# COTYPE AND ABSOLUTELY SUMMING HOMOGENEOUS POLYNOMIALS IN $\mathcal{L}_p$ SPACES

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Abstract. In this paper we lift to homogeneous polynomials and multilinear mappings a linear result due to Lindenstrauss and Pełczyński for absolutely summing mappings. We explore the notion of cotype to obtain stronger results and provide various examples of situations in which we have the space of absolutely summing polynomials different from the whole space. Among other consequences, these results enable us to obtain answers to some open questions about absolutely summing polynomials and multilinear mappings on $\mathcal{L}_\infty$ spaces.

## 1. Introduction

The theory of absolutely summing multilinear mappings was first sketched by A. Pietsch in 1983 [15] and it was rapidly developed thereafter ([2],[6],[11],[9]). In this paper, the definitions of absolutely summing polynomials and multilinear mappings we will work with were outlined by Pietsch and first explored by Alencar and Matos [1] and have been broadly used (see [2],[4],[6],[14]).

In the seminal paper [7] "Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications", Lindenstrauss and Pełczyński provide a beautiful Theorem which states that if $E$ is an infinite dimensional Banach space with unconditional Schauder basis, $\dim F = \infty$ and every linear mapping from $E$ into $F$ is absolutely $(1;1)$-summing, then $E$ is isomorphic to $l_1(\Gamma)$ and $F$ is isomorphic to a Hilbert space. We will refine this statement by exploring the cotype of $F$, not only for the linear cases, but also and mainly for polynomial and multilinear mappings. As corollaries we obtain several negative results, showing, in particular, that various of the known Coincidence Theorems (see e.g. [2],[11]) for polynomials and multilinear mappings cannot be improved in many natural ways.

## 2. Background and Notation

Throughout this paper $E, E_1, ..., E_n, F, X, Y$ will stand for Banach spaces. The scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$.

The Banach space of all continuous $n$-linear mappings from $E_1 \times ... \times E_n$ into $F$ endowed with the canonical norm will be denoted by $\mathcal{L}(E_1, ..., E_n; F)$. The Banach space of all continuous $n$-homogeneous polynomials $P$ from $E$ into $F$ with the norm $\|P\| = \sup\{\|Px\|; \|x\| \leq 1\}$ will be denoted by $\mathcal{P}(^n E, F)$.

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For the natural isometry

$$\Psi : \mathcal{L}(E_1, ..., E_n; F) \to \mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F))$$

we will use the following convention: If $T \in \mathcal{L}(E_1, ..., E_n; F)$ then $\Psi(T) = T_1$ and if $T \in \mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F))$, then $\Psi^{-1}(T) = T_0$.

The linear space of all sequences $(x_j)_{j=1}^{\infty}$ in $E$ such that

$$\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty} \|x_j\|^p)^{\frac{1}{p}} < \infty$$

will be denoted by $l_p(E)$. We will also denote by $l_p^w(E)$ the linear subspace of $l_p(E)$ composed by the sequences $(x_j)_{j=1}^{\infty}$ in $E$ such that $(< \varphi, x_j >)_{j=1}^{\infty} \in l_p(\mathbb{K})$ for every continuous linear functional $\varphi : E \to \mathbb{K}$. We define $\|.\|_{w,p}$ in $l_p^w(E)$ by

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} := Sup_{\varphi \in B_{E'}} (\sum_{j=1}^{\infty} |< \varphi, x_j >|^p)^{\frac{1}{p}}.$$

The case $p = \infty$ is the case of bounded sequences and in $l_\infty(E)$ we use the sup norm. One can see that $\|.\|_p$ ($\|.\|_{w,p}$) is a $p$-norm in $l_p(E)$( $l_p^w(E)$) for $p < 1$ and a norm in $l_p(E)$( $l_p^w(E)$) for $p \geq 1$. In any case, they are complete metrizable linear spaces.

Recall that if $2 \leq q \leq \infty$ and $(r_j)_{j=1}^{\infty}$ are the Rademacher functions, $E$ has cotype $q$ if there exists $C_q(E) \geq 0$ such that, no matter how we choose $k \in \mathbb{N}$ and $x_1, ..., x_k \in E$,

$$(\sum_{j=1}^{k} \|x_j\|^q)^{\frac{1}{q}} \leq C_q(E) (\int_0^1 \|\sum_{j=1}^{k} r_j(t) x_j\|^2 dt)^{\frac{1}{2}}.$$

To cover the case $q = \infty$ we replace $(\sum_{j=1}^{k} \|x_j\|^q)^{\frac{1}{q}}$ by $max_{j \leq n} \|x_j\|$. We will define the cotype of $E$ by

$$\cot E = \inf\{2 \leq q \leq \infty; E \text{ has cotype } q\}.$$

The concept of absolutely summing polynomials and multilinear mappings we will work with is the following natural generalization of the linear case.

**Definition 1.** *(Alencar-Matos) A continuous multilinear mapping*

$$T : E_1 \times ... \times E_n \to F$$

*is absolutely* $(p; q_1, ..., q_n)$*-summing (or* $(p; q_1, ..., q_n)$*-summing) if*

$$(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^{\infty} \in l_p(F)$$

*for all* $(x_j^{(s)})_{j=1}^{\infty} \in l_{q_s}^w(E)$, $s = 1, ..., n$. *A continuous* $n$*-homogeneous polynomial* $P : E \to F$ *is absolutely* $(p; q)$*-summing (or* $(p; q)$*-summing) if*

$$(P(x_j))_{j=1}^{\infty} \in l_p(F)$$

*for all* $(x_j)_{j=1}^{\infty} \in l_q^w(E)$.

In order to avoid trivialities we assume that $p \geq \frac{q}{n}$ in the polynomial case and $\frac{1}{p} \leq \frac{1}{q_1} + ... + \frac{1}{q_n}$ in the $n$-linear case. We will denote the space of absolutely $(p; q_1, ..., q_n)$-summing $n$-linear mappings from $E_1 \times ... \times E_n$ into $F$ by $\mathcal{L}_{as(p;q_1,...,q_n)}(E_1, ..., E_n; F)$. When $q_1 = ... = q_n = q$, we write $\mathcal{L}_{as(p;q)}(E_1, ..., E_n; F)$. Analogously, the space of all absolutely $(p; q)$-summing polynomials from $E$ into $F$ is denoted by $\mathcal{P}_{as(p;q)}(^nE; F)$.

For $n$-homogeneous polynomials and $n$-linear mappings, the polynomials ($n$-linear mappings) $(\frac{p}{n}; p)$-summing will be called $p$-dominated polynomials ($n$-linear mappings), as it can be seen in Matos and Tonge-Meléndez [9][11]. For the $p$-dominated polynomials ($n$-linear mappings) several natural versions of linear results are applicable, such as Factorization Theorems, Domination Theorem, Extrapolation Theorems, etc. (see [9],[11],[13]).

As in the linear case, we have a characterization Theorem which plays a prominent role in the theory.

**Theorem 1.** *(Matos [9]) Let $P$ be an $m$-homogeneous polynomial from $E$ into $F$. Then the following statements are equivalent:*

*(1) $P$ is absolutely $(p; q)$-summing.*

*(2) There exists $L > 0$ such that*

$$(\sum_{j=1}^{k} \|P(x_j)\|^p)^{\frac{1}{p}} \leq L\|(x_j)_{j=1}^k\|_{w,q}^m \; \forall k \in \mathbb{N} \text{ and } x_j \in E.$$

*(3) There exists $L > 0$ such that*

$$(\sum_{j=1}^{\infty} \|P(x_j)\|^p)^{\frac{1}{p}} \leq L\|(x_j)_{j=1}^{\infty}\|_{w,q}^m \forall (x_j)_{j=1}^{\infty} \in l_q^w(E). \tag{2.1}$$

The infimum of the $L > 0$ for which inequality (2.1) always holds is a norm for the case $p \geq 1$ or a $p$-norm for the case $p < 1$([9]) on the space of absolutely $(p; q)$-summing polynomials. In any case, we have complete topological metrizable spaces. This norm ($p$-norm) will be denoted by $\|.\|_{as(p;q)}$.

The characterization for the multilinear case and the definition of the norm ($p$-norm) follow the same reasoning. The forthcoming Theorem constitutes the definitive crucial joining of absolutely summing linear mappings and cotype.

**Theorem 2.** *(Maurey-Talagrand) $E$ has cotype $q > 2$ if, an only if,*

$$id : E \to E \text{ is } (q; 1)\text{-summing.}$$

*If $E$ has cotype 2, then $id : E \to E$ is $(2; 1)$-summing. The converse is not true.*

As a consequence of Theorem 2 and the Generalized Hölder Inequality one can prove the following result:

**Theorem 3.** *(Botelho [2]) If $Y$ has cotype $q$, then*

$$\mathcal{L}(^nX; Y) = \mathcal{L}_{as(q;1)}(^nX; Y) \text{ for all Banach space } X.$$

*If $X$ has cotype $q$, then*

$$\mathcal{L}(^nX;Y) = \mathcal{L}_{as(\frac{q}{n};1)}(^nX;Y) \text{ for all Banach space } Y.$$

In the next sections, among other results, we will prove that, in general, we cannot expect a result stronger than Theorem 3.

## 3. Absolutely summing polynomials from Banach spaces with unconditional Schauder basis

The remarkable works of Maurey-Pisier [10] and Lindenstrauss-Pełczyński [7] will play a fundamental role in this paper. We start with the following Theorem, which proof has inspired our results.

**Theorem 4.** *(Lindenstrauss-Pełczyński* [7]*, Th. 4.2) If $X$ has an unconditional Schauder basis,* $\dim X = \dim Y = \infty$ *and every bounded linear operator from $X$ into $Y$ is absolutely* $(1;1)$*-summing, then $X$ is isomorphic to* $l_1(\Gamma)$ *and $Y$ is a Hilbert space.*

This result and the Multilinear Grothendieck-Pietsch domination Theorem lead us to interesting, although restrict, initial results as we will see below.

**Example 1.** *Adapting an idea of* [8] *one can proof, for instance, that if $X$ has an unconditional Schauder basis, then*

$$\mathcal{L}(^nX;\mathbb{K}) \neq \mathcal{L}_{as(\frac{1}{n};1)}(^nX;\mathbb{K})$$

*and thus,*

$$\mathcal{L}(^nX;Y) \neq \mathcal{L}_{as(\frac{1}{n};1)}(^nX;Y) \text{ for every Banach space } Y.$$

Indeed, if we had $\mathcal{L}(^2X;\mathbb{K}) = \mathcal{L}_{as(\frac{1}{2};1)}(^2X;\mathbb{K})$, then given $S : X \to X'$, we could define $T_S : X \times X \to \mathbb{K}$ such that $(T_S)_1 = S$. By hypothesis, $T_S$ would be $(\frac{1}{2};1,1)$-summing. Hence, by the Grothendieck-Pietsch domination Theorem,

$$\|T_S(x,y)\| \leq C(\int_{B_{X'}} \mid \varphi(x) \mid d\mu_1)(\int_{B_{X'}} \mid \psi(y) \mid d\mu_2)$$

and

$$\|(T_S)_1(x)\| = Sup_{\|y\|\leq 1}\|T_S(x,y)\| \leq$$
$$\leq Sup_{\|y\|\leq 1} C(\int_{B_{X'}} \mid \varphi(x) \mid d\mu_1)(\int_{B_{X'}} \mid \psi(y) \mid d\mu_2) \leq$$
$$\leq C(\int_{B_{X'}} \mid \varphi(x) \mid d\mu_1).$$

Then $\|S(x)\| \leq C(\int_{B_{X'}} \mid \varphi(x) \mid d\mu_1)$ and then

$$\mathcal{L}(X;X') = \mathcal{L}_{as(1;1)}(X;X')$$

(contradiction by Theorem 4). The general case follows by a standard inductive process. Observe that the natural isometry between homogeneous polynomial and symmetric multilinear mappings is not enough to yield, mutatis mutandis, a polynomial version to the example above.

It is also easy to prove the following result:

**Proposition 1.** *If $E$ is an infinite dimensional Hilbert space, then*

$$\mathcal{L}(^nE;F) \neq \mathcal{L}_{as(\frac{r}{n};r)}(^nE;F) \text{ for all } r \geq 1 \text{ and every } F.$$

Proof. It suffices to consider the case $F = \mathbb{K}$. Since $E$ is Hilbert, $E'$ is also Hilbert and

$$\mathcal{L}(E;E') \neq \mathcal{L}_{as(1;1)}(E;E') = \mathcal{L}_{as(r;r)}(E;E') (\text{see [5], page 224})$$

and the proof of the last example yields the case $n = 2$. The general case is obtained by a standard inductive process. Q.E.D.

The same simple construction give us many other results. However, the previous negative results, albeit interesting, are confined to the dominated cases (which have Grothendieck-Pietsch domination Theorem as a fundamental gun) and they do not give us the full story. In general, the spaces of $p$-dominated homogeneous polynomials and multilinear mappings are small and negative results are not surprising. We will present new negative results which significantly improve the last ones. Our approach consists in lifting Theorem 4 to polynomial and multilinear versions and refining them by exploring the properties of cotype.

Our definition of Schauder basis is the same as in [7] and does not ask for a separate space, but it is clear that in the following proofs there is no loss of generality if we restrict ourselves to the separate cases.

**Theorem 5.** *Let $X$ and $Y$ be infinite dimensional Banach spaces. Suppose that $X$ has an unconditional Schauder basis. If $q$ is such that $\frac{1}{m} < q < 2$ and*

$$\mathcal{P}_{as(q;1)}(^mX;Y) = \mathcal{P}(^mX;Y) \tag{3.1}$$

*then regardless of the unconditional normalized Schauder basis $\{x_n\}$ of $X$, the natural mapping*

$$\varphi : X \to l_\infty : x = \sum a_i x_i \to (a_i)_{i=1}^\infty \tag{3.2}$$

*is such that $\varphi(X) \subset l_{\frac{2mq}{2-q}}$. If, in particular, $\frac{1}{m} \leq q \leq 1$ and (3.1) holds, then, for any unconditional normalized Schauder basis $\{x_n\}$ of $X$ , we obtain $\varphi(X) \subset l_{mq}$, which is a better estimate than (3.2).*

Proof. Consider $q$ such that $\frac{1}{m} \leq q < 2$. By hypothesis, there exists $K > 0$ such that $\|P\|_{as(q;1)} \leq K\|P\|$ for all continuous $m$-homogeneous polynomial $P : X \to Y$.

By the main Lemma of Dvoretzky-Rogers Theorem, for every $n$, there are normalized $y_1, ..., y_n$ in $E$ such that

$$\| \sum_{j=1}^n \lambda_j y_j \| \leq 2(\sum_{j=1}^n | \lambda_j |^2)^{1/2}.$$

Let $\{\mu_i\}_{i=1}^n$be such that $\sum\limits_{j=1}^n \mid \mu_j \mid^s= 1$ with $s = \frac{2}{q}$. Define $P : X \to Y$ by

$$Px = \sum_{j=1}^n \mu_j^{1/q} a_j^m y_j, \text{ if } x = \sum_{j=1}^\infty a_j x_j.$$

Since $\{x_n\}$ is an unconditional basis, there exists $\rho > 0$ such that $\| \sum\limits_{j=1}^\infty \varepsilon_j a_j x_j \| \leq \rho \| \sum\limits_{j=1}^\infty a_j x_j \| = \rho\|x\|$ for all $\varepsilon_j = 1$ or $-1$.

Hence $\| \sum\limits_{j=1}^n \varepsilon_j a_j x_j \| \leq \rho\|x\|$ for all $n$ and any $\varepsilon_j = 1$ or $-1$. We have

$$\|Px\| = \| \sum_{j=1}^n \mu_j^{1/q} a_j^m y_j \| \leq 2 (\sum_{j=1}^n \mid \mu_j^{1/q} a_j^m \mid^2)^{1/2}$$

$$\leq 2 (\sum_{j=1}^n \mid \mu_j \mid^{2/q} \rho^{2m} \|x\|^{2m})^{1/2} \leq$$

$$\leq 2\rho^m \|x\|^m (\sum_{j=1}^n \mid \mu_j \mid^{2/q})^{1/2} = 2\rho^m \|x\|^m (\sum_{j=1}^n \mid \mu_j \mid^s)^{1/2} \leq 2\rho^m \|x\|^m. \tag{3.3}$$

Then $\|P\| \leq 2\rho^m$ and $\|P\|_{as(q;1)} \leq 2K\rho^m$. Therefore

$$[\sum_{j=1}^n (\mid a_j^m \mu_j^{1/q} \mid)^q]^{1/q} = (\sum_{j=1}^n \|P a_j x_j\|^q)^{1/q} \leq \|P\|_{as(q;1)} \|(a_j x_j)_{j=1}^n\|_{w,1}^m =$$

(3.4)

$$= \|P\|_{as(q;1)} \max_{\varepsilon_j \in \{1,-1\}} \{ \| \sum_{j=1}^n \varepsilon_j a_j x_j \| \}^m \leq \|P\|_{as(q;1)} (\rho\|x\|)^m \leq 2K\rho^{2m}\|x\|^m.$$

Recall that (3.4) holds whenever $\sum\limits_{j=1}^n \mid \mu_j \mid^s= 1$. Hence

$$[\sum_{j=1}^n ( \mid a_j \mid^{\frac{s}{s-1} mq})]^{1/(\frac{s}{s-1})} = [\sum_{j=1}^n (\mid a_j^{mq} \mid^{\frac{s}{s-1}})]^{1/(\frac{s}{s-1})} = \|(a_j^{mq})_{j=1}^n\|_{\frac{s}{s-1}}$$

$$= Sup\{ \mid \sum_{j=1}^n \mu_j a_j^{mq} \mid ; \sum_{j=1}^n \mid \mu_j \mid^s = 1 \} \leq$$

$$\leq Sup\{ \sum_{j=1}^n (\mid \mu_j \mid . \mid a_j^{mq} \mid); \sum_{j=1}^n \mid \mu_j \mid^s = 1 \}$$

and by (3.4) we get

$$[\sum_{j=1}^n (\mid a_j \mid^{\frac{s}{s-1} mq})]^{1/(\frac{s}{s-1})} \leq (2K\rho^{2m}\|x\|^m)^q$$

and then

$$[\sum_{j=1}^{n}(\mid a_j \mid^{\frac{s}{s-1}mq})]^{1/(\frac{s}{s-1}mq)} \leq (2K\rho^{2m}\|x\|^m)^{1/m}.$$

Since $\frac{s}{s-1}mq = \frac{2mq}{2-q}$, and $n$ is arbitrary, the first part of our Theorem is proved. If, in particular, $\frac{1}{m} \leq q \leq 1$, let us define $S : X \to Y$ by

$$Sx = \sum_{j=1}^{n} a_j^m y_j, \text{if } x = \sum_{j=1}^{\infty} a_j x_j.$$

We have

$$\|Sx\| = \|\sum_{j=1}^{n} a_j^m y_j\| \leq 2(\sum_{j=1}^{n} \mid a_j^m \mid^2)^{1/2} = 2[(\sum_{j=1}^{n} \mid a_j \mid^{2m})^{1/2m}]^m \leq$$

$$\leq 2[(\sum_{j=1}^{n} \mid a_j \mid^{\frac{s}{s-1}mq})^{1/\frac{s}{s-1}mq}]^m \leq 2(2K\rho^{2m}\|x\|^m)$$

since $m \geq \frac{1}{2}.\frac{s}{s-1}mq = \frac{mq}{2-q}$. Then $\|S\| \leq 2(2K\rho^{2m})$ and $\|S\|_{as(r;1)} \leq 2K(2K\rho^{2m})$. Hence

$$\sum_{j=1}^{n} \mid a_j \mid^{qm} = \sum_{j=1}^{n} \|Sa_j x_j\|^q \leq \|S\|_{as(r;1)}^q \max_{\varepsilon_j \in \{1,-1\}} \{\|\sum_{j=1}^{n} \varepsilon_j a_j x_j\|\}^{qm} \leq$$

$$\leq [2K(2K\rho^{2m})]^q (\rho\|x\|)^{qm}$$

and consequently, since $n$ is arbitrary, we obtain $\sum_{j=1}^{\infty} \mid a_j \mid^{qm} < \infty$ whenever $x = \sum_{j=1}^{\infty} a_j x_j \in X$. Q.E.D.

It is worthwhile observing that $\dim Y = \infty$ is unavoidable in our approach since we ought to have "enough dimension " to apply the Dvoretzky-Rogers Lemma for sufficiently large $n$.

In this paper $\varphi$ will always denote the natural mapping (3.2) from a Banach space with a normalized unconditional Schauder basis $\{x_n\}$ into $l_\infty$.

**Corollary 1.** *If $q < 2$, $r \geq 1$ and $m \in \mathbb{N}$, we have*

$$\mathcal{P}(^m c_0; Y) \neq \mathcal{P}_{as(q;r)}(^m c_0; Y)$$

*regardless of the infinite dimensional Banach space $Y$. When $Y$ is finite dimensional the statement is not valid since it is well known that $\mathcal{P}(^2 c_0; \mathbb{K}) = \mathcal{P}_{as(1;1)}(^2 c_0; \mathbb{K})$.*

*A standard localization argument can be used to obtain the Corollary above for $\mathcal{L}_\infty$ spaces in the place of $c_0$.*

Theorems 3 and 5 furnish interesting special corollaries.

**Corollary 2.** *If $Y$ is an infinite dimensional Banach space, $\frac{1}{m} \leq q \leq 1$ and $p \geq 2$, then*

$$\mathcal{P}(^m l_p; Y) = \mathcal{P}_{as(q;1)}(^m l_p; Y) \Leftrightarrow qm \geq p.$$

**Corollary 3.** *If $Y$ is an infinite dimensional Banach space, $1 \leq p \leq 2$ and $\frac{1}{m} \leq q \leq 1$, then*
$\mathcal{P}(^m l_p; Y) = \mathcal{P}_{as(q;1)}(^m l_p; Y) \Rightarrow qm \geq p$ *and* $qm \geq 2 \Rightarrow \mathcal{P}(^m l_p; Y) = \mathcal{P}_{as(q;1)}(^m l_p; Y)$.

## 4. Absolutely summing polynomials from Banach spaces with unconditional Schauder basis into Banach spaces with finite cotype

In this section we will explore cotype properties to obtain significant improvements for the Theorem 5.

The following definition and Lemma can be found in [10].

**Definition 2.** *We say that $Y$ finitely factors $(ff)$ the formal inclusion $l_q \rightarrow l_\infty$ for $0 < \delta < 1$ if for every $n$ there are $y_1, ..., y_n$ such that*

$$(1-\delta)\|a\|_\infty \leq \|\sum_{k \leq n} a_k y_k\| \leq \|a\|_q$$

*for all $a = (a_k)_{k=1}^n \in l_q^n$.*

Note that $(1-\delta) \mid a_k \mid \leq \|a_k y_k\| \leq \mid a_k \mid$ and then $(1-\delta) \leq \|y_k\| \leq 1$ for all $k$.

**Lemma 1.** *(Maurey-Pisier) For any infinite dimensional Banach space $Y$ we have*

$$\inf\{2 \leq q \leq \infty; Y \text{ has cotype } q\} =$$
$$= sup\{2 \leq q \leq \infty; \exists \delta \text{ such that } Y\ ff\ l_q \rightarrow l_\infty\}.$$

The next Theorem shows that we can push a little further if we properly take advantage of the cotype of the range.

**Theorem 6.** *Let $X$ denote an infinite dimensional Banach space with unconditional Schauder basis and $Y$ an infinite dimensional Banach space with finite cotype $p$. If $\mathcal{P}(^m X; Y) = \mathcal{P}_{as(q;1)}(^m X; Y)$ and $\frac{1}{m} \leq q < p$, we conclude that for any unconditional normalized Schauder basis $\{x_n\}$ for $X$, $\varphi(X) \subset l_{\frac{mqp}{p-q}}$. When, in particular, $\frac{1}{m} \leq q \leq \frac{p}{2}$, we have $\varphi(X) \subset l_{mq}$, which is a better estimate.*

Proof. By hypothesis, there exists a positive $K$ such that $\|P\|_{as(q;1)} \leq K\|P\|$ for all $P \in \mathcal{P}(^m X; Y)$.

Let $\{\mu_i\}_{i=1}^n$ be such that $\sum_{j=1}^n \mid \mu_j \mid^s = 1$, with $s = \frac{p}{q}$. Define $P : X \rightarrow Y$ by

$$Px = \sum_{j=1}^n \mu_j^{1/q} a_j^m y_j, \text{ se } x = \sum_{j=1}^\infty a_j x_j$$

where the $y_j$ are given by the foregoing Lemma 1 and Definition 2. Since $\{x_n\}$ is an unconditional basis, there exists a positive $\rho$ such that

$$\|\sum_{j=1}^{\infty}\varepsilon_j a_j x_j\| \leq \rho\|\sum_{j=1}^{\infty} a_j x_j\| = \rho\|x\| \text{ for any } \varepsilon_j = 1 \text{ or } \varepsilon_j = -1.$$

Hence $\|\sum_{j=1}^{n}\varepsilon_j a_j x_j\| \leq \rho\|x\|$ for all $n$ and any $\varepsilon_j = 1$ or $-1$ and then we have

$$\|Px\| = \|\sum_{j=1}^{n}\mu_j^{1/q}a_j^m y_j\| \leq (\sum_{j=1}^{n} \mid \mu_j^{1/q}a_j^m \mid^p)^{1/p} \leq$$

$$\leq \rho^m\|x\|^m(\sum_{j=1}^{n} \mid \mu_j \mid^{p/q})^{1/p} \leq \rho^m\|x\|^m(\sum_{j=1}^{n} \mid \mu_j \mid^{p/q})^{1/p} =$$

$$= \rho^m\|x\|^m(\sum_{j=1}^{n} \mid \mu_j \mid^{s})^{1/p} \leq \rho^m\|x\|^m. \tag{4.1}$$

We arrive at $\|P\| \leq \rho^m$ and $\|P\|_{as(q;1)} \leq K\rho^m$. We can now achieve the estimate below:

$$[\sum_{j=1}^{n}(\mid a_j^m\mu_j^{1/q}(1-\delta)\mid)^q]^{1/q} \leq [\sum_{j=1}^{n}(\|a_j^m\mu_j^{1/q}y_j\|)^q]^{1/q} =$$

$$(\sum_{j=1}^{n}\|Pa_jx_j\|^q)^{1/q} \leq \|P\|_{as(q;1)}\,\|(a_jx_j)_{j=1}^n\|_{w,1}^m =$$

$$= \|P\|_{as(q;1)} \max_{\varepsilon_j\in\{1,-1\}}\{\|\sum_{j=1}^{n}\varepsilon_j a_j x_j\|\}^m \leq$$

$$\leq \|P\|_{as(q;1)}\,(\rho\|x\|)^m \leq K\rho^{2m}\|x\|^m. \tag{4.2}$$

However, it is crucial that (4.2) holds whenever $\sum_{j=1}^{n} \mid \mu_j \mid^s = 1$. Hence, since

$$\frac{1}{s} + \frac{1}{\frac{s}{s-1}} = 1$$

we have

$$[\sum_{j=1}^{n}(\mid a_j \mid^{\frac{s}{s-1}mq})]^{1/(\frac{s}{s-1})} = [\sum_{j=1}^{n}(\mid a_j^{mq} \mid^{\frac{s}{s-1}})]^{1/(\frac{s}{s-1})} = \|(a_j^{mq})_{j=1}^n\|_{\frac{s}{s-1}}$$

$$= Sup\{\mid \sum_{j=1}^{n}\mu_j a_j^{mq} \mid; \sum_{j=1}^{n} \mid \mu_j \mid^s = 1\} \leq$$

$$\leq Sup\{\sum_{j=1}^{n}(\mid \mu_j \mid\mid a_j^{mq} \mid); \sum_{j=1}^{n} \mid \mu_j \mid^s = 1\}.$$

Again, by (4.2), it follows that

$$[\sum_{j=1}^{n}(\mid a_j \mid^{\frac{s}{s-1}mq})]^{1/(\frac{s}{s-1})} \leq [(1-\delta)^{-1}K\rho^{2m}\|x\|^m]^q, \tag{4.3}$$

and then

$$[\sum_{j=1}^{n} \mid a_j \mid^{\frac{s}{s-1}mq}]^{1/(\frac{s}{s-1})mq} \leq [(1-\delta)^{-1}K\rho^{2m}\|x\|^m]^{1/m}.$$

Since $\frac{s}{s-1}mq = \frac{mpq}{p-q}$ and $n$ is arbitrary, the first part of the Theorem is proved. Now, if $\frac{1}{m} \leq q \leq \frac{p}{2}$, define $S : X \to Y$ by

$$Sx = \sum_{j=1}^{n} a_j^m y_j \text{ if } x = \sum_{j=1}^{\infty} a_j x_j.$$

We obtain

$$\|Sx\| = \|\sum_{j=1}^{n} a_j^m y_j\| \leq (\sum_{j=1}^{n} \mid a_j^m \mid^p)^{1/p} =$$

$$= [(\sum_{j=1}^{n} \mid a_j \mid^{mp})^{1/mp}]^m \leq [(\sum_{j=1}^{n} \mid a_j \mid^{\frac{s}{s-1}mq})^{1/\frac{s}{s-1}mq}]^m =$$

$$= [(1-\delta)^{-1}K\rho^{2m}\|x\|^m]$$

since

$$2q \leq p \Rightarrow p - q \geq q \Rightarrow 1 \geq \frac{q}{p-q} \Rightarrow$$

$$\Rightarrow mp \geq mp\frac{q}{p-q} = \frac{s}{s-1}mq.$$

Thus $\|S\| \leq (1-\delta)^{-1}K\rho^{2m}$ and $\|S\|_{as(q;1)} \leq (1-\delta)^{-1}K^2\rho^{2m}$ and hence

$$\sum_{j=1}^{n} \mid a_j^m(1-\delta) \mid^q \leq \sum_{j=1}^{n} \|a_j^m y_j\|^q = \sum_{j=1}^{n} \|S a_j x_j\|^q \leq$$

$$\leq \|S\|_{as(q;1)}^q \max_{\varepsilon_j \in \{1,-1\}} \{\|\sum_{j=1}^{n} \varepsilon_j a_j x_j\|\}^{mq} \leq ((1-\delta)^{-1}K^2\rho^{2m})^q(\rho\|x\|)^{mq}.$$

Consequently, since $n$ is arbitrary, we have $\sum_{j=1}^{\infty} \mid a_j \mid^{mq} < \infty$ whenever $x = \sum_{j=1}^{\infty} a_j x_j \in X$ . Q.E.D.

**Corollary 4.** *If $Y$ has finite cotype and $P(^m l_t; Y) = \mathcal{P}_{as(q;1)}(^m l_t; Y)$ for some $q$ such that $\frac{1}{m} \leq q \leq \frac{\cot Y}{2}$, then $t \leq mq$.*

By Theorems 3 and 6 we obtain the next Corollary.

**Corollary 5.** *If $Y$ is infinite dimensional and has finite cotype then*

$$\mathcal{P}(^m c_0; Y) = \mathcal{P}_{as(q;1)}(^m c_0; Y) \Leftrightarrow q \geq \cot Y.$$

*It is clear that one can replace $c_0$ by any $\mathcal{L}_\infty$ space.*

**Corollary 6.** *If $\mathcal{P}(^m l_t; l_p) = \mathcal{P}_{as(q;1)}(^m l_t; l_p)$ for some $q$ such that $\frac{1}{m} \leq q < \cot l_p$ then*

$$t \leq \frac{mq \max\{p,2\}}{\max\{p,2\} - q}.$$

## 5. Absolutely summing polynomials from Banach spaces with unconditional Schauder basis into Banach spaces without finite cotype

If $Y$ does not have finite cotype, a slight modification of the proof of Theorem 6 gives us the strongest negative result we could ever wish.

**Theorem 7.** *If $Y$ does not have finite cotype and $X$ is an $\mathcal{L}_\infty$ space, then*

$$\mathcal{P}(^m X; Y) \neq \mathcal{P}_{as(r;s)}(^m X; Y)$$

*regardless of the $r > 0$ and $s \geq 1$.*

Proof. It suffices to prove that $\mathcal{P}(^m c_0; Y) \neq \mathcal{P}_{as(q;1)}(^m c_0; Y)$ for every $q \geq \frac{1}{m}$.
Suppose that there exists $q \geq \frac{1}{m}$ such that

$$\mathcal{P}(^m c_0; Y) = \mathcal{P}_{as(q;1)}(^m c_0; Y).$$

As before, we can find a positive $K$ such that $\|P\|_{as(q;1)} \leq K\|P\|$ regardless of the $P \in \mathcal{P}(^m c_0; Y)$.

Since $Y$ does not have finite cotype, $Y$ contains $l_\infty^n$ $\lambda-$uniformly for all $n$ and some $\lambda > 1$. Hence there are $y_1, ..., y_n$ in $Y$ such that

$$\lambda^{-1}\|a\|_\infty \leq \|\sum_{k \leq n} a_k y_k\| \leq \|a\|_\infty.$$

Notice that $\lambda^{-1} \leq \|y_k\| \leq 1$ for all $k$.

Choose an arbitrary $s > 1$. Let $\{\mu_i\}_{i=1}^n$be such that $\sum\limits_{j=1}^{n} \mid \mu_j \mid^s = 1$. Define $P : X \to Y$ by

$$Px = \sum_{j=1}^{n} \mu_j^{1/q} a_j^m y_j \text{ if } x = \sum_{j=1}^{\infty} a_j x_j.$$

As usual, since $\{x_n\}$ is an unconditional basis, there exists a positive $\rho$ such that

$$\|\sum_{j=1}^{\infty} \varepsilon_j a_j x_j\| \leq \rho\|\sum_{j=1}^{\infty} a_j x_j\| = \rho\|x\| \text{ for any } \varepsilon_j = 1 \text{ or } -1.$$

Hence $\| \sum\limits_{j=1}^{n} \varepsilon_j a_j x_j \| \leq \rho \|x\|$ for all $n$ and any $\varepsilon_j = 1$ or $-1$. We get

$$\|Px\| = \| \sum_{j=1}^{n} \mu_j^{1/q} a_j^m y_j \| \leq \| (\mu_j^{1/q} a_j^m)_{j=1}^n \|_\infty \leq (\sum_{j=1}^{n} \mid \mu_j^{1/q} a_j^m \mid^{sq})^{1/sq} \leq$$

$$\leq \rho^m \|x\|^m (\sum_{j=1}^{n} \mid \mu_j \mid^s)^{1/sq} \leq \rho^m \|x\|^m (\sum_{j=1}^{n} \mid \mu_j \mid^s)^{1/sq} \leq \rho^m \|x\|^m. \tag{5.1}$$

So, inequality (5.1) gives us $\|P\| \leq \rho^m$ and $\|P\|_{as(q;1)} \leq K\rho^m$.Then

$$[\sum_{j=1}^{n} (\mid a_j^m \mu_j^{1/q} \lambda^{-1} \mid)^q]^{1/q} \leq [\sum_{j=1}^{n} (\|a_j^m \mu_j^{1/q} y_j\|)^q]^{1/q} = (\sum_{j=1}^{n} \|Pa_j x_j\|^q)^{1/q} \leq$$

$$\leq \|P\|_{as(q;1)} \|(a_j x_j)_{j=1}^n\|_{w,1}^m = \|P\|_{as(q;1)} \max_{\varepsilon_j \in \{1,-1\}} \{\| \sum_{j=1}^{n} \varepsilon_j a_j x_j \|\}^m \leq$$

$$\leq \|P\|_{as(q;1)} (\rho \|x\|)^m \leq K\rho^{2m} \|x\|^m. \tag{5.2}$$

Recalling that (5.2) holds whenever $\sum\limits_{j=1}^{n} \mid \mu_j \mid^s = 1$ we obtain

$$[\sum_{j=1}^{n} (\mid a_j \mid^{\frac{s}{s-1} mq})]^{1/\frac{s}{s-1}} = [\sum_{j=1}^{n} (\mid a_j^{mq} \mid^{\frac{s}{s-1}})]^{1/\frac{s}{s-1}} = \|(a_j^{mq})_{j=1}^n\|_{\frac{s}{s-1}} =$$

$$= Sup\{\mid \sum_{j=1}^{n} \mu_j a_j^{mq} \mid ; \sum_{j=1}^{n} \mid \mu_j \mid^s = 1\} \leq$$

$$\leq Sup\{\sum_{j=1}^{n} (\mid \mu_j \mid\mid a_j^{mq} \mid) ; \sum_{j=1}^{n} \mid \mu_j \mid^s = 1\}.$$

By (5.2) it follows that $\sum\limits_{j=1}^{n} (\mid a_j \mid^{\frac{s}{s-1} mq})]^{1/\frac{s}{s-1}} \leq (\lambda K \rho^{2m} \|x\|^m)^q$ and then

$$[\sum_{j=1}^{n} (\mid a_j \mid^{\frac{s}{s-1} mq})]^{1/\frac{s}{s-1} mq} \leq (\lambda K \rho^{2m} \|x\|^m)^{1/m}.$$

Since $n$ is arbitrary, we have thus shown that

$$\sum_{j=1}^{\infty} \mid a_j \mid^{\frac{s}{s-1} mq} < \infty$$

whenever $x = \sum\limits_{j=1}^{\infty} a_j x_j \in c_0$ and it is far from the truth. Q.E.D.

As a simple outcome of the last Theorem we obtain the following Corollary.

**Corollary 7.** *If $Y$ does not have finite cotype then*

$$\mathcal{P}(^m l_t;Y) = \mathcal{P}_{as(q;1)}(^m l_t;Y) \Rightarrow t \leq mq. \tag{5.3}$$

*and*

$$max\{t,2\} \leq mq \Rightarrow \mathcal{P}(^m l_t;Y) = \mathcal{P}_{as(q;1)}(^m l_t;Y) \tag{5.4}$$

Proof. The proof of (5.3) is the same as the last proof. It suffices to realize that since $s > 1$ is arbitrary, we have

$$x = \sum_{j=1}^{\infty} a_j x_j \in l_t \Rightarrow (a_j)_{j=1}^{\infty} \in l_{\frac{s}{s-1}mq}$$

and thus

$$t \leq \lim_{s\to\infty} \frac{s}{s-1} mq = mq.$$

The proof of (5.4) is straightforward by Theorem 3. Q.E.D.
With another immediate consequence of a slight variation of the proof of Theorem 7 we get the next result.

**Theorem 8.** *If $E$ is an infinite dimensional Banach space with normalized unconditional Schauder basis $\{x_n\}$ and $\mathcal{P}(^m E;F) = \mathcal{P}_{as(q;1)}(^m E;F)$ for some $F$ with $\cot F = \infty$, then for any $x \in E$, $x = \sum_{n=1}^{\infty} a_n x_n$ we have $(a_n)_{n=1}^{\infty} \in l_{mq}$.*

## 6. Multilinear mappings from Banach spaces with unconditional Schauder basis

It is clear that every negative polynomial result $\mathcal{P}(^n E;F) \neq \mathcal{P}_{as(r;s)}(^n E;F)$ furnishes a multilinear negative result $\mathcal{L}(^n E;F) \neq \mathcal{L}_{as(r;s)}(^n E;F)$. However, the reader shall realize that the same reasoning we have used for polynomials can be adjusted to other multilinear cases, as we sketch in the Theorem below.

**Theorem 9.** *Let $Y$ be an infinite dimensional Banach space and $E_1,...,E_m$ denote infinite dimensional Banach spaces with unconditional Schauder basis. If $q$ is such that $\frac{1}{m} \leq q < 2$ and $\mathcal{L}_{as(q;1)}(E_1,...,E_m;Y) = \mathcal{L}(E_1,...,E_m;Y)$ we conclude that for any normalized unconditional Schauder basis $\{x_j^1\},...,\{x_j^m\}$ for $E_1,...,E_m$, respectively, the natural mapping*

$$\psi : E_1 \times ... \times E_m \to l_\infty : (\sum a_i^{(1)} x_i^1, ..., \sum a_i^{(m)} x_i^m) \to (a_i^{(1)}...a_i^{(m)})_{i=1}^{\infty}$$

*is such that $\psi(E_1 \times ... \times E_m) \subset l_{\frac{2q}{2-q}}$. If, in particular, $\frac{1}{m} \leq q \leq 1$ and*

$$\mathcal{L}_{as(q;1)}(E_1 \times ... \times E_m;Y) = \mathcal{L}(E_1 \times ... \times E_m;Y)$$

*we conclude that $\psi(X) \subset l_q$.*

Proof. For each $k = 1,...,m$, consider $a_k = \sum_i a_i^{(k)} x_i^k$. For each natural $n$ it suffices to define $T : E_1 \times ... \times E_m \to Y$ by

$$T(a_1,...,a_m) = \sum_{j=1}^{n} \mu_j^{1/q} a_j^{(1)}...a_j^{(m)} y_j,$$

where the $y_j$ are given by the main Lemma of Dvoretzky Rogers Theorem, and proceed as before.

**Corollary 8.** *If $E_1$ is an $\mathcal{L}_\infty$ space and $E_2$ is an infinite dimensional Banach space, then $\mathcal{L}(E_1, E_2; F) \neq \mathcal{L}_{as(\frac{r}{2};r)}(E_1, E_2; F)$ for every $1 \leq r < 2$ and every Banach space $F$.*

Proof. If we had

$$\mathcal{L}(E_1, E_2; F) = \mathcal{L}_{as(\frac{r}{2};r)}(E_1, E_2; F)$$

we would have( Lemma 3.4 of [2])

$$\mathcal{L}(E_1; \mathcal{L}(E_2; F)) = \mathcal{L}_{as(r;r)}(E_1; \mathcal{L}(E_2; F))$$

for $r < 2$. However, by Theorem 9, we know that

$$\mathcal{L}(E_1; \mathcal{L}(E_2; F)) \neq \mathcal{L}_{as(r;1)}(E_1; \mathcal{L}(E_2; F))$$

for all $r < 2$. Q.E.D.

For an infinite dimensional $F$, the last Corollary is a particular case of a result of Botelho (Theorem 3.5 of [2]). However, for an arbitrary Banach space $F$ it is a new result which states the impossibility of improving ( for $r < 2$ ) a Theorem of Tonge-Melendez [11] and Botelho-Floret [2], which asserts that every continuous bilinear mapping from an $\mathcal{L}_\infty$ space into $\mathbb{K}$ is 2-dominated.

The multilinear versions of Theorems 6 and 7 are also straightforward.

**Remark 1.** *The argument of localization works in $\mathcal{L}_p$ spaces not only if $p = \infty$. Using a Theorem of Pełczyński and Rosenthal (Corollary 2.1 of [12]) it is not hard to prove that if $1 \leq p \leq \infty$ $X$ and $Z$ are $\mathcal{L}_p$ spaces and*

$$\mathcal{L}(^n X; Y) = \mathcal{L}_{as(r;s_1,...,s_n)}(^n X; Y)$$

*then*

$$\mathcal{L}(^n Z; Y) = \mathcal{L}_{as(r;s_1,...,s_n)}(^n Z; Y).$$

*Therefore it is clear that every negative result that we have stated for $l_p$ (e.g. Corollaries 4, 6, 7) can be extended to $\mathcal{L}_p$ spaces, no matter they do not have an unconditional Schauder basis.*

## 7. Final applications

The following simple result, which proof we will omit, added to our negative results will provide some simple answers to interesting questions about absolutely summing polynomials.

**Proposition 2.** *If $\mathcal{L}(E_1, ..., E_n; F) = \mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1, ..., E_n; F)$ then*

$$\mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F) = \mathcal{L}_{as(r;s_1,...,s_t)}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F).$$

*and reciprocally.*

The multilinear version of Theorem 7 and Proposition 2 yield the results below.

**Proposition 3.** *Let $n \geq 3$, $r_1, ..., r_n \in [1, \infty]$ and $X_1, ..., X_n$ be infinite dimensional $\mathcal{L}_\infty$ spaces such that*

$$\mathcal{L}(X_1, ..., X_n; \mathbb{K}) = \mathcal{L}_{as(s;r_1,...,r_n)}(X_1, ..., X_n; \mathbb{K}).$$

*Then, no more than one of the $r_j$ can be $\infty$.*

Proof. There is no loss of generality if we admit $r_{n-1} = r_n = \infty$. Hence, by Proposition 2 we would have

$$\mathcal{L}(X_1, ..., X_{n-2}; \mathcal{L}(X_{n-1}, X_n; \mathbb{K})) = \mathcal{L}_{as(s;r_1,...,r_t,)}(X_1, ..., X_{n-2}; \mathcal{L}(X_{n-1}, X_n; \mathbb{K}))$$

which is impossible by Theorem 7 since $\mathcal{L}(X^{(n-1)}, X^{(n)}; \mathbb{K})$ has only infinite cotype.

**Proposition 4.** *Let $n \geq 2$ and $r_1, ...., r_n \geq 1$. If $X$ is an infinite dimensional $\mathcal{L}_\infty$ space and $\mathcal{L}(^n X; F) = \mathcal{L}_{as(s;r_1,...,r_n)}(^n X; F)$, with $\dim F = \infty$, then $r_j \neq \infty$ for all $j$.*

Proof. It suffices to observe that $\cot \mathcal{L}(X; F) = \infty$ and use the multilinear version of Theorem 7. We will make the case $n = 2$ to be more precise. If $\mathcal{L}(^2 X; F) = \mathcal{L}_{as(s;r,\infty)}(^2 X; F)$ then Proposition 2 give us

$$\mathcal{L}(X; \mathcal{L}(X; F)) = \mathcal{L}_{as(s;r)}(X; \mathcal{L}(X; F))$$

and that is impossible. Q.E.D.

The same reasoning applies for $\mathcal{L}(X_1, ..., X_n; F)$ with $X_1, ..., X_n$ infinite dimensional $\mathcal{L}_\infty$ spaces.

As a final application, we will list some two recent positive results and our negative results will be able to show that they cannot be generalized in some natural ways.

**Theorem 10.** *(D.Perez [14]) If each $X_j$ is an $\mathcal{L}_{\infty,\lambda_j}$ space, then every continuous multilinear mapping from $X_1 \times ... \times X_n$ into $\mathbb{K}$ is $(1; 2, ..., 2)$ -summing and*

$$\|T\|_{as(1;2,...,2)} \leq K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

Proof. It is an interesting application of an Inequality of Grothendieck. [14]

It is clear that, in particular, Theorem 9 applied to $E_1, ..., E_n = c_0$ and a standard localization argument imply that Theorem 10 cannot be improved to an infinite dimensional $F$ in the place of the scalar field $\mathbb{K}$.

**Theorem 11.** *If each $X_j$ is an $\mathcal{L}_{\infty,\lambda_j}$ space and $F$ has cotype $q \neq \infty$, then every continuous $n$-linear mapping from $X_1 \times ... \times X_n$ into $F$ is $(q; 2, ..., 2)$- summing and*

$$\|T\|_{as(q;2,...,2)} \leq C_q(F) K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

*In particular, if $X$ is an $\mathcal{L}_{\infty,\lambda}$ space and $F$ has cotype $q \neq \infty$, then*

$$\mathcal{P}(^n X; F) = \mathcal{P}_{as(q;2)}(^n X; F).$$

Proof. Let $(f_j^{(1)})_{j=1}^{\infty} \in l_2^w(X_j), ..., (f_j^{(n)})_{j=1}^{\infty} \in l_2^w(X_n)$. Since for every $R \in \mathcal{L}(X_1, ..., X_n; \mathbb{K})$ we have

$$\|R\|_{as(1;2,...,2)} \leq K_G 3^{\frac{n-2}{2}} \|R\| \prod_{j=1}^{n} \lambda_j$$

then,

$$(\sum_{j=1}^{\infty} \|T(f_j^{(1)}, ..., f_j^{(n)})\|^q)^{\frac{1}{q}} \leq C_q(F) \|(T(f_j^{(1)}, ..., f_j^{(n)}))_{j=1}^{\infty}\|_{w,1} =$$

$$= C_q(F) \sup_{y' \in B_{F'}} \sum_{j=1}^{\infty} |< y', T(f_j^{(1)}, ..., f_j^{(n)}) >| =$$

$$= C_q(F) \sup_{y' \in B_{F'}} \sum_{j=1}^{\infty} | \ (y' \circ T)(f_j^{(1)}, ..., f_j^{(n)}) \ | \leq$$

$$\leq C_q(F) C \|T\| \|(f_j^{(1)})_{j=1}^{\infty}\|_{w,2} ... \|(f_j^{(n)})_{j=1}^{\infty}\|_{w,2}$$

where $C = K_G 3^{\frac{n-2}{2}} \prod_{j=1}^{n} \lambda_j$. Q.E.D.

As a consequence of the last Theorem, we have the following result, answering a question posed by Botelho in [3].

**Corollary 9.** *If $n \geq 2$ and each $X_j$ is a $\mathcal{L}_{\infty,\lambda_j}$ space then*

$$\mathcal{L}(X_1, ..., X_n; \mathbb{K}) = \mathcal{L}_{as(2;2,...,2,\infty)}(X_1, ..., X_n; \mathbb{K}) \tag{7.1}$$

*and*

$$\|T\|_{as(2;2,...,2,\infty)} \leq C_2(X_n') K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

Proof. Let $T : X_1 \times ... \times X_n \to \mathbb{K}$ be a continuous $n$-linear mapping. Then

$T_1 : X_1 \times ... \times X_{n-1} \to X_n{}'$ is $(2; 2, ..., 2)$-summing since $X_n{}'$ has cotype 2. Hence

$$(\sum_{j=1}^{\infty} \|T_1(x_j^{(1)}, ..., x_j^{(n-1)})\|^2)^{1/2} \leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2}$$

and

$$(\sum_{j=1}^{\infty} \underset{x_j^{(n)} \in B_{X_j}}{Sup} \|T_1(x_j^{(1)}, ..., x_j^{(n-1)})(x_j^{(n)})\|^2)^{1/2} \leq$$

$$\leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2}$$

If $(x_j^{(n)})_{j=1}^{\infty} \in l_{\infty}(X_n)$ is non zero, we have

$$(\sum_{j=1}^{\infty} \|T_1(x_j^{(1)}, ..., x_j^{(n-1)})(\frac{x_j^{(n)}}{\parallel (x_j^{(n)})_{j=1}^{\infty} \parallel_{\infty}})\|^2)^{1/2} \leq$$

$$\leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} \dots \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} .$$

Hence

$$(\sum_{j=1}^{\infty} \|T(x_j^{(1)},...,x_j^{(n)})\|^2)^{1/2} \leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} \dots \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} \parallel (x_j^{(n)})_{j=1}^{\infty} \parallel_{\infty}$$

where $C = C_2(X_n')K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j$.

The case $(x_j^{(n)})_{j=1}^{\infty} = 0$ does not offer any trouble. Q.E.D.

The above result generalizes the case $n = 2$ due to Botelho ([2] and [3]). In [3] is asked whether

$$\mathcal{L}(^2C(K);\mathbb{K}) = \mathcal{L}_{as(2;2,\infty)}(^2C(K);\mathbb{K})$$

could be improved to some infinite dimensional Banach space $F$ in the place of $\mathbb{K}$ or not.

In particular, Proposition 4 answers negatively this question and not only for the bilinear case, but for any $n$-linear case of (7.1). As we can see, the answer extends to several other cases. In fact, we do not need that every space on the domain is a $\mathcal{L}_\infty$ space. For instance, in the bilinear case, we have the following straightforward result.

**Proposition 5.** *If $E$ and $F$ are infinite dimensional then*

$$\mathcal{L}(C(K), E; F) \neq \mathcal{L}_{as(s;r,\infty)}(C(K), E; F)$$

*for all $r \geq 1$ and $s > 0$.*

We can also point some other interesting remarks about the coincidence result (7.1). We shall observe that it is also impossible to improve (7.1) with $(s; t_1, t_2, ..., t_{n-1}, \infty)$ and $s < 2$ and $t_1, ..., t_{n-1} \geq 1$ . In fact, if it was possible, we would obtain that every $T : C(K) \times ... \times C(K) \to C(K)'$ would be $(s; t_1, t_2, ..., t_{n-1})$ -summing (contradiction by Theorem 9 and a simple localization argument, since $c_0$ and $C(K)$ are $\mathcal{L}_\infty$ spaces).

Other questions raised in [3] were:

- $\mathcal{P}(^2C(K);F) \neq \mathcal{P}_{as(\frac{r}{2};r)}(^2C(K);F)$ for every infinite dimensional Banach space $F$ and every $r < \infty$ ?
- $\mathcal{P}(^nC(K);F) \neq \mathcal{P}_{as(\frac{r}{n};r)}(^nC(K);F)$ for every $n > 2$, $r < \infty$ and every Banach space $F$ ?

It is worth remarking that the questions are for the dominated cases, and our answers, albeit partial in some situations, go beyond the dominated cases and sometimes furnish much completer results, such as when $\cot F = \infty$.

For $n > 2$, Corollary 1 gives a partial answer when $\frac{r}{n} < 2$. But Corollary 5 achieves a more general result when $\cot F < \infty$ since it asserts that whenever $r < n \cot F$ we have

$$\mathcal{P}(^nC(K);F) \neq \mathcal{P}_{as(\frac{r}{n};r)}(^nC(K);F).$$

Finally, Theorem 7 shows that when $F$ does not have finite cotype, we have for any $s>0$ and $r\geq 1$

$$\mathcal{P}(^nC(K);F)\neq \mathcal{P}_{as(r;s)}(^nC(K);F)$$

which is a complete answer and goes beyond the dominated case.

**Remark 2.** *This paper forms a portion of the author´s doctoral thesis which is being written at UNICAMP under supervision of M.Matos. The author thanks Professor M.Matos and Professor J. Mujica for the suggestions.*

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