Convolution of Robust Functions[1]

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Abstract- Recently Zheng [1,2], in the setting of global optimization, introduced the concepts of robust set and robust function as a generalization of open set and upper semicontinuous (u.s.c) function, respectively. The aims of this paper are to study the structure of robust sets defined on a normed space X as well as to extend some multivalued convergence results obtained by the author in [3,4] and Greco et al. [6] for semicontinuous functions to the class of robust functions. More precisely, we introduce the concepts of level-convergence and epigraphic convergence on $\mathcal{R}(X)$, the space of nonnegative robust functions on a normed space X and, on one hand, we study its properties and relationships, and on the other, we present some results on level-approximation and epi-approximation of functions by using convolution of robust functions.

Keywords- Normed spaces, robust sets, Hausdorff pseudometric, Kuratowski limits, Level-convergence, Convolution of functions.

1. INTRODUCTION

The study of level convergence, hypographic convergence and epigraphic convergence of functions and its applications has been done by many authors, including Román-Flores [3,4,5] in the setting of convergence of fuzzy sets on finite-dimensional spaces, level-convergence of functions on regular topological spaces and compactness of spaces of fuzzy sets on a metric space, respectively, Greco et al. [6] in variational convergence of fuzzy sets on metric spaces and Attouch [7] in calculus of variations.

The principal tools of this convergence are based in the Hausdorff metric and Kuratowski limits, and one of the most important properties of the hypo-convergence (epi-convergence) is the preservation of maximum (minimum) points

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in hypo-convergent (epi-convergent) sequences of functions. This explains the success of these convergence schemes in global optimization theory, see [7].

The aims of this paper are, on one hand, to study structural properties of robust sets and robust functions on a normed space X, and on the other, to study several types of convolution of functions and its applications to level-approximation of functions. In particular, we study some connections between level-convergence and epi-convergence of robust functions.

This paper is organized as follows. In Section 2, we give the basic material that will be used in the article. In Section 3, we introduce the concept of inf-convolution of functions and its applications to level-approximation of functions (approximation in D-pseudometric). In this direction we prove that $(\mathcal{R}(X), D)$ the space of robust functions on X, is a dense subspace of $(\mathcal{F}(X), D)$, the space of nonnegative functions with non-empty levels on X. In Section 4, we give a characterization for the existence of proper local minimum points of robust functions. In Section 5, we introduce the concept of sum-convolution of functions and its applications to epi-approximation of functions (approximation in D_e-pseudometric). In this context we prove that $(\mathcal{R}_e(X), D_e)$ the space of epi-robust functions on X, is a dense subspace of $(\mathcal{F}(X), D_e)$. In Section 6 we prove that, in general, D-convergence is stronger than D_e-convergence on $\mathcal{F}(X)$. Finally, in Section 7, we prove the equivalence between L-convergence and D_E-convergence of functions on $\mathcal{R}(X)$, under condition of level-continuity of the limit function. Furthermore, some examples and applications are presented.

2. PRELIMINARIES

In the sequel, X will be assumed to be a normed vector space.

Definition 2.1 ([1]). *A set $A \subseteq X$ is said to be robust iff $\overline{A} = \overline{IntA}$.*

Remark 2.2. We observe that an open set is robust. Actually, the concept of robustness is a generalization of that of openness.

We recall that if $A, B \subseteq X$, then $A + B = \{a + b/\ a \in A,\ b \in B\}$, with the convention $A + \emptyset = \emptyset + A = \emptyset$.
An important class of robust sets is the family of convex subsets of X.

PROPOSITION 2.3.*([8]). Let K be a convex subset of a normed space X. The closure $\overline{K}$ of K and the interior $IntK$ of K are convex. Moreover, if $IntK \neq \emptyset$, then $\overline{K} = \overline{IntK}$ and $IntK = Int\overline{K}$.*

As a direct consequence of the above result we obtain

COROLLARY 2.4. *Let K be a convex subset of a normed space X with $IntK \neq \emptyset$. Then K is robust.*

PROPOSITION 2.5*([8, pp.62, Proposition 5]). Let A and B be two nonempty subsets of X. If A is open, then $A+B$ is open.*

COROLLARY 2.6*([9]). Let A and B be nonempty sets in X. If $intA \neq \emptyset$ then $intA + B \subseteq int(A+B)$.*

THEOREM 2.7. *Let A and B be subsets in X. Then*

i) If A is robust then $A+B$ is robust.

ii) λA is robust, $\forall \lambda > 0$.

PROOF.
i) We have the following cases
a) If $A = \emptyset$ or $B = \emptyset$ then $A + B = \emptyset$ is robust
b) If $A \neq \emptyset$ and $B \neq \emptyset$ then it is clear that $\overline{Int(A+B)} \subseteq \overline{A+B}$. On other hand, if $x \in \overline{A+B}$ then there exists a sequence $(a_n + b_n) \subseteq A+B$ such that $a_n + b_n \to x$.

Because $a_n \in A \subseteq \overline{A} = \overline{IntA}$ then, for each $n \in \mathbb{N}$, there exists a sequence $(a_{ni}) \subseteq IntA$ such that
$\lim_{i\to\infty} a_{ni} = a_n$.

Thus, if $\epsilon > 0$ is given we can construct a subsequence $(a_{ni_n}) \subseteq IntA$ such that $\mid a_n - a_{ni_n} \mid < \epsilon/2$, $\forall n$. So, we have that the sequence $a_{ni_n} + b_n \in IntA + B$ and

$$\begin{aligned} \mid & a+b-(a_{ni_n}+b_n) \mid \leq \mid a+b-(a_n+b_n) \mid + \mid a_n+b_n-(a_{ni_n}+b_n) \mid \\ \leq & \mid a-a_n \mid + \mid b-b_n \mid + \mid a_n - a_{ni_n} \mid < \epsilon \end{aligned}$$

for every n sufficiently large.
Therefore, due Corollary 2.6, $a+b \in \overline{IntA+B} \subseteq \overline{int(A+B)}$.

So, $\overline{A+B} = \overline{Int(A+B)}$ and $A+B$ is robust.

ii) A straightfoward calculus shows that
a) $\overline{\lambda A} = \lambda\overline{A}$, for every real number $\lambda \neq 0$ and
b) $\lambda IntA = Int(\lambda A)$, for every real number $\lambda > 0$.
Thus, if A is a robust set and $\lambda > 0$ then

$$\overline{\lambda A} = \lambda\overline{A} = \lambda\overline{IntA} = \overline{\lambda IntA} = \overline{Int(\lambda A)}.$$

Therefore λA is robust and the proof is completed. ■

3. CONVOLUTION OF ROBUST FUNCTIONS

The concept of robust functions has been studied by many authors in the setting of global optimization, including Zheng [1,2]. Also, in [2, Prop. 2.4] the author shows that a bounded robust function can be uniformly approximated by a sequence of robust step functions.

Our main result in this section is to show a level-approximation result for robust functions by using $\bigtriangledown$-convolution. More specifically, we will prove that the space of robust functions is a dense subspace of $(\mathcal{F}(X), D)$.

DEFINITION 3.1. *An extended pseudometric on $\mathcal{Z}$ is a function $p : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ such that*

(i) $p(x,y) \leq p(x,z) + p(z,y)$, *for all x, y, z in $\mathcal{Z}$;*

(ii) $p(x,y) = p(y,x)$, *for all x, y in $\mathcal{Z}$;*

(iii) $p(x,x) = 0$, *for all x in $\mathcal{Z}$.*

Let $\mathcal{P}(X) = \{A/\ A \subseteq X\}$ and $\mathcal{P}_0(X) = \{A \in \mathcal{P}(X)/\ A \neq \emptyset\}$. If $A \in \mathcal{P}_0(X)$ we define the "ϵ-dilatation of A" as the set

$$N(A, \epsilon) = \{x \in X/\ d(x, A) < \epsilon\},$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$.

If $A, B \in \mathcal{P}_0(X)$ define

$$H(A,B) = inf\{\epsilon > 0/\ A \subseteq N(B,\epsilon)\ and\ B \subseteq N(A,\epsilon)\},$$

where, as usual, $inf\emptyset = +\infty$.

Thus, we allow $+\infty$ as a possible value for H.

Proposition 3.2. *([10]). H is an extended pseudometric on $\mathcal{P}_0(X)$.*

Remark 3.3. An equivalent formula for H (see [8]) is given by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

H is called the Hausdorff extended pseudometric on $\mathcal{P}_0(X)$ derived from the norm.

Remark 3.4. If X is a Banach space and

$$\mathcal{B}(X) = \{A \in \mathcal{P}_0(X)/A\ is\ closed\ and\ bounded\}$$

then $(\mathcal{B}(X), H)$ is a separable and complete metric space and, in this case, H is called the Hausdorff metric on $\mathcal{B}(X)$ (see [10]).

Proposition 3.5. *([10]). If $A_1, A_2, B_1, B_2 \in \mathcal{P}_0(X)$ and $\lambda > 0$ then*

i) $H(\lambda A_1, \lambda B_1) = \lambda H(A_1, B_1)$;

ii) $H(A_1 + A_2,\ B_1 + B_2) \leq H(A_1, B_1) + H(A_2, B_2)$

Let $\mathcal{F}(X) = \{f : X \to [0,\infty]\ /\ \{f < \alpha\} \in \mathcal{P}_0(X),\ \forall \alpha > 0\}$ and define the class of non-negative robust function on X as

$$\mathcal{R}(X) = \{f \in \mathcal{F}(X)/\ \{f < \alpha\}\ is\ robust,\ \forall \alpha > 0\}.$$

where $\{f < \alpha\} = \{x \in X/\ f(x) < \alpha\}$ is the α-level of f.

Remark 3.6. We observe that, due Remark 2.2, an u.s.c. function is robust, so is a continuous function.

If $f, g \in \mathcal{F}(X)$ we can define a generalized Hausdorff extended pseudometric by mean

$$D(f,g) = \sup_{\alpha>0} H(\{f<\alpha\},\{g<\alpha\}).$$

DEFINITION 3.7. *Let f and g be in $\mathcal{F}(X)$. Then, the inf-convolution is defined by*

$$[f \triangle g](x) = \operatorname*{Inf}_{y\in X} \{f(x-y) \vee g(y)\}$$

where $\vee \doteqdot$ maximum.

Several important applications justify the study of $\triangle$-convolution (see [11,12]).

PROPOSITION 3.8.*(Rockafellar [11, p. 40]). Let f and g in $\mathcal{F}(X)$. Then, for all $\alpha > 0$, one has*

$$\{f \triangle g < \alpha\} = \{f<\alpha\} + \{g<\alpha\}.$$

As a direct consequence of Theorem 2.7 and Proposition 3.8, we obtain

PROPOSITION 3.9. *Let f and g be in $\mathcal{F}(X)$ and suppose that f is robust. Then $f \triangle g$ is robust.*

THEOREM 3.10. *For each $f \in \mathcal{F}(X)$ there exists a sequence $(f_p) \in \mathcal{R}(X)$ such that $D(f, f_p) \leq 1/p$ for $p = 1, 2, \ldots$.*

PROOF.

Let $g_p = I_{B[\mathbf{0},1/p]}$ the indicator function of the closed ball $B[\mathbf{0}, 1/p]$, *i.e.* :

$$I_{B[0,1/p]}(x) = \begin{cases} 0 & \text{if} \quad x \in B[\mathbf{0}, 1/p] \\ +\infty & \text{if} \quad x \notin B[\mathbf{0}, 1/p]. \end{cases}$$

Then $\{g_p < \alpha\} = B[\mathbf{0}, 1/p]$, $\forall \alpha > 0$. Therefore, due convexity of $B[\mathbf{0}, 1/p]$ and Corollary 2.4, we have that g_p is robust for every $p \in \mathbb{N}$.

Now, let $f_p = g_p \triangle f$ be. Then, due Proposition 3.9, f_p is robust for each $p \in \mathbb{N}$ and

$$\begin{aligned} H(\{f &< \alpha\},\{f_p<\alpha\}) = H(\{f<\alpha\},\{g_p<\alpha\}+\{f<\alpha\}) \\ &= H(\{\mathbf{0}\}+\{f<\alpha\},\{g_p<\alpha\}+\{f<\alpha\}) \end{aligned}$$

$$
\begin{aligned}
&\leq H(\{\mathbf{0}\}, \{g_p < \alpha\}) + H(\{f < \alpha\}, \{f < \alpha\}) \quad (by\ Prop.3.5) \\
&= H(\{\mathbf{0}\}, B[\mathbf{0}, 1/p]) \\
&= 1/p.
\end{aligned}
$$

So, taking supremum in $\alpha > 0$, we obtain $D(f, f_p) \leq 1/p$, for every p, and the proof is complete. ∎

COROLLARY 3.11. *$(\mathcal{R}(X), D)$ is a dense subspace of $(\mathcal{F}(X), D)$.*

4. ROBUSTNESS AND PROPER LOCAL MINIMUM POINTS

In this section we shall prove a characterization, via level sets, of non-existence of proper local minimum points for robust functions. This result generalize an analogous ones for upper semicontinuous functions obtained for the author in [3,4].

DEFINITION 4.1. *If $f : X \to [0, \infty]$ is a function in $\mathcal{F}(X)$, then $x_0 \in X$ is said to be a proper local minimum point of f if $f(x_0) > 0$ and there is a neighborhood U at x_0 such that $f(x_0) \leq f(x)$, for every $x \in U$.*

THEOREM 4.2. *Let $f \in \mathcal{F}(X)$ be a robust function. Then are equivalents:*

i) *f has no proper local minimum points,*

ii) $\overline{\{f \leq \alpha\}} = \overline{Int\{f < \alpha\}}$.

PROOF.
ii) $\to$ *i)*. Suppose that x_0 is a proper local minimum point of f. Then $f(x_0) > 0$ and there is a neighborhood U at x_0 such that $0 < \alpha_0 = f(x_0) \leq f(x)$, for every $x \in U$. So, $x_0 \in \{f \leq \alpha_0\}$ and $U \cap \{f < \alpha_0\} = \emptyset$. Consequently, $x_0 \in \overline{\{f \leq \alpha_0\}} \setminus \overline{\{f < \alpha_0\}}$.

But, due robustness of f, $\overline{\{f < \alpha_0\}} = \overline{Int\{f < \alpha_0\}}$, therefore

$$x_0 \in \overline{\{f \leq \alpha_0\}} \setminus \overline{Int\{f < \alpha_0\}}.$$

i) $\to$ *ii)*. Suppose that there exists $\alpha_0 > 0$ such that $\overline{\{f \leq \alpha_0\}} \neq \overline{Int\{f < \alpha_0\}}$.

Then, due robustness of f, $\overline{\{f \le \alpha_0\}} \neq \emptyset$. In fact,

$$\overline{\{f \le \alpha_0\}} = \emptyset \Rightarrow \overline{\{f < \alpha_0\}} \subseteq \overline{\{f \le \alpha_0\}} = \emptyset \Rightarrow \{f < \alpha_0\} = \emptyset$$

which is impossible due $\{f < \alpha_0\} \in \mathcal{P}_0(X)$.

Moreover,

$$\begin{aligned} \{f = \alpha_0\} = \emptyset \;\; &\Rightarrow \;\; \{f \le \alpha_0\} = \{f < \alpha_0\} \\ &\Rightarrow \;\; \overline{\{f \le \alpha_0\}} = \overline{\{f < \alpha_0\}} \\ &\Rightarrow \;\; \overline{\{f \le \alpha_0\}} = \overline{Int\{f < \alpha_0\}} \;\; (due\ f \in \mathcal{R}(X)) \end{aligned}$$

in contradiction with our hypothesis.
Therefore, must be $\{f = \alpha_0\} \neq \emptyset$. Consequently, there exists $x_0 \in X$ such that $f(x_0) = \alpha_0$ and $x_0 \notin \overline{Int\{f < \alpha_0\}} = \overline{\{f < \alpha_0\}}$. So, there exists a neighborhood U of x_0 such that $U \cap \{f < \alpha_0\} = \emptyset$ which implies that $f(x) \ge \alpha_0 = f(x_0)$ for every $x \in U$.

Therefore x_0 is a proper local minimum point of f and the proof is complete. ■

Remark 4.3. We observe that, under conditions of Theor. 4.2., any local minimum of f is a global minimum.

Example 4.4. Consider the function $f : \mathbb{R} \to [0, +\infty]$ as

$$f(x) = \begin{cases} 0 & if \;\; x < 0 \\ x & if \;\; 0 \le x < 1 \\ 1 & if \;\; 1 \le x < 2 \\ x - 1 & if \;\; x \ge 2. \end{cases}$$

Then it is clear that, due continuity, f is a robust function. Furthermore, x is a proper local minimum point for every $x \in (1, 2]$. On other hand, for the level $\alpha = 1$ we have $\{f \le 1\} = \overline{\{f \le 1\}} = [0, 2]$, whereas

$$\begin{aligned} \{f < 1\} = [0, 1) \;\; &\Rightarrow \;\; Int\{f < 1\} = (0, 1) \\ &\Rightarrow \;\; \overline{Int\{f < 1\}} = [0, 1]. \end{aligned}$$

Therefore $\overline{\{f \le 1\}} \neq \overline{Int\{f < 1\}}$.

Remark 4.5. In relation to Theor. 4.2, analogous optimization results have been obtained for a more restricted class of functions. For instance:

(a) Martinez-Legaz [13, Cor. 2.19, p. 117]: "If $X = \mathbb{R}^n$ then the function $f \in \mathcal{F}(X)$ is upper semicontinuous and has no local maximum points in its domain if and only if $\{f < \alpha\} = int\{f \leq \alpha\}$, for every $\alpha > 0$".

(b) Román-Flores et al. [4]: "If X is a regular topological space and $f \in \mathcal{F}(X)$ is an upper semicontinuous function with $\sup\limits_{x \in X} f(x) = M$, then are equivalents:

i) f is without proper local maximum points

ii) $\{f \geq \alpha\} = \overline{\{f > \alpha\}}$, $\forall\, \alpha \in (0, M)$.

iii) f is level-continuous.

We recall that an upper semicontinuous function $f \in \mathcal{F}(X)$ is level-continuous if, and only if,

$$\alpha_p \to \alpha \Rightarrow H(\{f \geq \alpha_p\} \to H(\{f \geq \alpha\}), \text{ for every } \alpha \in \left(0, \sup_{x \in X} f(x)\right).$$

5. EPI-APPROXIMATION OF ROBUST FUNCTIONS

Analogously to the methods developed in Section 3 for approximation of robust functions via level sets and $\triangle$-convolution, we can to approximate robust functions via its epigraphs (epi-convergence). The fundamental variational property of epi-convergence can be established as follows: If $f_n,\ f : X \to \overline{\mathbb{R}},\ n = 1, 2, ...,$ is a sequence of real (extended) functions which satisfies the condition that there exists a relative compact subset K of X such that, for every $n = 1, 2, ...,$

$$\underset{x \in X}{inf}\ f_n(x) = \underset{x \in K}{inf}\ f_n(x),$$

then $f = epi - lim\ f_n$ implies

$$\underset{x \in X}{inf}\ f_n(x) \to \underset{x \in X}{inf}\ f(x)\ as\ n \to \infty,$$

and every cluster point x of a minimizing sequence $(x_n \in Argmin\ f_n,\ n = 1, 2, ...)$ minimizes f, where $Argmin\ f = \{y/\ f(y) \leq inf\ f\}$. For details see [7].

DEFINITION 5.1. *Let $f \in \mathcal{F}(X)$ be. The epigraph of f is defined by*

$$epi(f) = \{(x, \alpha) \in X \times [0, +\infty]/\ f(x) < \alpha\}.$$

REMARK 5.2. If $f(x) = +\infty$ then, as a direct consequence of above definition we obtain $(x, \alpha) \notin epi(f)$, for any $\alpha \in [0, +\infty]$.

DEFINITION 5.3. *Let f and g be in $\mathcal{F}(X)$. Then, the sum-convolution is defined by*

$$(f \Box g)(x) = \underset{y \in X}{Inf}\ \{f(x - y) + g(y)\}.$$

PROPOSITION 5.4. *Let f and g in $\mathcal{F}(X)$. Then,*

$$epi(f \Box g) = epi(f) + epi(g).$$

(For details, see [8], [11, [12]).

Let $\mathcal{R}_e(X)$ be the class of all f in $\mathcal{F}(X)$ such that $epi(f)$ is robust in the product space $X \times [0, +\infty]$ endowed with the topology induced by the usual metric

$$\rho((x, \alpha), (y, \beta)) = max\{\|x - y\|, \mid \alpha - \beta \mid\}.$$

If $f \in \mathcal{R}_e(X)$ then we say that f is epi-robust.
The next proposition shows that every robust function has a robust epigraph.

THEOREM 5.5. $\mathcal{R}(X) \subseteq \mathcal{R}_e(X)$.
PROOF.
Let $f \in \mathcal{R}(X)$ be. It is sufficient to show that $\overline{epi(f)} \subseteq \overline{Int\ epi(f)}$.
In fact, if $(x, \alpha) \in \overline{epi(f)}$ then there exists a sequence $((x_n, \alpha_n)) \subseteq epi(f)$ such that $lim\ (x_n, \alpha_n) = (x, \alpha)$ as $n \to \infty$. Therefore $f(x_n) < \alpha_n$ and, due robustness of f,

$$x_n \in \{f < \alpha_n\} \subseteq \overline{\{f < \alpha_n\}} = \overline{Int\{f < \alpha_n\}},\ \forall n.$$

Thus, for each $n \in \mathbb{N}$ there exists a sequence $(x_{ni}) \subseteq Int\{f < \alpha_n\}$ such that $\underset{i \to \infty}{lim}\ x_{ni} \to x_n$. Thus, for each n we can choose i_n, with $i_n < i_{n+1}$, such that $\|x_n - x_{ni_n}\| < 2^{-n}$ for every n. So,

$$\|x - x_{ni_n}\| \leq \|x - x_n\| + \|x_n - x_{ni_n}\| \to 0 \text{ as } n \to \infty.$$

On other hand, due $f(x_{ni_n}) < \alpha_n$ for every n, we can construct a sequence (ϵ_n) such that $\epsilon_n \to 0$, $0 < \epsilon_n < \alpha_n$, $B(x_{ni_n}, \epsilon_n) \subseteq Int\{f < \alpha_n\}$ and $f(x_{ni_n}) < \alpha_n - \epsilon_n < \alpha_n$, $\forall n$.
We claim that $B\left((x_{ni_n}, \alpha_n + \epsilon_n), \epsilon_n\right) \subseteq epi(f)$. In fact,

$$(z, \lambda) \in B\left((x_{ni_n}, \alpha_n + \epsilon_n), \epsilon_n\right) \Rightarrow max\{\|z - x_{ni_n}\|, \mid \lambda - (\alpha_n + \epsilon_n) \mid\} < \epsilon_n.$$

Then:
a) $\|z - x_{ni_n}\| < \epsilon_n \Rightarrow z \in B(x_{ni_n}, \epsilon_n) \subseteq Int\{f < \alpha_n\} \Rightarrow f(z) < \alpha_n$.
b) $\mid \lambda - (\alpha_n + \epsilon_n) \mid < \epsilon_n \Rightarrow -\epsilon_n < \lambda - (\alpha_n + \epsilon_n) \Rightarrow \alpha_n < \lambda$.

So, from a) and b) we obtain $f(z) < \lambda$, that is to say $(z, \lambda) \in epi(f)$ and, consequently, $(x_{ni_n}, \alpha_n + \epsilon_n) \in Int\ epi(f)$ for every n.
Finally, due $\lim_{n\to\infty} (x_{ni_n}, \alpha_n + \epsilon_n) = (x, \alpha)$, we conclude that $(x, \alpha) \in \overline{Int\ epi(f)}$. ■

The following example shows that, actually, the inclusion in above theorem is proper.

EXAMPLE 5.6. Let $x_0 \in X$ be and $f = \chi_{\{x_0\}}$ the indicator function of $\{x_0\}$, i.e.:

$$\chi_{\{x_0\}}(x) = \begin{cases} 1 & \text{if} \quad x = x_0 \\ 0 & \text{if} \quad x \neq x_0. \end{cases}$$

Then

$$\{f = \chi_{\{x_0\}} < \alpha\} = \begin{cases} X & \text{if} \quad \alpha > 1 \\ \{x_0\} & \text{if} \quad 0 < \alpha \leq 1. \end{cases}$$

Thus, if $0 \leq \alpha_0 \leq 1$ we have $\{f < \alpha_0\} = \{x_0\}$ and, consequently, $\overline{Int\{f < \alpha\}} = \emptyset$ whereas $\overline{\{f < \alpha\}} = \{x_0\}$. Therefore f is not a robust function.
Nevertheless, $epi(f) = X \setminus \{x_0\} \times (0, +\infty] \cup \{x_0\} \times (1, +\infty]$ which is a robust set in the product space $X \times [0, +\infty]$.

PROPOSITION 5.7. *Let $f \in \mathcal{F}(X)$ be and $g \in \mathcal{R}_e(X)$. Then $f \Box g \in \mathcal{R}_e(X)$.*
PROOF.
By Proposition 5.4 we have $epi(f \Box g) = epi(f) + epi(g)$, and on other hand, due robustness of $epi(g)$ then, by Theor. 2.7 we obtain $f \Box g \in \mathcal{R}_e(X)$. ■

Now, if $f, g \in \mathcal{F}(X)$ we can define an epi-generalized Hausdorff extended pseudometric by mean

$$D_e(f, g) = H(epi(f), epi(g)),$$

where H is the generalized Hausdorff extended pseudometric induced by the distance ρ on the product space $X \times [0, +\infty]$.
Our main result in this section is to show an epi-approximation result on $\mathcal{F}(X)$ by using $\Box$-convolution. More specifically, we will prove that the space of epi-robust functions is a dense subspace of $(\mathcal{F}(X), D_e)$.

THEOREM 5.8. *For each* $f \in \mathcal{F}(X)$ *there exists a sequence* $(f_p) \in \mathcal{R}_e(X)$ *such that* $D_e(f, f_p) \leq 1/p$ *for* $p = 1, 2, ...$

PROOF.
Consider the sequence (g_p) defined as the indicator function of the closed ball $B[\mathbf{0}, 1/p]$, *i.e.* :

$$g_p(x) = \begin{cases} 0 & \text{if} \quad x \in B[\mathbf{0}, 1/p] \\ +\infty & \text{if} \quad x \notin B[\mathbf{0}, 1/p]. \end{cases}$$

Then $epi(g_p) = B[\mathbf{0}, 1/p] \times (0, +\infty]$ which is a robust set and, consequently, $g_p \in \mathcal{R}_e(X)$, $\forall p$. Now, defining $f_p = f \Box g_p$, due Proposition 5.7 and Proposition 5.4, we have that $f_p \in \mathcal{R}_e(X)$ and $epi(f_p) = epi(f) + epi(g_p)$, respectively, for every $p \in \mathbb{N}$.
We claim that $D_e(f, f_p) \leq 1/p$, $\forall p$.
In fact, if $(x, \alpha) \in epi(f)$ then, because $(\mathbf{0}, 1/p) \in epi(g_p)$, we obtain

$$(x, \alpha + 1/p) \in epi(f) + epi(g_p) = epi(f_p).$$

Therefore

$$\begin{aligned} d\left((x, \alpha), epi(f_p)\right) &= \inf_{z \in epi(f_p)} \rho((x, \alpha), z) \\ &\leq \rho((x, \alpha), (x, \alpha + 1/p) \\ &= \frac{1}{p}. \end{aligned}$$

Thus,

$$\sup_{(x,\alpha) \in epi(f)} d\left((x, \alpha), epi(f_p)\right) \leq \frac{1}{p}. \tag{1}$$

Conversely, if $(y, \beta) \in epi(f_p)$ then $(y, \beta) = (y_1, \beta_1) + (y_2, \beta_2)$, with $(y_1, \beta_1) \in epi(f)$ and $(y_2, \beta_2) \in epi(g_p)$. So, due Remark 5.2, we observe that $(y_2, \beta_2) \in epi(g_p)$ implies $y_2 \in B[\mathbf{0}, 1/p]$ and, consequently, $\| y_2 \| \leq 1/p$. On other hand,

$(y_1, \beta_1) \in epi(f)$ implies $f(y_1) < \beta_1$, therefore $(y_1, \beta_1 + \beta_2 + 1/p) \in epi(f)$. Thus,

$$\begin{aligned} d\left((y,\beta), epi(f)\right) &= \inf_{z \in epi(f)} \rho((y,\beta), z) \\ &\leq \rho((y,\beta), (y_1, \beta_1 + \beta_2 + 1/p)) \\ &= max\{\|y_2\|, 1/p\} \\ &\leq 1/p. \end{aligned}$$

Thus,

$$\sup_{(y,\beta)\in epi(f_p)} d\left((y,\beta), epi(f)\right) \leq \frac{1}{p}. \tag{2}$$

Therefore, from (1), (2) and Remark 3.3 we obtain

$$\begin{aligned} D_e(f, f_p) &= H\left(epi(f), epi(f_p)\right) \\ &= max\left\{ \sup_{(x,\alpha)\in epi(f)} d\left((x,\alpha), epi(f_p)\right), \sup_{(y,\beta)\in epi(f_p)} d\left((y,\beta), epi(f)\right) \right\} \\ &\leq 1/p, \end{aligned}$$

and the proof is complete. ■

Corollary 5.9. *$(\mathcal{R}_e(X), D_e)$ is a dense subspace of $(\mathcal{F}(X), D_e)$.*

6. D-CONVERGENCE IS STRONGER THAN D_e-CONVERGENCE

An interesting problem is to compare D-convergence and D_e-convergence on $\mathcal{F}(X)$. This problem, in a more restricted context, has been studied in [3,4], where the authors proves some results on equivalence of convergences for bounded an upper semicontinuous functions (which are a particular case of robust functions). The following examples shows that, in general, D-convergence and D_e-convergence are not equivalents.

EXAMPLE 6.1. Consider $f_n, f : \mathbb{R} \to [0, +\infty)$ defined by

$$f_n(x) = \begin{cases} \dfrac{1}{n} & if \quad x \notin [-1/n, 1/n] \\ 0 & if \quad -\dfrac{1}{n} \leq x \leq \dfrac{1}{n} \end{cases}$$

$$f(x) = 0, \ \forall x \in \mathbb{R}.$$

Then f_n, f are robust functions with $epi(f_n) = \mathbb{R} \setminus [-1/n, 1/n] \times (1\backslash n, +\infty] \cup [-1/n, 1/n] \times (0, +\infty]$ and $epi(f) = \mathbb{R} \times (0, +\infty]$. Also, because $H(epi(f_n), epi(f)) \leq 1/n$, it is easy to see that f_n D_e-converges to f. Nevertheless, for $\alpha = 1/n$ we have $\{f_n < 1/n\} = [-1/n, 1/n]$ whereas $\{f < 1/n\} = \mathbb{R}$. Therefore

$$\begin{aligned} D(f, f_n) &= \sup_{\alpha > 0} H(\{f < \alpha\}, \{g < \alpha\}) \\ &\geq H(\{f < 1/n\}, \{f_n < 1/n\}) \\ &= +\infty \end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, f_n does not converges to f in D-pseudometric.

EXAMPLE 6.2. Consider $f_n, f : \mathbb{R} \to [0, +\infty]$ defined by

$$f_n(x) = \begin{cases} 0 & if \quad 0 \leq x \leq 1 \\ \frac{1}{n}(x-1) + 1 - \frac{1}{n} & if \quad 1 < x \leq 2 \\ +\infty & if \quad x \notin [0, 2]. \end{cases}$$

$$f(x) = \begin{cases} 0 & if \quad 0 \leq x \leq 1 \\ 1 & if \quad 1 < x \leq 2 \\ +\infty & if \quad x \notin [0, 2]. \end{cases}$$

Then, it is clear that $H(epi(f_n), epi(f)) \leq 1/n$, therefore f_n D_e-converges to f. On other hand, for $\alpha = 1$ we have $\{f_n < 1\} = [0, 2)$ whereas $\{f < 1\} = [0, 1]$, therefore $H(\{f_n < 1\}, \{f < 1\}) = 1$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} D(f, f_n) &= \sup_{\alpha > 0} H(\{f < \alpha\}, \{g < \alpha\}) \\ &\geq H(\{f < 1\}, \{f_n < 1\}) \\ &= 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, f_n does not converges to f in D-pseudometric.

Remark 6.3. We note that in Example 6.1 the α-level sets of f_n and f are not bounded subsets of $\mathbb{R}$, whereas in Example 6.2 the α-level sets of f_n and f are bounded subsets of $\mathbb{R}$. Also, it is important to remark that, in the first case, the limit function f has no proper local minimum points whereas, in the second case, the limit function f possesses proper local minimum points.

Theorem 6.4. *Let $f_n, f \in \mathcal{R}(X)$ be. Then, $f_n \stackrel{D}{\to} f$ implies f_n D_e-converges to f.*
Proof.
If we suppose that $f_n \stackrel{D}{\to} f$ then given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$D(f_n, f) = \sup_{\alpha>0} H(\{f < \alpha\}, \{g < \alpha\}) < \epsilon,$$

for all $n \geq N$. Now, if $(y, \beta) \in epi(f_n)$ we have $f_n(y) < \beta$ which implies $y \in \{f_n < \beta\}$. On other hand, as

$$\begin{aligned} H(\{f_n < \beta\}, \{f < \beta\} &= max\left\{ \sup_{x\in\{f_n<\beta\}} d\left(x, \{f < \beta\}\right), \sup_{z\in\{f<\beta\}} d\left(z, \{f_n < \beta\}\right) \right\} \\ &< \epsilon \end{aligned}$$

we conclude that

$$\sup_{x\in\{f_n<\beta\}} d\left(x, \{f < \beta\}\right) < \epsilon$$

and, consequently, $d\left(y, \{f < \beta\}\right) < \epsilon$.

Therefore, there exists $z \in \{f < \beta\}$ such that $\|y - z\| < \epsilon$. Thus, $(z, \beta) \in epi(f)$ and

$$\rho((y, \beta), (z, \beta)) = \|y - z\| < \epsilon,$$

which implies that $d((y, \beta), epi(f)) < \epsilon$. As (y, β) is arbitrary in $epi(f_n)$ we obtain

$$\sup_{(y,\beta)\in epi(f_n)} d\left((y, \beta), epi(f)\right) \leq \epsilon.$$

In a similar way, we can prove that

$$\sup_{(x,\alpha)\in epi(f)} d\left((x, \alpha), epi(f_n)\right) \leq \epsilon,$$

which implies that $H(epi(f_n), epi(f)) \leq \epsilon$ for all $n \geq N$ and, consequently, f_n D_e-converges to f. ■

7. D_E-CONVERGENCE AND L-CONVERGENCE

In order to establish a type of reverse implication of above Theorem 6.4 we need to introduce the concepts of "Kuratowski convergence" of sets and D_E-convergence and L-convergence on $\mathcal{F}(X)$, and for this, we shall work with non-strict epigraphs and levels.

DEFINITION 7.1. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets in $\mathcal{P}_0(X)$. Define*

$$\begin{aligned} lim\ inf\ A_n &= \{x \in X /\ x = \lim_{n\to\infty} x_n,\ x_n \in A_n,\ \forall n\} \\ lim\ sup\ A_n &= \{x \in X /\ x = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ \forall k\} \end{aligned}$$

If $lim\ inf\ A_n = lim\ sup\ A_n = A$, then we say A is the limit of the sequence $\{A_n\}_n$ and the sequence $\{A_n\}_n$ converges to A (in the Kuratowski sense), and we write $A = lim\ A_n$ (or $A_n \overset{K}{\to} A$).

PROPOSITION 7.2. *If $\{A_n\}_n$ is a sequence of subsets in $\mathcal{P}_0(X)$, then*

i) $lim\ inf\ A_n \subseteq lim\ sup\ A_n$;

ii) $lim\ inf\ A_n$ and $lim\ sup\ A_n$ are closed subsets of X;

iii) $lim\ inf\ A_n = lim\ inf\ \overline{A_n}$ and $lim\ sup\ A_n = lim\ sup\ \overline{A_n}$;

iv) $lim\ sup\ A_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} A_k}$;

iv) *$lim\ inf\ A_n = \bigcap_{F} \overline{\bigcup_{k\in F} A_k}$, where F denotes an arbitrary cofinal subset of $\mathbb{N}$ and the intersection is over all such F.*

For more details see [7,10].

REMARK 7.3. We recall that F is a cofinal subset of $\mathbb{N}$ if $\forall n \in \mathbb{N}$, $\exists m \in F$ such that $m > n$.

PROPOSITION 7.4.*([6]). Let $\{A_n\}_n$ be a sequence of subsets in $\mathcal{P}_0(X)$ and suppose that there exists a compact set $K \subseteq X$ such that $A_n, A \subseteq K$ for all $n \in \mathbb{N}$. Then $A_n \overset{H}{\to} A$ if and only if*

$$lim\ sup\ A_n \subseteq A \subseteq lim\ inf\ A_n.$$

DEFINITION 7.5. *If $f \in \mathcal{F}(X)$ then we define*

$$Epi(f) = \{(x, \alpha)/\ f(x) \leq \alpha\}.$$

DEFINITION 7.6. *If $f \in \mathcal{F}(X)$ then we define $L_\alpha f = \{x \in X/\ f(x) \leq \alpha\}$.*

REMARK 7.7. It is well known that, if $f \in \mathcal{F}(X)$, then $Epi(f)$ is closed iff $L_\alpha f$ is closed (see [1]).

PROPOSITION 7.8.*([1, Theor. 3.3]). If $f \in \mathcal{F}(X)$ then f is robust if and only if $Epi(f)$ is robust.*

DEFINITION 7.9.*(L-convergence). If $f_n, f \in \mathcal{F}(X)$ then we said that f_n L-converges to f (for short: $f_n \overset{L}{\to} f$) iff $L_\alpha f_n \overset{K}{\to} L_\alpha f$, $\forall \alpha > 0$.*

DEFINITION 7.10.*(D_E-convergence). If $f_n, f \in \mathcal{F}(X)$ then we said that f_n D_E-converges to f (for short: $f_n \overset{D_E}{\to} f$) iff $Epi(f_n) \overset{K}{\to} Epi(f)$.*

REMARK 7.11. We want to observe that in Examples 6.1 in above section, the limit function f is lower semicontinuous (i.e., $Epi\,(f)$ is closed) and $f_n \overset{D_E}{\to} f$. Also, in this case, we have $f_n \overset{L}{\to} f$.

EXAMPLE 7.12. Consider $f_n, f : \mathbb{R} \to [0, +\infty]$ defined by

$$f_n(x) = \begin{cases} 0 & if \quad 0 \leq x \leq 1 \\ \frac{1}{n}(1-x) + 1 + \frac{1}{n} & if \quad 1 < x \leq 2 \\ +\infty & if \quad x \notin [0,2]. \end{cases}$$

$$f(x) = \begin{cases} 0 & if \quad 0 \leq x \leq 1 \\ 1 & if \quad 1 < x \leq 2 \\ +\infty & if \quad x \notin [0,2]. \end{cases}$$

Then, it is clear that $Epi(f)$ is closed and

$$\begin{aligned} Epi(f) &= [0,1]\times[0,+\infty]\cup(1,2]\times[1,+\infty] \\ &= \bigcap_{n=1}^{\infty}\overline{\bigcup_{k\geq n} Epi(f_k)}, \end{aligned}$$

therefore f_n D_E-converges to f. On other hand, for $\alpha = 1$ we have $L_1 f = [0,2]$ whereas $L_1 f_n = [0,1]\cup\{2\}$ and, consequently, $H(L_1 f_n, L_1 f) = 1$ for all $n \in \mathbb{N}$. Thus, because $L_1 f_n$ and $L_1 f$ are contained in $[0,2]$ which is a compact subset of $\mathbb{R}$ then, due Proposition 7.4, $L_1 f_n$ does not converges (in the Kuratowski sense) to $L_1 f$, and this implies that f_n does not L-converges to f.

Summarizing, we have the following result:

THEOREM 7.13. *Let $f_n, f \in \mathcal{R}(X)$ be with $Epi(f)$ closed and suppose that f has no proper local minimum points. Then, the following conditions are equivalents:*

i) $f_n \overset{L}{\to} f$

ii) $f_n \overset{D_E}{\to} f$

PROOF.
i)→ii). In order to prove that $f_n \overset{D_E}{\to} f$, it is sufficient to show that

$$lim\ sup\ Epi(f_n) \subseteq Epi(f) \subseteq lim\ inf\ Epi(f_n).$$

Let $(x,\alpha) \in lim\ sup\ Epi(f_n)$. Then

$$(x,\alpha) \in \bigcap_{p\geq 1}\overline{\bigcup_{k\geq p} End(f_k)}\ . \tag{3}$$

If we suppose that $f(x) > \alpha$, then there exists $\epsilon > 0$ such that $f(x) > \alpha + \epsilon > \alpha$. So, due $f_n \overset{L}{\to} f$, we obtain that $x \notin L_{\alpha+\epsilon} f = \bigcap_{p\geq 1}\overline{\bigcup_{k\geq p} L_{\alpha+\epsilon} f_k}$. This implies that $\exists p_0$ such that $x \notin \overline{\bigcup_{k\geq p_0} L_{\alpha+\epsilon} f_k}$ and, therefore, there exists a neighborhood $U(x) = U$ of x such that

$$U \cap [\bigcup_{k\geq p_0} L_{\alpha+\epsilon} f_k] = \emptyset. \tag{4}$$

Now, we assure that $[U \times (0, \alpha + \epsilon)] \cap [\bigcup_{k \geq p_0} Epi(f_k)] = \emptyset$. In fact,

$$(y, \beta) \in U \times (0, \alpha + \epsilon) \cap [\bigcup_{k \geq p_0} Epi(f_k)] \Rightarrow \begin{cases} \beta < \alpha + \epsilon \quad and \\ \exists k_0 \geq p_0 \; such \; that \; (y, \beta) \in Epi(f_{k_0}). \end{cases}$$

Therefore, $f_{k_o}(y) \leq \beta < \alpha + \epsilon$.. But, due (4), $y \in U$ implies $y \notin \bigcup_{k \geq p_0} L_{\alpha+\epsilon} f_k$.
That is, $f_k(y) > \alpha + \epsilon$, $\forall k \geq p_0$, which is absurd.
Thus, $U \times (\alpha - \epsilon, \infty)$ is an open in the product topology which nonintersecting $\bigcup_{k \geq p_0} Epi(f_k)$.
So, because $(x, \alpha) \in U(x) \times (0, \alpha + \epsilon)$, we obtain that $(x, \alpha) \notin \overline{\bigcup_{k \geq p_0} Epi(f_k)}$.
Therefore, $(x, \alpha) \notin \bigcap_{p \geq 1} \overline{\bigcup_{k \geq p} Epi(f_k)}$, in contradiction with (3). So, must be $f(x) \geq \alpha$ and, consequently, $(x, \alpha) \in Epi(f)$.
On the other hand, let $(x, \alpha) \in Epi(f)$. Then $f(x) \leq \alpha$ and, due $f_n \xrightarrow{L} f$, we obtain that

$$x \in lim \; sup \; L_\alpha f_n = \bigcap_F \overline{\bigcup_{k \in F} L_\alpha f_k}. \tag{5}$$

If we suppose that $(x, \alpha) \notin \underline{lim \; inf} \; Epi(f_n)$, then there exists F_0 cofinal such that $(x, \alpha) \notin \overline{\bigcup_{k \in F_0} Epi(f_k)}$.
Therefore, there exists a neighborhood V of (x, α) such that

$$V \cap [\bigcup_{k \in F_0} Epi(f_k)] = \emptyset. \tag{6}$$

Without loss of generality, we can to suppose that V is a basic open of the product topology, that is, $V = U \times (\theta, \eta)$ where U is an open in X and (θ, η) is an open interval in $\mathbb{R}^+$containing α. We note that if $y \in U$, then $V = U \times (\theta, \eta)$ containing the segment $\{y\} \times (\theta, \eta)$.
Now, we assure that the projection $p_X(V)$ is an open set in X which nonintersecting $\bigcup_{k \in F_0} L_\alpha f_k$ (we recall that p_X is an open mapping). In fact, if we suppose that $p_X(V) \cap [\bigcup_{k \in F_0} L_\alpha f_k] \neq \emptyset$, then there exists $y \in p_X(V)$ such that $f_{k_0}(y) \leq \alpha$, for some $k_0 \in F_0$.
Therefore, $y \in U$ and there is $\beta \geq \alpha$ such that $(y, \beta) \in V = U \times (\theta, \eta)$.

But then, $(y,\beta) \in V \cap Epi(f_{k_0}) \subseteq V \cap [\bigcup_{k\in F_0} Epi(f_k)]$, in contradiction with (6).
Thus, because $p_X(V) \cap [\bigcup_{k\in F_0} L_\alpha f_k] = \emptyset$ and $x \in p_X(V)$, we conclude that $x \notin \overline{\bigcup_{k\in F_0} L_\alpha f_k}$ which, due (5), is absurd.
Summarizing, we must have $(x,\alpha) \in lim\ inf\ Epi(f_n)$.
Therefore, $lim\ Epi(f_n) = Epi(f)$, which implies that $f_n \xrightarrow{\Gamma} f$, completing the first part of our proof.
ii)→i). In order to prove that $f_n \xrightarrow{L} f$, it is sufficient to show that

$$lim\ sup\ L_\alpha f_n \subseteq L_\alpha f \subseteq lim\ inf\ L_\alpha f_n,\ \forall \alpha > 0.$$

For this, let $\alpha \in [0,\infty)$ be and suppose that

$$x \in lim\ sup\ L_\alpha f_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} L_{\alpha_k} f}. \tag{7}$$

If $f(x) > \alpha$, then $(x,\alpha) \notin Epi(f) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} Epi(f_k)}$.
Therefore, $\exists n_0$ such that$(x,\alpha) \notin \overline{\bigcup_{k\geq n_0} Epi(f_k)}$.

Consequently, there exists a neighborhood V of (x,α) such that

$$V \cap [\bigcup_{k\geq n_0} Epi(f_k)] = \emptyset. \tag{8}$$

Also, without loss of generality, we can suppose that V is an basic open of the product topology, that is, $V = U \times (\theta,\eta)$. But then, the projection $U = p_X(V)$ is a neighborhood of x which nonintersecting $\bigcup_{k\geq n_0} L_\alpha f_k$. In fact, if $y \in U \cap \bigcup_{k\geq n_0} L_\alpha f_k$ then there exists $\beta \geq \alpha$ such that $(y,\beta) \in V$, and $\exists k_0 \geq n_0$ such that $f_{k_0}(y) \leq \alpha \leq \beta$, that is, $(y,\beta) \in Epi(f_{k_0})$.
Thus, $(y,\beta) \in V \cap [\bigcup_{k\geq n_0} Epi(f_k)]$ which contradicts (8). So, $U \cap [\bigcup_{k\geq n_0} L_\alpha f_k] = \emptyset$ but, because $x \in U$, we conclude that $x \notin \overline{\bigcup_{k\geq n_0} L_\alpha f_k}$, in contradiction with (7).
Hence $f(x) \leq \alpha$ and, consequently, $x \in L_\alpha f$. Therefore, $lim\ sup\ L_\alpha f_n \subseteq L_\alpha f$.

On the other hand, let $x \in L_\alpha f$ be and suppose that $f(x) < \alpha$. Then there is $\epsilon > 0$ such that $f(x) < \alpha - \epsilon$. So, due $f_n \xrightarrow{\Gamma} f$, we have

$$(x, \alpha - \epsilon) \in Epi(f) = lim\ inf\ Epi(f_n) = \bigcap_F \overline{\bigcup_{k \in F} Epi(f_k)}. \tag{9}$$

Now, if we suppose that $x \notin lim\ inf\ L_\alpha f_n$, then $\exists F_0$ cofinal such that $x \notin \overline{\bigcup_{k \in F_0} L_\alpha f_k}$ and, therefore, $\exists U = U(x)$ such that

$$U \cap [\bigcup_{k \in F_0} L_\alpha f_k] = \emptyset. \tag{10}$$

We assure that $[U \times (0, \alpha)] \cap \bigcup_{k \in F_0} Epi(f_k) = \emptyset$.
In fact, if $(y, \beta) \in [U \times (0, \alpha)] \cap \bigcup_{k \in F_0} Epi(f_k)$ then $f_{k_0}(y) \leq \beta \leq \alpha$ for some $k_0 \in F_0$, and this implies $y \in U \cap L_\alpha f_{k_0} \subseteq U \cap [\bigcup_{k \in F_0} L_\alpha f_k]$, in contradiction with (10).

Thus, because $(x, \alpha - \epsilon) \in U \times (0, \alpha)$, we obtain that $(x, \alpha - \epsilon) \notin \overline{\bigcup_{k \in F_0} Epi(f_k)}$ and, therefore, $(x, \alpha - \epsilon) \notin lim\ inf\ Epi(f_n) = Epi(f)$ which, due (9), is absurd.
So, necessarily, we must have $x \in lim\ inf\ L_\alpha f_n$ and, consequently, $\{f < \alpha\}$ is contained in $lim\ inf\ L_\alpha f_n$.
Finally, we observe that:

a) $Epi(f)$ closed iff f lower semicontinuous ($i.e.$: $L_\alpha f = \{f \leq \alpha\}$ closed, $\forall \alpha$)

b) If f is a lower semicontinuous and robust function in $\mathcal{F}(X)$ then, due Theor. 4.2, the condition "f has no proper local minimum points" is equivalent to $\{f \leq \alpha\} = \overline{Int\{f < \alpha\}}$, $\forall \alpha$.

Thus, because f has no proper local minimum points, $lim\ inf\ L_\alpha f_n$ is closed and $\{f < \alpha\} \subseteq lim\ inf\ L_\alpha f_n$, we have

$$\begin{aligned} \{f < \alpha\} \subseteq lim\ inf\ L_\alpha f_n &\Rightarrow \overline{\{f < \alpha\}} \subseteq lim\ inf\ L_\alpha f_n \\ &\Rightarrow \overline{Int\{f < \alpha\}} \subseteq lim\ inf\ L_\alpha f_n \\ &\Rightarrow L_\alpha f \subseteq lim\ inf\ L_\alpha f_n. \end{aligned}$$

Consequently, $f_n \stackrel{L}{\to} f$ and the proof is complete.■

REMARK 7.14. The Example 7.12 shows that the condition "f has no proper local minimum points" in above Theorem 7.13 can not avoided.

REMARK 7.15. If $f \in \mathcal{F}(X)$ we say that f is level continuous if

$$\alpha_p \to \alpha \Rightarrow L_{\alpha_p} f \stackrel{K}{\to} L_\alpha f, \ \forall \alpha > 0.$$

There exists an interesting connection between level-continuity and existence of proper local minimum points and the following result is the dual version for lower semicontinuous functions obtained by the author in [4] for upper semicontinuous functions (compare with Remark 4.5 (b) in this paper):

If $f \in \mathcal{F}(X)$ is a lower semicontinuous function, then are equivalents:

i) f has no proper local maximum points

ii) $\{f \leq \alpha\} = \overline{\{f < \alpha\}}, \ \forall \ \alpha > 0.$

iii) f is level-continuous.

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