MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR RESONANT ELLIPTIC PROBLEMS

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Abstract. In this paper we establish the existence of multiple solutions for the semilinear elliptic problem

$$\begin{array}{lll} -\Delta u = g(x,u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(x,0) = 0$ and which is asymptotically linear at infinity. We considered both cases, resonant and nonresonant. We use critical groups to distinguish the critical points.

1. Introduction

Let us consider the problem

$$\begin{array}{lll} -\Delta u = g(x,u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a open bounded domain with smooth boundary $\partial\Omega$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $g(x,0) = 0$, which implies that (1) possesses the trivial solution $u = 0$. We will be interested in nontrivial solutions. Assume that

$$\limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \ell, \quad \ell \in \mathbb{R}.$$

The classical solutions of the problem (1) correspond to critical points of the functional F defined on $H = H_0^1(\Omega)$, by

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \quad u \in H, \tag{2}$$

where $G(x,t) = \int_0^t g(x,s)ds$. Under the above assumptions $F \in C^2$.

Denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots$ the eigenvalues of $(-\Delta, H_0^1)$. We write $a(x) \not\leq b(x)$ to indicate that $a(x) \leq b(x)$ with strict inequality holding on a set of positive measure.

1991 *Mathematics Subject Classification.* 35J65.

Key words and phrases. Cerami condition, multiplicity of solution, double resonance, sign changing solution.

The author was supported by CAPES/Brazil.

We first assume the followings hypotheses on g.

$$\begin{cases} \frac{g(x,t)}{t} \text{ is strictly increasing with respect to } t \geq 0, \text{ a.e. in } \Omega, \quad \text{and} \\ \frac{g(x,t)}{t} \text{ is strictly decreasing with respect to } t \leq 0, \text{ a.e. in } \Omega. \end{cases} \tag{3}$$

$$\lambda_j \not\leq L(x) = \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} = K(x) \not\leq \lambda_{j+1}, \quad \text{uniformly in } \Omega. \tag{4}$$

$$\lambda_j \not\leq L(x) \quad \text{and} \quad \lim_{|t|\to\infty} [tg(x,t) - 2G(x,t)] = \infty, \quad \text{a.e. } x \in \Omega. \tag{5}$$

Theorem 1.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x,0) = 0$, which satisfies* (3)*. Suppose that there exist $k \geq 2$ and $m \geq 1$ such that*

$$\lambda_{k-1} \leq g'(x,0) < \lambda_k \leq \lambda_{k+m} \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{k+m+1},$$

where the limits are uniform for x in Ω. If either (4) *or* (5) *hold with $j = k + m$. Then problem* (1) *has at least two nontrivial solutions.*

Theorem 1.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x,0) = 0$. Suppose that there exists $m \geq 1$ such that*

$$g'(x,0) < \lambda_1 < \lambda_{m+1} \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{m+2},$$

where the limits are uniform for x in Ω. Suppose that either (4) *or* (5) *hold with $j = m+1$. Moreover if* (5) *hold, assume that there exists $C(x) \in L^1(\Omega)$ such that $tg(x,t) - 2G(x,t) \geq C(x)$ $\forall\, t \in \mathbb{R}$, a.e. $x \in \Omega$. Then problem* (1) *has at least three nontrivial solutions.*

Theorem 1.3. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x,0) = 0$, which satisfies* (3)*. Suppose that there exists $m \geq 2$ such that*

$$g'(x,0) < \lambda_1 < \lambda_{m+1} \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{m+2},$$

where the limits are uniform for x in Ω. If either (4) *or* (5) *hold with $j = m + 1$. Then problem* (1) *has at least four nontrivial solutions, one of those changing sign, another one positive and a third one negative.*

Theorem 1.4. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(x,0) = 0$, which satisfies* (3)*. If there exists $k \geq 2$ such that*

$$\lambda_{k-1} \leq g'(x,0) < \lambda_k < \lim_{t\to\pm\infty} \frac{g(x,t)}{t} < \lambda_{k+1},$$

where the limits are uniform for x in Ω. If 0 *is an isolated critical point then problem* (1) *has exactly two nontrivial solutions.*

Remark 1.1. We call (1) a resonant or double resonant problem when it happens, respectively, that

$$\lim_{|t|\to\infty} \frac{g(x,t)}{t} = \lambda_j,$$

$$\lambda_j \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{j+1},$$

for some $j \geq 1$, uniform for a.e. $x \in \Omega$ (cf. [2, 5]). Multiplicity for double resonant problems were treated by recent papers [23, 24, 25]. In [23], the author considered only the autonomous case and assume strong resonant hypotheses. Theorem 1.5 below, gives a example of a function that satisfies the hypotheses of Theorem 1.1 and does not satisfy the hypotheses in [24, 25]. Under the conditions of Theorem 1.2, but assuming resonance only at one eigenvalue, Dancer & Zhang [13] proved that problem (1) has at least one sign-changing solution, one positive solution, and one negative solution.

Now consider the autonomous problem

$$\begin{aligned} -\Delta u = g(u) \quad & \text{in} \quad \Omega \\ u = 0 \qquad & \text{on} \quad \partial\Omega, \end{aligned} \tag{6}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(0) = 0$. Castro & Lazer [6] and Ambrosetti & Mancini [1] proved that if $g \in C^2$, $tg''(t) > 0$ a.e. in $\mathbb{R}$, and

$$\lambda_{k-1} < g'(0) < \lambda_k < \lim_{t\to\pm\infty} g'(t) < \lambda_{k+1}$$

for some $k \geq 1$, then (6) has exactly two nontrivial solutions. In Mizoguchi [19], it was shown that if $g \in C^2$, $tg''(t) > 0$ a.e. in $\mathbb{R}$, and

$$\lambda_{k-1} \leq g'(0) < \lambda_k \leq \lambda_{k+1} < \lim_{t\to\pm\infty} g'(t) < \lambda_{k+2},$$

then there exist at least two nontrivial solutions of the problem (6). Our next result extends the previous results in this autonomous case.

Theorem 1.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(0) = 0$, which satisfies*

$$\begin{cases} g(t) \text{ is convex if } t \geq 0, & \text{and} \\ g(t) \text{ is concave if } t \leq 0. \end{cases}$$

If there exist $k \geq 2$ and $m \geq 1$ such that

$$\lambda_{k-1} \leq g'(0) < \lambda_k \leq \lambda_{k+1} \leq \cdots \leq \lambda_{k+m} < \ell_\pm = \lim_{t\to\pm\infty} g'(t) \leq \lambda_{k+m+1},$$

then problem (6) *has at least two nontrivial solutions.*

In fact, the above hypothesis on the convexity of g implies that (see Proposition 3.1)

$$\lim_{|t|\to\infty} [tg(t) - 2G(t)] = \infty.$$

Hence the previous theorem is corollary of Theorem 1.1.

Remark 1.2. In [4], Bartsch, Chang & Wang showed that if $g'(t) > g(t)/t \ \forall \ t \neq 0$ and

$$g'(0) < \lambda_1 < \lambda_2 \leq \lambda_k < \lim_{|t|\to\infty} g'(t) < \lambda_{k+1}, \ (k > 2),$$

then problem (6) has at least four nontrivial solution, two of these solutions change sign, one is positive and another one is negative. They observe that the nonresonance at infinity in the above result can be removed using arguments like in [3] and [9]. In [9], the author assumes a Ladesmam-Lazer condition and that $|g(t) - \lambda_{k+1}t|$ is bounded for all $t \in \mathbb{R}$. In [3], the authors suppose that $|g(t) - \lambda_{k+1}t| \leq c(|t|^r + 1)$, for some $r \in (0,1)$, with the purpose of computing the critical groups at infinity. The next result is a corollary of Theorem 1.3.

Theorem 1.6. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, $g(0) = 0$, which satisfies*

$$\begin{cases} g(t) \ \textit{is convex if} \ t \geq 0, & \textit{and} \\ g(t) \ \textit{is concave if} \ t \leq 0. \end{cases}$$

Suppose that there exist $k > 2$ such that

$$g'(0) < \lambda_1 < \lambda_2 \leq \lambda_k < \lim_{t\to\pm\infty} \frac{g(t)}{t} = \lambda_{k+1}.$$

Then problem (6) *has at least four nontrivial solutions, one of those change sign, one is positive and another one is negative.*

Remark 1.3. The functional in the nonresonant case satisfies the Palais-Smale Condition, (PS) in short, and the difficulty in the resonant case is the lack of a (PS) condition. But if the function g satisfies

$$\lim_{|t|\to\infty} [tg(x,t) - 2G(x,t)] = \infty, \quad \text{uniformly in } \Omega,$$

then in [11], Costa & Magalhães showed that this condition is sufficient to obtain a weak version of the (PS) condition, namely the (C) condition, which was introduced by Cerami in [7]. The (C) condition was used by Bartolo, Benci & Fortunato in [2] to prove a general minimax theorem (see [20] for this results with the (PS) condition). The so called Second Deformation Lemma, proved by Chang (see [9]), has a version with the Cerami condition replacing the usual (PS) condition, as proved by Silva & Teixeira in [21].

In Section 2, we collect some results on Morse Theory, with the functional satisfying the Cerami condition. In Section 3, we prove some lemmas about the geometry of the functional and a compactness condition. In Section 4 we prove the main theorems.

2. Remarks on Critical Point Theory

In this section some classical definitions and results in Morse Theory are recalled. These results will be used in the proofs of main theorems. In [9] the (PS) condition is used, whereas we use here a weaker compactness condition on the functionals. This results can be found in [18] with others hypotheses.

Let H be a Hilbert space and $f : H \to \mathbb{R}$ be a functional of class C^1. Denote the set of critical points of f by K. Given $c \in \mathbb{R}$, we set $f_c = \{x \in H : f(x) \leq c\}$ and $K_c = f^{-1}(c) \cap K$.

Definition 2.1. *Given $f \in C^1(H, \mathbb{R})$ and $c \in \mathbb{R}$, we say that f satisfies the Cerami condition at level $c \in \mathbb{R}$, denoted by $(C)_c$, if every sequence $\{x_n\} \subset H$ satisfying*

$$f(x_n) \to c \quad \text{and} \quad (1 + ||x_n||)||f'(x_n)|| \to 0, \quad n \to \infty,$$

has a converging subsequence. If f satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$, we say that it satisfies the (C) condition.

It is clear that a functional satisfying the (PS) condition also satisfies the (C) condition.

Definition 2.2. *$f \in C^1(H, \mathbb{R})$ is said to possess the deformation property if it satisfies the following condition*
(i) for every $a < b$ such that $K \cap f^{-1}(a, b) = \emptyset$, then f_a is a strong deformation retract of $f_b \setminus K_b$.

The next result is a version of a deformation lemma (for references see [9]) proved in [22] (see also [21]).

Proposition 2.1 (Deformation Lemma)**.** *Suppose that $f \in C^1(H, \mathbb{R})$ satisfies the (C) condition and a is the only possible critical value of f in the interval $[a, b)$. Assume that the connected components of K_a are only isolated points. Then, f_a is a strong deformation retract of $f_b \setminus K_b$.*

This lemma is an important tool in Critical Point Theory. Now we state some known results, which are also true under the (C) condition instead of the usual (PS). For proofs of these results assuming (PS) see [9]. Where further references can be found. These proofs can be early adapted for the case when (C) condition is assumed.

Let $Y \subset X$ be topological spaces, denote by $H_*(X, Y)$ the singular relative homology groups with coefficients in $\mathbb{Z}$.

Definition 2.3. *Let x_0 be an isolated critical point of f, and let $c = f(x_0)$. We call*

$$C_p(f, x_0) = H_p(f_c \cap U_{x_0}, (f_c \setminus \{x_0\}) \cap U_{x_0})$$

the p^{th} critical group of f at x_0, $p = 0, 1, 2, \ldots$, where U_{x_0} is a neighborhood of x_0 such that $K \cap (f_c \cap U_{x_0}) = \{x_0\}$.

Theorem 2.1. *Assume that $\alpha \in H_j(f_b, f_a)$ is nontrivial, and*

$$c = \inf_{\tau \in \alpha} \sup_{x \in |\tau|} f(x). \tag{7}$$

Suppose that f possesses the deformation property. Then there exists $x_0 \in K_c$ such that $C_j(f, x_0) \neq 0$.

Definition 2.4. *Let D be a j-topological ball in H, and S be a subset in H. We say that ∂D and S homologically link, if $\partial D \cap S = \emptyset$ and $|\tau| \cap S \neq \emptyset$, for each singular j chain τ with $\partial \tau = \partial D$ where $|\tau|$ is the support of τ.*

The following proposition provides examples of sets homologically linking, their proofs are a consequence of Example 2, 3 and Theorem 1.2 in Chapter II of [9].

Proposition 2.2. *Let H_1 and H_2 be two closed subspaces of a Hilbert space H. Suppose that*

$$H = H_1 \oplus H_2, \quad \dim H_1 < \infty.$$

Then, if $D_1 = B_R \cap H_1$ and $S_1 = H_2$, ∂D_1 and S_1 homologically link.

Proposition 2.3. *Let H_1 and H_2 be two closed subspaces of a Hilbert space H. Suppose that*

$$H = H_1 \oplus H_2, \quad \dim H_1 < \infty.$$

Let $e \in H_2$, $||e|| = 1$, and $R,\ r,\ \rho > 0$ with $\rho < R$. Set $D_2 = \{x + se\ ;\ x \in H_1 \cap B_r,\ s \in [0, R]\}$ and $S_2 = H_2 \cap \partial B_\rho$. Then ∂D_2 and S_2 homologically link.

Theorem 2.2 (Theorem 1.1', Chapter II, [9]). *Assume that ∂D and S homologically link, where D is a j-topological ball. If $f \in C(H, \mathbb{R}^1)$ satisfies*

$$f(x) > a \quad \forall x \in S \quad \text{and} \quad f(x) \le a \quad \forall\ x \in \partial D,$$

then $H_j(f_b, f_a) \neq 0$ for $b > \mathrm{Max}\{f(x)\ ;\ x \in D\}$.

We intend to compute the critical groups of an isolated critical point. For this purpose we present the Shifting Theorem. First, consider the Splitting Theorem

Theorem 2.3 (Splitting Theorem). *Suppose that U is a neighborhood of x_0 in a Hilbert space H and that $f \in C^2(U, \mathbb{R})$. Assume that x_0 is the only critical point of f and that $A = d^2 f(x_0)$ with kernel N. If 0 is either an isolated point of the spectrum $\sigma(A)$ or not in $\sigma(A)$, then there exists a ball B_δ, $\delta > 0$, centered at 0, an ordering-preserving local homomorphism ϕ defined on B_δ, and a C^1 mapping $h : B_\delta \cap N \to N^\perp$ such that*

$$f \circ \phi(z + y) = \frac{1}{2}(Az, z) + f(h(y) + y), \quad \forall x \in B_\delta,$$

where $y = P_N x$, $z = P_{N^\perp} x$, and P_N is the orthogonal projection onto the subspace N.

We call $\mathcal{N} = \phi(U \cap N)$. The following theorem sets up the relationship between the critical points of f and those of $\tilde{f} := f|_{\mathcal{N}}$. It is proved in [9].

Theorem 2.4 (Shifting Theorem). *Suppose the hypotheses of the Splitting Theorem. Assume that the Morse index of f at x_0 is μ, then we have*

$$C_p(f, x_0) = C_{p-\mu}(\tilde{f}, x_0), \quad p = 0, 1, \ldots.$$

In addition, if $d^2 f(x_0)$ has finite dimensional kernel, then we have

Corollary 2.1. *Suppose that N is finite dimensional with dimension ν and x_0 is*

(i) *a local minimum of $\tilde{f}$, then*

$$C_p(f, x_0) = \delta_{p\mu}\mathbb{Z},$$

(ii) *a local maximum of* $\tilde{f}$, *then*

$$C_p(f, x_0) = \delta_{p(\mu+\nu)}\mathbb{Z},$$

(iii) *neither a local maximum nor a local minimum of* $\tilde{f}$, *then*

$$C_p(f, x_0) = 0 \quad \text{for} \quad p \leq \mu, \quad \text{and} \quad p \geq \mu + \nu.$$

3. Preliminary Lemmas

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $g(x, 0) = 0$. Suppose that there exist $k \geq 2$ and $m \geq 1$ such that

$$\begin{gathered} \lambda_{k-1} \leq g'(x, 0) < \lambda_k \\ \lambda_{k+m} \nleq L(x) = \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{k+m+1}, \end{gathered} \tag{8}$$

where the limits are uniform for x in Ω.

Let $H = H_0^1(\Omega)$ and denote the norms in $H_0^1(\Omega)$ and $L^2(\Omega)$ by $||\cdot||$ and $|\cdot|_2$, respectively. Let H_1, H_2 and H_3 be the subspaces of H spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_1, \ldots, \lambda_{k-1}\}$, $\{\lambda_k, \ldots, \lambda_{k+m}\}$ and $\{\lambda_{k+m+1}, \ldots\}$, respectively.

The next result is similar to Lemma 1 in [19]. The proof given here is a variation of the one found in [19]. Let F be defined as in (2).

Lemma 3.1. *Under the assumptions above and the hypothesis* (3), *the following statements hold:*

(i) *There are* $r > 0$ *and* $a > 0$ *such that* $F(u) \geq a$ *for all* $u \in H_2 \oplus H_3$ *with* $||u|| = r$;
(ii) $F(u) \to -\infty$, *as* $||u|| \to \infty$, *for* $u \in H_1 \oplus H_2$;
(iii) $F(u) \geq 0$ *for all* $u \in H_3$; *and*
(iv) $F(u) \leq 0$ *for all* $u \in H_1$.

Proof. By (3) and (8), we can take the positive numbers α, δ satisfying that

$$\lambda_{k-1} \leq \frac{g(x,t)}{t} \leq \alpha < \lambda_k$$

for all $t \in \mathbb{R}$ with $|t| \leq \delta$. Moreover, there exists $\ell \in \mathbb{R}$ such that $\frac{g(x,t)}{t} < \ell$ for all $t \in \mathbb{R}$ from (3) and (8). Let $H^\ell = \ker(-\Delta - \ell I)$. Then $H_2 \oplus H_3 = V \oplus W$, where $V = H_2 \oplus H^\ell$. For $u \in H_2 \oplus H_3$, put $u = v + w$, $v \in V$ and $w \in W$. Since V is spanned by a finite number of eigenfunctions which are L^∞-functions. Then there exists $r > 0$ such that

$$\sup_{x\in\Omega}|v(x)| \leq \frac{\gamma - \ell}{\gamma - \alpha} \cdot \delta$$

if $||v|| \leq r$, where $\gamma > \ell$ and $|\nabla w|_2^2 \geq \gamma |w|_2^2$ for all $w \in W$. Suppose that $||v|| \leq r$. If $|v(x) + w(x)| \leq \delta$, then

$$\begin{aligned}\frac{1}{2}\lambda_k|v|^2 &+ \frac{1}{4}\gamma|w|^2 - G(x,v+w)\\ &\geq \frac{1}{2}\lambda_k|v|^2 + \frac{1}{4}\gamma|w|^2 - \frac{1}{2}\alpha(v+w)^2\\ &= \frac{1}{2}\lambda_k|v|^2 + \frac{1}{4}\gamma|w|^2 - \frac{1}{2}\alpha v^2 - \frac{1}{2}\alpha w^2 - \alpha vw\\ &= -\frac{1}{4}\alpha|w|^2 + \frac{1}{4}(\gamma-\alpha)w^2 + \frac{1}{2}(\lambda_k-\alpha)v^2 - \alpha vw\\ &\geq -\frac{1}{4}\alpha|w|^2 + \frac{1}{2}(\lambda_k-\alpha)v^2 - \alpha vw\\ &\geq -\frac{1}{4}\ell|w|^2 + \frac{1}{2}(\lambda_k-\alpha)v^2 - \alpha vw.\end{aligned}$$

If $|v(x)+w(x)| > \delta$, we have

$$|G(x,v+w)| \leq \frac{1}{2}\ell(v+w)^2 - \frac{1}{2}(\ell-\alpha)\delta^2$$

and hence

$$\begin{aligned}\frac{1}{2}\lambda_k|v|^2 &+ \frac{1}{4}\gamma|w|^2 - G(x,v+w)\\ &\geq \frac{1}{2}\lambda_k|v|^2 + \frac{1}{4}\gamma|w|^2 - \frac{1}{2}\ell(v+w)^2 + \frac{1}{2}(\ell-\alpha)\delta^2\\ &= \frac{1}{2}\lambda_k|v|^2 + \frac{1}{4}\gamma|w|^2 - \frac{1}{2}\ell|v|^2 - \frac{1}{2}\ell|w|^2 - \ell vw + \frac{1}{2}(\ell-\alpha)\delta^2\\ &= -\frac{1}{4}\ell|w|^2 + \frac{1}{2}(\lambda_k-\alpha)|v|^2 - \alpha vw\\ &+ \frac{1}{4}(\gamma-\ell)|w|^2 + (\alpha-\ell)vw + \frac{1}{2}(\alpha-\ell)|v|^2 + \frac{1}{2}(\ell-\alpha)\delta^2\\ &\geq -\frac{1}{4}\ell|w|^2 + \frac{1}{2}(\lambda_k-\alpha)|v|^2 - \alpha vw,\end{aligned}$$

(in order to see the last inequality, consider

$$\frac{1}{4}(\gamma-\ell)|w|^2 + (\alpha-\ell)vw + \frac{1}{2}(\alpha-\ell)|v|^2 + \frac{1}{2}(\ell-\alpha)\delta^2$$

as a quadratic form in w and prove that it is positively defined).
Therefore, we obtain

$$\begin{aligned}F(u) &= \frac{1}{2}||v+w||^2 - \int_\Omega G(x,u)dx\\ &\geq \frac{1}{4}||w||^2 - \frac{1}{4}\ell|w|_2^2 + \frac{1}{2}(\lambda_k-\alpha)|v|_2^2\\ &\geq \min\Big\{\frac{1}{4}\Big(1-\frac{\ell}{\gamma}\Big), \frac{\lambda_k-\alpha}{2\ell}\Big\}||u||^2.\end{aligned}$$

This implies statement (i). By the hypothesis $\lambda_{k+m} \not\leq L(x)$, the Proposition 2 in [12] states that there exists $\delta_1 > 0$ such that

$$||u||^2 - \int_\Omega L(x)u^2 dx \leq -\delta_1 ||u||^2 \quad \forall\, u \in H_1 \oplus H_2.$$

To show (ii), let δ_1 be given above and $\epsilon > 0$ be such that $\epsilon < \lambda_1\delta_1$. By the definition of $L(x)$, there exists $M = M(\epsilon)$ such that

$$2F(x,t) \geq (L(x) - \epsilon)t^2 - M \quad \forall\, t \in \mathbb{R}, \quad a.e. \;\; x \in \Omega.$$

Therefore, for $u \in H_1 \oplus H_2$ we have

$$\begin{aligned} 2F(u) &\leq ||u||^2 - \int_\Omega L(x)u^2 dx + \epsilon|u|_2^2 + M|\Omega| \\ &\leq \Big(-\delta_1 + \frac{\epsilon}{\lambda_1}\Big)||u||^2 + M|\Omega| \to -\infty, \quad \text{as} \;\; ||u|| \to \infty, \end{aligned}$$

since $\epsilon/\lambda_1 - \delta_1 < 0$. We prove (iii) and (iv) by straightforward calculations. □

Let u_0 be a critical point of F, defined by (2). The Morse index $\mu(u_0)$ of u_0 measures the dimension of the maximal subspace of $H = H_0^1(\Omega)$ on which $F''(u_0)$ is negative definite. We denote the dimension of the kernel of $F''(u_0)$ by $\nu(u_0)$. The next lemma evaluates $\nu(u_0)$ for a nonzero critical point of F. Similar ideas used in the proof below can be seen in [19] and [6].

Lemma 3.2. *Under the hypotheses of Lemma 3.1, $\nu(u_0) \leq m$ provided that u_0 is a nonzero critical point of F defined in* (2).

Proof. Let u_0 be a nonzero critical point of F, that is, a nontrivial weak solution of problem (6). We denote $g(x, u_0) = g(u_0)$ and $g'(x, u_0) = g'(u_0)$. Note that $F''(u_0)u = 0$ if and only if

$$\begin{aligned} -\Delta u = g'(u_0)u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

From $g(x, 0) = 0$, the problem (6) can be rewritten in the form

$$\begin{aligned} -\Delta u = q(x)u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}$$

where $q(x) = g(u_0)/u_0$ if $u_0(x) \neq 0$ and $q(x) = g'(0)$ if $u_0(x) = 0$. It is a standard result that u_0 is a classical solution of (6). Then u_0 cannot vanish identically on every open subset of Ω, by the unique continuation property (see [15]). Let $\alpha_1 < \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$ and $\beta_1 < \beta_2 \leq \cdots \leq \beta_n \leq \cdots$ be eigenvalues of the problems

$$\begin{aligned} -\Delta u = \alpha q(x)u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{9}$$

and

$$\begin{aligned} -\Delta u = \beta g'(u_0)u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{10}$$

respectively. Let $\{\psi_n\}$ and $\{\phi_n\}$ denote the corresponding eigenfunctions of problems (9) and (10) satisfying

$$\int_\Omega q\psi_n\psi_m dx = \delta_{nm}$$

and

$$\int_\Omega g'(u_0)\phi_n\phi_m dx = \delta_{nm}$$

for all $n, m \in \mathbb{N}$.

Claim: $\beta_n < \alpha_n$, for all $n \in \mathbb{N}$.

In fact, by (3) we have $g'(u_0(x)) \geq \frac{g(u_0(x))}{u_0(x)}$ and again by the unique continuation property

$$m\Big(\Big\{x \in \Omega \ ; \ g'(u_0(x)) > \frac{g(u_0(x))}{u_0(x)}\Big\}\Big) > 0.$$

Then we use Proposition 1.12 A in [14], and so the *claim* is proved.
Next, suppose that $\{\nu_n\}$ and $\{\delta_n\}$ denote the eigenvalues of the problems

$$\begin{array}{lll} -\Delta u = \nu\lambda_{k+m+1}u & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array} \tag{11}$$

and

$$\begin{array}{lll} -\Delta u = \delta\lambda_{k-1}u & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array} \tag{12}$$

respectively. Immediately, this implies $\nu_n = \lambda_n/\lambda_{k+m+1}$ and $\delta_n = \lambda_n/\lambda_{k-1}$. By (3) and (4), we have

$$\lambda_{k-1} < \frac{g(u_0(x))}{u_0(x)} \leq g'(u_0(x)) < \lambda_{k+m+1}$$

for all $x \in \Omega$ such that $u_0(x) \neq 0$. By a method similar to the proof of $\beta_n < \alpha_n$, we obtain

$$\alpha_{k-1} < \delta_{k-1} = 1, \quad 1 = \nu_{k+m+1} < \alpha_{k+m+1} \quad \text{and} \quad 1 = \nu_{k+m+1} < \beta_{k+m+1}.$$

From $u_0 \neq 0$, 1 is an eigenvalue of (9). Therefore, it holds that $\alpha_k = 1$, or $\alpha_{k+1} = 1$, ..., or $\alpha_{k+m} = 1$.
If $\alpha_{k+m} = 1$, the fact

$$\beta_{k+m} < \alpha_{k+m} = 1 = \nu_{k+m+1} < \beta_{k+m+1}$$

implies that 1 is not an eigenvalue of (10), i.e. $\nu(u_0) = 0$.
If $\alpha_{k+m-1} = 1$, the fact

$$\beta_{k+m-1} < \alpha_{k+m-1} = 1 = \nu_{k+m+1} < \beta_{k+m+1}$$

implies that $\nu(u_0) \leq 1$.
Analogously, if $\alpha_{k+m-2} = 1$ then $\nu(u_0) \leq 2$, ..., if $\alpha_k = 1$ then $\nu(u_0) \leq m$.
This completes the proof of the lemma. $\square$

Now we observe a compactness condition for the functional F defined by (2), in the resonant case.

Consider $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and $G(x,t) = \int_0^t g(x,s)ds$ such that

$$\lambda_j \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{j+1}, \quad \text{uniformly in } \Omega\,; \tag{13}$$

there exists $C(x) \in L^1(\Omega)$ such that

$$tg(x,t) - 2G(x,t) \geq C(x), \quad \forall\, t \in \mathbb{R}, \text{ a.e } x \in \Omega\,; \tag{14}$$

and

$$\lim_{|t|\to\infty} [tg(x,t) - 2G(x,t)] = \infty, \quad \text{a.e } x \in \Omega. \tag{15}$$

In [11] it was shown that the assumptions (13), (14) and (15) are enough to prove that functional the F, defined by (2), satisfies the Cerami condition (see [16]). Note that the hypothesis (3) implies (14) with $C(x) = 0$.

In order to prove that Theorems 1.5 and 1.6 follow from Theorems 1.1 and 1.3, respectively, we have to prove that the function g satisfies (15).

Proposition 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonlinear function of class C^1, $g(0) = 0$, which satisfies*

$$\begin{cases} g(t) \text{ is convex if } t \geq 0 \quad \text{and} \\ g(t) \text{ is concave if } t \leq 0. \end{cases}$$

Moreover, assume that $g(t)/t$ is bounded. Then

$$\lim_{|t|\to\infty} [tg(t) - 2G(t)] = \infty. \tag{16}$$

Proof. Fix $t > 0$, and note that

$$\frac{1}{2}[tg(t) - 2G(t)] = \int_0^t \Big(\frac{g(t)}{t}s - g(s)\Big)ds$$

The convexity of g gives that $(g(t)/t)s > g(s)$ for $s \in (0,t)$. Denote by A_t the region of the plane between the line $s \mapsto (g(t)/t)s$ and $s \mapsto g(s)$ in $(0,t)$. Let $s(t) \in (0,t)$ defined by

$$\frac{g(t)}{t}s(t) - g(s(t)) = \max_{s\in(0,t)} \Big(\frac{g(t)}{t}s - g(s)\Big),$$

and the triangle $\triangle_t$ with vertices $(0,0)$, $(s(t), g(s(t)))$ and $(t, g(t))$. We have $\triangle_t \subset A_t$ by convexity of g, hence

$$|\triangle_t| \leq \frac{1}{2}[tg(t) - 2G(t)].$$

Therefore the Proposition follows of

Claim: $|\triangle_t| \to \infty$, as $t \to \infty$.

In fact, the height of $\triangle_t$, with reference to base $b_t = [(0,0),(t,g(t))]$, is

$$h(t) = \Big[\frac{g(t)}{t}s(t) - g(s(t))\Big]\cos\Big(\arctan\Big(\frac{g(t)}{t}\Big)\Big).$$

Hence $\liminf_{t\to\infty} h(t) > 0$, since $g(t)/t$ is bounded; and $b_t \to \infty$ as $t \to \infty$. The claim is proved. The argument with $t < 0$ is entirely similar and the proof of proposition is complete. □

Lemma 3.3. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(x,t)/t$ is nondecreasing for $t > 0$, $g(x,t) = 0$ for all $t \le 0$, and*

$$\lambda_j \le L(x) = \lim_{t\to\infty} \frac{g(x,t)}{t} \le \lambda_{j+1}, \quad j \ge 2. \tag{17}$$

Then the C^{2-0}-functional $F_+ : H_0^1 \to \mathbb{R}$ defined by

$$F_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,t)dx,$$

satisfies the (PS) condition.

Proof. Let $\{u_n\} \in H_0^1$ be a sequence such that $\{F_+(u_n)\}$ is bounded, and $||F_+'(u_n)|| \to 0$ as $n \to \infty$. It follows that for all $\varphi \in H_0^1$ we have

$$< F_+'(u_n), \varphi >= \int_\Omega \nabla u \nabla \varphi - \int_\Omega g(x,u_n)\varphi dx \to 0, \quad as\ n \to \infty. \tag{18}$$

Set $\varphi = u_n$; we have

$$||u_n||^2 \le \int_\Omega g(x,u_n)u_n dx + O(||u_n||) \le \lambda_{j+1}|u_n|_2^2 + O(||u_n||).$$

Therefore, we need to show that $\{|u_n|_2\}$ is bounded, which implies that $\{||u_n||\}$ is bounded. Since Ω is bounded and g is subcritical, then if $\{||u_n||\}$ is bounded, by the compactness of Sobolev embedding and by the standard processes we know that there exists a subsequence of $\{u_n\}$ in H_0^1 which converges strongly, hence the Lemma is proved.

Assume by contradiction that $|u_n|_2 \to \infty$ as $n \to \infty$. Let $v_n = u_n/|u_n|_2$. Then $|v_n|_2 = 1$ and $\{||v_n||\}$ is bounded. We can assume that $v_n \to v$ weakly in H_0^1, strongly in L_2 and a.e. in Ω. Thus, $u_n(x) \to \infty$ a.e. in Ω. From (18) it follows that

$$\int_\Omega [\nabla v \nabla \varphi - L(x)v^+\varphi]dx, \qquad \forall\ \varphi \in H_0^1, \tag{19}$$

where $v^+(x) = \max\{0, v(x)\}$. By the regularity theory we have

$$-\Delta v = L(x)v^+ \quad \text{in } \Omega.$$

By the maximum principle and by the unique continuation property, $v = v^+ \ge 0$ and $L \equiv \lambda_j$ or $L \equiv \lambda_{j+1}$. Since, $j \ge 2$, $v \equiv 0$, which contradicts $|v|_2 = 1$. The proof is completed. □

4. Proofs of main Theorems

It follows from [11] that the functional F, defined by (2), satisfies the (C) condition (or the (PS) condition on the nonresonant case). Then we can use the theorems in Section 2. Without loss of generality, we assume that F has only a finite number of critical points.

Proof of Theorem 1.1. The cases (4) and (5) are considered simultaneously.

Let H_i, $i = 1,\ 2,\ 3$, be as in Lemma 3.1. Consider

$$S_1 = B_r \cap (H_2 \oplus H_3) \ \text{ and } \ D_1 = \{v + te\ ;\ v \in H_1,\ 0 \le t \le R,\ ||v + te|| \le R\},$$

where B_r denotes the closed ball with radius r centered of 0, and $e \in H_2$ is chosen such that

$$F(u) > 0 \ \ \forall\ u \in S_1 \ \text{ and } \ F(u) \le 0 \ \ \forall\ u \in \partial D_1, \tag{20}$$

this is possible by (i) and (iii) in Lemma 3.1. Since ∂D_1 and S_1 homologically link and D_1 is a k-topological ball, by (20) we have $H_k(F_b, F_0) \neq 0$, where $b > \max\{F(u) \mid u \in D\}$ (see Theorem 2.2 in Section 2). Hence we can conclude, by Theorem 2.1, that there exist u_1 critical point of F, such that

$$C_k(F, u_1) \neq 0. \tag{21}$$

Next, set $S_2 = H_3$ and $D_2 = B_R \cap (H_1 \oplus H_2)$. By (ii) and (iv) in Lemma 3.1, we have

$$F(u) \ge 0 \ \ \forall\ u \in S_2 \ \text{ and } \ F(u) < 0 \ \ \forall\ u \in \partial D_2. \tag{22}$$

Again, since ∂D_2 and S_2 homologically link and D_2 is a $(k+m)$-topological ball, we have that there exist u_2 critical point of F, such that

$$C_{k+m}(F, u_2) \neq 0. \tag{23}$$

Now we have to prove that $u_1 \neq u_2$, and are nontrivial. Note that 0 is a critical point of F and $\mu(0) + \nu(0) \le k - 1$. By Shifting Theorem, $C_p(F, 0) = 0$ for all $p \ge k$. So u_1 and u_2 are nontrivial, by (21) and (23). Again by Shifting Theorem (Corollary 2.1) we have, either

(i) $C_p(F, u_1) = \delta_{p\mu(u_1)}$, or
(ii) $C_p(F, u_1) = \delta_{p(\mu(u_1)+\nu(u_1))}$, or
(iii) $C_p(F, u_1) = 0$ if $p \notin (\mu(u_1), (\mu(u_1) + \nu(u_1))$.

If (i) or (ii) hold, then $C_{k+m}(F, u_1) = 0$ by (21). If (iii) hold then $k > \mu(u_1)$ by (21) and hence $k + m > \mu(u_1) + \nu(u_1)$ by Lemma 3.2, again $C_{k+m}(F, u_1) = 0$ by (iii). Therefore $u_1 \neq u_2$ by (23). The proof of Theorem 1.1 is finished. □

Proof of Theorem 1.2. Set

$$g_+(x,t) = \begin{cases} g(x,t), & t \ge 0, \\ 0, & t \le 0, \end{cases}$$

and consider the problem

$$\begin{aligned} -\Delta u &= g_+(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\ \text{on} \quad \partial\Omega, \end{aligned} \tag{24}$$

Define
$$F_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G_+(x,u)dx, \quad u \in H_0^1(\Omega).$$
Then $F_+ \in C^{2-0}$ and, by Lemma 3.3, satisfies (PS) condition.

Since $g'(x,0) < \lambda_1$, $u = 0$ is a strictly local minimum of F_+. Let $\varphi_1 > 0$ to be the first eigenfunction of (Δ, H_0^1), and consider $\gamma > \lambda_1$ such that $G_+(x,t) \geq (\gamma/2)t^2 - C$ for $t > 0$. Then
$$\begin{aligned} F_+(s\varphi_1) &= \frac{s^2}{2}\int_\Omega |\nabla \varphi_1|^2 dx - \int_\Omega G_+(x, s\varphi_1)dx \\ &\leq \frac{\lambda_1 s^2}{2}\int_\Omega \varphi_1^2 dx - \frac{\gamma s^2}{2}\int_\Omega \varphi_1^2 dx + C \\ &= \frac{s^2(\lambda_1-\gamma)}{2}\int_\Omega \varphi_1^2 dx + C \to -\infty, \quad as\ s \to \infty. \end{aligned}$$
By the mountain pass theorem, F_+ has a nontrivial critical point u_+. By the maximum principle, $u_+ > 0$. Therefore u_+ is a critical point of the functional F defined by (2). Similarly, we get a negative critical point u_- of F. Moreover, as in [10], we have
$$\text{rank } C_p(F_\pm|_{C_0^1}, u_\pm) = \delta_{p1}.$$
Thus,
$$\text{rank } C_p(F|_{C_0^1}, u_\pm) = \text{rank } C_p(F_\pm|_{C_0^1}, u_\pm) = \delta_{p1} \ \ \forall\ p = 0,\ 1,\ 2, ...$$
By the proof of the previous theorem, there exists a nontrivial solution u such that
$$C_{m+1}(F,u) \neq 0, \quad \text{where } m \geq 1.$$
By Theorem 1 in [8], we have
$$C_{m+1}(F|_{C_0^1}, u) = C_{m+1}(F,u).$$
Therefore u is a third nontrivial solution. □

Proof of Theorem 1.3. By the proof of the previous theorem, problem (1) has at least three nontrivial solutions one is positive, another is negative and a third solution u is such that
$$C_{m+1}(F,u) \neq 0, \quad m \geq 2.$$
So the Theorem follows of next claim.

Claim: (1) has a sign changing solution w such that
$$C_p(F,w) = \delta_{p2}\mathbb{Z}.$$
Proof: We use the notation as in [4].

Let $P = \{u \in X = C_0^1(\Omega); u \geq 0\}$, $D = P\cup(-P)$, $\dot{D}$ and φ_i the normalized eigenfunction associated to λ_i, $i = 1, 2$; we have $\varphi_1 \in \overset{\circ}{P}$.

The main ingredient in the proof of the *Claim* is the negative gradient flow φ^t of F in H, that is,

$$\frac{d}{dt}\varphi^t = -\nabla F \circ \varphi^t, \quad \varphi^0 = \mathrm{id}.$$

We have that $\varphi^t(u) \in X$ for $u \in X$ and φ^t induces a continuous (local) flow on X which we continue to denote by φ^t. The main order related property of φ^t is that P and $-P$ are positively invariant (by $g(x,t)t \geq 0$). F has the retracting property on X (see [13]).

Now the proof follows as in Theorem 3.6 in [4]. We sketch it briefly for completeness. Here we denote by $F^a = \{u \in X;\ F(u) \leq a\}$.

As $k > 2$ by (ii) in Lemma 3.1 there exists $R > 0$ such that $F(u) < 0$ for any $u \in \mathrm{span}\{\varphi_1, \varphi_2\}$ with $||u|| \geq R$. Now we set

$$B = \{s\varphi_1 + \varphi_2\ ;\ |s| \leq R,\ 0 \leq t \leq R\}$$

and

$$\partial B = \{s\varphi_1 + \varphi_2\ ;\ |s| = R \text{ or } t \in \{0, R\}\}.$$

We have $\partial B \subset F^0 \cup D$. Let $\beta = \max F(B)$ so that $(B, \partial B) \hookrightarrow (F^\beta \cup D, F^0 \cup D)$. Let $\xi_\beta \in H_2(F^\beta \cup D, F^0 \cup D)$ be the image of $1 \in \mathbb{Z} = H_2(B, \partial B)$ under the homomorphism

$$\mathbb{Z} = H_2(B, \partial B) \to H_2(F^\beta \cup D, F^0 \cup D)$$

induced by the inclusion. For $\gamma \leq \beta$ let

$$j_\gamma : H_2(F^\gamma \cup D, F^0 \cup D) \to H_2(F^\beta \cup D, F^0 \cup D)$$

be also induced by the inclusion. Now we define

$$\Gamma = \{\gamma \leq \beta\ ;\ \xi_\beta \in \text{image } (j_\gamma)\}$$

and $c = \inf \Gamma$. It is a critical value by the next lemma and standard deformation arguments.

Lemma 4.1. $\xi_\beta \neq 0$.

In fact, let $e_1 \in \overset{\circ}{P}$ be the first eigenvalue of

$$\begin{aligned} -\Delta u - g'(x,0)u &= \lambda u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

and set $X_1 = \mathrm{span}\{e_1\}$, $X_2 = X_1^\perp \cap X$. We have $\inf F(X_2 \cap \partial B_\rho) \geq \alpha > 0$ for some $\rho > 0$ small. This implies

$$(B, \partial B) \subset (F^\beta \cup D, F^0 \cup D) \subset (X, X \setminus X_2 \cap \partial B_\rho).$$

Therefore the lemma follows of that the homeomorphism

$$H_2(B, \partial B) \to H_2(X, X \setminus X_2 \cap \partial B_\rho)$$

induced by inclusion is nontrivial (it is showed in [4]).

As a consequence of previous lemma we have $0 \notin \Gamma$ because $j_0 = 0$. As $F^0 \cup D$ is a strong deformation retract of $F^\gamma \cup D$ for $\gamma > 0$ small enough (see Remark 5.1 in Appendix), we have $c > 0$. Clearly $\beta \in \Gamma$, hence $c \in (0, \beta]$.

We choose $\epsilon > 0$ small enough. Consider the commutative diagram

$$\begin{tikzcd}
H_2(F^{c-\epsilon} \cup D, F^0 \cup D) \arrow[d, "j"] \arrow[dr, "j_{c-\epsilon}"] & \\
H_2(F^{c+\epsilon} \cup D, F^0 \cup D) \arrow[r, "j_{c+\epsilon}"] \arrow[d] & H_2(F^{\beta} \cup D, F^0 \cup D) \\
H_2(F^{c+\epsilon} \cup D, F^{c-\epsilon} \cup D) &
\end{tikzcd}$$

Since $c + \epsilon \in \Gamma$ there exists $\xi_{c+\epsilon} \in H_2(F^{c+\epsilon} \cup D, F^0 \cup D)$ with $j_{c+\epsilon}(\xi_{c+\epsilon}) = \xi_\beta$. Now $\xi_{c+\epsilon} \notin$ image $(j_{c-\epsilon})$ because $c - \epsilon \notin \Gamma$. Therefore the exactness of the left column yields $H_2(F^{c+\epsilon} \cup D, F^{c-\epsilon} \cup D) \neq 0$. This implies that there exists a critical point w such that $w \notin D$ and $C_2(F, w) \neq 0$ (see the Appendix, above).

Let $w_+ = \max\{w, 0\}$ and $w_- = w_+ - w$. By (3) we have

$$\begin{aligned}
\langle F''(w)w_+, w_+\rangle &= \int_\Omega (|\nabla w_+|^2 - g'(x,w)w_+^2) \\
&= \int_\Omega (w_+ g(x,w) - g'(x,w)w_+^2) \\
&= \int_\Omega w_+^2\Big(\frac{g(x,w)}{w_+} - g'(x,w)\Big) \\
&= \int_\Omega w_+^2\Big(\frac{g(x,w_+)}{w_+} - g'(x,w_+)\Big) < 0.
\end{aligned}$$

Similarly $\langle F''(w)w_-, w_-\rangle < 0$. As w_+ and w_- are orthogonal, we have $\langle F''(w)u, u\rangle < 0$ for all $u \in \operatorname{span}\{w_+, w_-\}$, that is, the Morse index of w is 2. By the Shifting Theorem we have $C_p(F, w) = \delta_{p2}\mathbb{Z}$. □

Proof of Theorem 1.4. Let $a < b$ such that $F(K) \in (a, b)$ (see [1]), then by the hypothesis

$$\lambda_{k-1} \leq g'(x,0) < \lambda_k < \lim_{t\to\pm\infty} \frac{g(x,t)}{t} < \lambda_{k+1}, \tag{25}$$

where the limits are uniformly in Ω. It is proved in [17], that

$$C_p(F, 0) = \delta_{p,k-1}\mathbb{Z}$$

and

$$H_p(F_b, F_a) = \delta_{pk}\mathbb{Z}.$$

Moreover, note that if u_0 is a nontrivial critical point of F by (25), the Lemma 3.2 states that u_0 is nondegenerate and the Morse index of u_0 is k. Therefore

$$C_p(F, u) = \delta_{pk}\mathbb{Z}.$$

Let m the number of nontrivial critical points of F, by the Morse identity, we have

$$(-1)^k = (-1)^{k-1} + m(-1)^k.$$

It follows that $m = 2$. Then problem (1) has exactly two solutions. $\square$

5. Appendix

In this section we prove that if $u_1, ..., u_r$ be all the sign changing critical points of F at the level c, then we can choose $\epsilon > 0$ such that

$$H_*(F^{c+\epsilon} \cup \dot{D}, F^{c-\epsilon} \cup \dot{D}) \simeq \bigoplus_{i=0}^{r} C_*(F, x_i).$$

Again $F^a = \{u \in X;\ F(u) \leq a\}$.

Let $N' \subset N$ be two closed neighborhoods of $\{u_1, ..., u_r\}$ satisfying

$$\mathrm{dist}(N', \partial N) \geq \frac{7}{8}\delta, \quad \delta > 0.$$

By the (C) condition there exist constants b and $\overline{\epsilon}$ positive, such that

$$||F'(u)|| \geq b \ \ \forall\ x \in F^{c+\overline{\epsilon}} \setminus (F^{c-\overline{\epsilon}} \cup N'),$$

$$0 < \overline{\epsilon} < \mathrm{Min}\Big\{\frac{1}{4}\delta b^2, \frac{1}{8}\delta b\Big\}.$$

Define a smooth function:

$$p(s) = \begin{cases} 0 & \text{for } s \notin [c - \overline{\epsilon}, c + \overline{\epsilon}] \\ 1 & \text{for } s \in [c - \epsilon, c + \epsilon], \end{cases}$$

with $0 \leq p(s) \leq 1$ and $0 < \epsilon < \frac{\overline{\epsilon}}{2}$. Let $A = \overline{H \setminus (N')_{\frac{\delta}{8}}}$, where $(N')_\delta = \{u \in H;\ \mathrm{dist}(u, N') \leq \delta\}$, and $B = N'$. Let

$$d(u) = \frac{\mathrm{dist}(u, B)}{\mathrm{dist}(u, A) + \mathrm{dist}(u, B)}.$$

We see that $0 \leq d(u) \leq 1$, $d = 0$ on N' and $d = 1$ outside $(N')_{\frac{\delta}{8}}$. Define

$$q(s) = \begin{cases} 1 & 0 \leq s \leq 1 \\ 1/s & s \geq 1. \end{cases}$$

Denote $h(u) = d(u)p(F(u))q(||F'(u)||)$. Consider the ODE

$$\begin{aligned} &\dot{\sigma}(\tau) = -h(\sigma(\tau))F'(\sigma(\tau)), \\ &\sigma(0) = u_0 \qquad \forall\ u_0 \in X. \end{aligned} \tag{26}$$

The global existence and uniqueness of the flow $\sigma(t)$ on $\mathbb{R}$ are known. Let

$$\eta(u, t) = \sigma(t), \quad \text{with} \quad \sigma(0) = u.$$

Then $\eta \in C([0, 1] \times X, X)$ satisfies

$$\eta(1, F^{c+\epsilon} \setminus N) \subset F^{c-\epsilon}.$$

This result can be found in [9], Theorem 3.3 in Chapter I. We use it to prove the next result.

Lemma 5.1. *Suppose that there are only finitely many sign changing critical points $u_1, ..., u_r$, of F at the level c. Then we can choose $\epsilon > 0$ and neighborhoods $N_i \subset X \setminus D$ of u_i with the following properties:*

(i) $N_i \cap N_j = \emptyset$ *for* $i \neq j$*;*
(ii) $u_i = N_i \cap K$*;*
(iii) $F^{c-\epsilon} \cup N_i$ *is positively invariant under* φ^t*; and*
(iv) *there exists* $T > 0$ *with* $\varphi^T(F^{c+\epsilon}) \subset F^{c-\epsilon} \cup N_1 \cup \cdots \cup N_r$*.*

Proof. Let be $u_0 \in F^{-1}[c-\epsilon, c+\epsilon] \cap X$. By the (C) condition, we have that there is a $\delta > \epsilon$ such that $0 < h(u)$ is bounded when $u \in F^{-1}[c-\delta, c+\delta] \cap H$. Let

$$\omega(\tau, u_0) = \int_0^\tau h(\eta(\zeta, u_0))d\zeta, \qquad \tau \in [0,1], \tag{27}$$

let $t = \omega(\tau, u_0) : [0,1] \to [0, \infty)$, and let $\varphi(t, u_0) = \eta(\tau, u_0)$. Then

$$\frac{d\varphi}{dt} = \frac{d\eta}{d\tau}\frac{d\tau}{dt} = -F'(\eta(\tau, u_0)) = -F'(\varphi(t, u_0)).$$

Now we choose the $(N_i)'s$ satisfying (i), (ii) and (iii), ϵ as in above result and we define $T = \max\{\omega(1, u_0)\ ;\ u_0 \in F^{-1}[c-\epsilon, c+\epsilon] \cap X\} < \infty$. Hence, by the previous result, we have

$$\varphi^T(F^{c+\epsilon} \setminus N) \subset F^{c-\epsilon},$$

and using (iii) we have (iv). □

Setting $N = N_1 \cup \cdots \cup N_r$ properties (iii) and (iv), in the above lemma, imply that $F^{c-\epsilon} \cup N \cup \dot{D}$ is a strong deformation retract of $F^{c+\epsilon} \cup \dot{D}$, hence

$$H_*(F^{c-\epsilon} \cup N \cup \dot{D}, F^{c-\epsilon} \cup \dot{D}) \simeq H_*(F^{c+\epsilon} \cup \dot{D}, F^{c-\epsilon} \cup \dot{D}).$$

The excision property of homology implies

$$\begin{aligned} H_*(N, N \cap F^{c-\epsilon}) &\simeq H_*(F^{c-\epsilon} \cup N, F^{c-\epsilon}) \\ &\simeq H_*(F^{c-\epsilon} \cup N \cup \dot{D}, F^{c-\epsilon} \cup \dot{D}). \end{aligned}$$

Now properties (i) and (ii) yield

$$H_*(N, N \cap F^{c-\epsilon}) \simeq \bigoplus_{i=0}^{r} H_*(N_i, N_i \cap F^{c-\epsilon}) \simeq \bigoplus_{i=0}^{r} C_*(F, x_i).$$

How we want to prove.

Remark 5.1. The same idea, in the Lemma 5.1, can be used to show that $F^0 \cup D$ is a strong deformation retract of $F^\gamma \cup D$ for $\gamma > 0$ small enough. In fact, we can prove that the flow used in [22] have the same orbits of the flow φ^t.

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