_{i-2}-B^-_{i-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\\
\qquad -(h_{i+2}-h_{i-1})+(h_{i+3}-h_{i-2})+-\ldots\pm(h_{2i-1}-h_2)\\
\qquad \pm(h_{2i}-h_1)\pm[(h_{2i+1}-h_0)+(e^+-e^-)]\\
\\
h_{i+1} \geq -\left[-(B^+_{i-1}-B^-_{i-1})+(B^+_{i-2}-B^-_{i-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\right.\\
\qquad -(h_{i+2}-h_{i-1})+(h_{i+3}-h_{i-2})+-\ldots\pm(h_{2i-1}-h_2)\\
\qquad \left.\pm(h_{2i}-h_1)\pm[(h_{2i+1}-h_0)+(e^+-e^-)]\right]
\end{array}\right.\\
\\
n=2i\left\{
\begin{array}{l}
h_i \geq -(B^+_{i-1}-B^-_{i-1})+(B^+_{i-2}-B^-_{i-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\\
\qquad -(h_{i+1}-h_{i-1})+(h_{i+2}-h_{i-2})+-\ldots\\
\qquad \pm(h_{2i-2}-h_2)\pm(h_{2i-1}-h_1)\pm[(h_{2i}-h_0)+(e^+-e^-)]\\
\\
h_i \geq -\left[-(B^+_{i-1}-B^-_{i-1})+(B^+_{i-2}-B^-_{i-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\right.\\
\qquad -(h_{i+1}-h_{i-1})+(h_{i+2}-h_{i-2})+-\ldots\\
\qquad \left.\pm(h_{2i-2}-h_2)\pm(h_{2i-1}-h_1)\pm[(h_{2i}-h_0)+(e^+-e^-)]\right]
\end{array}\right.\\
\qquad\qquad\vdots\\
\left\{
\begin{array}{l}
h_j \geq -(B^+_{j-1}-B^-_{j-1})+(B^+_{j-2}-B^-_{j-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\\
\qquad -(h_{n-(j-1)}-h_{j-1})+(h_{n-(j-2)}-h_{j-2})+-\ldots\\
\qquad \pm(h_{n-1}-h_1)\pm[(h_n-h_0)+(e^+-e^-)]\\
\\
h_{n-j} \geq -\left[-(B^+_{j-1}-B^-_{j-1})+(B^+_{j-2}-B^-_{j-2})+-\ldots\pm(B^+_2-B^-_2)\pm(B^+_1-B^-_1)\right.\\
\qquad -(h_{n-(j-1)}-h_{j-1})+(h_{n-(j-2)}-h_{j-2})+-\ldots\\
\qquad \left.\pm(h_{n-1}-h_1)\pm[(h_n-h_0)+(e^+-e^-)]\right]
\end{array}\right.\\
\qquad\qquad\vdots\\
\left\{
\begin{array}{l}
h_2 \geq -(B^+_1-B^-_1)-(h_{n-1}-h_1)+(h_n-h_0)+(e^+-e^-)\\
h_{n-2} \geq -\left[-(B^+_1-B^-_1)-(h_{n-1}-h_1)+(h_n-h_0)+(e^+-e^-)\right]
\end{array}\right.\\
\\
\left\{
\begin{array}{l}
h_1 \geq h_0-1+e^-\\
h_{n-1} \geq h_n-1+e^+
\end{array}\right.
\end{array}\right.
\tag{7}
$$

Furthermore, the Poincaré-Hopf equality must be considered in the odd-dimensional case $n = 2i + 1$:

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1}(-1)^j h_j \tag{8}$$

where

$$\mathcal{B}^+ = \frac{(-1)^i}{2}B_i^+ \pm B_{i-1}^+ \pm \ldots - B_1^+$$
$$\mathcal{B}^- = \frac{(-1)^i}{2}B_i^- \pm B_{i-1}^- \pm \ldots - B_1^-$$

Moreover, in the even dimensional case $n = 2 \mod 4$, the condition that

$$h_i - \sum_{j=1}^{i-1}(-1)^{j+1}(B_j^+ - B_j^-) - \sum_{j=0}^{i-1}(-1)^j(h_{2i-j} - h_j) + (e^- - e^+) \text{ be even} \tag{9}$$

must be imposed.

The Poincaré-Hopf inequalities for isolating blocks will be the collection of constraints (7)–(9).

In [1] it is shown that

Proposition 2.1 *The systems* (10) *and* (11) *have nonnegative integral solutions* $(h_1^c\, h_1^d, \ldots, h_{n-1}^c\, h_{n-1}^d)$ *if and only if the Poincaré-Hopf inequalities* (7), (8), (9), *for isolating blocks are satisfied.*

$$n = 2i+1 \left\{ \begin{array}{l} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \;\; j = 1, \ldots, 2i \\ e^+ - 1 - h_{2i}^d = 0 \\ \left\{ \begin{array}{l} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ \vdots \\ \frac{-(B_i^+ - B_i^-)}{2} + h_i^d - h_{i+1}^c = 0 \end{array} \right. \end{array} \right. \tag{10}$$

$$n = 2i \left\{ \begin{array}{l} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \;\; j = 1, \ldots, 2i-1, \beta^i = 0 \textit{ if } j \neq i \textit{ and } 2i \neq 0 \mod 4 \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \left\{ \begin{array}{l} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{array} \right. \end{array} \right. \tag{11}$$

3 Equivalence results

In this section we work with a particular case of the Proposition 2.1 by taking $B_j^- = B_j^+$. Hence system (10) reduces to (12) and system (11) reduces to (43) in the case $n = 0 \mod 4$ or to (70) in the case $n = 2 \mod 4$. By Proposition 2.1, the systems (12), (43) and (70) have a nonnegative integral solution $(h_1^c\, h_1^d, \dots, h_{n-1}^c\, h_{n-1}^d)$ if and only if the Poincaré-Hopf inequalities (2), (3) and (4) are satisfied.

In the following we show that (12) (resp., (70)) has nonnegative integral solutions if and only if there exist nonnegative integral Betti number vectors that satisfy the Morse inequalities (14) (resp., (73)). The result involving (43) and (47), corresponding to the case $n = 0 \mod 4$, involve the additional hypothesis that the pre-assigned index data $(h_0, \dots, h_n)$ satisfy the condition $\sum_{j=0}^{n}(-1)^{j+1}h_j$ be even.

3.1 Case n odd

Suppose there are nonnegative integers $(h_0, h_1, \dots, h_{2i+1}, h_1^c, h_1^d, \dots, h_{2i}^c, h_{2i}^d)$, where $i \geq 1$, that satisfy the linear system

$$\left\{\begin{array}{rcll} h_1^c &=& h_0 - 1 & \\ h_j^c + h_j^d &=& h_j, & \text{for } j = 1, \dots, 2i \\ h_{2i}^d &=& h_{2i+1} - 1 & \\ h_j^d - h_{j+1}^c + h_{2i-j}^d - h_{2i-j+1}^c &=& 0, & \text{for } j = 1, \dots, i-1 \\ h_i^d - h_{i+1}^c &=& 0. & \end{array}\right. \tag{12}$$

Consider as fixed the nonnegative integers $(h_0, h_1, \dots, h_{2i+1})$ that form part of a solution of (12). Then, for this fixed $(h_0, h_1, \dots, h_{2i+1})$, there exists a nonnegative integral $(h_1^c, h_1^d, h_2^c, h_2^d, \dots, h_{2i}^c, h_{2i}^d)$ that solves (12) and thus the equivalent system below, obtained by multiplying the odd equations in (12) by -1:

$$\left\{\begin{array}{rcll} -h_1^c &=& -(h_0 - 1) & \\ (-1)^{j+1}(h_j^c + h_j^d) &=& (-1)^{j+1}h_j, & \text{for } j = 1, \dots, 2i \\ h_{2i}^d &=& h_{2i+1} - 1 & \\ (-1)^j(h_j^d - h_{j+1}^c + h_{2i-j}^d - h_{2i-j+1}^c) &=& 0, & \text{for } j = 1, \dots, i-1 \\ (-1)^i(h_i^d - h_{i+1}^c) &=& 0. & \end{array}\right. \tag{13}$$

It can be shown, see [1], that system (13) constitutes a network-flow problem. The coefficient matrix of (13) is the node-arc incidence matrix of a digraph. Each equation in (13) represents flow balance in the corresponding node. Variables represent flow along the arcs. The network contains a chain of $i-1$ cycles of length four. The arcs in the j-th cycle are associated with variables h^d_{j+1}, h^c_{2i-j}, h^d_{2i-j} and h^c_{j+1}, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle. The nodes in the j-th cycle are associated with equations $j+2$, $2i+1-j$, $2i+2+j$ and $2i+3+j$, of (13). Thus the node associated with the $(2i+2+j)$-th equation of (13) constitutes the intersection of cycles $j-1$ and j. The arc sequence associated with $(h^c_1, h^d_1, h^c_{2i}, h^d_{2i})$, the variables still unaccounted for, form a nonoriented path that is adjacent to the first cycle. The intersection of this path and the first cycle is the node associated with equation $2i+3$ of (13). Arcs corresponding to flow variables $(h^c_1, h^d_1, h^c_2, h^d_2, \ldots, h^c_{2i}, h^d_{2i})$, in this order, form an Eulerian nonoriented path covering the whole digraph. This path has a zig-zag shape in the planar embedding of the digraph exemplified in Figure 1 for the case $i = 3$. Inside each node is written the right-hand-side of the associated flow balance equation.

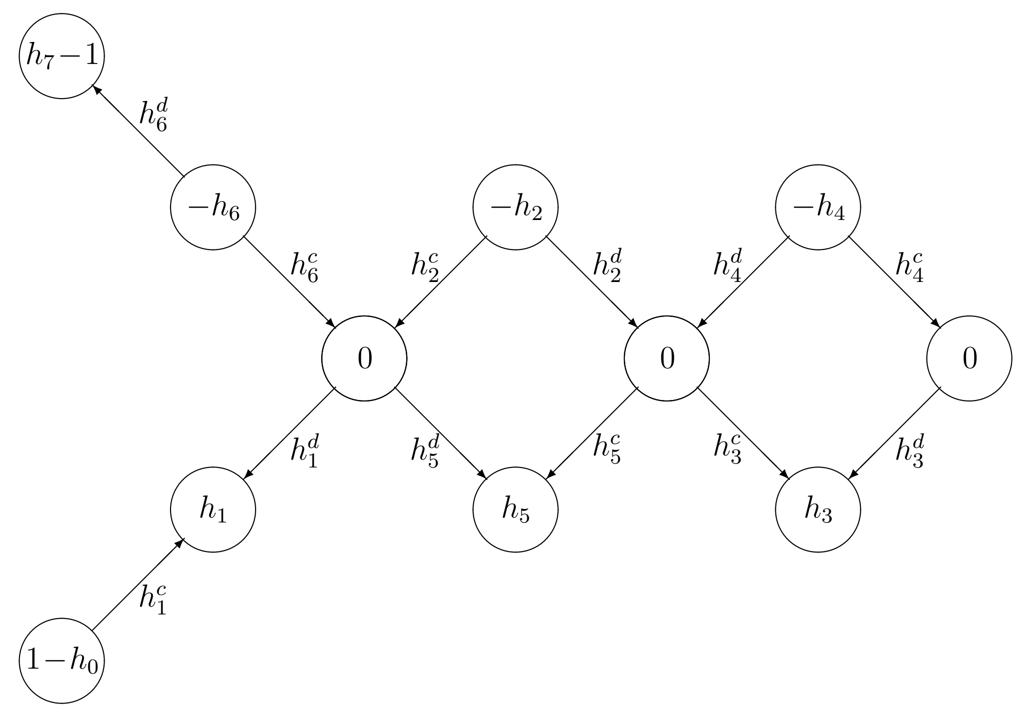


Figure 1: Network instance, for $i = 3$.

Proposition 3.1 *If, for a given set of nonnegative integers $(h_0, h_1, \ldots, h_{2i}, h_{2i+1})$ the system (13) has a nonnegative integral solution, then there exist nonnegative integers $(\gamma_0, \gamma_1, \ldots, \gamma_{2i}, \gamma_{2i+1})$*

satisfying

$$\sum_{j=0}^{2i+1-k}(-1)^{j+1}\gamma_j \begin{cases} = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j, & \textit{if } k=0 \\ \geq \displaystyle\sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, & \textit{if } 1\leq k\leq 2i+1,\ k \textit{ odd} \\ \leq \displaystyle\sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, & \textit{if } 1\leq k\leq 2i+1,\ k \textit{ even} \end{cases} \tag{14}$$

and

$$\gamma_0 = \gamma_{2i+1} = 1 \tag{15}$$

$$\gamma_j = \gamma_{2i+1-j}, \quad \textit{for } 1\leq j\leq i \tag{16}$$

Proof: Let $h^{cd} = (h_1^c, h_1^d, h_2^c, h_2^d, \ldots, h_{2i}^c, h_{2i}^d)$ be a nonnegative integral solution of (13). We assume, without loss of generality, that h^{cd} satisfies the equations

$$h_j^c h_{2i+1-j}^d = 0, \quad \text{for } j = 2,\ldots,i, \tag{17}$$

since, given an arbitrary flow h^{cd} satisfying (13), one can reduce to zero at least one of the flows in (17) by sending through the $(j-1)$-th cycle a circulation of value $\min\{h_j^c, h_{2i+1-j}^d\}$ in the direction opposite to that of the arc associated with h_j^c. The integral circulation thus constructed is a solution of the homogeneous version of system (13), whose addition to the flow (a solution of system (13)) results in yet another nonnegative integral flow satisfying (13) and (17).

We claim that the vector γ defined by

$$\gamma_0 = \gamma_{2i+1} = 1$$
$$\gamma_j = \begin{cases} h_j^d - h_{j+1}^c, & \text{if } 1\leq j < i \\ h_i^d, & \text{if } j = i \\ h_{i+1}^c, & \text{if } j = i+1 \\ -h_{j-1}^d + h_j^c, & \text{if } i+2\leq j\leq 2i \end{cases} \tag{18}$$

is a nonnegative integral solution of (14), (15) and (16).

Integrality of γ follows easily from the integrality of h^{cd}. Furthermore, equation $2i+2+j$ of (13) implies

a) If $1 \leq j < i$

$$\begin{aligned} (-1)^j (h^d_j - h^c_{j+1} + h^d_{2i-j} - h^c_{2i-j+1}) &= \\ (-1)^j (\gamma_j - \gamma_{2i+1-j}) &= 0. \end{aligned} \tag{19}$$

b) If $j = i$

$$\begin{aligned} (-1)^i (h^d_i - h^c_{i+1}) &= \\ (-1)^i (\gamma_i - \gamma_{i+1}) &= 0. \end{aligned} \tag{20}$$

Equations (19), (20) and the definition of γ_0 and γ_{2i+1} given in (18) imply that the γ defined satifies the boundary and symmetry conditions (15)–16).

Given that (16) is already established, the nonnegativity of γ is established if we show that either γ_j or γ_{2i+1-j}, for $0 \leq j \leq i$, is nonnegative. This is trivially true for $j = 0$ and i. Consider $1 \leq j \leq i-1$. By definition and (16),

$$\gamma_j = h^d_j - h^c_{j+1} = -h^d_{2i-j} + h^c_{2i+1-j} = \gamma_{2i+1-j}.$$

From (17) we have that $h^c_{j+1} h^d_{2i-j} = 0$. If $h^c_{j+1} = 0$, then $\gamma_{2i+1-j} = \gamma_j = h^d_j - h^c_{j+1} = h^d_j \geq 0$. If $h^d_{2i-j} = 0$, then $\gamma_j = \gamma_{2i+1-j} = -h^d_{2i-j} + h^c_{2i+1-j} = h^c_{2i+1-j} \geq 0$. Therefore the vector γ defined above is nonnegative. In the following it will be shown that it also satisfies (14).

Adding equations 1 through $2i+2$ of (13) we obtain

$$\begin{aligned}
-h_1^c + \sum_{j=1}^{2i}(-1)^{j+1}(h_j^c + h_j^d) + h_{2i}^d &= \\
-h_1^c + h_1^c + h_1^d + \sum_{j=2}^{2i-1}(-1)^{j+1}(h_j^c + h_j^d) - h_{2i}^c - h_{2i}^d + h_{2i}^d &= \\
h_1^d + \sum_{j=2}^{2i-1}(-1)^{j+1}(h_j^c + h_j^d) - h_{2i}^c &= \\
\sum_{j=1}^{2i-1}(-1)^{j+1}h_j^d + \sum_{j=2}^{2i}(-1)^{j+1}h_j^c &= \\
\sum_{j=1}^{2i-1}(-1)^{j+1}h_j^d + \sum_{j=1}^{2i-1}(-1)^{j}h_{j+1}^c &= \\
\sum_{j=1}^{2i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) &= -(h_0 - 1) + \sum_{j=1}^{2i}(-1)^{j+1}h_j + h_{2i+1} - 1 \\
&= \sum_{j=0}^{2i+1}(-1)^{j+1}h_j. \qquad (21)
\end{aligned}$$

The alternate sum of γ's, according to (18), gives

$$\begin{aligned}
\sum_{j=0}^{2i+1}(-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}h_i^d + (-1)^{i+2}h_{i+1}^c + \\
&\quad \sum_{j=i+2}^{2i}(-1)^{j+1}(-h_{j-1}^d + h_j^c) + 1 \\
&= \sum_{j=1}^{i}(-1)^{j+1}(h_j^d - h_{j+1}^c) + \sum_{j=i+1}^{2i-1}(-1)^{j}(-h_j^d + h_{j+1}^c) \\
&= \sum_{j=1}^{2i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c). \qquad (22)
\end{aligned}$$

Equations (21) and (22) imply that

$$\sum_{j=0}^{2i+1}(-1)^{j+1}\gamma_j = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j, \qquad (23)$$

that is, γ satisfies the first equation in (14).

Now consider the sum of equations 1 through $2i+2-\ell$ of (13), where $1 \le \ell \le 2i$:

$$\begin{aligned}
-h_1^c + \sum_{j=1}^{2i+1-\ell}(-1)^{j+1}(h_j^c+h_j^d) &= \\
h_1^d + \sum_{j=2}^{2i+1-\ell}(-1)^{j+1}(h_j^c+h_j^d) &= \\
\sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j^d + \sum_{j=2}^{2i+1-\ell}(-1)^{j+1}h_j^c + (-1)^{2i+2-\ell}h_{2i+1-\ell}^d &= \\
\sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{2i+2-\ell}h_{2i+1-\ell}^d &= -(h_0-1) + \sum_{j=1}^{2i+1-\ell}(-1)^{j+1}h_j.
\end{aligned}$$

The last equality implies

$$-1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{2i+2-\ell}h_{2i+1-\ell}^d = \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j. \tag{24}$$

We consider three cases, when calculating the partial alternate sum of the first $2i+2-\ell$ components of γ:

a) $1 \le \ell \le i$

$$\begin{aligned}
\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}h_i^d + (-1)^{i+2}h_{i+1}^c + \\
&\quad \sum_{j=i+2}^{2i+1-\ell}(-1)^{j+1}(-h_{j-1}^d + h_j^c) \\
&= -1 + \sum_{j=1}^{i}(-1)^{j+1}(h_j^d - h_{j+1}^c) + \sum_{j=i+1}^{2i-\ell}(-1)^{j}(-h_j^d + h_{j+1}^c) \\
&= -1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c). \qquad (25)
\end{aligned}$$

Substituting (25) in (24) we obtain

$$\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+2-\ell}h_{2i+1-\ell}^d = \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j. \tag{26}$$

Taking into account the fact that $h_j^d \geq 0$ for all $j = 1, \ldots, 2i$, equation (26) implies

$$\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}\gamma_j \begin{cases} \geq \displaystyle\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ odd} \\ \leq \displaystyle\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ even} \end{cases} \tag{27}$$

which means γ defined in (18) satisfies the inequalities in (14), for $1 \leq \ell \leq i$.

b) $\ell = i + 1$

$$\begin{aligned} \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{i+1}h_i^d \\ &= -1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{2i+2-\ell}h_{2i+1-\ell}^d. \end{aligned} \tag{28}$$

From (28) and (24) we conclude that, for $\ell = i + 1$,

$$\sum_{j=0}^{i}(-1)^{j+1}\gamma_j = \sum_{j=0}^{i}(-1)^{j+1}h_j, \tag{29}$$

and thus γ also satisfies (14) for $\ell = i + 1$.

c) $i + 2 \leq \ell \leq 2i$

$$\begin{aligned} \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{2i+1-\ell} (-1)^{j+1}(h_j^d - h_{j+1}^c) \\ &= -1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{2i+2-\ell}(h_{2i+1-\ell}^d - h_{2i+2-\ell}^c). \end{aligned} \tag{30}$$

Using (30) and (24) we obtain

$$\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}\gamma_j + (-1)^{2i+2-\ell}h_{2i+2-\ell}^c = \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}h_j, \tag{31}$$

which implies (27), given the nonnegativity of h_j^c, for all j.

The first equation in (13), $-h_1^c = -(h_0 - 1)$ and the fact that $h_1^c \geq 0$ imply $h_0 \geq 1 = \gamma_0$, so the last inequality in (14) is also true.

Thus we have established that the γ defined in (18) satisfies constraints in (14). ■

The following proposition establishes the converse of Proposition 3.1, i.e., if (14)–(16) has a nonnegative integral solution then so does (12).

Proposition 3.2 *If, for a given set of nonnegative integers $(h_0, h_1, \ldots, h_{2i}, h_{2i+1})$, there is a nonnegative integral $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i}, \gamma_{2i+1})$ that satisfies (14)–(16), then system (12) has a nonnegative integral solution.*

Proof: We may assume without loss of generality that the nonnegative integral solution γ of (14)–(16) saturates the inequality corresponding to $k = i + 1$ in (14),

$$\sum_{j=0}^{i}(-1)^{j+1}\gamma_j = \sum_{j=0}^{i}(-1)^{j+1}h_j,$$

since, if we fix at their current values all γ's except γ_i and γ_{i+1}, which we let free to vary, then all inequalities in (14–16) are satisfied as long as the following conditions hold:

$$\begin{aligned} \gamma_i &= \gamma_{i+1} && (32)\\ (-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}\gamma_j &\leq (-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}h_j && (33)\\ \gamma_i, \gamma_{i+1} &\geq 0. && (34) \end{aligned}$$

This is due to the fact that all linear inequalities in (14) except the above either contain the difference $\pm(\gamma_i - \gamma_{i+1})$ or do not contain neither γ_i nor γ_{i+1}. It follows that

$$0 \leq \gamma_i = \gamma_{i+1} \leq (-1)^{i+1}\left(\sum_{j=0}^{i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j\right) = h_i + (-1)^{i+1}\sum_{j=0}^{i-1}(-1)^{j+1}(h_j - \gamma_j). \tag{35}$$

The interval $\left[0, h_i + (-1)^{i+1}\sum_{j=0}^{i-1}(-1)^{j+1}(h_j - \gamma_j)\right]$ is nonempty (it contains the current value of γ_i) and has integral-valued endpoints, since γ is by assumption integral and nonnegative. Thus we may assume that γ_i (and, therefore, γ_{i+1}) is at the right end side of the above interval.

Now define the vector h^{cd} according to

$$\begin{aligned}
h^d_{2i+1-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma_j), \text{ for } 1 \le \ell \le i && (36)\\
h^c_{2i+2-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma_j), \text{ for } i+2 \le \ell \le 2i+1 && (37)\\
h^d_\ell &= \gamma_\ell + h^c_{\ell+1}, \qquad \text{for } 1 \le \ell \le i-1 && (38)\\
h^c_\ell &= \gamma_\ell + h^d_{\ell-1}, \qquad \text{for } i+2 \le \ell \le 2i && (39)\\
h^d_i &= \gamma_i && (40)\\
h^c_{i+1} &= \gamma_{i+1}. && (41)
\end{aligned}$$

We claim that the vector h^{cd} thus defined is a nonnegative integral solution of (12). Integrality follows easily from the integrality of γ and h. Rewriting (14) as

$$(-1)^k \sum_{j=0}^{2i+1-k} (-1)^{j+1}\gamma_j \begin{cases} = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j, & \text{if } k = 0 \\ \le (-1)^k \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, & \text{if } 1 \le k \le 2i+1 \end{cases} \tag{42}$$

we conclude that the components of h^{cd} defined in (36) and (37) are nonnegative. This fact, on the other hand, together with the hypotheses $h \ge 0$ and $\gamma \ge 0$, imply that the components defined in (38)–(41) are also nonnegative.

We must now verify that h^{cd} satisfies the constraints in (12). Equation (37) for $\ell = 2i+1$ and (15) imply

$$h^c_1 = (-1)^{2i+1}(-1)^1(h_0 - \gamma_0) = h_0 - 1,$$

thus h^{cd} satisfies the first equation in (12). Equation (36) for $\ell = 1$, (15) and the first equation of (14) imply

$$\begin{aligned}
h^d_{2i} &= (-1)^1 \sum_{j=0}^{2i}(-1)^{j+1}(h_j - \gamma_j)\\
&= -\left(\sum_{j=0}^{2i+1}(-1)^{j+1}(h_j - \gamma_j) - (h_{2i+1} - \gamma_{2i+1})\right)\\
&= h_{2i+1} - \gamma_{2i+1}\\
&= h_{2i+1} - 1,
\end{aligned}$$

which implies the $(2i+2)$-th equation of (12).

The last i equations of (12),

$$\begin{aligned} h^d_j - h^c_{j+1} + h^d_{2i-j} - h^c_{2i+1-j} &= (\gamma_j + h^c_{j+1}) - h^c_{j+1} + h^d_{2i-j} - (\gamma_{2i+1-j} + h^d_{2i-j}) \\ &= \gamma_j - \gamma_{2i+1-j} = 0, \qquad \text{for } 1 \le j \le i-1 \\ h^d_i - h^c_{i+1} &= \gamma_i - \gamma_{i+1} = 0, \end{aligned}$$

are validated using (16).

Let $i+2 \le \ell \le 2i$. Summing the appropriate equations in (37) we obtain

$$\begin{aligned} h^c_{2i+2-\ell} + h^c_{2i+1-\ell} &= (-1)^\ell \left(\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma_j) - \sum_{j=0}^{2i-\ell} (-1)^{j+1}(h_j - \gamma_j) \right) \\ &= (-1)^\ell (-1)^{2i+2-\ell}(h_{2i+1-\ell} - \gamma_{2i+1-\ell}) \\ &= h_{2i+1-\ell} - \gamma_{2i+1-\ell}, \end{aligned}$$

which implies, using (38), the $(2i+2-\ell)$-th equation of (12):

$$\gamma_{2i+1-\ell} + h^c_{2i+2-\ell} + h^c_{2i+1-\ell} = h^d_{2i+1-\ell} + h^c_{2i+1-\ell} = h_{2i+1-\ell}.$$

Thus equations $(1+j)$ of (12), for $1 \le j \le i-1$, are satisfied by h^{cd}.

The $(1+i)$-th equation of (12) follows from (37), (40) and (35):

$$\begin{aligned} h^c_i + h^d_i &= (-1)^{i+2} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j) \ + \ \gamma_i \\ &= (-1)^{i+2} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j) + h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j) \\ &= h_i, \end{aligned}$$

and the $(1+(i+1))$-th equation of (12) follows from (36), (41) and (35):

$$\begin{aligned} h^c_{i+1} + h^d_{i+1} &= \gamma_{i+1} + (-1)^i \sum_{j=0}^{i+1} (-1)^{j+1}(h_j - \gamma_j) \\ &= h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j) + (-1)^i \sum_{j=0}^{i+1} (-1)^{j+1}(h_j - \gamma_j) \\ &= h_i + (-1)^i \left((-1)^{i+1}(h_i - \gamma_i) + (-1)^{i+2}(h_{i+1} - \gamma_{i+1}) \right) \\ &= h_i - (h_i - \gamma_i) + (h_{i+1} - \gamma_{i+1}) \\ &= h_{i+1}. \end{aligned}$$

Finally, we show that h^{cd} satisfies the remaining equations of (12): equations $(i+2+j)$, for $1 \leq j \leq i-1$. Let $1 \leq \ell \leq i-1$. Using (36) we have

$$\begin{aligned} h^d_{2i+1-\ell} + h^d_{2i-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell} (_1)^{j+1}(h_j - \gamma_j) + (-1)^{ell+1} \sum_{j=0}^{2i-\ell} (_1)^{j+1}(h_j - \gamma_j) \\ &= (-1)^\ell (-1)^{2i+2-\ell}(h_{2i+1-\ell} - \gamma_{2i+1-\ell}). \end{aligned}$$

Therefore, taking into account definition (39)

$$h^d_{2i+1-\ell} + \gamma_{2i+1-\ell} + h^d_{2i-\ell} = h^d_{2i+1-\ell} + h^c_{2i+1-\ell} = h_{2i+1-\ell}.$$

■

3.2 Case $\mathbf{n = 0 \mod 4}$

Assume $n = 2i$, where $i \geq 2$ is even. Suppose there are nonnegative integers $(h_0, h_1, \ldots, h_{2i+1}, h^c_1, h^d_1, \ldots, h^c_{2i}, h^d_{2i})$ that satisfy the linear system

$$\left\{ \begin{array}{rcll} h^c_1 &=& h_0 - 1 & \\ h^c_j + h^d_j &=& h_j, & \text{for } j = 1, \ldots, i-1 \\ h^c_i + \beta + h^d_i &=& h_i & \\ h^c_j + h^d_j &=& h_j, & \text{for } j = i+1, \ldots, 2i-1 \\ h^d_{2i-1} &=& h_{2i} - 1 & \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} &=& 0, & \text{for } j = 1, \ldots, i-1 \end{array} \right. \tag{43}$$

Fix $(h_0, h_1, \ldots, h_{2i})$ at nonnegative integer values such that (43) has a nonnegative integral solution $(h^c_1, h^d_1, h^c_2, h^d_2, \ldots, h^c_{2i-1}, h^d_{2i-1})$. Then this latter vector satisfies (44) below, equivalent to (43), obtained by multiplying by -1 the odd equations up to $i+1$ and the even equations thereafter:

$$\left\{ \begin{array}{rcll} -h^c_1 &=& -(h_0 - 1) & \\ (-1)^{j+1}(h^c_j + h^d_j) &=& (-1)^{j+1} h_j, & \text{for } j = 1, \ldots, i-1 \\ -h^c_i - \beta - h^d_i &=& -h_i & \\ (-1)^j(h^c_j + h^d_j) &=& (-1)^j h_j, & \text{for } j = i+1, \ldots, 2i-1 \\ h^d_{2i-1} &=& h_{2i} - 1 & \\ (-1)^j(h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j}) &=& 0, & \text{for } j = 1, \ldots, i-1. \end{array} \right. \tag{44}$$

System (44) may be decomposed, see [1], into two independent systems: (45) and (46). System (45) is a network-flow problem defined on a digraph whose incidence matrix is the coefficient matrix of (13) for $n = 2i - 1$. Thus for the case $i = 4$ the digraph has precisely the structure depicted in Figure 1, but of course with different labels on arcs and nodes. In the general case, the digraph with incidence matrix given by the coefficient matrix of (45) has $i - 2$ cycles of length four plus a nonoriented path also of length four. The j-th cycle contains arcs associated with variables h^d_{j+1}, h^c_{2i-1-j}, h^d_{2i-1-j} and h^c_{j+1}, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle. Nodes of the j-th cycle are associated with equations $j + 2$, $2i + 2 + j$, $2i - j$ and $2i + 1 + j$.

$$\left\{ \begin{array}{rcll} -h^c_1 & = & -(h_0 - 1) & \\ (-1)^{j+1}(h^c_j + h^d_j) & = & (-1)^{j+1} h_j, & \text{for } j = 1, \ldots, i-1 \\ (-1)^{j}(h^c_j + h^d_j) & = & (-1)^{j} h_j, & \text{for } j = i+1, \ldots, 2i-1 \\ h^d_{2i-1} & = & h_{2i} - 1 & \\ (-1)^j(h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j}) & = & 0, & \text{for } j = 1, \ldots, i-2 \\ -h^d_{i-1} + h^c_{i+1} & = & -\sum_{j=0}^{i-1}(-1)^{j+1}h_j - \sum_{j=i+1}^{2i}(-1)^j h_j & \end{array} \right. \tag{45}$$

$$\left\{ \begin{array}{rcl} -h^c_i - \beta - h^d_i & = & -h_i \\ h^c_i - h^d_i & = & \sum_{j=0}^{i-1}(-1)^{j+1}h_j + \sum_{j=i+1}^{2i}(-1)^j h_j \end{array} \right. \tag{46}$$

Proposition 3.3 *If, for a given set of nonnegative integers* $(h_0, h_1, \ldots, h_{2i})$ *the system* (44) *has a nonnegative integral solution, then there exist nonnegative integers* $(\gamma_0, \gamma_1, \ldots, \gamma_{2i})$ *satisfying*

$$\sum_{j=0}^{2i-k}(-1)^{j+1}\gamma_j \left\{ \begin{array}{ll} = \sum_{j=0}^{2i}(-1)^{j+1}h_j, & \text{if } k = 0 \\ \leq \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq 2i,\ k \text{ odd} \\ \geq \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq 2i,\ k \text{ even} \end{array} \right. \tag{47}$$

and

$$\gamma_0 = \gamma_{2i} = 1 \tag{48}$$

$$\gamma_j = \gamma_{2i-j}, \quad \text{for } 1 \leq j \leq i-1. \tag{49}$$

The vector γ *further satisfies* γ_i *even if and only if* $\sum_{j=0}^{2i}(-1)^j h_j$ *is even.*

Proof: If γ satisfies (47) and (49) then

$$\begin{aligned}\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j &= \\ 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j + (-1)^{i+1}\gamma_i &= \sum_{j=0}^{2i}(-1)^{j+1}h_j.\end{aligned}$$

Therefore if $\gamma_i = \sum_{j=0}^{2i}(-1)^j h_j - 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j$ is even, then $\sum_{j=0}^{2i}(-1)^j h_j$ is also even.

On the other hand, if $\sum_{j=0}^{2i}(-1)^j h_j$ is even then so is $\Gamma = \sum_{j=0}^{2i}(-1)^{j+1}h_j - 2\sum_{j=i}^{2i}(-1)^{j+1}h_j = \sum_{j=0}^{i-1}(-1)^{j+1}h_j + \sum_{j=i}^{2i}(-1)^j h_j$ and, in this case, we are able to construct, as shown in the sequence, a solution γ satisfying (47), (49) and such that γ_i is even.

Let $h^{cd} = (h_1^c, h_1^d, \ldots, h_i^c, \beta, h_i^d, \ldots, h_{2i-1}^c, h_{2i-1}^d)$ be a nonnegative integral solution of (44). If Γ is even, then it is easy to see from (46) that β is even. We assume, without loss of generality, that h^{cd} satisfies the equations

$$h_j^c h_{2i-j}^d = 0, \quad \text{for } j = 2, \ldots, i. \tag{50}$$

For $2 \leq j \leq i-1$, the trick is the same as in the proof of Proposition 3.1, that is, to add to an arbitrary solution h^{cd} of (44) the appropriate circulation along the $(j-1)$-th cycle. However, when $j = i$, the homogeneous system solution $\tilde{h}^{cd}$ that is added to h^{cd} is such that $(\tilde{h}_i^c, \tilde{\beta}, \tilde{h}_i^d) = \min\{h_i^c, h_i^d\}(-1, 2, -1)$ and the remaining components are zero. Notice that both types of addition preserve nonnegativity and integrality of the solution of (44), while the last one, the only one that may modify the value of component β, preserves its parity, as should be expected.

We claim that the vector γ defined by

$$\begin{aligned}&\gamma_0 = \gamma_{2i} = 1\\ &\gamma_j = \begin{cases} h_j^d - h_{j+1}^c, & \text{if } 1 \leq j \leq i-1\\ \beta, & \text{if } j = i\\ -h_{j-1}^d + h_j^c, & \text{if } i+1 \leq j \leq 2i-1\end{cases}\end{aligned} \tag{51}$$

is a nonnegative integral solution of (47)–(49).

The integrality of h^{cd} implies the integrality of γ. Equation $2i+1+j$, for $1 \leq j \leq i-1$, of (44) implies

$$\begin{aligned}(-1)^j(h_j^d - h_{j+1}^c + h_{2i-1-j}^d - h_{2i-j}^c) &= \\ (-1)^j(\gamma_j - \gamma_{2i-j}) &= 0,\end{aligned} \tag{52}$$

which means that γ satisfies (49). Condition (48) is true by definition.

Nonnegativity of γ_j is trivial for $j = 0$, i and $2i$, and, for $1 \leq j \leq 2i-1$, follows from (49) and (50), as in the proof of Proposition 3.1. It remains to be shown that it also satisfies (47). The proof is analogous to that of Proposition 3.1, thus it is convenient to define the system (53), equivalent to (43), obtained by multiplying odd equations by -1:

$$\left\{\begin{array}{rcll} -h_1^c &=& -(h_0-1) & \\ (-1)^{j+1}(h_j^c+h_j^d) &=& (-1)^{j+1}h_j, & \text{for } j=1,\ldots,i-1 \\ -h_i^c-\beta-h_i^d &=& -h_i & \\ (-1)^{j+1}(h_j^c+h_j^d) &=& (-1)^{j+1}h_j, & \text{for } j=i+1,\ldots,2i-1 \\ -h_{2i-1}^d &=& -(h_{2i}-1) & \\ (-1)^{j+1}(h_j^d-h_{j+1}^c+h_{2i-1-j}^d-h_{2i-j}^c) &=& 0, & \text{for } j=1,\ldots,i-1. \end{array}\right. \tag{53}$$

Adding equations 1 through $2i+1$ of (53) we obtain

$$\begin{aligned} -h_1^c+\sum_{j=1}^{2i-1}(-1)^{j+1}(h_j^c+h_j^d)-h_{2i-1}^d-\beta &= \\ h_1^d+\sum_{j=2}^{2i-2}(-1)^{j+1}(h_j^c+h_j^d)+h_{2i-1}^c-\beta &= \\ \sum_{j=1}^{2i-2}(-1)^{j+1}h_j^d+\sum_{j=2}^{2i-1}(-1)^{j+1}h_j^c-\beta &= \\ \sum_{j=1}^{2i-2}(-1)^{j+1}h_j^d+\sum_{j=1}^{2i-2}(-1)^{j}h_{j+1}^c-\beta &= \\ \sum_{j=1}^{2i-2}(-1)^{j+1}(h_j^d-h_{j+1}^c)-\beta &= -(h_0-1)+\sum_{j=1}^{2i-1}(-1)^{j+1}h_j-(h_{2i}-1) \\ &= 2+\sum_{j=0}^{2i}(-1)^{j+1}h_j. \end{aligned} \tag{54}$$

The alternate sum of γ's, according to (51), gives

$$\begin{aligned}\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j &= -1+\sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d-h_{j+1}^c)-\beta+\sum_{j=i+1}^{2i-1}(-1)^{j+1}(-h_{j-1}^d+h_j^c)-1\\ &= -2+\sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d-h_{j+1}^c)+\sum_{j=i}^{2i-2}(-1)^{j+1}(h_j^d-h_{j+1}^c)-\beta\\ &= -2+\sum_{j=1}^{2i-2}(-1)^{j+1}(h_j^d-h_{j+1}^c)-\beta. \qquad (55)\end{aligned}$$

Comparing (54) and (55) we conclude that

$$\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j=\sum_{j=0}^{2i}(-1)^{j+1}h_j, \qquad (56)$$

that is, γ satisfies the first equation in (47).

When calculating the partial sum of equations 1 through $2i+1-\ell$ of (53), there are two cases to consider:

a) $1\le\ell\le i$

$$\begin{aligned}-h_1^c+\sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^c+h_j^d)-\beta &=\\ h_1^d+\sum_{j=2}^{2i-\ell}(-1)^{j+1}(h_j^c+h_j^d)-\beta &=\\ \sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j^d-\sum_{j=1}^{2i-1-\ell}(-1)^{j+1}h_{j+1}^c-\beta &=\\ \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}(h_j^d-h_{j+1}^c)+(-1)^{2i+1-\ell}h_{2i-\ell}^d-\beta &= -(h_0-1)+\sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j\\ &= 1+\sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \qquad (57)\end{aligned}$$

b) $i+1 \leq \ell \leq 2i$

$$\begin{aligned}
-h_1^c + \sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^c + h_j^d) &= \\
h_1^d + \sum_{j=2}^{2i-\ell}(-1)^{j+1}(h_j^c + h_j^d) &= \\
\sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j^d - \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}h_{j+1}^c &= \\
\sum_{j=1}^{2i-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) + (-1)^{2i+1-\ell}h_{2i+1-\ell}^c &= -(h_0-1) + \sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j \\
&= 1 + \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \qquad (58)
\end{aligned}$$

Likewise, we separate into two possibilities the partial sum $\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j$:

a) $1 \leq \ell \leq i$

$$\begin{aligned}
\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) - \beta + \sum_{j=i+1}^{2i-\ell}(-1)^{j+1}(-h_{j-1}^d + h_j^c) \\
&= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h_j^d - h_{j+1}^c) + \sum_{j=i}^{2i-1-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) - \beta \\
&= -1 + \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}(h_j^d - h_{j+1}^c) - \beta. \qquad (59)
\end{aligned}$$

Comparing (59) and (57) we conclude that

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+1-\ell}h_{2i-\ell}^d = \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \qquad (60)$$

Using the fact that $h_j^d \geq 0$ for all $j = 1, \ldots, 2i-1$, equation (60) implies

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j \begin{cases} \leq \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ odd} \\ \geq \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ even} \end{cases} \qquad (61)$$

which means γ defined in (51) satisfies the inequalities in (47), for $1 \leq \ell \leq i$.

b) $i+1 \leq \ell \leq 2i-1$

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j \quad = \quad -1+\sum_{j=1}^{2i-\ell}(-1)^{j+1}(h^d_j - h^c_{j+1}). \tag{62}$$

Comparing (62) and (58) we conclude that

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+1-\ell}h^c_{2i+1-\ell} = \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \tag{63}$$

Using the fact that $h^c_j \geq 0$ for all $j = 1, \ldots, 2i-1$, equation (63) implies that γ satifies (61) for $i+1 \leq \ell \leq 2i-1$.

Finally, from (58) with $\ell = 2i$ we have $-h^c_1 = 1 - h_0$ and since, by hypothesis, $h^c_1 \geq 0$, then $-h_0 \leq -1 = -\gamma_0$ holds, the last inequality in (47).

We have thus established that the γ defined in (51) satisfies constraints in (47). ∎

Proposition 3.4 *If, for a given set of nonnegative integers* $(h_0, h_1, \ldots, h_{2i}, h_{2i})$, *there is a nonnegative integral* $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i})$ *that satisfies* (47)–(49), *then system* (43) *has a nonnegative integral solution.*

Proof: The proof is by construction. Consider the vector h^{cd} defined by

$$\begin{aligned}
h^d_{2i-\ell} &= (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j - \gamma_j), \text{ for } 1 \leq \ell \leq i && (64)\\
h^c_{2i+1-\ell} &= (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j - \gamma_j), \text{ for } i+1 \leq \ell \leq 2i && (65)\\
h^d_\ell &= \gamma_\ell + h^c_{\ell+1}, \qquad \text{for } 1 \leq \ell \leq i-1 && (66)\\
h^c_\ell &= \gamma_\ell + h^d_{\ell-1}, \qquad \text{for } i+1 \leq j \leq 2i-1 && (67)\\
\beta &= \gamma_i. && (68)
\end{aligned}$$

Clearly, the vector h^{cd} given by (64)–(68) is integral, since so are γ and h. Rewriting (47) as

$$(-1)^{k+1}\sum_{j=0}^{2i-k}(-1)^{j+1}\gamma_j \left\{ \begin{array}{ll} = & \sum_{j=0}^{2i}(-1)^{j+1}h_j, \quad \text{if } k=0 \\ \leq & (-1)^{k+1}\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, \quad \text{if } 1 \leq k \leq 2i \end{array}\right. \tag{69}$$

we conclude that the components of h^{cd} defined in (64) and (65) are nonnegative. This fact, on the other hand, together with the hypotheses $h \geq 0$ and $\gamma \geq 0$, imply that the components defined in (66)–(68) are also nonnegative.

From (65) with $\ell = 2i$ we obtain

$$h_1^c = (-1)^{2i+1}(-1)^{0+1}(h_0 - \gamma_0) = h_0 - 1,$$

that is, the first equation in (43). On the other hand, (64) with $\ell = 1$ gives

$$\begin{aligned} h_{2i-1}^d &= (-1)^{1+1}\sum_{j=0}^{2i-1}(-1)^{j+1}(h_j - \gamma_j) \\ &= \sum_{j=0}^{2i}(-1)^{j+1}(h_j - \gamma_j) - (-1)^{2i+1}(h_{2i} - \gamma_{2i}) \\ &= h_{2i} - 1, \end{aligned}$$

which is the $(2i+1)$-th equation of (43).

The last $i-1$ equations of (43) are easily verified:

$$\begin{aligned} h_j^d - h_{j+1}^c + h_{2i-1-j}^d - h_{2i-j}^c &= (\gamma_j + h_{j+1}^c) - h_{j+1}^c + h_{2i-1-j}^d - (\gamma_{2i-j} + h_{2i-1-j}^d) \\ &= \gamma_j - \gamma_{2i-j} = 0, \qquad \text{for } 1 \leq j \leq i-1. \end{aligned}$$

Let $i+1 \leq \ell \leq 2i-1$. Summing the appropriate equations in (65) we obtain

$$\begin{aligned} h_{2i+1-\ell}^c + h_{2i-\ell}^c &= (-1)^{\ell+1}\left(\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j - \gamma_j) - \sum_{j=0}^{2i-\ell-1}(-1)^{j+1}(h_j - \gamma_j)\right) \\ &= (-1)^{\ell+1}(-1)^{2i+1-\ell}(h_{2i-\ell} - \gamma_{2i-\ell}) \\ &= h_{2i-\ell} - \gamma_{2i-\ell}, \end{aligned}$$

which implies, using (66), the $(2i+1-\ell)$-th equation of (43):

$$\gamma_{2i-\ell} + h_{2i+1-\ell}^c + h_{2i-\ell}^c = h_{2i-\ell}^d + h_{2i-\ell}^c = h_{2i-\ell}.$$

Thus equations $(1+j)$ of (43), for $1 \leq j \leq i-1$, are satisfied by h^{cd}.

Verifying the $(i+1)$-th equation of (43):

$$\begin{aligned} h_i^c + \beta + h_i^d &= (-1)^{i+2}\sum_{j=0}^{2i-(i+1)}(-1)^{j+1}(h_j - \gamma_j) + \gamma_i + (-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}(h_j - \gamma_j) \\ &= (-1)^{i+1}(-1)^{i+1}(h_i - \gamma_i) + \gamma_i \\ &= h_i. \end{aligned}$$

Finally, letting $1 \le \ell \le i-1$, we have

$$\begin{aligned} h^d_{2i-\ell} + h^d_{2i-1-\ell} &= (-1)^{\ell+1}\left(\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j-\gamma_j) - \sum_{j=0}^{2i-1-\ell}(-1)^{j+1}(h_j-\gamma_j)\right) \\ &= (-1)^{\ell+1}(-1)^{2i+1-\ell}(h_{2i-\ell}-\gamma_{2i-\ell}) \\ &= h_{2i-\ell}-\gamma_{2i-\ell}, \end{aligned}$$

which implies, using (66)

$$h^d_{2i-\ell} + h^d_{2i-1-\ell} + \gamma_{2i-\ell} = h^d_{2i-\ell} + h^c_{2i-\ell} = h_{2i-\ell},$$

which completes the verification that h^{cd} satisfies (43). ■

3.3 Case $\mathbf{n = 2 \mod 4}$

In this section we assume $n = 2i$, where $i \ge 3$ is odd. Suppose there are nonnegative integers $\{h_0, h_1, \dots, h_{2i+1}, h^c_1, h^d_1, \dots, h^c_{2i}, h^d_{2i}\}$ that satisfy the linear system

$$\left\{ \begin{aligned} h^c_1 &= h_0 - 1 \\ h^c_j + h^d_j &= h_j, \qquad \text{for } j = 1, \dots, 2i-1 \\ h^d_{2i-1} &= h_{2i} - 1 \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} &= 0, \qquad \text{for } j = 1, \dots, i-1 \end{aligned} \right. \tag{70}$$

If system (70) has a solution, then so does system (71), obtained from (70) by adding variable δ in the $i+1$-th equation, since any solution of (70) may be transformed into a solution of (71) by setting δ to zero.

$$\left\{ \begin{aligned} h^c_1 &= h_0 - 1 \\ h^c_j + h^d_j &= h_j, \qquad \text{for } j = 1, \dots, i-1 \\ h^c_i + \delta + h^d_i &= h_i \\ h^c_j + h^d_j &= h_j, \qquad \text{for } j = i+1, \dots, 2i-1 \\ h^d_{2i-1} &= h_{2i} - 1 \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} &= 0, \qquad \text{for } j = 1, \dots, i-1. \end{aligned} \right. \tag{71}$$

Notice that system (71) is of the same type as system (43). Furthermore, in order for (70) to have a solution, we must have, see [1], $\sum_{j=0}^{i-1}(-1)^{j+1}h_j + \sum_{j=i}^{2i}(-1)^j h_j$ even. If we multiply by -1 the odd equations with index in the range $\{1,\dots,i\}$, and the even equations with higher indices, the resulting system, equivalent to (71), may be decomposed into the two independent linear systems. The first is given by (45) and the second by (72). Therefore the first system is a network-flow problem defined on a digraph whose incidence matrix is the coefficient matrix of (13) for $n = 2i-1$. This digraph has $i-2$ cycles of length four plus a nonoriented path also of lenght four. The j-th cycle contains arcs associated with variables h^d_{j+1}, h^c_{2i-1-j}, h^d_{2i-1-j} and h^c_{j+1}, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle. Nodes of the j-th cycle are associated with equations $j+2$, $2i+2+j$, $2i-j$ and $2i+1+j$.

$$\left\{\begin{array}{rcl} -h^c_i - \delta - h^d_i & = & -h_i \\ -h^c_i + h^d_i & = & \sum_{j=0}^{i-1}(-1)^{j+1}h_j + \sum_{j=i+1}^{2i}(-1)^j h_j \end{array}\right. \tag{72}$$

Proposition 3.5 *If, for a given set of nonnegative integers $(h_0, h_1, \dots, h_{2i})$, the system (70) has a nonnegative integral solution, then there exist nonnegative integers $(\gamma_0, \gamma_1, \dots, \gamma_{2i})$ satisfying*

$$\sum_{j=0}^{2i-k}(-1)^{j+1}\gamma_j \left\{\begin{array}{ll} = \sum_{j=0}^{2i}(-1)^{j+1}h_j, & \textit{if } k=0 \\ \leq \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \textit{if } 1 \leq k \leq 2i,\ k \textit{ odd} \\ \geq \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \textit{if } 1 \leq k \leq 2i,\ k \textit{ even} \end{array}\right. , \tag{73}$$

$$\begin{array}{l} \gamma_0 = \gamma_{2i} = 1 \\ \gamma_j = \gamma_{2i-j}, \quad \textit{if } 1 \leq j \leq i-1 \end{array} \tag{74}$$

and such that γ_i is even.

Proof: Let $h^{cd} = (h^c_1, h^d_1, \dots, h^c_i, 0, h^d_i, \dots, h^c_{2i-1}, h^d_{2i-1})$ be a nonnegative integral solution of (71) obtained by taking a nonnegative integral solution of (70) and extending it to a solution of (71) by setting δ to zero. Employing the circulation-based argument worked out in the proof of Proposition 3.1, we may justify the assumption that h^{cd} satisfies the equations

$$h^c_j h^d_{2i-j} = 0, \quad \text{for } j = 2, \dots, i-1. \tag{75}$$

When $j = i$, the homogeneous system solution $\tilde{h}^{cd}$ that is added to h^{cd} is such that $(\tilde{h}^c_i, \tilde{\delta}, \tilde{h}^d_i) = \min\{h^c_i, h^d_i\}(-1, 2, -1)$ and the remaining components are zero. Thus the value of δ in the solution considered is either zero or a positive even number.

The remainder of the proof is analogous to the proof of Proposition 3.3 from (51) on. ∎

Proposition 3.6 *If, for a given set of nonnegative integers $(h_0, h_1, \dots, h_{2i}, h_{2i})$, there is a nonnegative integral $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{2i})$, with γ_i even, that satisfies (73)–(74), then system (70) has a nonnegative integral solution.*

Proof: First we construct a nonnegative integral solution $\tilde{h}^{cd} = (h_1^c, h_1^d, \dots, h_i^c, \delta, h_i^d, \dots, h_{2i-1}^c, h_{2i-1}^d)$ to (71), with δ even:

$$
\begin{aligned}
h_{2i-\ell}^d &= (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j-\gamma_j), \text{ for } 1 \le \ell \le i && (76)\\
h_{2i+1-\ell}^c &= (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j-\gamma_j), \text{ for } i+1 \le \ell \le 2i && (77)\\
h_\ell^d &= \gamma_\ell + h_{\ell+1}^c, \qquad \text{for } 1 \le \ell \le i-1 && (78)\\
h_\ell^c &= \gamma_\ell + h_{\ell-1}^d, \qquad \text{for } i+1 \le j \le 2i-1 && (79)\\
\delta &= \gamma_i. && (80)
\end{aligned}
$$

Since the equations of (71) fall in the same pattern as (43), the proof of Proposition 3.4 may be easily adapted to show that $\tilde{h}^{cd}$ is a nonnegative integral solution of (71). The assumption that γ_i is even implies that δ is even.

Finally, suppose $\delta = 2k$, where k is a nonnegative integer. We claim $h^{cd} = (h_1^c, h_1^d, \dots, h_i^c + k, h_i^d + k, \dots, h_{2i-1}^c, h_{2i-1}^d)$ is a nonnegative integral solution of (70). In order to see that we only need to check the equations containing h_i^c and h_i^d:

$$
\begin{aligned}
h_i &= h_i^c + \delta + h_i^d = h_i^c + 2k + h_i^d = (h_i^c + k) + (h_i^d + k),\\
0 &= h_{i-1}^d - h_i^c + h_i^d - h_{i+1}^c = h_{i-1}^d - h_i^c - k + k + h_i^d - h_{i+1}^c = h_{i-1}^d - (h_i^c + k) + (h_i^d + k) - h_{i+1}^c.
\end{aligned}
$$

■

4 Polytopes

In this section we study the Morse polyhedron restricted to the nonnegative orthant $\mathcal{P}$, i.e., the set of nonnegative $\gamma = (\gamma_0, \dots, \gamma_n)$ satisfying the Morse inequalities (1) for a pre-assigned index data $(h_0, \dots, h_n)$, and the duality conditions $\gamma_k = \gamma_{n-k}$, for $k = 0, \dots, n$ and $\gamma_{n/2}$ even if n is even.

First of all we point out that $\mathcal{P}$ is a polytope, i.e., a bounded polyhedron. This follows from the nonnegative restriction on γ and the fact that taking all possible pairs of consecutive inequalities in (1) we conclude $\gamma_j \le h_j$, for $j = 1, \dots, n$.

We will show that this polytope has integral vertices, which implies $\mathcal{P}$ is the convex hull of the integral vectores in $\mathcal{P}$. Or, equivalently, $\mathcal{P}$ is the convex hull of the collection of Betti number vectors which satisfy the Morse inequalities.

4.1 Case n odd

Propositions 3.1 and 3.2 not only establish that (12) has a solution if and only if (14)–(16) also has one, they also show how to construct a nonnegative γ satisfying (14)–(16) from an appropriate nonnegative h^{cd} satisfying (12) and vice-versa. This suggest a kinship between the polyhedra defined by the two systems of inequalities.

The polyhedron made up of the nonnegative h^{cd} satisfying (12) was studied in [1], where it was shown that it is in fact an integral polytope, that is, a limited polyhedron with integral vertices. Moreover, from the development in [1] we may extract a “recipe” for constructing all nonnegative integral h^{cd} satisfying.

We show in this section that similar properties hold for the polytope of nonnegative γ satisfying (14)–(16). Furthermore, the knowledge gained about the generation of feasible h^{cd} in [1], the construction exhibited in the proof of Proposition 3.2, and the characteristics that will be established for this polytope, will lead to a mechanism to construct all γ therein.

In order to simplify the exposition, conditions (15)–(16) will be used to eliminate more than half the variables, namely γ_0, γ_{i+1}, $\ldots$, γ_{2i+1}. Using equations (15)–(16), we have that

$$\sum_{j=0}^{2i+1-k} (-1)^{j+1}\gamma_j = -1 + \sum_{j=1}^{\min\{k-1,2i+1-k\}} (-1)^{j+1}\gamma_j.$$

Also note that, conditions (15) and (16) imply that the constraints corresponding to $k = 0$, 1 and $2i + 1$ in (14) represent, in fact, constraints on $h = (h_0, \ldots, h_n)$. There is a 1-to-1 correspondence between the nonnegative $\gamma = (\gamma_0, \ldots, \gamma_{2i+1})$ satisfying (14)–(16) and the nonnegative $\gamma^r = (\gamma_1, \ldots, \gamma_i)$ satisfying (81) below. Thus, instead of $\mathcal{P}$, we may consider the polytope $\mathcal{P}^r = \{\gamma^r \in \mathbb{R}^i \mid \text{constraints in (81)}\}$.

$$\begin{aligned} 0 = \textstyle\sum_{j=0}^{2i+1}(-1)^{j+1}h_j,\ \ 0 \le h_0 - 1,\ \ 0 \le h_{2i+1} - 1 \\ \textstyle\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j \le 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j,\ \text{for } 2 \le k \le 2i,\ k \text{ even} \\ \textstyle\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j \ge 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j,\ \text{for } 2 \le k \le 2i,\ k \text{ odd} \\ \gamma^r \ge 0. \end{aligned} \tag{81}$$

Proposition 4.1 *Polytope $\mathcal{P}^r$ given by (81) satisfies the following properties:*

1. *The vertices of $\mathcal{P}^r$ are integral.*
2. *Each vertex of $\mathcal{P}^r$ belongs to one of the faces: $\mathcal{F}_t = \{\gamma \in \mathcal{P}^r \mid \sum_{j=1}^{i}(-1)^{j+1}\gamma_j = 1 + \sum_{j=1}^{i}(-1)^{j+1}h_j\}$ or $\mathcal{F}_0 = \{\gamma \in \mathcal{P}^r \mid \gamma_i = 0\}$.*
3. *Each (integral) nonnegative γ^r in $\mathcal{F}_t$ corresponds to an (integral) nonnegative h^{cd} satisfying (12).*

Proof:

1. Let A be the coefficient matrix of the system of inequalities in $\mathcal{P}^r$, excepting the nonnegativity inequalities. Variable γ_j shows up only in constraints $3+j$ through $3+2i-j$ with coefficient $(-1)^{j+1}$. Thus the $0,1$ matrix $\tilde{A} = (\tilde{a}_{ij}) = (|a_{ij}|)$ has the consecutive ones property, which implies it is totally unimodular, see [4, 5]. Since A is obtained from $\tilde{A}$ by myltiplying even columns by -1, A is also totally unimodular. Finally note that the right-hand-side elements in the inequalities that define polytope $\mathcal{P}^r$ are clearly integral. Thus the polytope $\mathcal{P}^r$ defined by (81) has integral vertices, or equivalently, it is the convex hull of the integral vectors satisfying the inequalities in (81).

2. Let $\bar{\gamma}^r \in \mathcal{P}^r$ and let $M\bar{\gamma}^r = q$ be the constraints that are tight at $\bar{\gamma}^r$. Then $\bar{\gamma}^r$ is vertex of $\mathcal{P}$ if and only if the rank of M is i. Now $\bar{A}$ contains precisely i columns, which implies it cannot contain a column of zeros. Since there are only two inequalities containing γ_i, one of these must be tight at $\bar{\gamma}$, otherwise the i-th column of M will be the zero vector. Therefore if $\bar{\gamma}$ is a vertex it must belong to one of the faces $\mathcal{F}_t = \{\gamma \in \mathcal{P}^r \mid (-1)^{i+1}\sum_{j=1}^{i}(-1)^{j+1}\gamma_j \le (-1)^{i+1}(1 + \sum_{j=1}^{i}(-1)^{j+1}h_j)\}$ or $\mathcal{F}_0 = \{\gamma \in \mathcal{P}^r \mid \gamma_i = 0\}$.

3. If (integral) $\gamma^r \in \mathcal{P}^r$ belongs to $\mathcal{F}_t$, then the nonnegative γ given by $\gamma_0 = \gamma_{2i+1} = 1$ and $\gamma_{2i+1-j} = \gamma_j$, for $j = 1, \ldots, i$ satisfies (14)–(16) and saturates the inequality corresponding to $k = i+1$ in (14). But then h^{cd} given by equations (36)–(41) is a (integral) nonnegative vector satisfying (12), as shown in the proof of Proposition 3.2. ∎

Suppose the system of equations (13) admits nonnegative solutions. Let $\tilde{h}^{cd}$ be the[1] nonnegative integral solution that also satisfies the complementarity conditions given in (17). Let $\tilde{\gamma} = (\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{2i+1})$ be the corresponding integral nonnegative solution of (14)–(16) given by (18). Its restriction $\tilde{\gamma}^r$ plays a special role in $\mathcal{P}^r$, as shown in the next proposition.

Proposition 4.2 *Suppose the system of equations (13) admits nonnegative solutions. Then the polytope $\mathcal{P}^r$ may be rewritten as*

$$\mathcal{P}^r = \{0 \le \gamma^r \in \mathbb{R}^i \mid (-1)^{k+1} \sum_{j=0}^{k} (-1)^{j+1}\gamma_j \le (-1)^{k+1} \sum_{j=0}^{k} (-1)^{j+1}\tilde{\gamma}_j, \ \textit{for } 1 \le k \le i\}. \tag{82}$$

Furthermore, $\tilde{\gamma}^r$ is a vertex of $\mathcal{P}^r$ and also its maximum vector, componentwise.

Proof: If (13) admits nonnegative solutions, the constraints $0 = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j$, $0 \le h_0 - 1$ and $0 \le h_{2i+1} - 1$ are redundant and may be dropped. Now notice that

$$\min\{k-1, 2i+1-k\} = \begin{cases} k-1, & \text{if } 2 \le k \le i \\ i = k-1 = 2i+1-k, & \text{if } k = i+1 \\ 2i+1-k, & \text{if } i+2 \le k \le 2i \end{cases}$$

Therefore the partial sum $\sum_{j=1}^{\ell}(-1)^{j+1}\gamma_j$, for $1 \le \ell \le i-1$, appears twice in (81): when $k = \ell+1$ and when $k = 2i+1-\ell$. Since $\ell+1$ and $2i+1-\ell$ are either both odd or both even, we may collect these two inequalities as follows:

$$\sum_{j=1}^{\ell}(-1)^{j+1}\gamma_j \begin{cases} \ge 1 + \max\{\sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell}(-1)^{j+1}h_j\}, & \text{if } 1 \le \ell \le i-1, \ \ell \text{ even} \\ \le 1 + \min\{\sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell}(-1)^{j+1}h_j\}, & \text{if } 1 \le \ell \le i-1, \ \ell \text{ odd.} \end{cases}$$

The fact that $\tilde{\gamma}$ is a nonnegative integral solution of (14)–(16) implies that $\tilde{\gamma}^r \in \mathcal{P}^r$. Furthermore, using (16), equations (26) and (31), for $2 \le \ell \le i$ and $k = 2i+2-\ell$ (thus $i+2 \le k \le 2i$), we

[1] If the complementarity conditions (17) are satisfied then the subgraph induced by the support of $\tilde{h}^{cd}$ is a forest, which implies it is the unique solution with such a support.

obtain

$$\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{2i+2-\ell}\tilde{h}^d_{2i+1-\ell} = \sum_{j=0}^{\ell-1}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{2i+2-\ell}\tilde{h}^d_{2i+1-\ell} = \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j \tag{83}$$

$$\sum_{j=0}^{2i+1-k}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{2i+2-k}\tilde{h}^c_{2i+2-k} = \sum_{j=0}^{\ell-1}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{\ell}\tilde{h}^c_{\ell} = \sum_{j=0}^{\ell-1}(-1)^{j+1}h_j. \tag{84}$$

Since, by (17), $\tilde{h}^c_\ell \tilde{h}^d_{2i+1-\ell} = 0$, at least one of the two inequalities above involving the partial sum $\sum_{j=0}^{\ell-1}(-1)^{j+1}\tilde{\gamma}_j$, must be satisfied as equality. Thus $\tilde{\gamma}^r$ satisfies:

$$\sum_{j=0}^{\ell-1}(-1)^{j+1}\tilde{\gamma}_j = \begin{cases} \max\{\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell-1}(-1)^{j+1}h_j\}, & \text{if } 2 \le \ell \le i,\ \ell \text{ odd} \\ \min\{\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell-1}(-1)^{j+1}h_j\}, & \text{if } 2 \le \ell \le i,\ \ell \text{ even} \end{cases} \tag{85}$$

which imply (82) for $k \neq i$, taking into account that $\tilde{\gamma}_0 = 1$. Finally, the inequality in (81) corresponding to $k = i$ is satisfied as equality by $\tilde{\gamma}$, as seen in (29) and thus

$$\sum_{j=0}^{i}(-1)^{j+1}\tilde{\gamma}_j = \sum_{j=0}^{i}(-1)^{j+1}h_j.$$

Odd and even cases may be combined by the appropriate multiplication, producing the desired inequalities, that, together with the nonnegative constraints, provide an alternative definition of $\mathcal{P}^r$:

$$\begin{array}{rcll} (-1)^{\ell+1}\sum_{j=1}^{\ell}(-1)^{j+1}\gamma_j & \le & (-1)^{\ell+1}\sum_{j=1}^{\ell}(-1)^{j+1}\tilde{\gamma}_j, & \text{for } 1 \le \ell \le i \\ \gamma^r & \ge & 0 & \end{array} \tag{86}$$

This completes the proof of the equivalence between (81) and (82).

Finally, the first inequality in (86) gives $\gamma_1 \le \tilde{\gamma}_1$, and, if we add inequalities $j-1$ and j of (86) we obtain $\gamma_j \le \tilde{\gamma}_j$. Thus $\tilde{\gamma}^r$ is the maximum vector, componentwise, of $\mathcal{P}$. Then $\tilde{\gamma}^r$ must be a vertex of $\mathcal{P}^r$. ∎

Proposition 4.1 and the argument in the proof of Proposition 3.2 expressed in (35) imply that $\mathcal{F}_0$ is the projection of $\mathcal{F}_t$ onto the $\gamma_i = 0$ hyperplane, and $\mathcal{P}$ is the convex hull of $\mathcal{F}_t \cup \mathcal{F}_0$. Also by Proposition 4.1, each (integral) vector in $\mathcal{F}_t$ may be obtained from a corresponding h^{cd} satisfying (12) (or, equivalently, (13)), although this is not, in general, a 1-to-1 correspondence. Now, given the solution $\tilde{h}^{cd}$ we may construct all integral vectors in $\mathcal{F}_t$ by the successive addition of circulations thereto and computation of the corresponding γ^r. The following example illustrates these facts in a concrete setting.

Example Let $n = 2i + 1 = 7$, $(h_0, \ldots, h_7) = (2, 5, 11, 10, 5, 3, 3, 3)$. Using formulas developed in [1] for the general solution h^{cd} of (13) we may obtain the (particular) solution $\tilde{h}^{cd}$ that satisfies (17) is $\tilde{h}^{cd} = (1, 4, 3, 8, 5, 5, 5, 0, 3, 0, 1, 2)$, shown in Figure 2. All other nonnegative integral solutions of (13) may be obtained by adding integer multiples of the unit circulation along cycle 1, $h^{cd\,1} = (0, 0, 1, -1, 0, 0, 0, 0, -1, 1, 0, 0)$, and/or the unit circulation along cycle 2, $h^{cd\,2} = (0, 0, 0, 0, 1, -1, -1, 1, 0, 0, 0, 0)$. From $\tilde{h}^{cd}$ we obtain vector $\tilde{\gamma} = (1, 1, 3, 5, 5, 3, 1, 1)$, and, using Proposition 4.2, the polytope $\mathcal{P}^r$ is given by the inequalities

$$\begin{array}{ccccccc} \gamma_1 & & & & & \leq & 1 \\ \gamma_1 & - & \gamma_2 & & & \geq & -2 \\ \gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 3 \\ & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0 \end{array}$$

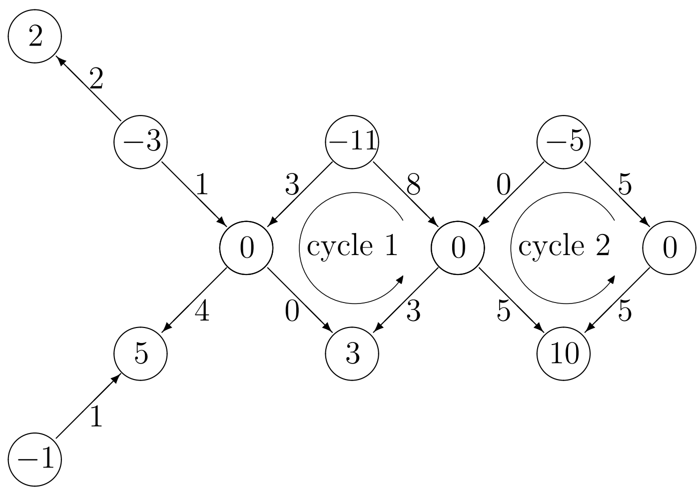


Figure 2: Solution $\tilde{h}^{cd}$ of example.

Figure 3 depicts four views of the polytope $\mathcal{P}^r$. In this example no inequality is redundant, and the polytope has six facets. Facet $\mathcal{F}_0$ is always hidden, but one can realize, perhaps with the aid of Figure 4, it is the projection of the top facet $\mathcal{F}_t$ on the $\gamma_3 = 0$ plane. The maximum vector $\tilde{\gamma}^r$ is labeled on the first view of $\mathcal{P}^r$.

In order to see the relationship between the solutions h^{cd} obtained as we add circulations and the corresponding γ^r, it is more convenient to look at the frame of polytope $\mathcal{P}^r$, depicted in Figure 4(a), together with the lattice determined by the integral vectors in the nonnegative orthant. The nonnegative integral points belonging to $\mathcal{P}^r$ are emphasized. Denote by $\gamma^1 = \tilde{\gamma}^r$, γ^2, $\ldots$, γ^7

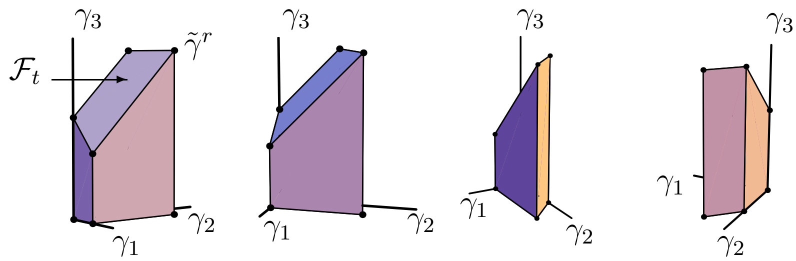


Figure 3: Four views of the polytope $\mathcal{P}^r$.

the integral vectors belonging to $\mathcal{F}_t$, in counterclockwise order. Notice that adding $h^{cd\,1}$ to h^{cd} has the effect of subtracting 1 from γ_1 and γ_2. Similarly, adding $h^{cd\,2}$ to h^{cd} has the effect of subtracting 1 from γ_2 and γ_3. Figure 4(b) focuses on $\mathcal{F}_t$ and the integral points therein. The following diagram summarizes the operations with circulations on h^{cd} needed to generate all points in $\mathcal{F}_t$. Notice there is not a 1-to-1 correspondence between nonnegative integral h^{cd} satisfying (13) and the nonnegative integral γ^r in $\mathcal{F}_t$. For instance, the vector $h^{cd(4)} + h^{cd\,2} = (1,4,4,7,8,2,2,3,2,1,1,2)$ is a valid nonnegative integral solution of (13), but the corresponding γ^r violates the nonnegative constraint $\gamma_2 \geq 0$.

$$\begin{tikzcd}
\tilde{\gamma}^r = \gamma^1 \arrow[r] \arrow[d, leftrightarrow] & \gamma^2 \arrow[r] \arrow[d, leftrightarrow] & \gamma^3 \arrow[r] \arrow[d, leftrightarrow] & \gamma^4 \arrow[r] \arrow[d, leftrightarrow] & \gamma^5 \arrow[r] \arrow[d, leftrightarrow] & \gamma^6 \arrow[r] \arrow[d, leftrightarrow] & \gamma^7 \arrow[r] \arrow[d, leftrightarrow] & \gamma^1 \arrow[d, leftrightarrow] \\
\tilde{h}^{cd} = h^{cd(1)} \arrow[r, "+h^{cd\,1}"] & h^{cd(2)} \arrow[r, "+h^{cd\,2}"] & h^{cd(3)} \arrow[r, "+h^{cd\,2}"] & h^{cd(4)} \arrow[r, "+h^{cd\,2}", "-h^{cd\,1}"'] & h^{cd(5)} \arrow[r, "-h^{cd\,2}"] & h^{cd(6)} \arrow[r, "-h^{cd\,2}"] & h^{cd(7)} \arrow[r, "-h^{cd\,2}"] & h^{cd(1)}
\end{tikzcd}$$

The knowledge of $\gamma^1, \dots, \gamma^7$ in fact makes it possible to compute all the integral points in $\mathcal{P}^r$, since they lie in one of the segments $[\gamma^i, \hat{\gamma}_i]$, for $i = 1, \dots, 7$, where $\hat{\gamma}^i_j = \gamma^i_j$, for $j \neq 3$, and $\hat{\gamma}_3 = 0$.

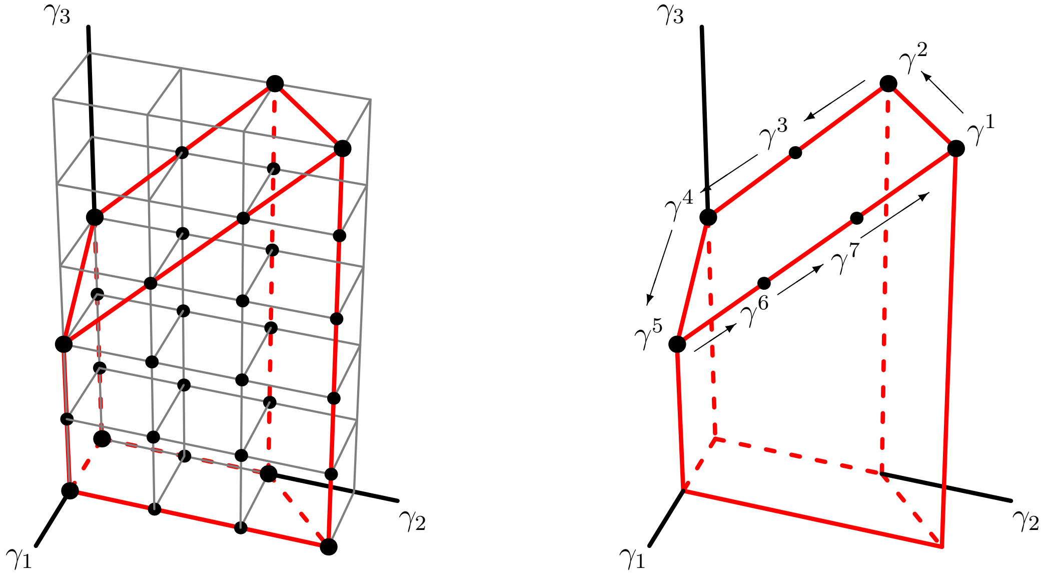
4.2 Case $\mathbf{n = 0} \mod \mathbf{4}$

Supppose $n = 2i$, where $i \geq 2$ is even. Consider the linear system in $\gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{2i}\}$ constituted by (47)–(49) and the nonnegative constraints (87) below.

$$\gamma \geq 0, \tag{87}$$

where the vector $(h_0, \dots, h_{2i})$ is such that $\sum_{j=0}^{2i} (-1)^j h_j$ is even.

The first equality in (47) may be used to eliminate γ_i from the system. This is accomplished by



(a) Lattice of integral points in $\mathcal{P}^r$.

(b) Integral points in facet $\mathcal{F}_t$ of $\mathcal{P}^r$.

Figure 4: Frame of polytope $\mathcal{P}^r$.

substituting

$$\gamma_i = (-1)^{i+1} \left(\sum_{j=0}^{2i} (-1)^{j+1} h_j - \sum_{j=0}^{i-1} (-1)^{j+1} \gamma_j - \sum_{j=i+1}^{2i} (-1)^{j+1} \gamma_j \right). \tag{88}$$

in the equations containing γ_i in (47), i.e., those corresponding to $k = 1, \ldots, i$. The constraint corresponding to a generic $k \in \{1, \ldots, i\}$ is thus transformed:

$$\begin{aligned}
& (-1)^{k+1} \left(\sum_{\substack{j=0 \\ j \neq i}}^{2i-k} (-1)^{j+1} \gamma_j + (-1)^{2(i+1)} \left(- \sum_{j=0}^{i-1} (-1)^{j+1} \gamma_j - \sum_{j=i+1}^{2i} (-1)^{j+1} \gamma_j \right) \right) \\
= \; & -(-1)^{k+1} \sum_{j=2i+1-k}^{2i} (-1)^{j+1} \gamma_j \\
\leq \; & (-1)^{k+1} \left(\sum_{j=0}^{2i-k} (-1)^{j+1} h_j - (-1)^{2(i+1)} \sum_{j=0}^{2i} (-1)^{j+1} h_j \right) \\
= \; & -(-1)^{k+1} \sum_{j=2i+1-k}^{2i} (-1)^{j+1} h_j
\end{aligned}$$

Using (49) the above constraint becomes

$$\begin{aligned}
-(-1)^{k+1} \sum_{j=2i+1-k}^{2i} (-1)^{j+1} \gamma_j &= -(-1)^{k+1} \sum_{j=2i+1-k}^{2i} (-1)^{j+1} \gamma_{2i-j} \\
&= -(-1)^{k+1} \sum_{j=0}^{k-1} (-1)^{j+1} \gamma_j \\
&\leq -(-1)^{k+1} \sum_{j=2i+1-k}^{2i} (-1)^{j+1} h_j,
\end{aligned}$$

and $\gamma_i \geq 0$ implies

$$\begin{aligned}
(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_j\right) &= \\
(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_{2i-j}\right) &= \\
(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{\ell=0}^{i-1}(-1)^{j+1}\gamma_\ell\right) &= \\
(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j\right) &\geq 0.
\end{aligned}$$

Therefore there is a 1-to-1 relationship between solutions $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i})$ of (47)–(87) and the solutions $\gamma^r = (\gamma_1, \ldots, \gamma_{i-1})$ of (89). Thus we may simplify our study of $\mathcal{P}$ by considering $\mathcal{P}^r = \{\gamma^r \in \mathbb{R}^{i-1} \mid \text{ constraints in (89)}\}$ instead.

$$\begin{aligned}
0 &\geq 1 - h_{2i} \\
\sum_{j=1}^{k}(-1)^{j+1}\gamma_j &\begin{cases} \leq 1 + \sum\limits_{j=2i-k}^{2i}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq i-1,\ k \text{ odd} \\ \geq 1 + \sum\limits_{j=2i-k}^{2i}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq i-1,\ k \text{ even} \end{cases} \\
(-1)^i\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j &\geq (-1)^i\left(1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right) \\
\sum_{j=1}^{2i-k}(-1)^{j+1}\gamma_j &\begin{cases} \leq 1 + \sum\limits_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } i+1 \leq k \leq 2i-1,\ k \text{ odd} \\ \geq 1 + \sum\limits_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } i+1 \leq k \leq 2i-1,\ k \text{ even} \end{cases} \\
0 &\geq 1 - h_0 \\
\gamma_j &\geq 0, \qquad \text{for } 1 \leq j \leq i-1
\end{aligned} \tag{89}$$

Proposition 4.3 *The polytope $\mathcal{P}^r$ defined by (89) has integral vertices and each (integral) γ^r in the polytope corresponds to an (integral) nonnegative h^{cd} satisfying (43). Each vertex of $\mathcal{P}^r$ belongs* *to one of three faces:*

$$
\begin{aligned}
\mathcal{F}_t &= \left\{\gamma \in \mathcal{P}^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \min\left\{\sum_{j=i+1}^{2i}(-1)^{j+1}h_j, \sum_{j=i+1}^{2i}(-1)^{j+1}h_j\right\}\right\} \\
\mathcal{F}_b &= \left\{\gamma \in \mathcal{P}^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right\} \\
\mathcal{F}_0 &= \{\gamma^r \in \mathcal{P}^r \mid \gamma_{i-1} = 0\}.
\end{aligned}
$$

Proof: The matrix of coefficients A corresponding to the inequalities in (89), excepting the nonnegativity ones, is a $0, \pm 1$ matrix. Its is easy to see that the $0, 1$ matrix $\tilde{A} = (|a_{ij}|)$ has the consecutive ones property and thus is totally unimodular. Since A is obtained from $\tilde{A}$ by multiplying the even columns by -1, it is also totally unimodular. Taking into account the fact that the righ-hand-side of (89) is integral, we conclude all vertices in $\mathcal{P}^r$ are integral.

The correspondence between integral points of $\mathcal{P}^r$ and h^{cd} satisfying (43) mimics the proof of item 3 of Proposition 4.1.

The argument for the last assertion is similar to the one given in the proof of item 2 of Proposition 4.1. In order for a $\bar{\gamma}^r$ in $\mathcal{P}^r$ to be a vertex, the set of saturated constraints at $\bar{\gamma}^r$ must include one of the four inequalities invoving γ_{i-1} in (89), repeated below for convenience. Notice we use the fact that i is even.

$$
\begin{aligned}
\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j &\leq 1 + \sum_{j=i+1}^{2i}(-1)^{j+1}h_j && (90) \\
\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j &\leq 1 + \sum_{j=i+1}^{2i}(-1)^{j+1}h_j && (91) \\
\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j &\geq 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j && (92) \\
\gamma_{i-1} &\geq 0 && (93)
\end{aligned}
$$

Inequalities (90)–(91) are equivalent to the inequality below.

$$
\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \leq 1 + \min\left\{\sum_{j=i+1}^{2i}(-1)^{j+1}h_j, \sum_{j=i+1}^{2i}(-1)^{j+1}h_j\right\}. \tag{94}
$$

Since one of the three inequalities (94), (92) or (93), must be tight at a vertex, the vertex must belong to $\mathcal{F}_t$, $\mathcal{F}_b$ or $\mathcal{F}_0$, respectively. ■

The next proposition is the analogue of Proposition 4.2 for the $n = 0 \mod 4$ case. As before we single out the vector $\tilde{\gamma}$ corresponding to the solution $\tilde{h}^{cd}$ of (44) satisfying (50).

Proposition 4.4 *Assume* (44) *has a nonnegative solution. Let* $\tilde{\gamma}$ *be the nonnegative integral solution of* (47)–(49) *corresponding to* $\tilde{h}^{cd}$*, the nonnegative integral solution of* (44) *that satisfies* (50)*. The polytope* $\mathcal{P}^r$ *may be recast as*

$$\mathcal{P}^r = \left\{ 0 \le \gamma^r \;\middle|\; \begin{array}{rcl} (-1)^{k+1}\displaystyle\sum_{j=0}^{k}(-1)^{j+1}\gamma_j & \le & (-1)^{k+1}\displaystyle\sum_{j=0}^{k}(-1)^{j+1}\gamma_j, \quad \textit{for } 1 \le k \le i-1 \\ \displaystyle\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j & \ge & \dfrac{1}{2}\displaystyle\sum_{j=0}^{2i}(-1)^{j+1}h_j \end{array} \right\} \tag{95}$$

Furthermore, $\tilde{\gamma}$ *(resp.,* $\tilde{\gamma}^r$*) is a vertex and the maximum vector in* $\mathcal{P}$ *(resp.* $\mathcal{P}^r$*), componentwise.*

Proof: If we assume (44) has a solution, the inequalities $0 \ge 1 - h_{2i}$ and $0 \ge 1 - h_0$ are redundant and may be eliminated. Grouping togheter the remaining inequalities in (47) we have

$$\begin{array}{rcll} \displaystyle\sum_{j=1}^{k}(-1)^{j+1}\gamma_j \left\{ \begin{array}{c} \le \\[2em] \ge \end{array} \right. & & \begin{array}{l} 1 + \min\left\{ \displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \displaystyle\sum_{j=0}^{k}(-1)^{j+1}h_j \right\} \\ 1 + \max\left\{ \displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \displaystyle\sum_{j=0}^{0}(-1)^{j+1}h_j \right\} \end{array} & \begin{array}{l} \text{if } 1 \le k \le i-1, \ k \text{ odd} \\[2em] \text{if } 1 \le k \le i-1, \ k \text{ even} \end{array} \\ \displaystyle\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j & \ge & 1 + \dfrac{1}{2}\displaystyle\sum_{j=0}^{2i}(-1)^{j+1}h_j & \\ \gamma_j & \ge & 0, & \text{for } 1 \le j \le i-1 \end{array} \tag{96}$$

Now $\tilde{\gamma}$ satisfies the first equation in (47) and (60). Using the first equation in (47) to eliminate $\tilde{\gamma}_i$ in (60), and then using (49), we have

$$\begin{aligned} & -\sum_{j=2i+1-k}^{2i}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{2i+1-k}h^d_{2i-k} \\ = & -\sum_{j=0}^{k-1}(-1)^{j+1}\tilde{\gamma}_j + (-1)^{2i+1-k}h^d_{2i-k} \\ = & -\sum_{j=2i+1-k}^{2i}(-1)^{j+1}h_j, \quad \text{for } 1 \le k \le i \end{aligned} \tag{97}$$

Equation (63) may be rewritten as

$$\sum_{j=0}^{k-1}(-1)^{j+1}\tilde{\gamma}_j + (-1)^k h_k^c = \sum_{j=0}^{k-1}(-1)^{j+1}h_j, \quad \text{for } 2 \leq k \leq i. \tag{98}$$

Equations (97)–(98), the facts that $\tilde{h}^{cd}$ satisfies (17), $\tilde{\gamma}_0 = 1$ and that $\tilde{\gamma}^r$ satisfies (89) imply

$$\min\left\{\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{k}(-1)^{j+1}h_j\right\} = \sum_{j=0}^{k}(-1)^{j+1}\tilde{\gamma}_j, \quad \text{for } 1 \leq k \leq i-1,\ k \text{ odd}$$

and

$$\max\left\{\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{0}(-1)^{j+1}h_j\right\} = \sum_{j=0}^{k}(-1)^{j+1}\tilde{\gamma}_j, \quad \text{for } 1 \leq k \leq i-1,\ k \text{ even.}$$

Substituting the above expressions in (96) we obtain (95).

Concerning the last assertion of the proposition, notice that the first inequality in (96) reads $\gamma_1 \leq \tilde{\gamma}_1$ and inequalities corresponding to $k = \ell - 1$ and $k = \ell$ imply $\gamma_\ell \leq \tilde{\gamma}_\ell$, for $2 \leq \ell 1 - 1$. Therefore $\tilde{\gamma}^r$ is the maximum vector, componenetwise, of $\mathcal{P}^r$. This implies it is a vertex of $\mathcal{P}^r$. Finally, (88) and (95) imply, using the fact that i is even,

$$\gamma_i = 2(\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=0}^{2i}(-1)^{j+1}h_j) \leq 2(\sum_{j=0}^{i-1}(-1)^{j+1}\tilde{\gamma}_j - \sum_{j=0}^{2i}(-1)^{j+1}h_j),$$

which gives an upper bound for γ_i, achieved by $\tilde{\gamma}_i$. Thus $\tilde{\gamma}$ is the maximum vector in $\mathcal{P}$, componentwise, and therefore a vertex thereof. ∎

The next example illustrates the relationship between the nonnegative integral solutions h^{cd} of (44) and the integral vectors γ^r of $\mathcal{P}^r$.

Example Let $n = 2i = 8$ and $(h_0, \ldots, h_8) = (3, 5, 7, 8, 5, 2, 2, 2, 2)$. Figure 5 gives an unorthodox representation for the constraints (44). Notice that the two arcs dangling down from the two rightmost nodes represent the same variable, namely h_4^d. This deviates from the network-flow framework. Thus, while $h^{cd\,1}$ and $h^{cd\,2}$, are associated with cycles 1 and 2 as in the last example, there is a third "circulation" $h^{cd\,3}$ whose support is given by $(h_4^c, \beta, h_4^d) = (1, -2, 1)$. Nevertheless, if we keep this anomaly in mind, we can still use the picture to quickly compute all the different solutions, which may be obtained by adding integer multiples of $h^{cd\,i}$, for $1 \leq i \leq 3$, to the solution $\tilde{h}^{cd}$ satisfying (50) depicted in Figure 5.

$$
\begin{array}{ccccccc}
\gamma_1 & & & & & \leq & 1 \\
\gamma_1 & - & \gamma_2 & & & \geq & -1 \\
\gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 1 \\
\gamma_1 & - & \gamma_2 & + & \gamma_3 & \geq & 0 \\
 & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0
\end{array}
$$

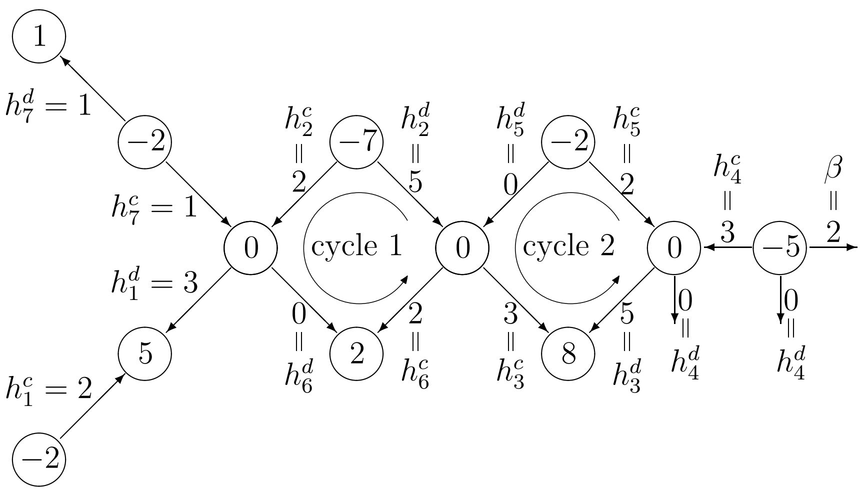
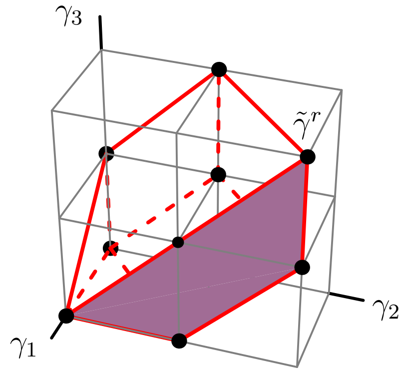


Figure 5: Solution $\tilde{h}^{cd}$ of example.

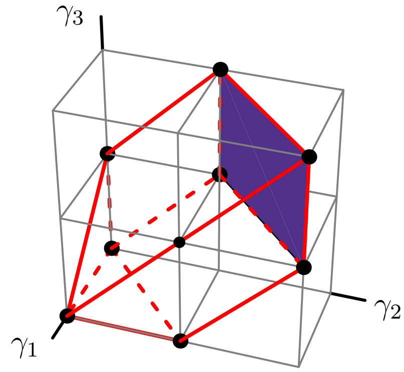
Polytope $\mathcal{P}^r$ corresponding to the given data is shown in Figure 6. All integral γ^r my be obtained starting at the maximum vector $\tilde{\gamma}^r$ and adding appropriate integer multiples of the vectors $(-1,-1,0)$, $(0,-1,-1)$, $(0,0,-1)$, $(1,0,-1)$ and $(-1,0,0)$. The corresponding operation on h^{cd} is shown below.

$$
\begin{tikzcd}
\bar{\gamma} \arrow[r, "{(-1,-1,0)}"] \arrow[d, leftrightarrow] & \hat{\gamma} \arrow[d, leftrightarrow] \\
\bar{h}^{cd} \arrow[r, "+h^{cd\,1}"] & \hat{h}^{cd}
\end{tikzcd}
\quad
\begin{tikzcd}
\bar{\gamma} \arrow[r, "{(0,-1,-1)}"] \arrow[d, leftrightarrow] & \hat{\gamma} \arrow[d, leftrightarrow] \\
\bar{h}^{cd} \arrow[r, "+h^{cd\,2}"] & \hat{h}^{cd}
\end{tikzcd}
\quad
\begin{tikzcd}
\bar{\gamma} \arrow[r, "{(0,0,-1)}"] \arrow[d, leftrightarrow] & \hat{\gamma} \arrow[d, leftrightarrow] \\
\bar{h}^{cd} \arrow[r, "+h^{cd\,3}"] & \hat{h}^{cd}
\end{tikzcd}
$$

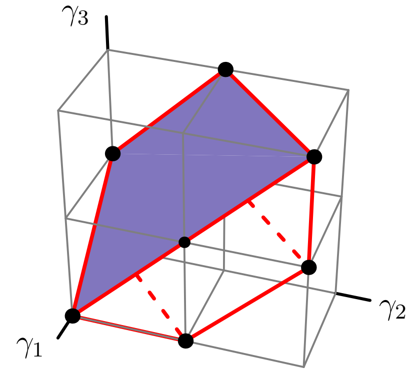
$$
\begin{tikzcd}
\bar{\gamma} \arrow[r, "{(1,0,-1)}"] \arrow[d, leftrightarrow] & \hat{\gamma} \arrow[d, leftrightarrow] \\
\bar{h}^{cd} \arrow[r, "+h^{cd\,2} - h^{cd\,1}"] & \hat{h}^{cd}
\end{tikzcd}
\quad
\begin{tikzcd}
\bar{\gamma} \arrow[r, "{(1,0,0)}"] \arrow[d, leftrightarrow] & \hat{\gamma} \arrow[d, leftrightarrow] \\
\bar{h}^{cd} \arrow[r, "-h^{cd\,1} + h^{cd\,2} - h^{cd\,3}"] & \hat{h}^{cd}
\end{tikzcd}
$$



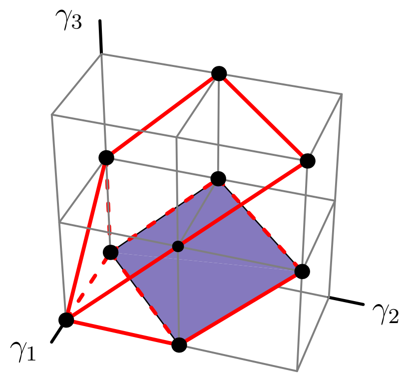
(a) $\mathcal{F}^1 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 = 1\}$.



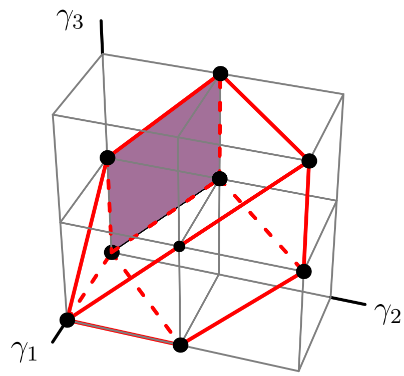
(b) $\mathcal{F}^2 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 = -1\}$.



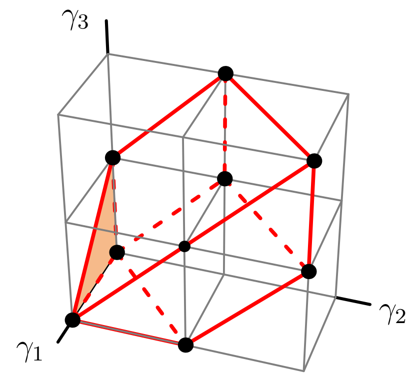
(c) $\mathcal{F}^3 = \mathcal{F}_t = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 + \gamma_3 = 1\}$.



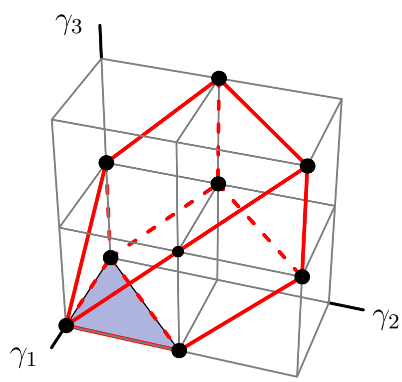
(d) $\mathcal{F}^4 = \mathcal{F}_b = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 + \gamma_3 = 0\}$.



(e) $\mathcal{F}^5 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 = 0\}$.



(f) $\mathcal{F}^6 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_2 = 0\}$.



(g) $\mathcal{F}^7 = \mathcal{F}_0 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_3 = 0\}$.

Figure 6: Facets of $\mathcal{P}^r$ and integer grid.

4.3 Case $\mathbf{n = 2 \mod 4}$

Suppose $n = 2i$, where $i \geq 3$ is odd, and $\sum_{j=o}^{2i}(-1)^{j+1}h_j$ is even. The constraints (73) are a copy of (47) and may be manipulated as in section 4.2 in order to obtain the smaller equivalent system (89) in $\gamma^r = (\gamma_1, \dots, \gamma_{i-1})$. Propositions 4.3 and 4.4 thus also hold for the case $n = 2 \mod 4$, with the slightly different construction rule for h^{cd} corresponding to a given γ satisfying (73) given at the end of the proof of Proposition 3.6.

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