COTYPE AND NON LINEAR ABSOLUTELY SUMMING MAPPINGS

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ABSTRACT. In this paper we study absolutely summing mappings on Banach spaces by exploring the cotype of their domains and ranges. It is proved that every n-linear mapping from $\mathcal{L}_\infty$-spaces into $\mathbb{K}$ is $(2;2,...,2,\infty)$-summing and also shown that every n-linear mapping from $\mathcal{L}_\infty$-spaces into F is $(q;2,...,2)$-summing whenever F has cotype q. We also give new examples of analytic summing mappings and polynomial and multilinear versions of a linear Extrapolation Theorem.

1. Introduction

In the fifties, A. Grothendieck´s seminal paper [8] "Resumé de la théorie métrique des produits tensoriels topologiques" provided the fundamentals of the absolutely summing operators theory. Subsequently, J. Lindenstrauss and A. Pełczyński [9] simplified Grothendieck´s tensorial notations leading to many interesting results. The multilinear theory of absolutely summing mappings was outlined by Pietsch [18] and has been developed by several authors (Alencar and Matos [1], Floret and Matos [7], Matos [12], Schneider [19], Tonge and Melendez [15], Botelho [2],[3], among others). Matos [12],[10], [11] also begun to study the concept of holomorphic absolutely summing mappings and a more general definition in such a way that the origin was not a distinguished point. The contribution of the notion of cotype to this theory is relevant and can be seen in [2],[3] and [7]. In this paper, we will generalize several results of [3] and [2] and also give new Coincidence Theorems and examples of absolutely summing holomorphic and analytic mappings.

2. Notation, general concepts and basic results

Throughout this paper $E, E_1, ..., E_n, F, X, Y$ will always denote Banach spaces and the scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$. We will denote by $C(K)$ the Banach space of continuous scalar valued functions on K(compact Hausdorff space) endowed with the sup norm.

The Banach space of all n-linear continuous mappings from $E_1 \times ... \times E_n$ into F endowed with the canonical norm will be denoted by $\mathcal{L}(E_1, ..., E_n;F)$ and the Banach space of all continuous n-homogeneous polynomials P from E into F with the norm $\|P\| = \sup\{\|Px\|; \|x\| \leq 1\}$ will be denoted by $\mathcal{P}(^nE, F)$. A mapping $f : E \to F$ will be said analytic at the point $a \in E$, if there exist a ball $B_\delta(a)$ and a sequence of polynomials $P_k \in \mathcal{P}(^kE, F)$ such that

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a) \text{ uniformly for } x \in B_\delta(a).$$

Henceforth δ_a will be called the radius of convergence of f around a. To emphasize the case $\mathbb{K} = \mathbb{C}$, we will sometimes use the term “holomorphic” in the place of “analytic”. Every analytic mapping in the whole space will be called entire mapping.

For the natural isometry

$$\Psi : \mathcal{L}(E_1, ..., E_n; F) \to \mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F))$$

we will use the following convention: If $T \in \mathcal{L}(E_1, ..., E_n; F)$ then $\Psi(T) = T_1$ and if $T \in \mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F))$, then $\Psi^{-1}(T) = T_0$.

For $p \in]0, \infty[$, the linear space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that

$$\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty} \|x_j\|^p)^{\frac{1}{p}} < \infty$$

will be denoted by $l_p(E)$. We will also denote by $l_p^w(E)$ the linear subspace of $l_p(E)$ formed by the sequences $(x_j)_{j=1}^{\infty}$ in E such that

$$(< \varphi, x_j >)_{j=1}^{\infty} \in l_p(\mathbb{K}),$$

for every continuous linear functional $\varphi : E \to \mathbb{K}$. We also define $\|.\|_{w,p}$ in $l_p^w(E)$ by

$$\|(x_j)_{j\in\mathbb{N}}\|_{w,p} := Sup_{\varphi \in B_{E'}} (\sum_{j=1}^{\infty} |< \varphi, x_j >|^p)^{\frac{1}{p}}.$$

The case $p = \infty$ is just the case of bounded sequences and in $l_\infty(E)$ we use the sup norm. The linear subspace of $l_p^w(E)$ of all sequences $(x_j)_{j=1}^{\infty} \in l_p^w(E)$, such that

$$\lim_{m\to\infty} \|(x_j)_{j=m}^{\infty}\|_{w,p} = 0,$$

is a closed linear subspace of $l_p^w(E)$ and will be denoted by $l_p^u(E)$. The case $p = 1$ motivates the name unconditionally p-summable sequences for the elements of $l_p^u(E)$ ([12]). One can see that $\|.\|_p$ ($\|.\|_{w,p}$) is a p-norm in $l_p(E)$($l_p^w(E)$) for $p < 1$ and a norm in $l_p(E)$($l_p^w(E)$) for $p \geq 1$. In any case, they are complete metrizable linear spaces.

Definition 1. *Let $2 \leq q \leq \infty$ and $(r_j)_{j=1}^{\infty}$ be the Rademacher functions. The Banach space E has cotype q, if there exists $C_q(E) \geq 0$, such that, for every $k \in \mathbb{N}$ and $x_1, ..., x_k \in E$,*

$$(\sum_{j=1}^{k} \|x_j\|^q)^{\frac{1}{q}} \leq C_q(E) (\int_0^1 \|\sum_{j=1}^{k} r_j(t) x_j\|^2 dt)^{\frac{1}{2}}.$$

To cover the case $q=\infty$, we replace $(\sum_{j=1}^{k}\|x_j\|^q)^{\frac{1}{q}}$ by

$$\max\{\|x_j\|; 1\leq j\leq k\}.$$

We will define the cotype of E by $\cot E=\inf\{2\leq q\leq\infty; E$ has cotype $q\}$.

Definition 2. *(Matos) A continuous n-linear mapping $T: E_1\times ...\times E_n\to F$ is absolutely $(p;q_1,...,q_n)$-summing (or $(p;q_1,...,q_n)$-summing) at $(a_1,...,a_n)\in E_1\times ...\times E_n$ if*

$$(T(a_1+x_j^{(1)},...,a_n+x_j^{(n)})-T(a_1,...,a_n))_{j=1}^{\infty}\in l_p(F)$$

for every $(x_j^{(s)})_{j=1}^{\infty}\in l_{q_s}^{w}(E)$, $s=1,...,n$. A continuous n-homogeneous polynomial $P: E\to F$ is absolutely $(p;q)$-summing (or $(p;q)$-summing) at $a\in E$ if

$$(P(a+x_j)-P(a))_{j=1}^{\infty}\in l_p(F)$$

for every $(x_j)_{j=1}^{\infty}\in l_q^w(E)$.

The space of all n-homogeneous polynomials $P: E\to F$ which are $(p;q)$-summing (at every point) will be denoted by $\mathcal{P}_{as(p;q)(E)}(^nE;F)$. The space of all n-homogeneous polynomials $P: E\to F$ which are $(p;q)$-summing (at the origin) will be denoted by $\mathcal{P}_{as(p;q)}(^nE;F)$. Analogously for n-linear mappings.

It must be noticed that the aforementioned definition, where the origin is not a privileged point, is actually a more restrictive definition. For example, if $n>1$ every n-linear mapping T from $l_1\times ...\times l_1$ into l_1 is absolutely $(1;1)$-summing at the origin, but we can always find $a\neq 0$ such that T is not absolutely $(1;1)$-summing at a [11]. Besides, the above definition turns possible to consider an absolutely summing holomorphy type in the sense of Nachbin (see [10]).

One can prove that if $r<s$ then the unique polynomial which is absolutely $(r;s)$-summing at every point is the trivial.

For n-homogeneous polynomials and n-linear mappings, the polynomials (n-linear mappings) $(\frac{p}{n};p)$-summing will be called p-dominated polynomials (n-linear mappings) (see [12],[15]). For the p-dominated polynomials (n-linear mappings) several natural versions of linear results still hold, as well as Factorization Theorems, Domination Theorem, etc. [12],[15],[19].

The following characterization will be useful:

Theorem 1. *(Matos [12]) Let P be an m-homogeneous polynomial from E into F. Then, the following statements are equivalent:*

(1) P is absolutely $(p;q)$-summing at 0.

(2) There exists $L>0$ such that

$$(\sum_{j=1}^{\infty}\|P(x_j)\|^p)^{\frac{1}{p}}\leq L\|(x_j)_{j=1}^{\infty}\|_{w,q}^{m}\forall(x_j)_{j=1}^{\infty}\in l_q^u(E).$$

(3) There exists $L > 0$ such that

$$(\sum_{j=1}^{k} \|P(x_j)\|^p)^{\frac{1}{p}} \leq L\|(x_j)_{j=1}^{k}\|_{w,q}^{m} \forall k \in \mathbf{N}, \forall x_1, ..., x_k.$$

(4) $(P(x_j))_{j=1}^{\infty} \in l_p(F)$ *for every* $(x_j)_{j=1}^{\infty} \in l_q^w(E)$.

The infimum of the possible constants $L > 0$ is a norm for the case $p \geq 1$ or a p-norm for the case $p < 1$([12] or [17] page 91) on the space of the absolutely $(p;q)$-summing polynomials. In any case, we have complete topological metrizable spaces. We will use the notation $\|.\|_{as(p;q)}$ for this norm ($p-$norm).

The characterization for the multilinear case and the definition of the norm (p-norm) follows the same reasoning.

The following Theorem plays an important role in our future results:

Theorem 2. *(Maurey-Talagrand* [20]*)If E has cotype p, then $id : E \to E$ is $(p;1)$-summing. The converse is true, except for $p = 2$.*

The next definition, due to Lindenstrauss and Pełczyński is of fundamental importance in the local study of Banach spaces and their properties:

Definition 3. *Let $1 \leq p \leq \infty$ and let $\lambda > 1$. The Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X for which there exists an isomorphism $v_E : F \to l_p^{\dim F}$ with $\|v_E\|\|v_E^{-1}\| < \lambda$. We say that X is an $\mathcal{L}_p$ space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda > 1$.*

3. Absolutely summing polynomials and multilinear mappings explored by the cotype of their ranges

The relation between cotype and absolutely summing linear mappings is clear by Theorem 2. For points different from the origin we have the straightforward following results:

Lemma 1. *Every continuous n-linear mapping $T : E_1 \times ... \times E_n \to F$ is such that*

$$(T(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - T(a_1, ..., a_n))_{j=1}^{\infty} \in l_1^w(F)$$

whenever $(x_j^{(1)})_{j=1}^{\infty} \in l_1^w(E_1), ..., (x_j^{(n)})_{j=1}^{\infty} \in l_1^w(E_n)$. The polynomial version is immediate.

Proof. We just need to invoke a well known, albeit unpublished, result of Defant and Voigt which states that every scalar valued n-linear mapping is absolutely $(1;1)$-summing at the origin (see [10], Theorem 1.6 or [12]), and explore multilinearity.

Theorem 3. *If F has cotype q, then every continuous n-linear mapping from $E_1 \times ... \times E_n$ into F is $(q;1)$-summing on $E_1 \times ... \times E_n$. The polynomial case is also valid.*

Proof. Since F has cotype q, Theorem 2 and Lemma 1 provide

$$(\sum_{j=1}^{\infty} \|T(a_1 + x_j^1, ..., a_n + x_j^n) - T(a_1, ..., a_n)\|^q)^{\frac{1}{q}} \leq$$

$$\leq \|(T(a_1 + x_j^1, ..., a_n + x_j^n) - T(a_1, ..., a_n))_{j=1}^{\infty}\|_{w,1} < \infty$$

whenever $(x_j^1)_{j=1}^{\infty} \in l_1^w(E_1), ..., (x_j^n)_{j=1}^{\infty} \in l_1^w(E_n)$.■

Theorem 3 generalizes to points different from the origin the following result:

Theorem 4. *(Botelho* [2]*) If F has cotype q then every continuous n-homogeneous polynomial from E into F is $(q;1)$-summing at the origin.*

In order to prove a new characterization of cotype in terms of absolutely summing polynomials we need the following Lemma:

Lemma 2. *If $\mathcal{P}_{as(r;s)(E)}(^nE;F) = \mathcal{P}(^nE;F)$ then $\mathcal{L}(E;F) = \mathcal{L}_{as(r;s)}(E;F)$.*

Proof. (Inspired on the proof of Dvoretzky Rogers Theorem for polynomials [11])

It is clear that $r \geq s$. Let us consider a continuous linear mapping $T : E \to F$. Define an n-homogeneous polynomial

$$P(x) = \varphi(x)^{n-1}T(x)$$

where φ is a non null continuous linear functional. Then, choosing $a \notin Ker(\varphi)$, we have

$$dP(a)(x) = (n-1)\varphi(a)^{n-2}\varphi(x)T(a) + \varphi(a)^{n-1}T(x).$$

It is not hard to see that $dP(a)$ is absolutely $(r;s)$-summing (see Matos [11]) and since φ is absolutely $(r;s)$-summing, it follows that T is absolutely $(r;s)$-summing.■

It is worth remarking that the converse of Lemma 2 does not hold. In fact,

$$\mathcal{L}(l_2;\mathbb{K}) = \mathcal{L}_{as(2;2)}(l_2;\mathbb{K}) \text{ and } \mathcal{P}(^2l_2;\mathbb{K}) \neq \mathcal{P}_{as(2;2)}(l_2;\mathbb{K}).$$

Now we have another characterization of cotype:

Theorem 5. *Let $n \geq 1$. E has cotype $q > 2$ if, and only if,*

$$\mathcal{P}(^nE;E) = \mathcal{P}_{as(q;1)(E)}(^nE;E).$$

Proof. If $\mathcal{P}(^nE;E) = \mathcal{P}_{as(q;1)(E)}(^nE;E)$ then, by Lemma 2, $id : E \to E$ is $(q;1)$-summing and consequently E has cotype q. Theorem 3 yields the converse.■

The following recent Theorem of D.Perez [16], that generalizes a 2- linear result of Floret-Botelho [2] and Tonge-Melendez[15], is an important instrument for other multilinear and holomorphic results, as we will see later.

Theorem 6. *(D.Perez [16]) If each X_j is an $\mathcal{L}_{\infty,\lambda_j}$ space, then every continuous n-linear mapping $(n \geq 2)$ from $X_1 \times ... \times X_n$ into $\mathbb{K}$ is $(1;2,...,2)$-summing at the origin and*

$$\|T\|_{as(1;2,...,2)} \leq K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

The polynomial version of this Theorem is immediate and will be useful for us in the last section of this paper.

Corollary 1. *If X is an $\mathcal{L}_{\infty,\lambda}$ space then every continuous scalar valued n-homogeneous polynomial $(n \geq 2)$ $P : X \to \mathbb{K}$ is $(1;2)$-summing at the origin and*

$$\|P\|_{as(1;2)} \leq K_G 3^{\frac{n-2}{2}} \|P\| \lambda^n.$$

We can explore last Theorem and the cotype of the range as follows:

Theorem 7. *If each X_j is an $\mathcal{L}_{\infty,\lambda_j}$ space and F has cotype $q \neq \infty$, then every continuous n-linear mapping from $X_1 \times ... \times X_n$ into F is $(q;2,...,2)$-summing at the origin and*

$$\|T\|_{as(q;2,...,2)} \leq C_q(F) K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

In particular, if X is an $\mathcal{L}_{\infty,\lambda}$ space and F has cotype $q \neq \infty$, then

$$\mathcal{P}(^n X; F) = \mathcal{P}_{as(q;2)}(^n X; F)$$

and

$$\|P\|_{as(q;2)} \leq C_q(F) K_G 3^{\frac{n-2}{2}} \|P\| \lambda^n. \tag{3.1}$$

Proof. Let $(f_j^{(1)})_{j=1}^{\infty} \in l_2^w(X_1), ..., (f_j^{(n)})_{j=1}^{\infty} \in l_2^w(X_n)$. Since

$$\mathcal{L}(X_1, ..., X_n; \mathbb{K}) = \mathcal{L}_{as(1;2,...,2)}(X_1, ..., X_n; \mathbb{K})$$

and for every $R \in \mathcal{L}(X_1, ..., X_n; \mathbb{K})$ we have

$$\|R\|_{as(1;2,...,2)} \leq K_G 3^{\frac{n-2}{2}} \|R\| \prod_{j=1}^{n} \lambda_j$$

then

$$(\sum_{j=1}^{\infty} \|T(f_j^{(1)}, ..., f_j^{(n)})\|^q)^{\frac{1}{q}} \leq C_q(F) \|(T(f_j^{(1)}, ..., f_j^{(n)}))_{j=1}^{\infty}\|_{w,1} =$$

$$= C_q(F) \sup_{y' \in B_{F'}} \sum_{j=1}^{\infty} \mid (y' \circ T)(f_j^{(1)}, ..., f_j^{(n)}) \mid =$$

$$\leq C_q(F) \sup_{y' \in B_{F'}} \|y' \circ T\|_{as(1;2,...,2)} \|(f_j^{(1)})_{j=1}^{\infty}\|_{w,2} \cdots \|(f_j^{(n)})_{j=1}^{\infty}\|_{w,2} \leq$$

$$\leq C_q(F) \sup_{y' \in B_{F'}} C \|y' \circ T\| \|(f_j^{(1)})_{j=1}^{\infty}\|_{w,2} \cdots \|(f_j^{(n)})_{j=1}^{\infty}\|_{w,2} \leq$$

$$\leq C_q(F) \sup_{y' \in B_{F'}} C\|y'\|\|T\|\|(f_j^{(1)})_{j=1}^{\infty}\|_{w,2}...\|(f_j^{(n)})_{j=1}^{\infty}\|_{w,2} \leq$$

$$\leq C_q(F) C\|T\|\|(f_j^{(1)})_{j=1}^{\infty}\|_{w,2}...\|(f_j^{(n)})_{j=1}^{\infty}\|_{w,2}$$

where $C = K_G 3^{\frac{n-2}{2}} \prod_{j=1}^{n} \lambda_j$.■

As a consequence of the last Theorem, we obtain a generalization of a bilinear result of Botelho ([2]), answering a question posed in [3]:

Theorem 8. *If $n \geq 2$ and each X_j is an $\mathcal{L}_{\infty,\lambda_j}$ space then every continuous n-linear mapping $T : X_1 \times ... \times X_n \to \mathbb{K}$ is $(2;2,...,2,\infty)$-summing at the origin and*

$$\|T\|_{as(2;2,...,\infty)} \leq C_2(X_n') K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j.$$

Proof. Let $T : X_1 \times ... \times X_n \to \mathbb{K}$ be a continuous n-linear mapping. Then

$T_1 : X_1 \times ... \times X_{n-1} \to X_n{}'$ is $(2;2,...,2)$-summing since $X_n{}'$ has cotype 2. So,

$$(\sum_{j=1}^{\infty} \|T_1(x_j^{(1)},...,x_j^{(n-1)})\|^2)^{1/2} \leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2}$$

and

$$(\sum_{j=1}^{\infty} \underset{x_j^{(n)} \in B_{X_n}}{Sup} \|T_1(x_j^{(1)},...,x_j^{(n-1)})(x_j^{(n)})\|^2)^{1/2} \leq$$

$$\leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} .$$

If $(x_j^{(n)})_{j=1}^{\infty} \in l_{\infty}(X_n)$ does not vanish, we have

$$(\sum_{j=1}^{\infty} \|T_1(x_j^{(1)},...,x_j^{(n-1)})(\frac{x_j^{(n)}}{\parallel (x_j^{(n)})_{j=1}^{\infty} \parallel_{\infty}})\|^2)^{1/2} \leq$$

$$\leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} .$$

Hence

$$(\sum_{j=1}^{\infty} \|T_1(x_j^{(1)},...,x_j^{(n-1)})(x_j^{(n)})\|^2)^{1/2} \leq$$

$$\leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} \parallel (x_j^{(n)})_{j=1}^{\infty} \parallel_{\infty}$$

and

$$(\sum_{j=1}^{\infty} \|T(x_j^{(1)},...,x_j^{(n)})\|^2)^{1/2} \leq C \parallel (x_j^{(1)})_{j=1}^{\infty} \parallel_{w,2} ... \parallel (x_j^{(n-1)})_{j=1}^{\infty} \parallel_{w,2} \parallel (x_j^{(n)})_{j=1}^{\infty} \parallel_{\infty}$$

where $C = C_2(X_n') K_G 3^{\frac{n-2}{2}} \|T\| \prod_{j=1}^{n} \lambda_j$.

The case $(x_j^{(n)})_{j=1}^{\infty}=0$ does not offer any trouble.■

For $n=2$, Theorem 8 has the following version:

Proposition 1. *If X is an $\mathcal{L}_\infty$ space, then every continuous 2 linear mapping $T:X\times E\rightarrow\mathbb{K}$ with* $\cot E'=q=2$ *is $(r;r,\infty)$-summing at the origin for every $r\geq 2$. If* $\cot E'=q>2$, *then T is $(r;r,\infty)$ and $(q;p,\infty)$-summing at the origin for every $r>q$ and $p<q$.*

Proof. (Case $q=2$) Let $T:X\times E\rightarrow\mathbb{K}$ be a continuous bilinear mapping. Then $T_1:X\rightarrow E'$ is $(r;r)$-summing since E' has cotype 2 [6]. Hence

$$(\sum_{j=1}^{\infty}\|T_1(x_j)\|^r)^{1/r}\leq C\parallel(x_j)_{j=1}^{\infty}\parallel_{w,r}$$

and thus

$$(\sum_{j=1}^{\infty}\underset{y_j\in B_E}{Sup}\|T_1(x_j)(y_j)\|^r)^{1/r}\leq C\parallel(x_j)_{j=1}^{\infty}\parallel_{w,r}.$$

If $(y_j)_{j=1}^{\infty}\in l_\infty(E)$ does not vanish, we have

$$(\sum_{j=1}^{\infty}\|T_1(x_j)(\frac{y_j}{\parallel(y_j)_{j=1}^{\infty}\parallel_\infty})\|^r)^{1/r}\leq C\parallel(x_j)_{j=1}^{\infty}\parallel_{w,r}.$$

Hence

$$(\sum_{j=1}^{\infty}\|T_1(x_j)(y_j)\|^r)^{1/r}\leq C\parallel(x_j)_{j=1}^{\infty}\parallel_{w,r}\parallel(y_j)_{j=1}^{\infty}\parallel_\infty$$

and

$$(\sum_{j=1}^{\infty}\|T(x_j,y_j)\|^r)^{1/r}\leq C\parallel(x_j)_{j=1}^{\infty}\parallel_{w,r}\parallel(y_j)_{j=1}^{\infty}\parallel_\infty.$$

The case $(y_j)_{j=1}^{\infty}=0$ does not offer any problem.

A linear result of Maurey (see [4], page 223, Th. 11.14 a) provides, through the same reasoning, the proof of the case $q>2$.■

Applying the same ideas we have the statement below:

Proposition 2. *If each X_j is an $\mathcal{L}_\infty$ space, then every continuous n-linear mapping $T:X_1\times...\times X_n\times E\rightarrow\mathbb{K}$ with* $\cot E'=q\geq 2, q\neq\infty$ *is $(q;2,...,2,\infty)$-summing at the origin.*

Theorem 8 can also be used to obtain other results. For example:

Theorem 9. *If each X_j is an $\mathcal{L}_\infty$ space and $T:X_1\times...\times X_n\rightarrow\mathbb{K}$ is a continuous n-linear mapping, then*

$n=2\Rightarrow T$ is $(r;r,r)$-summing on $X_1\times X_2$, for every $r\geq 2$.

$n\geq 3\Rightarrow T$ é $(r;2,...,2,r)$-summing on $X_1\times...\times X_n$ for every $r\geq 2$.

Proof. The case $n=2$ is the easiest and we will omit the proof. For the case $n=3$, let $(x_j)_{j=1}^{\infty}\in l_2^w(X_1),(y_j)_{j=1}^{\infty}\in l_2^w(X_2)$ and $(z_j)_{j=1}^{\infty}\in l_2^w(X_3)$. Then

$$(\sum_{j=1}^{\infty}\|T(a+x_j,b+y_j,c+z_j)-T(a,b,c)\|^r)^{\frac{1}{r}}=$$

$$=(\sum_{j=1}^{\infty}(\|T(a,y_j,z_j)\|^r)^{\frac{1}{r}}+(\sum_{j=1}^{\infty}(\|T(x_j,b,c)\|^r)^{\frac{1}{r}}+$$

$$+(\sum_{j=1}^{\infty}(\|T(x_j,y_j,c)\|^r)^{\frac{1}{r}}+(\sum_{j=1}^{\infty}(\|T(x_j,b,z_j)\|^r)^{\frac{1}{r}}+(\sum_{j=1}^{\infty}(\|T(a,b,z_j)\|^r)^{\frac{1}{r}}$$

$$+(\sum_{j=1}^{\infty}(\|T(a,y_j,c)\|^r)^{\frac{1}{r}}+(\sum_{j=1}^{\infty}(\|T(x_j,y_j,z_j)\|^r)^{\frac{1}{r}}<\infty$$

since every linear mapping is $(r;r)$ and $(r;2)$-summing, every such bilinear mapping is $(r;r,r)$, $(r;2,r)$ and $(r;2,2)$-summing and every such 3-linear mapping above is $(r;2,2,r)$-summing at the origin.

For $n>3$ we use an inductive principle.

Theorem 7 can be extended as follows:

Theorem 10. *If each X_j is an $\mathcal{L}_\infty$ space and $\cot F=q$, then every continuous n-linear mapping from $X_1\times...\times X_n$ into F is $(q;2,...,2)$-summing on $X_1\times...\times X_n$.*

Proof. If $q=2$, it is enough to use the last reasoning with Theorem 7 and the Dubinsky-Pełczyński-Rosenthal ([4] page 223, Th. 11.14 (a) or [6]) result which asserts that every linear mapping from an $\mathcal{L}_\infty$ space into F(with $\cot F=2$) is $(2;2)$-summing.

If $q>2$, we shall use the same reasoning with the Maurey ([4] page 223, Th. 11.14(b)) result which asserts that every linear mapping from an $\mathcal{L}_\infty$ space into F(with cot $F=q>2$) is $(q;p)$-summing for each $p<q$.■

4. R-FULLY ABSOLUTELY SUMMING MULTILINEAR MAPPINGS

The following definition is inspired in the work of Matos [13] which is being developed by M.L.V. Souza in his doctoral dissertation.

Definition 4. *A continuous n-linear mapping $T:E_1\times...\times E_n\to F$ will be said r-fully $(p;q_1,...,q_n)$-summing if*

$$\sum_{j_1,...,j_r=1}^{\infty}\left\|T(x_{j_1}^{(1)},...,x_{j_r}^{(r)},x_{j_r}^{(r+1)},...,x_{j_r}^{(n)}\right\|^p<\infty$$

whenever $(x_k^{(l)})_{k=1}^{\infty}\in l_{q_l}^w(E_l), l=1,...,n$. In this case we will write

$$T\in\mathcal{L}_{f(r)as(p;q_1,...,q_n)}(E_1,...,E_n;F).$$

When $r=1$, we have the $(p;q_1,...,q_n)$-summing mappings and when $r=n$ we call T just by fully $(p;q_1,...,q_n)$-summing, which is a concept introduced by Matos [13].

A natural question is: Does every n-linear mappings from $\mathcal{L}_\infty$ spaces into $\mathbb{K}$ is fully $(2;2,...,2)$-summing?

We will show in Corollary 2 that Theorem 8 give us partial answers.

Theorem 11. *If $\mathcal{L}(E_n;F)=\mathcal{L}_{as(q;r)}(E_n;F)$, then*

$$\mathcal{L}_{as(q,p_1,...,p_{n-1},\infty)}(E_1,...,E_n;F)\subset\mathcal{L}_{f(2)as(q;p_1,...,p_{n-1},r)}(E_1,...,E_n;F).$$

Proof. Let us consider $T\in\mathcal{L}_{as(q;p_1,...,p_{n-1},\infty)}(E_1,...,E_n;F)$. If

$$(x_k^{(1)})_{k=1}^\infty\in l_{p_1}^w(E_1),...,(x_k^{(n-1)})_{k=1}^\infty\in l_{p_{n-1}}^w(E_{n-1}),(y_k)_{k=1}^\infty\in l_r^w(E),$$

then, for each k fixed,

$$\sum_{j=1}^\infty\left\|T(x_k^{(1)},...,x_k^{(n-1)},y_j)\right\|^q\leq\left\|(y_j)_{j=1}^\infty\right\|_{w,r}^q\left\|T(x_k^{(1)},...,x_k^{(n-1)},.)\right\|_{as(q,r)}^q\leq$$

$$\leq\left\|(y_j)_{j=1}^\infty\right\|_{w,r}^q C\left\|T(x_k^{(1)},...,x_k^{(n-1)},.)\right\|^q\leq$$

$$\leq\left\|(y_j)_{j=1}^\infty\right\|_{w,r}^q C(\left\|T(x_k^{(1)},...,x_k^{(n-1)},z_k)\right\|^q+\frac{1}{2^k}).$$

where each z_k belongs to the unit ball B_{E_n}.

Therefore

$$\sum_{k=1}^\infty\sum_{j=1}^\infty\left\|T(x_k^{(1)},...,x_k^{(n-1)},y_j)\right\|^q\leq$$

$$\leq\|(y_j)\|_{w,r}^q C\sum_{k=1}^\infty(\left\|T(x_k^{(1)},...,x_k^{(n-1)},z_k)\right\|^p+\frac{1}{2^k})<\infty.\blacksquare$$

Corollary 2. *If each E_k is an $\mathcal{L}_\infty$ space, we have*

$$\mathcal{L}(E_1,...,E_n;\mathbb{K})=\mathcal{L}_{f(2)as(2;2,...,2,2)}(E_1,...,E_n;\mathbb{K}).$$

Proof. It suffices to realize that $\mathcal{L}(E_n;\mathbb{K})=\mathcal{L}_{as(2;2)}(E_n;\mathbb{K})$ and apply last Theorem and Theorem 8.$\blacksquare$

5. Other results

An important and broadly used result is the Generalized Hölder´s Inequality, which is a natural instrument to deal with absolutely summing multilinear mappings.

Theorem 12 (Generalized Hölder´s Inequality). *If $\frac{1}{p}\leq\frac{1}{p_1}+...+\frac{1}{p_n}$, then*

$$(\sum_{j=1}^\infty\mid a_j^{(1)}...a_j^{(n)}\mid^p)^{\frac{1}{p}}\leq(\sum_{j=1}^\infty\mid a_j^{(1)}\mid^{p_1})^{\frac{1}{p_1}}...(\sum_{j=1}^\infty\mid a_j^{(n)}\mid^{p_n})^{\frac{1}{p_n}}.$$

If $T : E_1 \times ... \times E_n \to F$ is a continuous multilinear mapping where at least one of the spaces which compose the Banach spaces of the domain has finite cotype, we can state the following result.

Theorem 13. *If $T : E_1 \times ... \times E_n \to F$ is a continuous multilinear mapping, $q_j = \cot E_j$, $j = 1, ..., n$, and at least one of the q_j finite, then, for any choice of $a_j \in [q_j, \infty]$, with at least one of the a_j finite, T is $(s; b_1, ..., b_n)$-summing at the origin, for any $s > 0$, such that $\frac{1}{s} \leq \frac{1}{a_1} + ... + \frac{1}{a_n}$, with $b_j = 1$, if $a_j < \infty$, and $b_j = \infty$ if $a_j = \infty$.*

Proof. Obvious, using Theorem 2, after some reasoning on how to optimize the use of the Generalized Hölder´s Inequality.

As a corollary, we have the a result due to Botelho [2].

Corollary 3. *If $T : E_1 \times ... \times E_n \to F$ is a continuous multilinear mapping and $q_j = \cot E_j < \infty$ for every $j = 1, ..., n$, then T is $(s; 1, ..., 1)$-summing at the origin for any $s > 0$ such that $\frac{1}{s} \leq \frac{1}{q_1} + ... + \frac{1}{q_n}$.*

Theorem 13 shows that even if just one of the spaces of the domain has finite cotype, the multilinear mapping is still well behaved. As an illustration we can see the example below.

Example 1. *If E has cotype p, then every $T : C(K) \times ... \times C(K) \times E \to F$ is $(p; \infty, ..., \infty, 1)$-summing at the origin.*

The following results show more about the mechanism of absolutely summing mappings.

Proposition 3. *If $\mathcal{L}(E_1, ..., E_n; F) = \mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1, ..., E_n; F)$ then*

$$\mathcal{L}(E_1, ..., E_t; F) = \mathcal{L}_{as(r;s_1,...,s_t)}(E_1, ..., E_t; F).$$

Proof. Given $T \in \mathcal{L}(E_1, ..., E_t; F)$ let us define

$$S(a_1, ..., a_n) = T(a_1, ..., a_t)\varphi_{t+1}(a_{t+1})...\varphi_n(a_n)$$

where $\varphi_{t+1}, ..., \varphi_n$ are non trivial bounded linear functionals. Let $b_{t+1}, ..., b_n$ be such that

$$\varphi_{t+1}(b_{t+1}) = ... = \varphi_n(b_n) = 1.$$

It follows that $T \in \mathcal{L}_{as(r;s_1,...,s_t)}(E_1, ..., E_t; F)$. In fact, if $(x_j^l)_{j=1}^{\infty} \in l_{s_l}^w(E_l)$ we have

$$\sum_{j=1}^{\infty} \|T(x_j^{(1)}, ..., x_j^{(t)})\|^r = \sum_{j=1}^{\infty} \|S(x_j^{(1)}, ..., x_j^{(t)}, b_{t+1}, ..., b_n)\|^r < \infty.\blacksquare$$

The next statement suggested by Matos extend the Lemma 3.2 of [2]:

Proposition 4. *If $\mathcal{L}(E_1, ..., E_n; F) = \mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1, ..., E_n; F)$, then*

$$\mathcal{L}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F)) = \mathcal{L}_{as(r;s_1,...,s_t)}(E_1, ..., E_t; \mathcal{L}(E_{t+1}, ..., E_n; F))$$

and conversely.

Proof: Suppose

$$\mathcal{L}(E_1,...,E_n;F)=\mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1,...,E_n;F).$$

Let $T:E_1\times...\times E_t\longrightarrow\mathcal{L}(E_{t+1},...,E_n;F)$ be a continuous multilinear mapping. We have

$$(\sum_{j=1}^{\infty}\|T(x_1^{(j)},...,x_t^{(j)})\|^r)^{\frac{1}{r}}=(\sum_{j=1}^{\infty}\underset{\substack{\|y_k\|\leq 1\\ k=t+1,...,n}}{Sup}\|T(x_1^{(j)},...,x_t^{(j)})(y_{t+1},...,y_n)\|^r)^{\frac{1}{r}}\leq$$

$$\leq(\sum_{j=1}^{\infty}\|T(x_1^{(j)},...,x_t^{(j)})(y_{t+1}^{(j)},...,y_n^{(j)})\|^r+\frac{1}{2^j})^{\frac{1}{r}}=$$

$$=(\sum_{j=1}^{\infty}\|T_0(x_1^{(j)},...,x_t^{(j)},y_{t+1}^{(j)},...,y_n^{(j)})\|^r+\frac{1}{2^j})^{\frac{1}{r}}<\infty$$

if $(x_1^{(j)})\in l_{s_1}^w(E_1),...,(x_t^{(j)})\in l_{s_t}^w(E_t)$.

On the other hand, suppose

$$\mathcal{L}(E_1,...,E_t;\mathcal{L}(E_{t+1},...,E_n;F)=\mathcal{L}_{as(r;s_1,...,s_t)}(E_1,...,E_t;\mathcal{L}(E_{t+1},...,E_n;F).$$

If $T:E_1\times...\times E_n\longrightarrow F$, and

$(x_1^{(j)})_{j=1}^{\infty}\in l_{s_1}^w(E_1),...,(x_t^{(j)})_{j=1}^{\infty}\in l_{s_t}^w(E_t),(y_{t+1}^{(j)})_{j=1}^{\infty}\in l_{\infty}(E_t),...,(y_n^{(j)})_{j=1}^{\infty}\in l_{\infty}(E_n)$,

we have

$$\sum_{j=1}^{\infty}\|T(x_1^{(j)},...,x_t^{(j)},y_{t+1}^{(j)},...,y_n^{(j)})\|^r)^{\frac{1}{r}}=(\sum_{j=1}^{\infty}\|T_1(x_1^{(j)},...,x_t^{(j)})(y_{t+1}^{(j)},...,y_n^{(j)})\|^r)^{\frac{1}{r}}\leq$$

$$\leq\|(y_{t+1}^{(j)})\|_{\infty}...\|(y_n^{(j)})\|_{\infty}\sum_{j=1}^{\infty}\|T_1(x_1^{(j)},...,x_t^{(j)})\|^r)^{\frac{1}{r}}<\infty. \tag{5.1}$$

We can see that it is also true that

$$T\in\mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1,...,E_n;F)\Rightarrow$$
$$\Rightarrow T_1\in\mathcal{L}_{as(r;s_1,...,s_t,)}(E_1,...,E_t;\mathcal{L}(E_{t+1},...,E_n;F))$$

and

$$T\in\mathcal{L}_{as(r;s_1,...,s_t,)}(E_1,...,E_t;\mathcal{L}(E_{t+1},...,E_n;F))\Rightarrow$$
$$\Rightarrow T_0\in\mathcal{L}_{as(r;s_1,...,s_t,\infty,...,\infty)}(E_1,...,E_n;F).\blacksquare$$

Remark 1. *The reader shall note that the converse of Proposition 3 cannot hold. In fact, we know that* $\mathcal{L}(E;\mathbb{K})=\mathcal{L}_{as(1;1)}(E;\mathbb{K})$. *If the converse of Proposition 3 held, we would have* $\mathcal{L}(E,E;\mathbb{K})=\mathcal{L}_{as(1;1,\infty)}(E,E;\mathbb{K})$ *and by Proposition 4*

$$\mathcal{L}(E;E')=\mathcal{L}_{as(1;1)}(E;E')$$

which is impossible, in general (see [9]*).*

Proposition 4 also furnishes an Inclusion Theorem for bilinear mappings.

Proposition 5. *(Inclusion for bilinear mappings)*
If $r > s$ *then* $\mathcal{L}_{as(s;s,\infty)}(E_1, E_2; F) \subset \mathcal{L}_{as(r;r,\infty)}(E_1, E_2; F)$.

Proof. If $r > s$ and $T \in \mathcal{L}_{as(s;s,\infty)}(E_1, E_2; F)$, then by Proposition 4, $T_1 : E_1 \to \mathcal{L}(E_2; F)$ is $(s; s)$-summing. By the Inclusion Theorem for linear mappings, T_1 will be $(r; r)$-summing and again by the Proposition 4, T will be $(r; r, \infty)$-summing at the origin.■

Example 2. *The famous Grothendieck´s Theorem, which asserts that every linear operator from an* $\mathcal{L}_1$ *space into an* $\mathcal{L}_2$ *space is* $(1; 1)$*-summing, and Proposition 4 lead us to conclude that if* E_1 *and* E_2 *are* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *spaces respectively, then*

$$\mathcal{L}(E_1, E_2; \mathbb{K}) = \mathcal{L}_{as(1;1,\infty)}(E_1, E_2; \mathbb{K}).$$

Thus, Proposition 5 yields

$$\mathcal{L}(E_1, E_2; \mathbb{K}) = \mathcal{L}_{as(r;r,\infty)}(E_1, E_2; \mathbb{K})$$

for every $r \geq 1$*. However, despite Grothendieck´s Theorem we know that*

$$\mathcal{L}(l_1, l_1; l_2) \neq \mathcal{L}_{as(1;1,\infty)}(l_1, l_1; l_2)$$

and furthermore

$$\mathcal{L}(l_1, l_1; \mathbb{K}) \neq \mathcal{L}_{as(1;1,\infty)}(l_1, l_1; \mathbb{K}).$$

The result below has the same spirit of the last Proposition.

Proposition 6. *If* $T : E_1 \times ... \times E_n \to F$ *is* p*-dominated, then* T *is* $(\frac{r}{n-1}; r, ..., r, \infty)$*-summing for every* $r \geq p$.

Proof. If $T : E_1 \times ... \times E_n \to F$ is p-dominated, then, by Grothendieck-Pietsch domination Theorem, if $T_1 : E_1 \times ... \times E_{n-1} \to \mathcal{L}(E_n; F)$ is such that $T_1 = \Psi(T)$, we obtain, for $r \geq p$,

$$\|T_1(x_1, ..., x_{n-1})\| = Sup_{\|y\| \leq 1} \|T_1(x_1, ..., x_{n-1})(y)\| =$$

$$= Sup_{\|y\| \leq 1} \|T(x_1, ..., x_{n-1}, y)\| \leq$$

$$\leq Sup_{\|y\| \leq 1} C(\int_{B_{E_1'}} | \varphi(x_1) |^r d\mu_1)^{\frac{1}{r}} ... (\int_{B_{E_n'}} | \varphi(y) |^r d\mu_n)^{\frac{1}{r}} \leq$$

$$\leq C(\int_{B_{E_1'}} | \varphi(x_1) |^r d\mu_1)^{\frac{1}{r}} ... (\int_{B_{E_{n-1}'}} | \varphi(x_{n-1}) |^r d\mu_{n-1})^{\frac{1}{r}}.$$

Thus, T_1 is r-dominated and, by Proposition 4, $T = (T_1)_0$
is $(\frac{r}{n-1} r; r, ..., r, \infty)$-summing.

Corollary 4. *If every* $T : E_1 \times ... \times E_n \to F$ *is* p*-dominated, then every* $T : E_{j_1} \times ... \times E_{j_r} \to F$*, with* $1 \leq r \leq n$ *and* $j_1, ..., j_r \in \{1, ..., n\}$ *mutually disjoint, is* $p-$*dominated.*

Proof. By Proposition 6, we have

$$\mathcal{L}(E_1,...,E_n;F)=\mathcal{L}_{as(\frac{p}{n-1};p,...,p,\infty)}(E_1,...,E_n;F)$$

and by Proposition 3 we obtain

$$\mathcal{L}(E_1,...,E_{n-1};F)=\mathcal{L}_{as(\frac{p}{n-1};p,...,p)}(E_1,...,E_{n-1};F).$$

The other cases use the same arguments.■
Similar reasoning furnishes the next Corollary.

Corollary 5. *If every $T:E_1\times...\times E_n\to F$ is p-dominated, then for every permutation $\pi:\{1,...,n\}\to\{1,...,n\}$ we have*

$$\mathcal{L}(E_{\pi(1)},...,E_{\pi(t)};\mathcal{L}(E_{\pi(t+1)},...,E_{\pi(n)};F))=$$
$$=\mathcal{L}_{as(\frac{p}{t},p,...,p)}(E_{\pi(1)},...,E_{\pi(t)};\mathcal{L}(E_{\pi(t+1)},...,E_{\pi(n)};F)).$$

The next result is essentially due to Botelho [2].

Corollary 6. *If some E_j is an $\mathcal{L}_\infty$ space, at least one other E_k is infinite dimensional and $\dim F=\infty$, then, regardless of the $p\geq 1$, we have*

$$\mathcal{L}(E_1,...,E_n;F)\neq\mathcal{L}_{as(\frac{p}{n};p)}(E_1,...,E_n;F).$$

Proof. There is no loss of generality in assuming $j=1$. If the equality held we would have

$$\mathcal{L}(E_1;\mathcal{L}(E_2,...,E_n;F))=\mathcal{L}_{as(p;p)}(E_1;\mathcal{L}(E_2,...,E_n;F))$$

which is a contradiction since $\mathcal{L}(E_2,...,E_n;F)$ has only infinite cotype (see [2],[5]).

6. Extrapolation Theorems

The linear theory of absolutely summing operators has some strong coincidence Theorems (see [4]). Many of them have their polynomial versions (see [10],[15]). We will give a polynomial and a multilinear version for the Maurey Extrapolation Theorem:

Theorem 14 (Polynomial Extrapolation Theorem)**.** *If $1<r<p<\infty$ and X is a Banach space such that*

(6.1) $\mathcal{P}_{as(\frac{p}{n};p)}(^nX;l_p)=\mathcal{P}_{as(\frac{r}{n};r)}(^nX;l_p)$ *and* $\mathcal{L}_{as,p}(X;l_p)=\mathcal{L}_{as,r}(X;l_p)$

then, for every Banach space Y we have

(6.2) $\mathcal{P}_{as(\frac{p}{n};p)}(^nX;Y)=\mathcal{P}_{as(\frac{1}{n};1)}(^nX;Y)$ *and* $\mathcal{L}_{as,p}(X;Y)=\mathcal{L}_{as,1}(X;Y)$.

Proof. As in the linear case, using a localization argument, we can prove that

$$\mathcal{P}_{as(\frac{p}{n};p)}(^nX;l_p)=\mathcal{P}_{as(\frac{r}{n};r)}(^nX;l_p)$$

is equivalent to

$$\exists D>0;\|P\|_{as(\frac{r}{2};r)}\leq D\,\|P\|_{as(\frac{p}{2};p)}\ \ \forall P\in\mathcal{P}_{(\frac{p}{2};p)}(^nX;L_p).$$

We want to prove that for each Banach space Y there exists $C > 0$ such that whenever $Q \in \mathcal{P}_{as(\frac{p}{n};p)}(^nX;Y)$, we have $\|Q\|_{as(\frac{1}{n};1)} \leq C\,\|Q\|_{as(\frac{p}{n};p)}$ since it will lead us to

$$\mathcal{P}_{as(\frac{p}{n};p)}(^nX;Y) = \mathcal{P}_{as(\frac{1}{n};1)}(^nX;Y).$$

Consider

$$\varphi : X \to C(B_{X^*}) : x \mapsto f_x$$

where $f_x(x^*) =< x^*, x >$. We will denote $K = B_{X^*}$. Let us denote by $P(K)$ the set of all probability measures on K with the weak star topology. For each $\mu \in P(K)$ define

$$j_\mu : X \subset C(K) \to L_p(\mu)$$

as the restriction of the canonical inclusion from $C(K)$ into $L_p(\mu)$.

Let $R : X \to Y$ be an n-homogeneous $(\frac{p}{n};p)$-summing polynomial. The polynomial version of Grothendieck-Pietsch Domination Theorem tells us that there exists $\mu_0 \in P(K)$ such that

$$\|Rx\| \leq C[\int_K |< \varphi, x >|^p \, d\mu_0(\varphi)]^{\frac{n}{p}} = C[\int_K \mid j_{\mu_0}(x)(\varphi) \mid^p \, d\mu_0(\varphi)]^{\frac{n}{p}} =$$

$$= C\|j_{\mu_0}(x)\|^n_{L_p(\mu_0)} \quad \text{for every } x \text{ in } X.$$

We must find $\lambda \in P(K)$ and a constant D (depending on X) such that

$$\|j_{\mu_0}(x)\|_{L_p(\mu_0)} \leq D\|j_\lambda(x)\|_{L_1(\lambda)} \ \forall x \in X, \tag{6.3}$$

and then the Theorem will be proved. Indeed, we will have

$$\|Rx\| \leq C\|j_{\mu_0}(x)\|^n_{L_p(\mu_0)} \leq CD\|j_\lambda(x)\|^n_{L_1(\lambda)} =$$

$$= C_1[\int_K \mid j_\lambda(x)(x^*) \mid d\lambda(\varphi)]^n = C_1[\int_K \mid x^*(x) \mid d\lambda(\varphi)]^n$$

and the Grothendieck-Pietsch Polynomial Domination Theorem yields that R is $(\frac{1}{n};1)$-summing.

In order to prove (6.3) it is enough to note that the hypothesis $\mathcal{L}_{as,p}(X;l_p) = \mathcal{L}_{as,r}(X;l_p)$ is sufficient to prove it (this is done in the proof of the linear Extrapolation Theorem. See Th. 3.17 of [4]).■

For the multilinear version, it is not difficult to prove that

$$\mathcal{L}_{as(\frac{p}{n};p)}(^nX;l_p) = \mathcal{L}_{as(\frac{r}{n};r)}(^nX;l_p)$$

implies $\mathcal{L}_{as(\frac{p}{n};p)}(X;l_p) = \mathcal{L}_{as(\frac{r}{n};r)}(X;l_p)$. The same reasoning give us the following statement:

Theorem 15. *If $1 < r < p < \infty$ and X is a Banach space such that*

$$\mathcal{L}_{as(\frac{p}{n};p)}(^nX;l_p) = \mathcal{L}_{as(\frac{r}{n};r)}(^nX;l_p)$$

then, for every Banach space Y, we have

$$\mathcal{L}_{as(\frac{p}{n};p)}(^nX;Y) = \mathcal{L}_{as(\frac{1}{n};1)}(^nX;Y).$$

7. Absolutely summing mappings

The concept of absolutely summing mapping (non necessarily multilinear or polynomial) and the first results and examples are due to M.Matos [12].

Definition 5. *(Matos) A mapping $f : E \to F$ is absolutely $(s;r)$-summing at $a \in E$ if $(f(a+x_j) - f(a))_{j=1}^{\infty} \in l_s(F)$ whenever $(x_j)_{j=1}^{\infty} \in l_r^u(E)$. We say that $f : E \to F$ is weakly absolutely (s,r)-summing at $a \in E$ if $(f(a+x_j) - f(a))_{j=1}^{\infty} \in l_s^w(F)$ whenever $(x_j)_{j=1}^{\infty} \in l_r^u(E)$.*

Since for every $(x_j)_{j=1}^{\infty} \in l_r^u(E)$ we have $\lim_{m\to\infty} \left\| (x_j)_{j=m}^{\infty} \right\|_{w,r} = 0$, it is clear that $\lim_{m\to\infty} \|x_m\| = 0$. Therefore, there is no loss of generality if, in the definition above, we restrict ourselves to $(x_j)_{j=1}^{\infty} \in l_r^u(E)$ with $\|x_j\| < \delta$ for all j and some δ.

It is possible to prove that if $f : E \to F$ is absolutely $(s;r)$-summing at $a \in E$ then f is continuous at a [10]. The behavior of f outside an open neighborhood of a is completely irrelevant.

In [3], Botelho proves for the complex case, using Cauchy integral formulas, that, if $\cot E = q$, every holomorphic entire mapping $f : E \to F$ such that $f(0) = 0$ is $(q;1)$-summing at the origin. We will prove that Cauchy integral formulas are not essential and this result still holds for the real case and for non zero points.

Lemma 3. *If $g : E \to F$ is analytic at $a \in E$, then there exists $\delta > 0$ such that*

$$\|(g(a+x_j) - g(a))_{j=1}^{\infty}\|_{w,1} \leq D\|(x_j)_{j=1}^{\infty}\|_{w,1}$$

whenever $\|(x_j)_{j=1}^{\infty}\|_{w,p} < \delta$.

Proof. If $g : E \to F$ is analytic at a and $C, c > 0$ are such that

$$\|\frac{1}{k!}\overset{\wedge}{d^k} g(a)\| \leq Cc^k \text{ for every } k$$

then, for each $\varphi \in F‘$, we have

$$\|\frac{1}{k!}\overset{\wedge}{d^k}\varphi g(a)\| = \|\varphi\frac{1}{k!}\overset{\wedge}{d^k} g(a)\| \leq Cc^k\|\varphi\| \text{ for all } k$$

and hence, by a result of Defant and Voigt (see [10], Theorem 1.6 or [12]),

$$\|\frac{1}{k!}\overset{\wedge}{d^k}\varphi g(a)\|_{as(1;1)} \leq e^k Cc^k\|\varphi\|.$$

Let us denote by $\epsilon_a > 0$ the radius of convergence of g around a.Thus, if $\|(x_j)_{j=1}^{\infty}\|_{w,1} \leq \delta = \min\{\frac{1}{2ec}, \epsilon_a\}$ we can write

$$\sum_{j=1}^{\infty} \mid \varphi g(a+x_j) - \varphi g(a) \mid \leq \sum_{k=1}^{\infty} \|\frac{1}{k!}\overset{\wedge}{d^k}\varphi g(a)\|_{as(1;1)} \|(x_j)_{j=1}^{\infty}\|_{w,1}^{k} =$$

$$= \|(x_j)_{j=1}^{\infty}\|_{w,1} \sum_{k=1}^{\infty} \|\frac{1}{k!}\overset{\wedge}{d^k}\varphi g(a)\|_{as(1;1)} \|(x_j)_{j=1}^{\infty}\|_{w,1}^{k-1} \leq$$

$$\leq \|(x_j)_{j=1}^{\infty}\|_{w,1} \sum_{k=1}^{\infty} \frac{e^k C c^k \|\varphi\|}{(2ec)^{k-1}} \leq D\|(x_j)_{j=1}^{\infty}\|_{w,1}$$

for every $\varphi \in B_{F'}$. Hence

$$\|(g(a+x_j)-g(a))_{j=1}^{\infty}\|_{w,1} \leq D\|(x_j)_{j=1}^{\infty}\|_{w,1}$$

whenever $\|(x_j)_{j=1}^{\infty}\|_{w,p} < \delta$.■

Proposition 7. *If F has cotype q and $g : E \to F$ is analytic at $a \in E$, then g is $(q;1)$-summing at a.*

Proof. Let $a \in E$. Since g is analytic at a, there exists δ such that

$$\|(g(a+x_j)-g(a))_{j=1}^{\infty}\|_{w,1} \leq D\|(x_j)_{j=1}^{\infty}\|_{w,1}.$$

Let $(x_j)_{j=1}^{\infty} \in l_1^u(E)$ and let $j_0 \in \mathbb{N}$ be such that $\|(x_j)_{j=j_0}^{\infty}\|_{w,1} < \delta$. Then

$$(\sum_{j=j_0}^{\infty} \| g(a+x_j)-g(a) \|^q)^{1/q} \leq C_q(F) \| (g(a+x_j)-g(a))_{j=j_0}^{\infty} \|_{w,1}$$

$$\leq D\|(x_j)_{j=j_0}^{\infty}\|_{w,1}.$$

Obviously,

$$(\sum_{j=1}^{j_0-1} \| g(a+x_j)-g(a) \|^q)^{1/q} < \infty.$$

Hence

$$(\sum_{j=1}^{\infty} \| g(a+x_j)-g(a) \|^q)^{1/q} < \infty$$

whenever $(x_j)_{j=1}^{\infty} \in l_1^u(E)$.■

In the real case, a slight variation of the Proposition 7 can be made as we see below:

Proposition 8. *Let $f : E \to F$ be an application of class C^k at $a \in E$. If $\cot F \leq q$ and $\cot E \leq kq$, then f is $(q;1)$-summing at a.*

Proof. Recall that if f is an application of class C^k at a, by Taylor's formula there exists $B_\delta(a)$ such that

$$\|f(a+x)-f(a)\| \leq \|df(a)(x)+\frac{\overset{\wedge}{d^2} f(a)}{2!}(x)+\ldots+\frac{\overset{\wedge}{d^k} f(a)}{k!}(x)\|+\|x\|^k \;\; \forall x \in B_\delta(a).$$

It is clear that we can consider $(x_j)_{j=1}^{\infty} \in l_1^u(E)$ so that $x_j \in B_\delta(a)$ for every j. Then,

$$(\sum_{j=1}^{m}\|f(a+x_j)-f(a)\|^q)^{1/q} \leq$$

$$\leq [\sum_{j=1}^{m}(\|df(a)(x_j)+\frac{\overset{\wedge}{d^2} f(a)}{2!}(x_j)+\ldots+\frac{\overset{\wedge}{d^k} f(a)}{k!}(x_j)\|+\|x_j\|^k)^q]^{1/q}.$$

Thus

$$(\sum_{j=1}^{m}\|f(a+x_j)-f(a)\|^q)^{1/q}\leq$$

$$\leq[\sum_{j=1}^{m}\|df(a)(x_j)+\frac{\overset{\wedge}{d^2}f(a)}{2!}(x_j)+...+\frac{\overset{\wedge}{d^k}f(a)}{k!}(x_j)\|^q]^{1/q}+[\sum_{j=1}^{m}(\|x_j\|^k)^q]^{1/q}.$$

Since $\cot E\leq kq$ and since $df(a),...,\overset{\wedge}{d^k}f(a)$ are $(q;1)$-summing, the proof is done.■

It is not difficult to achieve the following result:

Theorem 16. *If F has cotype q, X is an $\mathcal{L}_{\infty,\lambda}$ space and $f:X\rightarrow F$ is analytic at a, then f is absolutely $(q;2)$-summing at a.*

Proof.

There are $C\geq 0$ and $c>0$ such that

$$\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|\leq Cc^k \text{ for every } k.$$

Thus, by (3.1) in Theorem 7 we have

(7.1)

$$\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|_{as(q;2)}\leq C_q(F)K_G3^{\frac{k-2}{2}}\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|\lambda^k\leq\ C_q(F)K_G3^{\frac{k-2}{2}}Cc^k\lambda^k.$$

for every $k\geq 2$.

For $k=1$ we know that every linear mapping from X into F is $(q;2)$-summing and it is enough, in addiction with (7.1), to obtain positive C_1 and c_1 so that

$$\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|_{as(q;2)}\leq C_1c_1^k \text{ for every } k\geq 1.$$

If δ_a is the radius of convergence of f around a, then, whenever $(x_j)_{j=1}^{\infty}$ is such that $\|(x_j)_{j=1}^{\infty}\|_{w,1}\leq\min\{\frac{1}{2c_1},\delta_a\}$, we have

$$(\sum_{j=1}^{\infty}\|f(a+x_j)-f(a)\|^q)^{\frac{1}{q}}=\sum_{j=1}^{\infty}(\|\sum_{k=1}^{\infty}\frac{1}{k!}\overset{\wedge}{d^k}f(a)(x_j)\|^q)^{\frac{1}{q}}\leq$$

$$\leq\sum_{k=1}^{\infty}[\sum_{j=1}^{\infty}\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)(x_j)\|^q]^{\frac{1}{q}}\leq$$

$$\leq\sum_{k=1}^{\infty}\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|_{as(q;2)}\|(x_j)_{j=1}^{\infty}\|_{w,2}^k=$$

$$=\|(x_j)_{j=1}^{\infty}\|_{w,2}\sum_{k=1}^{\infty}\|\frac{1}{k!}\overset{\wedge}{d^k}f(a)\|_{as(q;2)}\|(x_j)_{j=1}^{\infty}\|_{w,2}^{k-1}\leq$$

$$\leq C_1 \|(x_j)_{j=1}^{\infty}\|_{w,2} \sum_{k=1}^{\infty} \frac{c_1^k}{2^{k-1} c_1^{k-1}} = C_2 \|(x_j)_{j=1}^{\infty}\|_{w,2}.$$

Remark 2. *Theorem 6 could induce us to postulate that every mapping, analytic at a, from an $\mathcal{L}_\infty$ space into $\mathbb{K}$ would be $(1;2)$-summing. However, it is not true since the only absolutely $(1;2)$-summing linear mapping is the trivial mapping.*

The next example follows the same line of thought of Lemma 3 and Theorem 16:

Example 3. *If X is an $\mathcal{L}_\infty$ space and $f : X \to \mathbb{K}$ is a mapping, analytic at a, so that $df(a) = 0$, then f is $(1;2)$-summing at a.*

The reader must note that the same reasoning of Theorem 16 lead us to the following useful Theorem:

Theorem 17. *If the mapping $f : E \to F$ is analytic at $a \in E$ and there are $C > 0$ and $c > 0$ such that for each natural n,*

$$\hat{d}^n f(a) \in \mathcal{P}_{as(s;r)}(^n E; F) \tag{7.2}$$

and

$$\|\frac{1}{n!}\hat{d}^n f(a)\|_{as(s;r)} \leq Cc^n, \tag{7.3}$$

then f is $(s;r)$-summing at a.

Remark 3. *For entire holomorphic mappings we have a completer result, due to Matos* [12].

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