Invariant almost Hermitian structures on flag manifolds

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Abstract

Let G be a complex semi-simple Lie group and form its maximal flag manifold $\mathbb{F} = G/P = U/T$ where P is a minimal parabolic subgroup, U a compact real form and $T = U \cap P$ a maximal torus of U. We study U -invariant almost Hermitian structures on $\mathbb{F}$. The $(1,2)$ -symplectic (or quasi-Kähler) structures are naturally related to the affine Weyl groups. A special form for them, involving abelian ideals of a Borel subalgebra, is derived. From the $(1,2)$-symplectic structures a classification of the whole set of invariant structures is provided, showing, in particular, that near Kähler invariant structures are Kähler, except in the A_2 case.

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1 Introduction

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and consider its maximal flag manifold $\mathbb{F} = G/P$ where G is a complex Lie group with Lie algebra $\mathfrak{g}$ and P a Borel (minimal parabolic) subgroup of G. For any maximal compact subgroup U of G we can write $\mathbb{F} = U/T$ where $T \subset U$ is a maximal torus. In this paper we study U-invariant almost Hermitian structures on $\mathbb{F}$. Such a structure is composed of a pair (J, Λ) with J an invariant almost complex structure and Λ an invariant Riemannian metric.

It will become clear at the end of the paper that the central point is a complete understanding of the class of $(1, 2)$-symplectic, or quasi-Kähler almost Hermitian structures. Thus we spend most of the time discussing these invariant structures.

We use the abbreviation *iacs* for invariant almost complex structure. An *iacs* J is said to be $(1, 2)$ -admissible if there exists a metric Λ such that the pair (J, Λ) is $(1, 2)$-symplectic. In this paper we give different characterizations of the $(1, 2)$-admissible *iacs*. The special case of the A_l series, when $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, and $\mathbb{F}$ is the manifold of complete flags of subspaces of $\mathbb{C}^n$, where considered by Cohen and the authors in [3] (see also [4]), using a method devised by Burstall and Salamon [2]. This method takes advantage of a natural bijection between invariant almost complex structures and tournaments. The combinatorics of tournament theory were used in [3] to derive a special form for $(1, 2)$-admissible *iacs*. With the aid of this form, the cone of the corresponding $(1, 2)$-symplectic metrics were determined. Tournament theory was also exploited in Mo and Negreiros [15], Negreiros [16] and Paredes [17].

In this paper we generalize the above mentioned results to arbitrary complex semi-simple Lie algebras. Our methods here are completely different. Instead of tournament theory, we use directly the geometrical combinatorics of root systems and their Weyl groups, obtaining independent proofs, when specializing to the A_l series.

In order to give an account of our results let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. An invariant almost complex structure on $\mathbb{F}$ is given by an assignment $\alpha \in \Pi \mapsto \varepsilon_\alpha \in \{\pm 1\}$, with $\varepsilon_{-\alpha} = -\varepsilon_\alpha$. Analogously, an invariant metric is given by $\lambda_\alpha > 0$ with $\lambda_{-\alpha} = \lambda_\alpha$, $\alpha \in \Pi$. Thus an invariant almost Hermitian structure is prescribed by a pair $(\{\varepsilon_\alpha\}, \{\lambda_\alpha\})$.

An easy computation shows that a pair $(\{\varepsilon_\alpha\}, \{\lambda_\alpha\})$ is almost Kähler (i.e., the fundamental Kähler 2-form Ω is symplectic) if and only if the set $\{\alpha : \varepsilon_\alpha = +1\}$ corresponds to a choice of positive roots in Π (this implies, in particular that almost Kähler structures are Kähler). By the well known equivalence between the possible choices of positive roots and Weyl chambers, we arrive that the set of *iacs* admitting an almost Kähler metric is in one-to-one correspondence with the set Weyl chambers in $\mathfrak{h}$, which in turn is in bijection with the Weyl group $\mathcal{W}$.

In the attempt of finding a similar geometric interpretation for the $(1,2)$-admissible *iacs* we were lead to consider the corresponding affine Weyl group, and the set of alcoves in $\mathfrak{h}$. With this in mind we fix a basic alcove A_0 and associate to an arbitrary alcove A an invariant almost complex structure $J(A) = \{\varepsilon_\alpha(A)\}$. The signs $\varepsilon_\alpha(A)$ are obtained by counting mod2 the number of hyperplanes $\{\alpha(\cdot) = k \in \mathbb{Z}\}$ separating A and A_0. We say that an *iacs* is affine if it has the type $J(A)$ for some alcove A.

The map $A \mapsto J(A)$ turns out to be the desired geometric description of the $(1,2)$-admissible *iacs*. Indeed in Section 3 we construct for any alcove A an invariant metric Λ, turning $(J(A), \Lambda)$ into a $(1,2)$-symplectic structure. Thus the affine *iacs* are $(1,2)$-admissible. On the other hand most of our efforts in this paper are directed towards the proof that any $(1,2)$ -admissible J is affine. To accomplish this we prove in Section 4 a result which has independent interest, namely that for any $(1,2)$-admissible J there exists a choice of positive roots Π^+ such that the set $\{\alpha > 0 : \varepsilon_\alpha = -1\}$ is an abelian ideal of Π^+. This very convenient form generalizes the stair-shaped incidence matrices of tournaments appearing in [3] in connection with *iacs* in the context of the A_l series.

In Section 5 we prove that for a given $(1,2)$ -admissible J there exists an alcove A such that $J = J(A)$, closing the connection between $(1,2)$-symplectic structures and the affine Weyl group. The technique here joins together the results by Shi [19] – characterizing the coordinates of an alcove – with the abelian ideal form admitted by the $(1,2)$-symplectic structures.

The abelian ideal form nearly gives a canonical form for the $(1,2)$-admissible *iacs*, in the sense that every equivalence class of *iacs* is represented by some J in this form, although some classes admit more than one J. In Section 6 we develop a formula relating two different abelian ideals representing the same equivalence class of almost Hermitian structures. Up to this section the affine *iacs* enters only as an additional description of the $(1,2)$-symplectic structures. The analysis of the equivalence classes is our first application of the affine description.

Our primarily goal was the study of the $(1,2)$-symplectic strucutures, seeking applications to harmonic maps through a theorem by Gray and independently by Lichnerowicz, which asserts that a holomorphic map from a Riemann surface whose target is a $(1,2)$-symplectic almost Hermitian manifold is automatically harmonic (see Gray [7], Lichnerowicz[14], Salamon [18]). However, having studied the $(1,2)$-symplectic structures we realized that in the invariant setting on $\mathbb{F}$ the $(1,2)$-symplectic is the main one among the sixteen classes of almost Hermitian manifolds. In fact, relying on Gray and Hervella [8] we show in Section 7 that these sixteen classes collapse down to four classes of invariant almost Hermitian structures with three possibilities for the *iacs*. These are the Kähler structures, the $(1,2)$-symplectic, the class of all invariant structures and a fourth one (named $W_1 \oplus W_3$) which includes every *iacs* but only some specific metrics, among them the Cartan-Killing ones. Most of the proofs in this section are direct consequences of the defining conditions for the classes. The only case which is more involved, requiring the results about the $(1,2)$ -symplectic structures, is the proof that invariant near Kähler strucutures are Kähler if the Lie algebra is not A_2.

In studing $(1,2)$-symplectic structures for the $\mathfrak{sl}(n,\mathbb{C})$ case through tournaments it was considered in [3] the concept of cone-free tournament. One of the issues there was the proof that *iacs* associated to such tournaments are $(1,2)$-admissible. When stated in terms of roots the cone-free property can be generalized to a condition on the rank three subsystems of the root system. In this general context it is possible to prove that cone-free *iacs* are affine, and thus $(1,2)$ -admissible. We do not prove this result here, leaving it to a forthcoming paper.

Now we discuss some links and forthcoming perspectives to our work. First, the intervenience of the affine Weyl group in the description of the $(1,2)$-symplectic strucutures suggest a relationship between them and the affine Kac-Moody algebra and hence to the loop groups. Indeed it easy to interpret the $(1,2)$-symplectic structures in terms of affine Lie algebras and embeddings of the flag manifolds into loop groups. There are also relations between $(1,2)$-symplectic structures and twistors theory (see Eells and Salamon [6]). We do not enter into these matters here, leaving them to another opportunity.

The abelian ideals of Π^+ (or the corresponding ideals of the Borel subalgebra) which appear extensively in our results, were studied recently by Kostant [13], connecting them with representation theory of Lie groups and algebras. One of the results reported in [13] says that the set of abelian ideals is in bijection with a subset of alcoves, suggesting a close relation with the invariant almost Hermitian structures, in particular with the $(1,2)$-symplectic ones.

In studing the classes of invariant almost Hermitian structures we arrived incidentally at a partial proof of a conjecture by Wolf and Gray [21] (see Conjecture 9.8), namely that a homogeneous space U/K of a compact Lie group U which is not a Hermitian symmetric space, and such that K has maximal rank in U, admits a near Kähler structure which is not Kähler if and only if the isotropy subalgebra is the fixed point set of an automorphism of order three. Our proof is partial in the sense that we consider only the maximal flag manifolds, that is, the case when the isotropy subgroup is the centralizer of a maximal torus of U. Further development of our methods to other flag manifolds are in progress, and eventually will lead to a complete proof of that conjecture.

2 Flag manifolds

Throughout the paper we assume that the Lie algebra $\mathfrak{g}$ is simple. There is no loss of generality in this hypothesis, since the full description of our objetcs in the semi-simple case can be easily done by the decomposition the Lie algebras into their simple ideals (cf. [20], Proposition 4.9). Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Given a Cartan subalgebra of $\mathfrak{g}$ denote by Π the set of roots of the pair $(\mathfrak{g},\mathfrak{h})$, so that

$$\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Pi}\mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}:\forall H\in\mathfrak{h},[H,X]=\alpha(H)X\}$ denotes the corresponding one-dimensional root space. The Cartan-Killing form $\langle X,Y\rangle=\mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$ of $\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$. Given $\alpha\in\mathfrak{h}^*$ we let H_α be given by $\alpha(\cdot)=\langle H_\alpha,\cdot\rangle$, and denote by $\mathfrak{h}_\mathbb{R}$ the subspace spanned over $\mathbb{R}$ by H_α, $\alpha\in\Pi$. Accordingly $\mathfrak{h}_\mathbb{R}^*$ stands for the real subspace of the dual $\mathfrak{h}^*$ spaned by the roots.

We fix once and for all a Weyl basis of $\mathfrak{g}$ which amounts to give $X_\alpha\in\mathfrak{g}_\alpha$ such that $\langle X_\alpha,X_{-\alpha}\rangle=1$, and $[X_\alpha,X_\beta]=m_{\alpha,\beta}X_{\alpha+\beta}$ with $m_{\alpha,\beta}\in\mathbb{R}$, $m_{-\alpha,-\beta}=-m_{\alpha,\beta}$ and $m_{\alpha,\beta}=0$ if $\alpha+\beta$ is not a root (see Helgason [9], Chapter IX).

Let $\Pi^+ \subset \Pi$ be a choice of positive roots, denote by Σ the corresponding simple system of roots and put $\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha$ for the Borel subalgebra generated by Π^+. We view the maximal flag manifold $\mathbb{F}$ of $\mathfrak{g}$ as the set of subalgebras conjugate to $\mathfrak{p}$. Thus, $\mathbb{F} = G/P$ where P is the normalizer of $\mathfrak{p}$ in G. Here G is any complex Lie group with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$. We can take $\mathfrak{u}$ to be the subspace spanned by $i\mathfrak{h}_\mathbb{R}$ and A_α, iS_α, $\alpha \in \Pi$, where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$. Denote by U the compact real form of G corresponding to $\mathfrak{u}$. By the transitive action of U on $\mathbb{F}$ we can write $\mathbb{F} = U/T$ where $T = P \cap U$ is a maximal torus of U. The Lie algebra of T is the real subspace $\mathfrak{t} = i\mathfrak{h}_\mathbb{R}$.

Denote by b_0 the origin of $\mathbb{F}$, viewed as a homogenous space either of G or of U. The tangent space of $\mathbb{F}$ at b_0 identifies naturally with the subspace $\mathfrak{q} = \mathfrak{u} \ominus \mathfrak{t} \subset \mathfrak{u}$, spanned by A_α, iS_α, $\alpha \in \Pi$. Analogously, the complex tangent space of $\mathbb{F}$ is identified to $\mathfrak{q}_\mathbb{C} = \mathfrak{g} \ominus \mathfrak{h} \subset \mathfrak{g}$, spanned by the root spaces. Clearly, the adjoint action of T on $\mathfrak{g}$ leaves $\mathfrak{q}$ invariant.

2.1 Invariant metrics

A U-invariant Riemannian metric ds^2 on $\mathbb{F}$ is completely determined by its value at the origin, that is, by an inner product $(\cdot,\cdot)$ in $\mathfrak{q}$, which is invariant under the adjoint action of T. Any such inner product has the form $(X,Y)_\Lambda = -\langle \Lambda X, Y\rangle$ with $\Lambda : \mathfrak{q} \to \mathfrak{q}$ positive-definite with respect to the Cartan-Killing form. The inner product $(\cdot,\cdot)_\Lambda$ admits a natural extension to a symmetric bilinear form on the complexification $\mathfrak{q}_\mathbb{C}$ of $\mathfrak{q}$. We do not change notation for these objects in $\mathfrak{q}$ and $\mathfrak{q}_\mathbb{C}$ either for the bilinear form $(\cdot,\cdot)_\Lambda$ or for the corresponding complexified map Λ. The T-invariance of $(\cdot,\cdot)_\Lambda$ is equivalent to the elements of the standard basis A_α, iS_α, $\alpha \in \Pi$, being eigenvectors of Λ, for the same eigenvalue. Thus, in the complex tangent space we have $\Lambda(X_\alpha) = \lambda_\alpha X_\alpha$ with $\lambda_\alpha > 0$ and $\lambda_{-\alpha} = \lambda_\alpha$.

We denote by ds^2_Λ the invariant metric given by Λ. In the sequel we abuse language and say that Λ itself is an invariant metric.

A special class of invariant metric is defined by choosing H in the positive Weyl chamber corresponding to Π^+ and putting

$$\Lambda_H = \{\lambda_\alpha = \alpha(H) : \alpha > 0\}.$$

We say that such a metric is of Borel type (see Borel [1]). A Borel type metric has the following intrinsic description. Let $\zeta : i\mathfrak{u} \ominus \mathfrak{h}_\mathbb{R} \to \mathfrak{u} \ominus i\mathfrak{h}_\mathbb{R}$ be given by $\zeta(X_\alpha) = -X_\alpha$ if $\alpha < 0$ and $\zeta(X_\alpha) = X_\alpha$ if $\alpha > 0$. Then an easy computation shows that

$$(X, Y)_{\Lambda_H} = \langle H, [X, \zeta Y] \rangle \qquad X, Y \in \mathfrak{q}.$$

(See Duistermmat, Kolk and Varadarajan [5].)

2.2 Invariant almost complex structures

A U-invariant almost complex structure $\mathcal{J}_*$ (abbreviated *iacs*) on $\mathbb{F}$ is completely determined by its value $J : \mathfrak{q} \to \mathfrak{q}$ in the tangent space at the origin. The map J satisfies $J^2 = -1$ and commutes with the adjoint action of T on $\mathfrak{q}$. We denote also by J its complexification to $\mathfrak{q}_\mathbb{C}$. The invariance of J entails that $J(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$ for all $\alpha \in \Pi$. The eigenvalues of J are $\pm i$ and the eigenvectors in $\mathfrak{q}_\mathbb{C}$ are X_α, $\alpha \in \Pi$. Hence $J(X_\alpha) = i\varepsilon_\alpha X_\alpha$ with $\varepsilon_\alpha = \pm 1$ satisfying $\varepsilon_\alpha = -\varepsilon_{-\alpha}$. As usual the eigenvectors associated to $+i$ are said to be of type $(1, 0)$ while the $-i$-eigenvectors are of type $(0, 1)$. Thus the $(1, 0)$ vectors are multiples of X_α, $\varepsilon_\alpha = +1$, and the $(0, 1)$ multiples of X_α, $\varepsilon_\alpha = -1$.

An *iacs* on $\mathbb{F}$ is completely prescribed by a set of signs $\{\varepsilon_\alpha\}_{\alpha\in\Pi}$ with $\varepsilon_{-\alpha} = -\varepsilon_{-\alpha}$. In the sequel we abuse language and say that an invariant almost complex structure on $\mathbb{F}$ is $J = \{\varepsilon_\alpha\}$.

Since $\mathbb{F}$ is a homogeneous space of a complex Lie group it has a natural structure of a complex manifold. The associated integrable almost complex structure J_c is given by $\varepsilon_\alpha = +1$ if $\alpha < 0$. The conjugate structure $-J_c$ is also integrable.

2.3 Equivalent structures

Let $\mathcal{W}$ be the Weyl group generated by the reflections with respect to the roots $\alpha \in \Pi$. It is well known that its action on $\mathfrak{h}^*$ leaves Π invariant. Also, $\mathcal{W}$ is isomorphic to $\mathrm{N}_U(\mathfrak{h})/T$, where $\mathrm{N}_U(\mathfrak{h})$ stands for the normalizer of $\mathfrak{h}$ in U. The group $\mathrm{N}_U(\mathfrak{h})$ acts on $\mathfrak{q}_\mathbb{C}$ by permuting the root spaces. Therefore, if J is an *iacs*, $\overline{w}J\overline{w}^{-1}$ is also an *iacs* if $\overline{w}$ is a representative of w in $\mathrm{N}_U(\mathfrak{h})$. Clearly the two *iacs* defined by J and $\overline{w}J\overline{w}^{-1}$ are equivalent in the sense that one is obtained from the other by a bi-holomorphic map. Since $\overline{w}J\overline{w}^{-1}$ depends only on w and not on the representative we have a well defined action of $\mathcal{W}$ on the set of *iacs*. We denote this action by $w \cdot J$. An easy computation shows that in terms of the signs ε_α, this action is given by

$$w \cdot J = w \cdot \{\varepsilon_\alpha\} = \{\varepsilon_{w^{-1}\alpha}\}.$$

Analogously, the Weyl group acts on the set of invariant metrics by $w \cdot \{\lambda_\alpha\} = \{\lambda_{w^{-1}\alpha}\}$. The two actions sum up to an action on the set of invariant almost Hermitian structures, which is denoted by $w \cdot (J, \Lambda) = (w \cdot J, w \cdot \Lambda)$.

In the sequel we say that $w \cdot J$ and $w \cdot \Lambda$ are equivalent to J and Λ, respectively. Of course, equivalent *iacs* as well equivalent invariant metrics share the same property. For instance if the pair (J, Λ) is $(1, 2)$-symplectic, the same holds to $w \cdot (J, \Lambda)$. Also, the *iacs* having the form $w \cdot J_c$, $w \in \mathcal{W}$, are provenient of complex structures. We call these the standard *iacs*.

2.4 Kähler form

It is easy to see that any invariant metric ds^2_Λ is almost Hermitian with respect to J, that is, $ds^2_\Lambda(JX, JY) = ds^2_\Lambda(X, Y)$ (cf. [21], Section 8, and [15]). Let $\Omega = \Omega_{J,\Lambda}$ stand for the corresponding Kähler form

$$\Omega(X, Y) = ds^2_\Lambda(X, JY) = -\langle \Lambda X, JY \rangle.$$

This form extends naturally to a U-invariant 2-form the complexication $\mathfrak{q}_\mathbb{C}$ of $\mathfrak{q}$, which we also denote by Ω. Its value on the basic vectors are:

$$\Omega(X_\alpha, X_\beta) = -i\lambda_\alpha \varepsilon_\beta \langle X_\alpha, X_\beta \rangle.$$

Since $\langle X_\alpha, X_\beta \rangle = 0$ unless $\beta = -\alpha$, Ω is not zero only on the pairs $(X_\alpha, X_{-\alpha})$, and $\Omega(X_\alpha, X_{-\alpha}) = i\lambda_\alpha \varepsilon_\alpha$. Relying on the invariance of Ω its exterior differential is easily computed from a standard formula: If $X, Y, Z \in \mathfrak{q}$ are regarded as vector fields in $\mathbb{F}$ then $d\Omega$ at the origin is given by

$$-\frac{1}{3} d\Omega(X, Y, Z) = -\Omega([X, Y], Z) + \Omega([X, Z], Y) - \Omega([Y, Z], X) \quad (1)$$

(see Kobayashi-Nomizu [12]).

Proposition 2.1 *$d\Omega(X_\alpha, X_\beta, X_\gamma)$ is zero unless $\alpha + \beta + \gamma = 0$. In this case*

$$d\Omega(X_\alpha, X_\beta, X_\gamma) = -i3m_{\alpha,\beta}(\varepsilon_\alpha \lambda_\alpha + \varepsilon_\beta \lambda_\beta + \varepsilon_\gamma \lambda_\gamma). \quad (2)$$

Proof: By the expression for the exterior derivative $-\frac{1}{3}d\Omega\left(X_\alpha, X_\beta, X_\gamma\right)$ is

$$-\Omega\left([X_\alpha, X_\beta], X_\gamma\right) + \Omega\left([X_\alpha, X_\gamma], X_\beta\right) - \Omega\left([X_\beta, X_\gamma], X_\alpha\right).$$

Using that $[X_\delta, X_\xi] = m_{\delta,\xi} X_{\delta+\xi}$ and the definition of Ω this expression becomes

$$-m_{\alpha,\beta}\varepsilon_\gamma\langle\lambda_{\alpha+\beta}X_{\alpha+\beta}, X_\gamma\rangle + m_{\alpha,\gamma}\varepsilon_\beta\langle\lambda_{\alpha+\gamma}X_{\alpha+\gamma}, X_\beta\rangle - m_{\beta,\gamma}\varepsilon_\alpha\langle\lambda_{\beta+\gamma}X_{\beta+\gamma}, X_\alpha\rangle.$$

Now, $\langle X_\delta, X_\xi\rangle \neq 0$ if and only if $\delta + \xi = 0$. Hence this sum is not zero only when $\alpha + \beta + \gamma = 0$. In this case the it reduces to

$$-m_{\alpha,\beta}\varepsilon_\gamma\lambda_{-\gamma} + m_{\alpha,\gamma}\varepsilon_\beta\lambda_{-\beta} - m_{\beta,\gamma}\varepsilon_\alpha\lambda_{-\alpha}$$

because $\langle X_\delta, X_{-\delta}\rangle = 1$. But $\alpha + \beta + \gamma = 0$ implies that

$$m_{\alpha,\beta} = m_{\beta,\gamma} = m_{\gamma,\alpha}$$

(see [9], Lemma III, 5.1). Since $m_{\alpha,\gamma} = -m_{\gamma,\alpha}$, we get (2). □

Remark: The above proposition provides an alternative of the computation of $d\Omega$, different from the proof of [15], which uses the moving frame method of Cartan.

Taking into account the expression for $d\Omega$ we make the following distinction between the triples of roots.

Definition 2.2 *Let $J = \{\varepsilon_\alpha\}$ be an iacs. A triple of roots α, β, γ with $\alpha + \beta + \gamma = 0$ is said to be*

1. *a $\{0, 3\}$-triple if $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon_\gamma$, and*
2. *a $\{1, 2\}$-triple otherwise.*

Recall that an almost Hermitian manifold is said to be $(1, 2)$ -symplectic (or quasi-Kähler) if

$$d\Omega\left(X, Y, Z\right) = 0$$

when one of the vectors X, Y, Z is of type $(1, 0)$ and the other two are of type $(0, 1)$.The structure is $(2, 1)$-symplectic if the roles of $(1, 0)$ and $(0, 1)$ are interchanged. In our invariant case, these two types of almost Hermitian manifolds are equal. In fact, we have the following criteria for an invariant pair (J, Λ) to be $(1, 2)$-symplectic, which follows immediately from formula (2), and the fact that X_α is of type $(1, 0)$ if $\varepsilon_\alpha = +1$ and $(0, 1)$ if $\varepsilon_\alpha = -1$ (cf. [21], Theorem 9.15).

Proposition 2.3 *The invariant pair* $(J = \{\varepsilon_\alpha\}, \Lambda = \{\lambda_\alpha\})$ *is* $(1,2)$*-symplectic if and only if*

$$\varepsilon_\alpha \lambda_\alpha + \varepsilon_\beta \lambda_\beta + \varepsilon_\gamma \lambda_\gamma = 0$$

for every $\{1,2\}$*-triple* $\{\alpha, \beta, \gamma\}$*.*

In the sequel we say that Λ is $(1,2)$-symplectic with respect to J if the invariant pair (J, Λ) is $(1,2)$-symplectic. Also, J is said to be $(1,2)$ -invariantly admissible or simply $(1,2)$-admissible if there exists Λ such that the invariant pair (J, Λ) is $(1,2)$-symplectic.

Now, recall that an almost Hermitian manifold is said to be almost Kähler if Ω is symplectic, that is $d\Omega = 0$. Also, the manifold is Kähler, if furthermore J is integrable. By formula (2) there are no $\{0,3\}$-triples for J if the invariant pair (J, Λ) is almost Kähler. In fact, $d\Omega = 0$ implies that $\varepsilon_\alpha \lambda_\alpha + \varepsilon_\beta \lambda_\beta + \varepsilon_\gamma \lambda_\gamma = 0$ when $\alpha + \beta + \gamma = 0$. Hence a $\{0,3\}$-triple would lead to $\lambda_\alpha + \lambda_\beta + \lambda_\gamma = 0$, which is impossible since $\lambda_\alpha > 0$. From this remark we can find the *iacs* taking part of an almost Kähler structure.

Proposition 2.4 *Suppose that the pair* (J, Λ) *is almost Kähler. Then the set* $P = \{\alpha : \varepsilon_\alpha = +1\}$ *is a choice of positive roots with respect to some lexicographic order in* $\mathfrak{h}_{\mathbb{R}}^*$*.*

Proof: Since there are no $\{0,3\}$-triples, the set P is closed, that is, $\alpha + \beta \in P$ if $\alpha, \beta \in P$ and $\alpha + \beta$ is a root. Also, $\Pi = P \cup (-P)$. Now, it is well known that these two properties imply that P is a choice of positive roots. □

Therefore, the *iacs* of an invariant almost Kähler structure are equivalent to the standard ones, which come from complex structures on $\mathbb{F}$. Note that the set of these *iacs* is in bijection with the Weyl group or the set of Weyl chambers in $\mathfrak{h}_{\mathbb{R}}$.

Corollary 2.5 *An invariant almost Hermitian structure on* $\mathbb{F}$ *is almost Kähler if and only if it is Kähler.*

3 Affine *iacs*

We have seen above that the almost Kähler (and Kähler) structures are in bijection with the set of Weyl chamber in $\mathfrak{h}_{\mathbb{R}}$. With the aim of describing the bigger class of $(1,2)$-symplectic structures we consider in this section the set of alcoves, or equivalently, the affine Weyl group associated with the root system Π.

We refer to Humphreys [10] as a basic source for the affine Weyl group. Consider the subspace $\mathfrak{h}_{\mathbb{R}}$. To conform with the usual notation we often identify $\mathfrak{h}_{\mathbb{R}}$ with its dual $\mathfrak{h}_{\mathbb{R}}^*$ and write $\langle x, \alpha \rangle$ instead of $\alpha(x)$, $x \in \mathfrak{h}_{\mathbb{R}}$, $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$. Given $\alpha \in \Pi$ and $k \in \mathbb{Z}$ define the affine hyperplane

$$H(\alpha, k) = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle = k\}.$$

The affine Weyl group $\mathcal{W}_a$ is the group of affine motions of $\mathfrak{h}_{\mathbb{R}}$ generated by the orthogonal reflections with respect to the hyperplanes $H(\alpha, k)$, $\alpha \in \Pi$, $k \in \mathbb{Z}$. It is well known that $\mathcal{W}_a$ is the semi-direct product of $\mathcal{W}$ by the group of translations by elements of the lattice $L = \mathbb{Z} \cdot \Pi^\vee$ spanned over $\mathbb{Z}$ by the co-roots

$$\Pi^\vee == \{\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} : \alpha \in \Pi\}.$$

(See [10], Proposition 4.2.) Another relevant group of affine motions is $\widehat{\mathcal{W}_a}$, which is the semi-direct product of $\mathcal{W}$ by the group of translations by the lattice

$$\widehat{L} = \{x \in \mathfrak{h}_{\mathbb{R}} : \forall \alpha \in \Pi,\ \langle \alpha, x \rangle \in \mathbb{Z}\}.$$

The complement $\mathcal{A}$ of the set of of hyperplanes $H(\alpha, k)$, $\alpha \in \Pi$, $k \in \mathbb{Z}$, is the union its connected components, each one of them is an open simplex called *alcove*. The affine group $\mathcal{W}_a$ leaves invariant the union of the hyperplanes $H(\alpha, k)$, $\alpha \in \Pi$, $k \in \mathbb{Z}$, hence $\mathcal{W}_a$ permutes the alcoves. The action of $\mathcal{W}_a$ on the set of alcoves is free and transitive so that $\mathcal{W}_a$ is in bijection with $\mathcal{A}$. The group $\widehat{\mathcal{W}_a}$ also acts transitively on the set of alcoves, but in general not freely.

Given an alcove A and a root α, there exists an integer $k_\alpha = k_\alpha(A)$ such that

$$k_\alpha < \langle x, \alpha \rangle < k_\alpha + 1.$$

Of course, $k_\alpha = [\alpha(x)]$ for any $x \in A$ where $[a]$ denotes the integer part of the real number a, that is, $[a]$ is the the largest integer such that $a - [a] > 0$. According to Shi [19], the integers $k_\alpha(A)$ are called the *coordinates* of the alcove A. An alcove is completely determined by its coordinates. However, it is not true that an arbitrary set of integers k_α, $\alpha \in \Pi$, form the coordinates of some alcove. Necessary and sufficient conditions for k_α, $\alpha \in \Pi$, to be the coordinates of an alcove where determined in [19]. We return to these conditions in Section 5 (see Proposition 5.2). For the moment we content ourselves with the following necessary conditions, which are easily obtained from the definition:

1. $k_{-\alpha} = -k_\alpha - 1$ and

2. either $k_\gamma = k_\alpha + k_\beta$ or $k_\gamma = k_\alpha + k_\beta + 1$ if $\gamma = \alpha + \beta$.

Now, with the aid of the coordinates of the alcoves we introduce the following class of *iacs*.

Definition 3.1 *Given an alcove A with coordinates k_α, the* iacs $J(A) = \{\varepsilon_\alpha(A)\}$ *is defined by $\varepsilon_\alpha(A) = (-1)^{k_\alpha}$. We say that J is an affine* iacs *if it has the form $J = J(A)$ for some alcove A.*

Note that $J(A)$ is indeed an *iacs*, since $k_{-\alpha} = -k_\alpha - 1$, so that $\varepsilon_{-\alpha}(A) = -\varepsilon_\alpha(A)$.

The definition of affine *iacs* has the following useful geometric interpretation: Giving a choice of positive roots $\Pi^+ \subset \Pi$, one has the basic alcove

$$A_0 = \{x \in \mathfrak{h}_{\mathbb{R}} : \forall\alpha > 0,\ 0 < \langle x, \alpha\rangle < 1\},$$

having coordinates $k_\alpha = 0$, $\alpha > 0$. If A is another alcove, and $\alpha \in \Pi^+$, denote by $q_\alpha(A)$ the number of hyperplanes of the form $H(\alpha, k)$ separating A of A_0. Since $\alpha > 0$, $q_\alpha(A) = |k_\alpha(A)|$. Therefore, $(-1)^{k_\alpha(A)} = (-1)^{q_\alpha(A)}$, so that the number of separating hyperplanes determines $J(A)$.

Before proceding we check that the map $A \mapsto J(A)$ which defines the affine *iacs* is well behaved under the Weyl group action.

Lemma 3.2 *The map $A \mapsto J(A)$ is equivariant with respect to the action of Weyl group $\mathcal{W}$, that is $J(wA) = w \cdot J(A)$, $w \in \mathcal{W}$. Here wA is the restriction to $\mathcal{W}$ of the action of $\mathcal{W}_a$ and $w \cdot \{\varepsilon_\alpha\} = \{\varepsilon_{w^{-1}\alpha}\}$ is the $\mathcal{W}$-action on the* iacs *defined before.*

Proof: Is immediate from the formula $k_\alpha(wA) = k_{w^{-1}\alpha}(A)$ whose proof is straighforward. □

The affine *iacs* are intimately related to the $(1,2)$ -admissible ones. Actually, one of the main purposes of this paper is to prove that these two classes of *iacs* coincide. We show next that affine *iacs* are $(1,2)$-admissible. This is the easy part of the proof that these properties are equivalent. The converse will be seen in later sections and requires several steps.

Theorem 3.3 *Let $J = J(A)$ be an affine invariant complex structure. Then J is $(1,2)$-invariantly admissible.*

Proof: Let $k_\alpha = k_\alpha(A)$ be the coordinates of A. Take $x \in A$ and define the invariant metric $\Lambda = \{\lambda_\alpha\}$ by

$$\lambda_\alpha = \varepsilon_\alpha(\alpha(x) - k_\alpha) + \frac{1-\varepsilon_\alpha}{2} = \begin{cases} \alpha(x) - k_\alpha & \text{if } \varepsilon_\alpha = +1 \\ 1 - \alpha(x) + k_\alpha & \text{if } \varepsilon_\alpha = -1. \end{cases}$$

Since $k_\alpha = [\alpha(x)]$, it follows that $\lambda_\alpha > 0$ for all α. Moreover, $\lambda_{-\alpha} = \lambda_\alpha$ is a consequence of $\varepsilon_{-\alpha} = -\varepsilon_\alpha$ and $k_{-\alpha} = -k_\alpha - 1$. Hence Λ is a well defined invariant metric. We claim that Λ is $(1,2)$ -symplectic with respect to J. To prove this take roots α, β and γ such that $\alpha + \beta + \gamma = 0$. A straighforward computation shows that

$$\varepsilon_\alpha\lambda_\alpha + \varepsilon_\beta\lambda_\beta + \varepsilon_\gamma\lambda_\gamma = \frac{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma - 3}{2} - (k_\alpha + k_\beta + k_\gamma). \tag{3}$$

By one of the necessary conditions satisfied by the coordinates stated above, $k_{-\gamma} = k_\alpha + k_\beta$ or $k_\alpha + k_\beta + 1$. Hence $k_\gamma = -(k_\alpha + k_\beta) - 1$ or $-(k_\alpha + k_\beta) - 2$, so that k_γ is determined by k_α, k_β and the mod2 cosets of k_α, k_β and k_γ. On the other hand, since J is affine, $\varepsilon_\delta = (-1)^{k_\delta}$ for any root δ. Therefore $k_\alpha + k_\beta + k_\gamma$ is either -1 or -2 and we can decide by one of these values as soon as we have $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma)$. With these remarks in mind we check that $\varepsilon_\alpha\lambda_\alpha + \varepsilon_\beta\lambda_\beta + \varepsilon_\gamma\lambda_\gamma = 0$ for the possible $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma)$ yielding $\{1,2\}$-triples. We list below the outcomes:

1. $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) = (+1, +1, -1)$. $\frac{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma - 3}{2} = -1$; $k_\alpha + k_\beta + k_\gamma = -1$.

2. $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) = (+1, -1, +1)$. $\frac{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma - 3}{2} = -1$; $k_\alpha + k_\beta + k_\gamma = -1$.

3. $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) = (+1, -1, -1)$. $\frac{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma - 3}{2} - 2$; $k_\alpha + k_\beta + k_\gamma = -2$.

4. $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) = (-1, -1, +1)$. $\frac{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma - 3}{2} = -2$; $k_\alpha + k_\beta + k_\gamma = -2$.

This concludes the proof that (J,Λ) is $(1,2)$-symplectic. $\square$

We finish this section by proving a homomorphic property of the affine *iacs* which might be useful in their future study.

Recall that the group $\widehat{\mathcal{W}_a}$ acts transitively on the set of alcoves and is the semi-direct product of $\mathcal{W}$ by the group of translations defined by the lattice $\widehat{L}$. Therefore, for every alcove A there exists $\lambda \in \widehat{L}$ and $w \in \mathcal{W}$ such that $A = t_\lambda w A_0$. Applying w^{-1} to this equality, we get

$$w^{-1}A = \left(w^{-1}t_\lambda w\right) A_0.$$

Now, $w^{-1}t_\lambda w = t_{w^{-1}\lambda}$, meaning that $\widehat{L}$ is stabilized by $\mathcal{W}$. Hence every alcove is in the $\mathcal{W}$ -orbit of some alcove obtained by translating the basic alcove A_0 by an element of $\widehat{L}$. Since the map $A \mapsto J(A)$ is equivariant, it follows that every affine *iacs* is equivalent to one of the form $J(t_\lambda A_0)$, $\lambda \in \widehat{L}$.

Lemma 3.4 *Let* $\lambda \in \widehat{L}$. *Then, the coordinates of* $t_\lambda A_0$ *are* $k_\alpha = \langle \lambda, \alpha\rangle$ *if* $\alpha > 0$. *Accordingly,* $k_\alpha = \langle \lambda, \alpha\rangle - 1$ *if* $\alpha < 0$.

Proof: Take $x \in A_0$. Then $\langle t_\lambda x, \alpha\rangle = \langle \lambda, \alpha\rangle + \langle x, \alpha\rangle$, so that $\langle \lambda, \alpha\rangle < \langle t_\lambda x, \alpha\rangle < \langle \lambda, \alpha\rangle + 1$ if $\alpha > 0$. $\square$

This lemma implies that $k_{\alpha+\beta} = k_\alpha + k_\beta$ if α, β and $\alpha+\beta$ are positive roots. Hence, $J(t_\lambda A)$ becomes a homomorphism when restricted to $\mathfrak{n}^+$, that is, $\varepsilon_{\alpha+\beta}(t_\lambda A_0) = \varepsilon_\alpha(t_\lambda A_0)\,\varepsilon_\beta(t_\lambda A_0)$ if $\alpha, \beta, \alpha+\beta \in \Pi^+$. Therefore, any affine *iacs* is equivalent to one satisfying this multiplicative property on the positive roots. We show next that this is also a sufficient condition for an *iacs* to be affine.

Proposition 3.5 *An* iacs $J = \{\varepsilon_\alpha\}$ *is affine if and only if there exists a choice of positive roots* Π^+ *such that* $\varepsilon_{\alpha+\beta} = \varepsilon_\alpha\varepsilon_\beta$ *when* $\alpha, \beta, \alpha+\beta \in \Pi^+$. *In other words, the restriction of* J *to* $\mathfrak{n}^+$ *is a homomorphism.*

Proof: It remains only to show that the multiplicative property on the positive roots imply that J is affine. For this we find $\lambda \in \widehat{L}$ such that $\varepsilon_\alpha = (-1)^{\langle \lambda, \alpha\rangle}$ if $\alpha > 0$. Since $\varepsilon_{\alpha+\beta} = \varepsilon_\alpha\varepsilon_\beta$ for positive roots, it is enough to have $\varepsilon_{\alpha_i} = (-1)^{\langle \lambda, \alpha_i\rangle}$ where $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ is the corresponding set of simple roots. Therefore the required λ is given by $\lambda = a_1\omega_1 + \cdots + a_l\omega_l$, where $\langle \mu_i, \alpha_j\rangle = \delta_{ij}$ and $a_i = 0$ if $\varepsilon_i = +1$ and $a_i = 1$ otherwise. $\square$

4 Abelian ideals

In this section we find a convenient representation for the $(1,2)$-admissible *iacs*, which generalizes the stair-shaped form of the incidence matrices of tournaments appearing in the context of [3]. We take a $(1,2)$-admissible *iacs* $J=\{\varepsilon_\alpha\}$ and let $\Lambda=\{\lambda_\alpha\}$ be a corresponding invariant $(1,2)$-symplectic metric.

Definition 4.1 *A root α is said to be J-decomposable (or simply decomposable) if there are roots β,γ such that $\alpha=\beta+\gamma$ with $\varepsilon_\alpha=\varepsilon_\beta=\varepsilon_\gamma$. The sum $\beta+\gamma$ is a J-decomposition of α. A root is J -indecomposable otherwise.*

Since $-\alpha=(-\beta)+(-\gamma)$ and $\varepsilon_{-\alpha}=-\varepsilon_\alpha$, it is clear that $\pm\alpha$ are simultaneously decomposable or indecomposable. We denote by $\mathcal{I}(J)$ or simply by $\mathcal{I}$ the set of J -indecomposable roots. In general, J-indecomposable roots may not exist. However, the presence of the $(1,2)$-symplectic metric Λ allows a treatment of $\mathcal{I}$ analogous to the usual construction of a simple system of roots. We start by noting that $\mathcal{I}\neq\emptyset$. In fact, let $\alpha=\beta+\gamma$ be a J-decomposition with $\varepsilon_\alpha=\varepsilon_\beta=\varepsilon_\gamma$. Then $\{-\alpha,\beta,\gamma\}$ is a $\{1,2\}$-triple. Since (J,Λ) is $(1,2)$-symplectic we have $\lambda_\alpha=\lambda_\beta+\lambda_\gamma$, so that $\lambda_\alpha>\lambda_\beta,\lambda_\gamma$. Therefore, the roots $\delta\in\Pi$ such that

$$\lambda_\delta=\min\{\lambda_\gamma:\gamma\in\Pi\}$$

are J-indecomposable. We have further that $\mathcal{I}$ spans $\mathfrak{h}^*$.

Lemma 4.2 *Every root α can be written (possibly in a not unique way) as*

$$\alpha=\alpha_1+\cdots+\alpha_s$$

with $\alpha_i\in\mathcal{I}$, and such that $\varepsilon_\alpha=\varepsilon_{\alpha_i}$, $i=1,\ldots,s$.

Proof: Supose that α is J-decomposable. Then $\alpha=\beta+\gamma$. If β and γ are indecomposable the result follows. Otherwise, decompose β and γ and so on. At each step $\lambda_\alpha=\lambda_\beta+\lambda_\gamma$. Hence the values of λ are strictly decreasing, so that the successive decompositions finally ends. Also, at each decomposition $\alpha=\beta+\gamma$ we have $\varepsilon_\alpha=\varepsilon_\beta=\varepsilon_\gamma$, implying the last statement. □

Now, put

$$\mathcal{I}^+=\{\alpha\in\mathcal{I}:\varepsilon_\alpha=+1\}.$$

Of course, $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$ if $\mathcal{I}^- = -\mathcal{I}^+ = \{\alpha \in \mathcal{I} : \varepsilon_\alpha = -1\}$. Since $\mathcal{I}$ spans $\mathfrak{h}^*$, it follows that $\mathcal{I}^+$ also spans $\mathfrak{h}^*$. Actually, the above lemma ensures that for an arbitrary root α, we have

$$\alpha = \varepsilon_\alpha \left(\alpha_1 + \cdots + \alpha_s\right). \tag{4}$$

with $\alpha_i \in \mathcal{I}^+$.

Soon it will become clear that in general $\mathcal{I}^+$ is not a basis of $\mathfrak{h}^*$. However, when this happens, that is, when $|\mathcal{I}^+| = \dim \mathfrak{h}$, the above lemma implies that $\mathcal{I}^+$ is a simple system of roots and J is equivalent to the standard *iacs* J_c so that (J, Λ) is Kähler. In any case $\mathcal{I}^+$ shares with the simple systems of roots the following useful property.

Lemma 4.3 *Let $\alpha, \beta \in \mathcal{I}^+$. Then $\alpha - \beta$ is not a root. Therefore, $\langle \alpha, \beta \rangle \leq 0$ if $\alpha, \beta \in \mathcal{I}^+$, $\alpha \neq \beta$.*

Proof: Suppose that $\alpha - \beta = \gamma \in \Pi$. If $\varepsilon_\gamma = +1$, we have the J-decomposition $\alpha = \beta + \gamma$. On the other hand there is the decomposition $\beta = \alpha + (-\gamma)$ if $\varepsilon_\gamma = -1$, leading to a contradiction. The last statement is a consequence of the Killing formula for the strings of roots. □

In order to understand the set $\mathcal{I}^+$ we make the following construction. Write

$$\mathcal{I}^+ = \{\alpha_1, \ldots, \alpha_m\}$$

where $m = |\mathcal{I}^+|$ and let V be an m-dimensional vector space with basis $\mathcal{B} = \{v_1, \ldots, v_m\}$. The bijection $v_i \in \mathcal{B} \leftrightarrow \alpha_i \in \mathcal{I}^+$, induces an onto linear map $P : V \to \mathfrak{h}^*$. Define the symmetric bilinear form $(x, y) = \langle Px, Py \rangle$, $x, y \in V$. Since the Cartan-Killing form is positive definite on $\mathfrak{h}^*$, we have

$$\ker P = \{x \in V : \forall y \in V, (x, y) = 0\}.$$

Also, $(x, x) = \langle Px, Px \rangle \geq 0$ so that $(\cdot, \cdot)$ is positive semi-definite, and satisfies $(u, u) > 0$ for $u \in \mathcal{B}$.

Now, let $\mathcal{W}_V$ be the group generated by the reflections

$$s_i\left(x\right) = x - \frac{2\left(x, v_i\right)}{\left(v_i, v_i\right)} v_i \qquad x \in V,$$

with respect to the basic elements $v_i \in \mathcal{B}$. According to [10], Sections 5.3 and 5.4, $\mathcal{W}_V$ is the geometric representation of the Coxeter group defined by the Killing-Cartan integers

$$\frac{2\,(v_i, v_j)}{(v_i, v_i)} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

Note that by Lemma 4.3 above these integers form a generalized Cartan matrix, so that they indeed define a Coxeter group. Since the form $(\cdot, \cdot)$ is positive semi-definite, $\mathcal{W}_V$ is a Coxeter group of affine type. Recall that the root system of $\mathcal{W}_V$ is defined to be the set

$$\widehat{\Pi} = \{\widehat{w}\,(u) : u \in \mathcal{B}, \widehat{w} \in \mathcal{W}_V\}.$$

The projection $P(\widehat{\Pi})$ is the root system in $\mathfrak{h}^*$ generated by $\mathcal{I}^+$. We denote it by $\Pi\,(\mathcal{I}^+)$.

Lemma 4.4 $\Pi\,(\mathcal{I}^+) \subset \Pi$.

Proof: Define the reflections $r_i\,(\alpha) = \alpha - (2\langle \alpha, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle)\,\alpha_i$ with respect to the roots in $\mathcal{I}^+$. A simple computation shows that $P \circ s_i = r_i \circ P$, so that for every $\widehat{w} \in \mathcal{W}_V$ there exists $w \in \mathcal{W}$ such that $P \circ \widehat{w} = w \circ P$. Hence, for any $u \in \widehat{\Pi}$, Pu has the form $w\alpha$, for some $w \in \mathcal{W}$ and $\alpha \in \mathcal{I}^+$, showing that $Pu \in \Pi$. $\square$

Our next objective is to prove the reverse inclusion, ensuring that $\Pi = \Pi\,(\mathcal{I}^+)$. For this we consider the case of G_2 separetely with the aim of simplifying some of the arguments envolving multiple-laced diagrams.

Regarding G_2, its proper subsystems are 1) the set of short roots, 2) the set of long roots, both isomorphic to A_2, and 3) the reducible ones, composed of two orthogonal roots. None of these subsystems can be $\Pi\,(\mathcal{I}^+)$. In fact, the long roots do not span G_2 over $\mathbb{Z}$, as is the case with $\Pi\,(\mathcal{I}^+)$, which spans Π. On the other hand, the set of short roots does not admit a generating set satisfying Lemma 4.3, since it violates the property that the difference of two roots is not a root. Furthermore, a pair of orthogonal roots do not span G_2 over $\mathbb{Z}$, as can be easily verified. Hence, we have $\Pi\,(\mathcal{I}^+) = \Pi$ in the G_2 root system.

For the general case we consider roots $\alpha, \beta \in \Pi\,(\mathcal{I}^+)$ and compare the strings of roots

$$\beta - p_{\mathcal{I}}\alpha, \ldots, \beta + q_{\mathcal{I}}\alpha \in \Pi\left(\mathcal{I}^+\right) \qquad \beta - p\alpha, \ldots, \beta + q\alpha \in \Pi$$

they form in each system $\Pi(\mathcal{I}^+)$ and Π. The strings are given by the well known Killing formula

$$p - q = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}.$$

Of course the right hand side of this formula is independent of the root system. However, the p's and q's can be different in the two root systems. Discarding G_2, there are the following possibilities:

1. $\langle \alpha, \beta \rangle \neq 0$ and the roots have the same length. In this case the Killing numbers are $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle = 2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle = \pm 1$, and the subspace spanned by α and β meets $\Pi(\mathcal{I}^+)$ and Π in an A_2-subsystem, Both strings depend only of the subsystem, so that they are the same, regardless the root system.

2. $\langle \alpha, \beta \rangle \neq 0$, and the roots α and β have different length ($\langle \beta, \beta \rangle / \langle \alpha, \alpha \rangle =$ 2 or 1/2). Again the strings are the same, since the subspace spanned by α and β meets both $\Pi(\mathcal{I}^+)$ and Π in the same B_2-subsystem.

3. $\langle \alpha, \beta \rangle = 0$, the subspace spanned by α and β meets the bigger root system Π in a B_2-subsystem, and α and β are short roots. In this case $\alpha \pm \beta \in \Pi$ but, in principle, it may happen that $\alpha \pm \beta$ are not $\Pi(\mathcal{I}^+)$. This is the only possibility for the strings to be different.

With this preparation we can prove that

Lemma 4.5 $\Pi(\mathcal{I}^+) = \Pi$.

Proof: It remains to check that $\Pi \subset \Pi(\mathcal{I}^+)$. This inclusion is proved by induction as follows. Write the set $\{\alpha \in \Pi : \varepsilon_\alpha = +1\}$ as $\{\alpha_1, \dots, \alpha_N\}$, ordered in such a way that

$$\lambda_{\alpha_1} \leq \cdots \leq \lambda_{\alpha_N}.$$

Then we show that $\alpha_i \in \Pi(\mathcal{I}^+)$ by induction on i. First, α_1 is J-indecomposable, since $\lambda_{\alpha_1} = \min\{\lambda_\gamma : \gamma \in \Pi\}$. Hence, $\alpha_1 \in \Pi(\mathcal{I}^+)$. Next, given $i = 1, \dots, N$ suppose by induction that $\alpha_j \in \Pi(\mathcal{I}^+)$ for all $j < i$. We can assume that α_i is J -decomposable, otherwise α_i is already in $\Pi(\mathcal{I}^+)$. Then $\alpha_i = \beta + \gamma$ with $\varepsilon_\beta = \varepsilon_\gamma = +1$. There are indices j and k such that $\beta = \alpha_j$ and $\gamma = \alpha_k$. Now, $\lambda_{\alpha_i} = \lambda_\beta + \lambda_\gamma$, hence $j, k < i$, so by the inductive hypothesis both $\beta, \gamma \in \Pi(\mathcal{I}^+)$. To prove that $\alpha_i \in \Pi(\mathcal{I}^+)$ we verify that the strings of roots determined by β and γ in Π and $\Pi(\mathcal{I}^+)$ are the same. According to the discussion above the only case to take care is when $\beta \pm \gamma \in \Pi$ and $\langle \beta, \gamma \rangle = 0$, that is, β and γ are short roots in the B_2-subsystem given by the intersection of Π with the subspace spanned by β and γ. There are the possibilities:

1. $\varepsilon_{\beta-\gamma} = +1$. Then $\beta = (\beta - \gamma) + \gamma$ is a J-decomposition, so that $\lambda_\beta = \lambda_{\beta-\gamma} + \lambda_\gamma$. Hence $\lambda_{\beta-\gamma} < \lambda_\beta < \lambda_{\alpha_i}$, and the inductive hypothesis implies that $\beta - \gamma \in \Pi(\mathcal{I}^+)$. Now, $\beta - \gamma$ and γ have different length. Hence the Killing formula implies that $\alpha_i = \beta + \gamma$ is also a root of $\Pi(\mathcal{I}^+)$.

2. $\varepsilon_{\beta-\gamma} = -1$, that is, $\varepsilon_{\gamma-\beta} = +1$. Interchanging the roles of β and γ we also conclude that $\alpha_i \in \Pi(\mathcal{I}^+)$.

Since the strings are equal, it follows that $\alpha_i \in \Pi(\mathcal{I}^+)$, showing the inductive step, and hence that $\Pi(\mathcal{I}^+) = \Pi$. □

We show next that the Coxeter graph of $\mathcal{W}_V$ is connected.

Lemma 4.6 *Suppose that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ with $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and $(u, v) = 0$ for all $u \in \mathcal{B}_1$ and $v \in \mathcal{B}_2$. Then either $\mathcal{B}_1$ or $\mathcal{B}_2$ is empty.*

Proof: Let V_i be the linear span of $\mathcal{B}_i$, $i = 1, 2$. We have $V = V_1 \oplus V_2$ and these subspaces are mutually orthogonal with respect to $(\cdot, \cdot)$. Since $\mathcal{W}_V$ is generated by the reflections with respect to the elements of $\mathcal{B}$, it follows that

$$\widehat{\Pi} = \left(V_1 \cap \widehat{\Pi}\right) \cup \left(V_2 \cap \widehat{\Pi}\right).$$

On the other hand, $\mathcal{I}^+ = p\mathcal{B}_1 \cup p\mathcal{B}_2$ is a disjoint union of subsets orthogonal with respect to the Cartan-Killing form in $\mathfrak{h}^*$. Also, $\mathfrak{h}^* = PV_1 + PV_2$ and $P\mathcal{B}_i$ spans PV_i, $i = 1, 2$. Hence, PV_1 is orthogonal to PV_2, so that $\mathfrak{h}^* = PV_1 \oplus PV_2$. Now, using the fact that $\Pi(\mathcal{I}^+) = \Pi$, we conclude that

$$\Pi = (PV_1 \cap \Pi) \cup (PV_2 \cap \Pi).$$

However, we are assuming that $\mathfrak{g}$ is simple, i.e., Π is irreducible. Therefore, either PV_1 or $PV_2 = 0$, implying that one of the subsets $\mathcal{B}_1$ or $\mathcal{B}_2$ is empty. □

The classification of the irreducible affine Coxeter groups is well known (see [10], [11]). In any one of them the radical of the corresponding quadratic form $(\cdot, \cdot)$ has dimension at most one:

$$\dim\{x \in V : \forall y \in V,\ (x, y) = 0\} \leq 1.$$

Hence $\ker P \leq 1$, so that $\dim V = \dim \mathfrak{h}$ or $\dim \mathfrak{h} + 1$, proving that

Proposition 4.7 *Either* $|\mathcal{I}^+| = \dim \mathfrak{h}$ *or* $|\mathcal{I}^+| = \dim \mathfrak{h} + 1$.

As mentioned above, $\mathcal{I}^+$ is a simple system of roots in case $|\mathcal{I}^+| = \dim \mathfrak{h}$, forcing J to be equivalent to the standard *iacs*. On the other hand if $|\mathcal{I}^+| = \dim \mathfrak{h} + 1$, $\mathcal{W}_V$ is a truly affine Coxeter group. The following description of an affine group from a finite Weyl group is well known (see [10], [11]):

Proposition 4.8 *In the space* V *of the geometric realization of the affine root system there are*

1. *a codimension* 1 *subspace* $U \subset V$ *(*$U \approx \mathfrak{h}^*$*),*
2. *a finite root system on* U*, denoted by* $\Pi(V)$*,*
3. *a simple system of roots* $\Sigma(V) \subset \Pi(V)$*, and*
4. *a generator* δ *of* $\ker P$ *(*1*-dimensional subspace complementing* U*)*

such that the basis $\mathcal{B} = \{v_1, \ldots, v_m\}$ *is given by*

$$\mathcal{B} = \Sigma(V) \cup \{\delta - \mu\}$$

where μ *is the highest root with respect to* $\Sigma(V)$*.*

We are now in postion to piece together all the previous discussion and arrive at the following characterization of the set of J-indecomposable roots.

Theorem 4.9 *As before let* $\mathcal{I}^+$ *be the set of* J*-indecomposable roots* α *such that* $\varepsilon_\alpha = +1$*. Then there exists a simple system of roots* $\Sigma \subset \Pi$ *such that either* $\mathcal{I}^+ = \Sigma$ *or*

$$\mathcal{I}^+ = \Sigma \cup \{-\mu\}$$

where μ *is the highest root with respect to* Σ*.*

Proof: $\mathcal{I}^+ = P\mathcal{B}$. Hence the theorem follows by the description of $\mathcal{B}$ in the above proposition. $\square$

Remark: In the light of Lemma 4.3 the statement of the above theorem is equivalent to the existence of a simple system of roots contained in $\mathcal{I}^+$. In fact, the only root γ which satisfies $\langle \gamma, \alpha \rangle \leq 0$ for all simple roots α is $-\mu$. We were not able to prove directly – without the intervenience of the affine Weyl groups – that $\mathcal{I}^+$ contains a simple system of roots. Nevertheless, we note that the condition of Lemma 4.3 alone is not enough to ensure that a set contains a simple system os roots, even if the set spans $\mathfrak{h}^*$. For instance, in a B_l root system, the set L given by the union of the set of long simple roots with the lowest root spans $\mathfrak{h}^*$ and satisfies $\langle \alpha, \beta \rangle \leq 0$ for all $\alpha, \beta \in L$. But there are no simple system of roots of B_l contained in L since the roots in L are long.

Definition 4.10 *Keep fixed a simple system of roots Σ with Π^+ the corresponding set of positive roots. A subset $M \subset \Pi^+$ is said to be an abelian ideal provided*

1. *M is abelian, that is, $\alpha + \beta$ is not a root if $\alpha, \beta \in M$.*
2. *One of the following equivalent conditions is satisfied.*
   - (a) *$\alpha + \gamma \in M$ if $\alpha \in M$ and $\gamma \in \Sigma$ are such that $\alpha + \gamma$ is a root.*
   - (b) *$\alpha + \gamma \in M$ if $\alpha \in M$ and $\gamma \in \Pi^+$ are such that $\alpha + \gamma$ is a root.*
   - (c) *Suppose that there are simple roots $\alpha_1, \ldots, \alpha_s$ and $\alpha \in M$ such that $\beta_k = \alpha + \alpha_1 + \cdots + \alpha_k$ is a root for all $k = 1, \ldots, s$. Then $\beta_k \in M$.*
   - (d) *Denote by μ the highest positive root and suppose that there are simple roots $\alpha_1, \ldots, \alpha_s$ such that $\alpha = \mu - \alpha_1 - \cdots - \alpha_s \in M$, and $\beta_k = \mu - \alpha_1 - \cdots - \alpha_k$ is a root for all $k = 1, \ldots, s$. Then $\beta_k \in M$.*

The equivalence of the conditions follow easily from the

Lemma 4.11 *Let α, β be positive roots such that $\alpha + \beta$ is a root. Then there are simple roots $\alpha_1, \ldots, \alpha_s$ such that $\beta = \alpha_1 + \cdots + \alpha_s$ and all intermediate sums $\alpha + \alpha_1 + \cdots + \alpha_k$, $k = 1, \ldots, s$, are roots.*

Proof: Follows by induction on the height of β, the well known fact that β is a consecutive sum of simple roots, and the following remark: If $\beta = \beta_1 + \beta_2$ then either $\alpha + \beta_1$ or $\alpha + \beta_2$ is a root. In turn, this remark is a consequence of the Jacobi identity. In fact,

$$0 \neq [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = [\mathfrak{g}_\alpha, [\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}]] = [\mathfrak{g}_\alpha, \mathfrak{g}_{\beta_1}], \mathfrak{g}_{\beta_2}] + [\mathfrak{g}_{\beta_1}, [\mathfrak{g}_\alpha, \mathfrak{g}_{\beta_2}]],$$

so that one of the terms on the right hand side must be $\neq 0$, implying that either $\alpha + \beta_1$ or $\alpha + \beta_2$ is a root. □

Now we are ready state the main result of this section establishing a special form for $(1, 2)$-symplectic invariant almost Hermitian structures.

Theorem 4.12 *Take a $(1, 2)$-symplectic invariant pair (J, Λ), $J = \{\varepsilon_\alpha\}$, $\Lambda = (\lambda_a)$. Let Σ be a simple system of J -indecomposable roots contained in $\mathcal{I}^+$, as ensured by Theorem 4.5. Denote by Π^+ the set of positive roots and by μ the highest root. Put*

$$M(J, \Sigma) = \{\alpha \in \Pi^+ : \varepsilon_\alpha = -1\}.$$

Then,

1. *$M(J, \Sigma)$ is an abelian ideal.*
2. *$M(J, \Sigma) \cap \Sigma = \emptyset$.*
3. *For $\alpha \in M(J, \Sigma)$ suppose that $\alpha = \mu - \alpha_1 - \cdots - \alpha_s$ with $\alpha_k \in \Sigma$ and $\mu - \alpha_1 - \cdots - \alpha_k$ roots for all $k = 1, \ldots, s$. Then*
$$\lambda_\alpha = \lambda_\mu + \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_s}.$$
4. *Let $\alpha \in \Pi^+ \setminus M(J, \Sigma)$ be such that $\alpha = \alpha_1 + \cdots + \alpha_s$ with $\alpha_1 + \cdots + \alpha_k$ roots for $k = 1, \ldots, s$. Then*
$$\lambda_\alpha = \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_s}.$$
5. *Take $\alpha \in M(J, \Sigma)$ and let $\beta \in \Pi^+$ be such that $\alpha + \beta$ is a root. Then $\lambda_{\alpha+\beta} = \lambda_\alpha + \lambda_\beta$.*

Proof: Let $\alpha \in M(J,\Sigma)$ and $\beta \in \Sigma$ be such that $\alpha+\beta$ is a root. If $\varepsilon_{\alpha+\beta} = +1$ then $\beta = (\alpha+\beta)+(-\alpha)$ is a J -decomposition of β contradicting the fact that β is indecomposable. Hence $\alpha+\beta \in M(J,\Sigma)$ and the first of the equivalent conditions of Definition 4.10 is satisfied. The expressions for λ_α follow easily from this condition and by successively adding a simple root, where at each step a $\{1,2\}$-triple is involved. Similarly, the last statement follows from second condition in Definition 4.10.

To see the abelian property, take $\alpha, \beta \in M(J,\Sigma)$ and suppose, by contradiction that $\gamma = \alpha+\beta$ is a root. Then $\gamma \in M(J,\Sigma)$ so that $\{-\gamma, \alpha, \beta\}$ is $\{1,2\}$-triple, implying that $\lambda_\gamma = \lambda_\alpha + \lambda_\beta$. Hence $\lambda_\gamma > \lambda_\alpha, \lambda_\beta$ contradicting the expression in the last statement, which was already proved. Finally, by construction $M(J,\Sigma)$ does not meet Σ. □

Definition 4.13 *We say that an* iacs J *satisfies the abelian ideal property with repect to* Σ *if* $M(J,\Sigma)$ *is an abelian ideal such that* $M(J,\Sigma) \cap \Sigma = \emptyset$. *In this case* J *has the abelian ideal form or pattern with respect to* Σ.

Remark: Notice that the cone of the invariant metrics Λ such that (J,Λ) is $(1,2)$-symplectic is $(l+1)$-dimensional ($l = \dim \mathfrak{h}$), unless in the Kähler case where J is the standard almost compex structure. In this the case metrics are those of Borel type. Also, it is not hard to see that if $M(J,\Sigma)$ is an abelian ideal with $M(J,\Sigma) \cap \Sigma = \emptyset$, then the expressions given in above theorem for Λ, indeed define a $(1,2)$-symplectic metric with respect to J, showing that J is $(1,2)$-admissible. In the next section this fact will be proved in another way, by showing that J is affine if $M(J,\Sigma)$ is an abelian ideal.

At this moment it is natural to ask whether the abelian ideal forms of Theorem 4.12 determine the equivalence classes of the $(1,2)$-symplectic structures under the $\mathcal{W}$-action. Of course, equivalent structures can be put in the same abelian ideal form. However, it is not true that two $J_1 \neq J_2$ satisfying the abelian ideal property with respect to the same Σ are not equivalent. Hence, the abelian ideal form is not a truly canonical form, in the sense that equivalence classes are not determined by them. We discuss these facts in Section 6, after we have established the correspondence between the $(1,2)$-admissible *iacs* with the affine ones.

For later reference we explicitate the following fact.

Proposition 4.14 *Suppose that J satisfies the abelian ideal property with respect to Σ. Then $\mathcal{I}(J) = \Sigma$ if $M(J,\Sigma) = \emptyset$ and $\mathcal{I}(J) = \Sigma \cup \{-\mu\}$ otherwise.*

Proof: Let $\alpha \in \Pi^+ \setminus M(J,\Sigma)$. If $\alpha \notin \Sigma$ then $\alpha = \beta + \gamma$ with $\beta, \gamma > 0$. Since $M(J,\Sigma)$ is an ideal, the sum $\alpha = \beta + \gamma$ is a J-decomposition, so that α is decomposable. On the other hand, let $\alpha \in M(J,\Sigma) \setminus \{\mu\}$. Then $\alpha = \mu - \beta$, with $\beta \in \Pi^+ \setminus M(J,\Sigma)$, because $M(J,\Sigma)$ is abelian. Hence, $\alpha = \mu + (-\beta)$ is a J-decomposition, concluding the proof. $\square$

5 $(1,2)$-Symplectic are affine

It was indicated before how to associate with an alcove A an affine *iacs* $J(A)$. Also, in Theorem 3.3 we exhibited an invariant metric which is $(1,2)$-symplectic with respect to $J(A)$. The purpose of this section is to prove that this construction exhausts the totality of $(1,2)$-invariantly admissible *iacs*. Starting with a$(1,2)$-admissible *iacs* J we find an alcove A such that $J = J(A)$. In finding A the metric does not show up, but only the fact that J can be put in the abelian ideal form described in Theorem 4.12. Thus our objective is to prove the following statement.

Theorem 5.1 *Let $J = \{\varepsilon_\alpha\}$ be an invariant almost complex structure. Keep fixed a simple system of roots Σ and assume that*

$$M(J,\Sigma) = \{\alpha > 0 : \varepsilon_\alpha = -1\}$$

is an abelian ideal. Then there exists an alcove A such that $J = J(A)$.

Remark: In Theorem 4.12 we obtained that $M(J,\Sigma)$ does not meet Σ. However, the proof that J is affine if it has the abelian ideal form does not require that $M(J,\Sigma) \cap \Sigma = \emptyset$.

The proof of the above theorem is based on the results of Shi [19] about the coordinates of an alcove. These results were stated with a specific normalization of our root system Π, which is viewed as the set of co-roots of another root system.

Thus we start with a root system $\widetilde{\Pi}$ normalized in such a way that $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \widetilde{\Pi}$ if it is simply-laced and $\langle \alpha, \alpha \rangle = 1$ for the short roots otherwise. Given $\alpha \in \widetilde{\Pi}$, let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the corresponding co-root. It is well known that the set $\widetilde{\Pi}^\vee$ of co-roots of $\widetilde{\Pi}$ is also a root system, and vice-versa, any root system is the set of co-roots of another system. We view our orginal root system Π as a set of co-roots:

$$\Pi = \widetilde{\Pi}^\vee = \{\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} : \alpha \in \widetilde{\Pi}\}$$

(e.g., if $\Pi = B_l$ then $\widetilde{\Pi} = C_l$ and vice-versa). If $\widetilde{\Pi}$ is simply-laced then $\Pi = 2\widetilde{\Pi}$ and both systems are isomorphic. However, if the Dynkin diagram of $\widetilde{\Pi}$ has multiple edges then the long roots of Π are the co-roots $\alpha^\vee$ with α running through the short roots of $\widetilde{\Pi}$ and reciprocally.

Now, consider the affine system associated to Π. The affine hyperplanes are defined by

$$H(\alpha^\vee, k) = \{x : \langle \alpha^\vee, x \rangle = k\} \qquad \alpha \in \widetilde{\Pi},\ \alpha^\vee \in \Pi, k \in \mathbb{Z}.$$

Given an alcove A and a root $\alpha \in \widetilde{\Pi}$ there are integers $k_\alpha = k_\alpha(A)$ such that $k_\alpha < \langle \alpha^\vee, x \rangle < k_\alpha + 1$. These integers define the alcove A, but there are redundancies in the inequalities, so that not every set of integers k_α is associated to an alcove. In fact, we have the following conditions.

Proposition 5.2 *A set of integers k_α, $\alpha \in \widetilde{\Pi}^+$, form the coordinates of an alcove if and only if for every pair of roots $\alpha, \beta \in \widetilde{\Pi}$ such that $\alpha + \beta \in \widetilde{\Pi}$, the following inequilities hold:*

$$\begin{aligned} |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 &\leq |\alpha + \beta|^2 (k_{\alpha+\beta} + 1) \\ &\leq |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1. \end{aligned} \tag{5}$$

Proof: See [19], Lemma 1.2 and Proposition 5.1. □

Now, recall the construction of the affine *iacs* $J(A) = \{\varepsilon(A)\}$ associated with the alcove A. We have $\varepsilon_\alpha = (-1)^{k_\alpha(A)}$, hence in order to prove Theorem 5.1 it is enough to find, for the given *iacs* $J = \{\varepsilon_\alpha\}$, a set of integers k_α satifying the inequalities (5) and such that $\varepsilon_\alpha = (-1)^{k_\alpha}$. Therefore, we get Theorem 5.1 as a consequence of the following construction.

Proposition 5.3 *Let $J = \{\varepsilon_\alpha\}$ be under the conditions of Theorem 5.1, and, for $\alpha > 0$, put*

$$k_\alpha = \begin{cases} 0, \ \alpha \notin M(J,\Sigma), \varepsilon_\alpha = +1 \\ 1, \ \alpha \in M(J,\Sigma), \varepsilon_\alpha = -1. \end{cases} \tag{6}$$

Then the inequalities (5) are satisfied by the integers k_α. Here we are using the convention $k_{\beta^\vee} = k_\beta$, $\beta \in \widetilde{\Pi}$.

We shall prove this proposition in several steps. Consider first the case where the diagram of $\widetilde{\Pi}$ is simply-laced, so that $\Pi = 2\widetilde{\Pi}$ and both root systems are equivalent. This implies that for $\alpha, \beta \in \widetilde{\Pi}$, $\alpha + \beta \in \widetilde{\Pi}$ if and only if $\alpha^\vee + \beta^\vee \in \Pi$. Furthermore, for any such triple the inequalities (5) reduce to

$$k_\alpha + k_\beta + 1 \leq k_{\alpha+\beta} + 1 \leq k_\alpha + k_\beta + 2$$

Now, we consider the possibilities for k_α, $\alpha > 0$, which are defined in (6) by means of the signs ε_α. We write $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_{\alpha+\beta}) = (\pm\pm\pm)$:

1. $(+++)$: Then, $k_\alpha = k_\beta = k_{\alpha+\beta} = 0$, so that the inequalities are $1 \leq 1 \leq 2$.

2. $(++-)$. Then $k_\alpha = k_\beta = 0$ and $k_{\alpha+\beta} = 1$, giving $1 \leq 2 \leq 2$.

3. $(+--)$. Then $k_\alpha = 0$, and $k_\beta = k_{\alpha+\beta} = 1$, and we have $2 \leq 2 \leq 3$.

The signs $(+-+)$ are not considered since by assumption $M(J,\Sigma)$ is an ideal. Analogously, $(---)$ and $(--+)$ do not show up by the abelian property of $M(J,\Sigma)$. This concludes the proof of Proposition 5.3 in the simply-laced case.

For the other diagrams we postpone the analysis of G_2 in order to simplify some of the arguments. Hence, in the discussion to follow we assume that $|\alpha|^2 = 1$ or 2 if $\alpha \in \widetilde{\Pi}$. We emphasize that the inequalities (5) are written in terms of triples of roots in $\widetilde{\Pi}$. However, the definition of k_α is based on the ideal $M(J,\Sigma) \subset \Pi$. Thus the first step consists in writing down the inequalities in terms of roots in Π.

Each pair of inequalities is given by a triple $(\alpha, \beta, \alpha+\beta)$ of roots in $\widetilde{\Pi}$. Writing l for long root and s for short root, there are the possibilities: (s,s,s), (l,l,l), (s,l,s) and (s,s,l). The case (l,l,s) never occurs. In fact, in a root system the sum of two long roots is never a short root (just look at B_2 or G_2). Appart from G_2 the only possibility for mixing l and s is in a B_2-subsystem.

Now, we translate these possibilities into triples in Π, by taking co-roots. We arrive at the cases (l,l,l), (s,s,s), (l,s,l) and (l,l,s). In the first two cases $\alpha^\vee+\beta^\vee=(\alpha+\beta)^\vee$. Hence they correspond to triple of roots $(u,v,u+v)$ in Π. The other two cases do not correspond to such triples in Π, but to triples as follows: Given a triple $(\alpha,\beta,\alpha+\beta)$ in $\widetilde{\Pi}$ of the type (s,l,s), we have $\alpha^\vee+2\beta^\vee=(\alpha+\beta)^\vee$ and reciprocally, a triple (u,v,w) of the type (l,s,l) in Π comes from (s,l,s) in $\widetilde{\Pi}$ if $u+2v=w$. Analogously, (s,s,l) triples in $\widetilde{\Pi}$ correspond to (l,l,s) triples (u,v,w) in Π satisfying $w=(u+v)/2$.

Having established these correspondences we write down the possible inequalities using triples in Π.

Proposition 5.4 *Let Π be a double-laced root system. A set of integers k_α, $\alpha\in\Pi^+$, form the coordinates of an alcove if the following inequalities are satisfied for the corresponding triples of roots in Π^+:*

1. $(\alpha,\beta,\alpha+\beta)=(l,l,l)$: $k_\alpha+k_\beta+1\leq k_{\alpha+\beta}+1\leq k_\alpha+k_\beta+2$
2. $(\alpha,\beta,\alpha+\beta)=(s,s,s)$: $2k_\alpha+2k_\beta+1\leq 2k_{\alpha+\beta}+2\leq 2k_\alpha+2k_\beta+5$
3. $(\alpha,\beta,\alpha+2\beta)=(l,s,l)$: $k_\alpha+2k_\beta+1\leq k_{\alpha+2\beta}+1\leq k_\alpha+2k_\beta+3$
4. $(\alpha,\beta,(\alpha+\beta)/2)=(l,l,s)$: $k_\alpha+k_\beta+1\leq 2k_{(\alpha+\beta)/2}+2\leq k_\alpha+k_\beta+3$

Now, the values of k_α, defined in Proposition 5.3 must be pluged into these inequalities. Since k_α is given by ε_α, we write the possibilities in terms of the signs. In the first two cases only the signs $(+++)$, $(++-)$ and $(-+-)$ appear, because $M(J,\Sigma)$ is an abelian ideal. The outcoming inequalities are depicted in the following table.

	$+++$	$++-$	$-+-$
lll	$1\leq1\leq2$	$1\leq2\leq2$	$2\leq2\leq3$
sss	$1\leq2\leq5$	$1\leq4\leq5$	$3\leq4\leq7$

The other cases are described below.

- The case $(\alpha,\beta,\alpha+2\beta)=(l,s,l)$. Take $\alpha,\beta\in\Pi^+$ such that $\alpha+2\beta\in\Pi^+$. Then $\varepsilon_\beta=+1$. In fact, $\varepsilon_\beta=-1$ entails $\alpha+\beta\in M(J,\Sigma)$, but then $\beta+(\alpha+\beta)$ is a sum of two roots in $M(J,\Sigma)$, contradicting the assumption on this set. Analogously, the case $\varepsilon_\alpha = -1$, $\varepsilon_\beta = +1$ and $\varepsilon_{\alpha+2\beta} = +1$ does not occur. It remains only the following three cases, with the correponding inequalities:

1. $(+++)$; $1 \leq 1 \leq 3$.
2. $(++-)$; $1 \leq 2 \leq 3$.
3. $(-+-)$; $2 \leq 2 \leq 4$.

- The case $(\alpha, \beta, (\alpha+\beta)/2) = (l, l, s)$. Take α and β positive roots such that $(\alpha+\beta)/2 \in \Pi^+$ is a positive root. We can identify the intersection of Π with the subspace spanned by α and β with the root system B_2, whose positive roots are

$$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}.$$

The identification is in such a way that $\alpha = \alpha_1$ and $\beta = \alpha_1 + 2\alpha_2$. Hence $(\alpha+\beta)/2$ becomes $\alpha_1 + \alpha_2$. Through this identification it is easy to see that $\beta = \alpha + (\alpha+\beta)/2$. This implies that $\varepsilon_{(\alpha+\beta)/2} = +1$ if $\varepsilon_\beta = +1$.

On the other hand, $\varepsilon_{(\alpha+\beta)/2} = -1$ if $\varepsilon_\alpha = \varepsilon_\beta = -1$. In fact, using the identification with B_2 we see that $(\alpha+\beta)/2 = \alpha + \alpha_2$ and $(\alpha+\beta)/2 = \beta - \alpha_2$. Hence $(\alpha+\beta)/2$ is bigger than α or β depending if α_2 is positive or negative in Π. In both cases $\varepsilon_{(\alpha+\beta)/2} = +1$ would contradict the fact that $M(J, \Sigma)$ is an ideal.

Therefore it remains only the following three cases, with the correponding inequalities:

1. $(+++)$: $1 \leq 2 \leq 3$.
2. $(-+-)$: $2 \leq 4 \leq 4$.
3. $(---)$: $3 \leq 4 \leq 5$.

This concludes the proof of Proposition 5.3 (and hence of Theorem 5.1) for the double-laced diagrams.

Now we consider G_2. Write its positive roots as

$$\begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \quad \alpha_1 + \alpha_2 \quad \alpha_1 + 2\alpha_2 \quad \alpha_1 + 3\alpha_2 \quad 2\alpha_1 + 3\alpha_2.$$

Then the possible J such that $M(J,\Sigma)$ is an abelian ideal are ${}^{+}_{+}++++$, ${}^{+}_{+}+++-$, ${}^{+}_{+}++--$ and ${}^{+}_{+}+---$. It is easily checked that these *iacs* correspond to alcoves, either by using inequalities (5) or by drawing the set of alcoves.

6 Equivalent $(1,2)$-symplectic *iacs*

In this section we look at the equivalence classes of $(1,2)$ -symplectic invariant structures under the action of the Weyl group. Since any structure can be put in abelian ideal form, it remains to determine when two invariant pairs (J_1,Λ_1) and (J_2,Λ_2) satisfying the abelian ideal property with respect to the same Σ are equivalent. Thus we fix Σ and check if there exists $w\in\mathcal{W}$ such that $J_2 = w\cdot J_1$. Having this in mind we develop here a formula for $M(w\cdot J,\Sigma)$ when both J and $w\cdot J$ satisfy the abelian ideal property with respect to Σ.

Recall that $w\cdot J = \{\varepsilon_{w^{-1}\alpha}\}$ if $J=\{\varepsilon_\alpha\}$. From this defining expression it follows immediately that a root α is J-decomposable if and only if $w^{-1}\alpha$ is $(w\cdot J)$-decomposable. Hence, $\mathcal{I}(w\cdot J) = w^{-1}\mathcal{I}(J)$. The following proposition characterizes those $w\in\mathcal{W}$ that do not destroy the abelian ideal property.

Proposition 6.1 *Fix a simple system of roots Σ and put $\widetilde{\Sigma} = \Sigma\cup\{-\mu\}$. Two invariant pairs (J_1,Λ_1) and (J_2,Λ_2), having the abelian ideal form with respect to Σ are equivalent if and only $(J_2,\Lambda_2) = (w\cdot J_1, w\cdot\Lambda_1)$ with $w\in\mathcal{W}$ satisfying $w\widetilde{\Sigma} = \widetilde{\Sigma}$.*

Proof: By Proposition 4.14

$$\mathcal{I}(J_1) = \mathcal{I}(J_2) = (\pm\Sigma)\cup\{\pm\mu\}.$$

Moreover, $\mathcal{I}(J_2) = w^{-1}\mathcal{I}(J_1)$. Hence, w and w^{-1} map the subset $(\pm\Sigma)\cup\{\pm\mu\}$ onto itself. We claim that $\mathcal{I}^+(J_1) = \mathcal{I}^+(J_2) = \Sigma\cup\{-\mu\}$ is also invariant under $w^{\pm 1}$. In fact, put $J_1 = \{\varepsilon_\alpha\}$ and $J_2 = \{\delta_\alpha\}$. Since the structures are in abelian ideal form, $\varepsilon_\alpha = \delta_\alpha = +1$ if α is simple. But $\delta_\alpha = \varepsilon_{w^{-1}\alpha}$ and $\varepsilon_\alpha = \delta_{w\alpha}$, so that $w\Sigma\subset\Sigma\cup\{-\mu\}$ and $w^{-1}\Sigma\subset\Sigma\cup\{-\mu\}$. Now, if $w^{-1}\Sigma\subset\Sigma$, $w=1$ and the claim follows. On the other hand, there exists $\alpha\in\Sigma$ such that $w^{-1}\alpha = -\mu$, that is, $w(-\mu)=\alpha$, which means that $\Sigma\cup\{-\mu\}$ is invariant under w, and hence under w^{-1}.

Conversely, suppose $\Sigma \cup \{-\mu\}$ is invariant under $w^{\pm 1} \in \mathcal{W}$. Then $\Sigma_1 = w^{-1}\Sigma$ is another choice of a simple system of roots within $\Sigma \cup \{-\mu\}$. Hence, by Theorem 4.12, J and $w \cdot J$ are in abelian ideal form with respect to both Σ and Σ_1. □

We denote by $\mathcal{W}_{\widetilde{\Sigma}}$ the subgroup of $\mathcal{W}$ leaving invariant $\widetilde{\Sigma}$. Due to the bijection of $\mathcal{W}$ with the set of simple systems of roots, it is clear that $\mathcal{W}_{\widetilde{\Sigma}}$ is in bijection with the set of simple systems of roots contained in $\widetilde{\Sigma}$. These systems are easily determined with the aid of the Coxeter graphs of the affine Weyl groups (extended Dynkin diagrams). In fact, we have the following characterization of the simple systems of roots contained in $\widetilde{\Sigma}$.

Lemma 6.2 *A subset $\Sigma_1 \subset \widetilde{\Sigma}$ is a simple system of roots if and only if Σ_1 is a subgraph of the extended diagram equal to the Dynkin diagram of Σ.*

Proof: Clearly, the condition is necessary, since $\widetilde{\Sigma}$ is the extended diagram. To prove sufficiency we must keep an eye at the extended Dynkin diagrams. Since they are easily accessible from textbooks (see [9], page 503 or [10], page 96), we do not reproduce them here. The subgraphs Σ_1 which are isomorphic to Σ are obtained by deleting from $\widetilde{\Sigma}$ either $-\mu$ or a simple root in a subset $\Delta \subset \Sigma$. Checking the coefficients of μ with respect to Σ (see [9], Table I, page 477 or [10], page 98), one sees that the coefficient of each $\alpha \in \Delta$ is 1.

Take a positive root $\beta = \sum_{\gamma \in \Sigma} n_\gamma \gamma$, $n_\gamma \geq 0$. If $\alpha \in \Delta$, $n_\alpha = 0$ or 1 because n_α is smaller than the coefficient of μ with respect to α. Now, it is easy to see that β is a linear combination of $(\Sigma \setminus \{\alpha\}) \cup \{-\mu\}$ with integer coefficients m_i, which are all ≥ 0 if $n_\alpha = 0$ and ≤ 0 if $n_\alpha = 1$. This implies that $(\Sigma \setminus \{\alpha\}) \cup \{-\mu\}$, $\alpha \in \Delta$, is a simple system of roots. □

By inspecting the table of the extended diagrams we find the following quantities of simple systems $\Sigma_1 \subset \widetilde{\Sigma}$:

$\widetilde{\Sigma}$	$\widetilde{A}_l$	$\widetilde{B}_l$	$\widetilde{C}_l$	$\widetilde{D}_l$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$	$\widetilde{G}_2$	$\widetilde{F}_4$
$\lvert\mathcal{W}_{\widetilde{\Sigma}}\rvert$	$l+1$	2	2	4	3	2	1	1	1

The numbers in this table are precisely the indices of connectivity of the affine groups $\mathcal{W}_a$. This index is either the order of $\widehat{\mathcal{W}}_a/\mathcal{W}_a$ or the order of the subgroup of $\widehat{\mathcal{W}}_a$ leaving invariant the basic alcove A_0 (see [10], page 98).

This suggests a relation between the latter subgroup and $\mathcal{W}_{\widetilde{\Sigma}}$. In fact, we have the following construction: Let P be the open parallelepiped

$$P = \{x \in \mathfrak{h}_{\mathbb{R}} : \forall \alpha \in \Sigma,\ 0 < \langle \alpha, x \rangle < 1\}.$$

Given $w \in \mathcal{W}$ there exists exactly one $\rho_w \in \widehat{L}$ such that $t_{\rho_w} w\,(A_0) \subset P$ (see [10], page 99), where t_λ is the affine translation by λ.

Put $\Sigma = \{\alpha_1 \ldots, \alpha_l\}$ and let $\{\omega_1, \ldots, \omega_l\}$ be defined by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. According to [10] (see page 99), $\rho_w = \sum a_i \omega_i$ with $a_i = 0$ if $w^{-1}\alpha_i > 0$ and $a_i = 1$ if $w^{-1}\alpha_i < 0$. Given $w \in \mathcal{W}_{\widetilde{\Sigma}}$, there is just one simple root, say α_w, such that $w^{-1}\alpha_w = -\mu$. For the other roots $\alpha \in \Sigma$, $w^{-1}\alpha \in \Sigma$, so that $w^{-1}\alpha > 0$. Hence, $\rho_w = \omega_i$ if $\alpha_w = \alpha_i$.

Lemma 6.3 *Take $w \in \mathcal{W}_{\widetilde{\Sigma}}$ and $\alpha > 0$. Then $w^{-1}\alpha > 0$ if and only if $\langle \rho_w, \alpha \rangle = 0$, and $w^{-1}\alpha < 0$ if and only if $\langle \rho_w, \alpha \rangle = 1$.*

Proof: The coefficient b_{α_w} of α_w in $\alpha = \sum_{\beta \in \Sigma} b_\beta \beta$ is $\langle \rho_w, \alpha \rangle$. As remarked above, $w \in \mathcal{W}_{\widetilde{\Sigma}}$ implies that the coefficient of the highest root μ in the direction of α_w is 1 (see the proof of Lemma 6.2). Hence, $\langle \rho_w, \alpha \rangle = 0$ or 1. Note that $w^{-1}\beta \in \Sigma$ if $\beta \neq \alpha_w$, $\beta \in \Sigma$. Hence, if $\langle \rho_w, \alpha \rangle = 0$ then $w^{-1}\alpha$ is a linear combination with positive integers of $w^{-1}(\Sigma \setminus \{\alpha_w\}) \subset \Sigma$, so that $w^{-1}\alpha > 0$. On the other hand, $\langle \rho_w, \alpha \rangle = 1$, implies that $w^{-1}\alpha$ has the form $-\mu + \gamma$ with γ a combination of $w^{-1}(\Sigma \setminus \{\alpha_w\})$, with coefficients necessarily smaller than the coefficients of μ. Therefore, at least one of the coefficients of $w^{-1}\alpha$ is < 0, implying that $w^{-1}\alpha < 0$. □

The next lemma establishes a relationship between $\mathcal{W}_{\widetilde{\Sigma}}$ and the subgroup of $\widehat{\mathcal{W}_a}$ leaving A_0 invariant.

Lemma 6.4 *If $w \in \mathcal{W}_{\widetilde{\Sigma}}$ then $t_{\rho_w} w\,(A_0) = A_0$.*

Proof: Take $x \in A_0$ and a positive root α. Then $\langle t_{\rho_w} w x, \alpha \rangle = \langle \rho_w, \alpha \rangle + \langle x, w^{-1}\alpha \rangle$. Suppose $w^{-1}\alpha > 0$. Then $0 < \langle x, w^{-1}\alpha \rangle < 1$, and by the above lemma, $\langle \rho_w, \alpha \rangle = 0$. Hence, $0 < \langle t_{\rho_w} w x, \alpha \rangle < 1$, so that $t_{\rho_w} w x \in A_0$. Similarly, $-1 < \langle x, w^{-1}\alpha \rangle < 0$ and $\langle \rho_w, \alpha \rangle = 1$ if $w^{-1}\alpha < 0$, concluding that $t_{\rho_w} w x \in A_0$ in each case. □

Remark: The above lemma becomes clear if one thinks of A_0 as defining a chamber of the geometric realization of the affine Weyl group. Since $\mathcal{W}_{\widetilde{\Sigma}}$ is the group of automorphisms of $\widetilde{\Sigma}$, it leaves invariant the basic chamber.

Returning to the equivalence question, let $J = J(A)$ be an affine *iacs*, and assume that it satisfies the abelian ideal property of Theorem 4.12, with $M(J,\Sigma)$ the corresponding abelian ideal. By Theorem 5.1 (and Proposition 5.3) we can assume that the coordinates $k_\alpha = k_\alpha(A)$, $\alpha > 0$, of A are $k_\alpha = 0$ if $\alpha \notin M(J,\Sigma)$ and $k_\alpha = 1$ if $\alpha \in M(J,\Sigma)$.

Fixing these notations we shall use the above lemmas to compute the coordinates of the alcove $\rho_w wA$ for $w \in \mathcal{W}_{\widetilde{\Sigma}}$. To this aim we note that the hyperplanes separating A_0 and A are $H(\alpha,1)$, $\alpha \in M(J,\Sigma)$. Applying the affine map $t_{\rho_w}w$, we see that the hyperplanes separating $t_{\rho_w}wA$ and $t_{\rho_w}wA_0 = A_0$ are

$$t_{\rho_w}wH(\alpha,1) = H(w\alpha, 1 + \langle w\alpha, \rho_w\rangle), \qquad \alpha \in M(J,\Sigma). \tag{7}$$

Lemma 6.5 *Take* $w \in \mathcal{W}_{\widetilde{\Sigma}}$ *and* $\alpha > 0$. *Then*

$$\langle w\alpha, \rho_w\rangle = \begin{cases} 0 & \text{if } w\alpha > 0 \\ -1 & \text{if } w\alpha < 0. \end{cases}$$

Proof: Let $\alpha_j \in \Sigma$ be such that $w\alpha_j = -\mu$. We have $\alpha_j = w^{-1}(-\mu)$, and since $w^{-1} \in \mathcal{W}_{\widetilde{\Sigma}}$ we conclude that the coefficient of μ in the direction of α_j is 1. Clearly, $w\alpha > 0$ if and only if $\langle \omega_j, \alpha\rangle = 0$, because $w\alpha_k \in \Sigma$ if $k \neq j$.

Now, $w^{-1}\alpha_w = -\mu$, so that no simple root α_k satisfies $w\alpha_k = \alpha_w$. This means that the only possibility for $w\alpha$ to have nonzero coefficient in the direction of α_w , that is, to have $\langle w\alpha, \rho_w\rangle \neq 0$ is when $\langle \omega_j, \alpha\rangle \neq 0$. Therefore, $\langle w\alpha, \rho_w\rangle = 0$ if $\langle \omega_j, \alpha\rangle = 0$, i.e., if $w\alpha > 0$. On the other hand, if $\langle \omega_j, \alpha\rangle \neq 0$, the only term which collaborates to the coefficient of α_w is $w\alpha_j = -\mu$. Hence, the coefficient of $w\alpha$ in the direction α_w is -1, concluding the proof. $\square$

By this lemma the hyperplanes given in (7) separating $t_{\rho_w}wA$ and $A_0 = t_{\rho_w}wA_0$ are rewritten as

$$\begin{array}{ll} H(w\alpha,1) & \text{if } w\alpha > 0 \\ H(w\alpha,0) & \text{if } w\alpha < 0 \end{array} \qquad \alpha \in M(J,\Sigma). \tag{8}$$

This implies the following expressions for the coordinates of $t_{\rho_w}wA_0$:

Lemma 6.6 *Keep the above notations. For $\beta > 0$,*

$$k_\beta\left(t_{\rho_w} wA_0\right) = \begin{cases} 0 & \text{if } \beta \notin \pm wM\left(J,\Sigma\right) \\ 1 & \text{if } \beta \in wM\left(J,\Sigma\right) \\ -1 & \text{if } \beta \in -wM\left(J,\Sigma\right) \end{cases}$$

Proof: The hyperplanes separating $t_{\rho_w} wA_0$ and A_0 have the form $H\left(w\alpha, k\right)$, $\alpha \in M\left(J,\Sigma\right)$, $k = 0, 1$. Therefore, if $\beta \notin \pm wM\left(J,\Sigma\right)$ no hyperplane of the form $H\left(\beta, k\right)$ separates $t_{\rho_w} wA_0$ and A_0, implying that $k_\beta\left(t_{\rho_w} wA_0\right) = 0$. Now, by (8), if $\beta = w\alpha > 0$, $\alpha \in M\left(J,\Sigma\right)$, then $H\left(\beta, 1\right)$ is the only separating hyperplane orthogonal to β, so that $k_\beta\left(t_{\rho_w} wA_0\right) = 1$. Finally, if $\beta = -w\alpha > 0$, the separating hyperplane is $H\left(\beta, 0\right) = H\left(w\alpha, 0\right)$. □

Now, we apply the following straighforward formula

$$k_\beta\left(t_\lambda A\right) = k_\beta\left(A\right) + \langle \lambda, \beta \rangle$$

to get the coordinates of the alcove wA_0.

Lemma 6.7 *Keep the above notations. For $\beta > 0$,*

$$k_\beta\left(wA_0\right) = \begin{cases} 0 & \text{if } \beta \notin \pm wM\left(J,\Sigma\right) \text{ and } \langle \beta, \rho_w \rangle = 0 \\ -1 & \text{if } \beta \notin \pm wM\left(J,\Sigma\right) \text{ and } \langle \beta, \rho_w \rangle = 1 \\ 1 & \text{if } \beta \in wM\left(J,\Sigma\right) \\ -2 & \text{if } \beta \in -wM\left(J,\Sigma\right) \end{cases}$$

Proof: The first two lines follow immediately from the previous lemma and the above formula. The other two cases are consequences of Lemma 6.5. □

Finally we describe the abelian ideal corresponding to $w \cdot J$ if $w \in \mathcal{W}_{\widetilde{\Sigma}}$ and J has the abelian ideal form with respect to Σ.

Proposition 6.8 *Let $J = J\left(A\right)$ be an affine* iacs, *satisfying the abelian ideal property with respect to Σ, with $M\left(J,\Sigma\right)$ the corresponding abelian ideal. Take $w \in \mathcal{W}_{\widetilde{\Sigma}}$. Then $w \cdot J$ has the abelian ideal property with respect to Σ, and $M\left(w \cdot J, \Sigma\right)$ is*

$$\left(wM\left(J,\Sigma\right) \cap \Pi^+\right) \cup \{\beta \in \Pi^+ : w^{-1}\beta \notin \pm M\left(J,\Sigma\right) \text{ and } \langle \beta, \rho_w \rangle = 1\}.$$

Proof: By the above lemma this is the set which corresponds to odd k_α. $\square$

¿From this expression for $M(w \cdot J, \Sigma)$ one is able to look at the abelian ideals which represent the same equivalence class, and eventually find convenient canonical forms for the $(1,2)$ -symplectic invariant almost Hermitian structures. We refrain ourselves to make here such a detailed analysis, but look at the case of the standard *iacs* $J_c = \{\varepsilon_\alpha\}$, $\varepsilon_\alpha = +1$ if $\alpha > 0$, when $M(J, \Sigma) = \emptyset$. By Proposition 6.8, $M(w \cdot J, \Sigma)$ is the set of positive roots having nonzero coefficient in the direction of α_w if $w \in \mathcal{W}_{\widetilde{\Sigma}}$. For example, in the A_l series with root α_{ij}, $1 \leq i \neq j \leq n = l+1$, any simple root $\alpha_{i,i+1}$ is α_w for some $w \in \mathcal{W}_{\widetilde{\Sigma}}$. Also, the set positive roots having coefficient in the $\alpha_w = \alpha_{i,i+1}$ is the "rectangle" $\{\alpha_{rs} : r \leq i, s \geq i+1\}$. Any such rectangle is a representative of the invariant Kähler structures. Note that they meet the set simple roots, so that the standard *iacs* cannot be put in the abelian ideal form of Theorem 4.12.

7 Classes of almost Hermitian structures

Following Grey and Hervella [8] the almost Hermitian structures are classified into sixteen classes, each one corresponding to an invariant subspace of a representation of $\mathrm{U}(n)$, say on a space W. This representation decomposes into four irreducible components $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. The possible combinations of these components (together with $\{0\}$) furnishes the different classes of almost Hermitian structures. This correspondence respects inclusion, since a class associated to an invariant subspace V_1 is contained in the class associated to V_2 if $V_1 \subset V_2$. We do not explain here the representation W neither its irreducible components. We just follow the numbering in [8] for the components, and their corresponding almost Hermitian classes. For some of the classes we use their defining property. When this happens we explicitate them. For instance, $\{0\}$ corresponds to Kähler metrics, $W_1 \oplus W_2$ to $(1,2)$-symplectic, and the co-symplectic class is given by $W_1 \oplus W_2 \oplus W_3$. As we shall see within the invariant almost Hermitian structures the sixteen classes collapse down to these three ones, together with another class, which includes every *iacs* but only some specific metrics, among them the Cartan-Killing ones.

To start with recall that we proved in Corollary 2.5 that almost Kähler structures are Kähler. In the notation of [8] the almost Kähler structure correponds W_2, so that $W_2 \approx \{0\}$.

The other cases require the Nijenhuis tensor N, which is defined by

$$\frac{1}{2}N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]. \tag{9}$$

In the invariant context with $J = \{\varepsilon_\alpha\}$, take roots α and β. An easy computation yields

$$-\frac{1}{2}N(X_\alpha, X_\beta) = m_{\alpha,\beta}\left(\varepsilon_\alpha\varepsilon_\beta + 1 - \varepsilon_\alpha\varepsilon_{\alpha+\beta} - \varepsilon_\beta\varepsilon_{\alpha+\beta}\right)X_{\alpha+\beta}. \tag{10}$$

Lemma 7.1 *Given three roots α, β and γ, $\left(N(X_\alpha,X_\beta), JX_\gamma\right)_\Lambda = 0$ unless $\alpha+\beta+\gamma = 0$. In this case,*

$$-\frac{1}{2}\left(N(X_\alpha,X_\beta), JX_\gamma\right)_\Lambda = i\lambda_\gamma m_{\alpha,\beta}\left(\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma + \varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma\right).$$

Proof: By (10), $-(1/2)\left(N(X_\alpha,X_\beta), JX_\gamma\right)_\Lambda$ is

$$i\lambda_{\alpha+\beta} m_{\alpha,\beta}\left(\varepsilon_\alpha\varepsilon_\beta + 1 - \varepsilon_\alpha\varepsilon_{\alpha+\beta} - \varepsilon_\beta\varepsilon_{\alpha+\beta}\right)\varepsilon_\gamma\langle X_{\alpha+\beta}, X_\gamma\rangle$$

which is zero unless $\gamma = -(\alpha+\beta)$. Now, the formula in the lemma follows because $\langle X_\alpha, X_{-\alpha}\rangle = 1$ and $\varepsilon_\gamma = -\varepsilon_{\alpha+\beta}$. □

With this lemma the Hermitian case, that is, when J is integrable, which means $N = 0$ is easily described. This case corresponds to $W_3 \oplus W_4$.

Proposition 7.2 *Let J be an* iacs *such with $N = 0$. Then the set $P = \{\alpha : \varepsilon_\alpha = +1\}$ is a choice of positive roots with respect to some lexicographic order in $\mathfrak{h}_{\mathbb{R}}^*$. Hence, if J is integrable, the pair (J,Λ) is Kähler.*

Proof: Take $\alpha, \beta \in P$ such that $\gamma = -(\alpha+\beta)$ is a root. By the above lemma we have $\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma + \varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma = 0$. This implies that $\varepsilon_\gamma = -1$ if $\varepsilon_\alpha = \varepsilon_\beta = +1$, so that $\varepsilon_{\alpha+\beta} = +1$. Therefore, P is closed under addition, and since $\Pi = P \cup -P$, it follows that P is a choice of positive roots. □

It follows by the inclusion among the classes, that those corresponding to W_3 and W_4 are also Kähler.

Next, we go directly to the co-symplectic structures $W_1 \oplus W_2 \oplus W_3$, which will help to solve many other cases.

Proposition 7.3 *Every invariant pair (J,Λ) is co-symplectic.*

Proof: By [8], Section 8, an almost Hermitian structure is co-symplectic if and only if the form

$$\theta(X) = \frac{1}{2n-1}\sum_i d\Omega(X, X_i, Y_i) \tag{11}$$

anihilates. Here $\{X_i\}$ is a basis of the tangent space and $\{Y_i\}$ is the dual basis with respect to the nondegenerate form Ω. In our case we take the basis to be $\{A_\alpha, iS_\alpha : \alpha \in \Pi^+\}$. Its dual is a multiple of $\{iS_\alpha, A_\alpha : \alpha \in \Pi^+\}$. Plugging these bases into (11), a straighforward computation shows that $\theta(X) = 0$ is equivalent to

$$\sum_{\alpha>0} d\Omega(X, X_\alpha, X_{-\alpha}) = 0$$

for all X. But this is true because $d\Omega(X_\beta, X_\gamma, X_\delta) = 0$ unless $\beta+\gamma+\delta = 0$, so that for every root γ, $d\Omega(X_\gamma, X_\alpha, X_{-\alpha}) = 0$. □

Proposition 7.4 *In a co-symplectic almost Hermitian manifold there are the following equivalences: 1) $W_1 \oplus W_3 \approx W_1 \oplus W_3 \oplus W_4$; 2) $W_1 \oplus W_4 \approx W_1$; 3) $W_1 \oplus W_2 \approx W_1 \oplus W_2 \oplus W_4$; 4) $W_2 \oplus W_3 \approx W_2 \oplus W_3 \oplus W_4$; 5) $W_3 \oplus W_4 \approx W_3$; 6) $W_2 \oplus W_4 \approx W_2$.*

Proof: Is a direct consequence of Table I in [8]. When $\delta\Omega = 0$ the corresponding defining conditions are the same. Note that in [8] the Kähler form is denoted by F and the Nijenhuis tensor by S. □

Therefore, in our invariant setting the classes $W_3 \oplus W_4$ and $W_2 \oplus W_4$ are Kähler. Also, $W_1 \oplus W_2 \oplus W_4$ is the same as $(1,2)$-symplectic $(W_1 \oplus W_2)$. Next we show that any invariant structure fall in the class $W_2 \oplus W_3 \approx W_2 \oplus W_3 \oplus W_4$. Consider the tensor $T(X,Y,Z) = (N(X,Y), JZ)_\Lambda$. The class corresponding to the subspace $W_2 \oplus W_3 \oplus W_4$ is formed by the almost Hermitian structures for which the symmetrizer $\mathfrak{S}T$ of T is zero.

Proposition 7.5 *Every invariant structure is in $W_2 \oplus W_3 \oplus W_4$.*

Proof: We must show that $\mathfrak{S}T = 0$ for any invariant pair (J, Λ). By Lemma 7.1 it is enough to show that $\mathfrak{S}T\left(X_\alpha, X_\beta, X_\gamma\right) = 0$ when $\alpha + \beta + \gamma = 0$, since these are the only triples of roots satisfying $T\left(X_\alpha, X_\beta, X_\gamma\right) \neq 0$. In view of the formula in Lemma 7.1 we must symmetrize only the component $\lambda_\gamma m_{\alpha,\beta}$, since $\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma + \varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma$ is already symmetric. Summing over the permutations we have

$$\lambda_\gamma m_{\alpha,\beta} + \lambda_\gamma m_{\beta,\alpha} + \lambda_\beta m_{\alpha,\gamma} + \lambda_\alpha m_{\beta,\gamma} + \lambda_\beta m_{\gamma,\alpha} + \lambda_\alpha m_{\gamma,\beta} = 0$$

because $m_{\xi,\eta} = -m_{\eta,\xi}$. □

Now, the defining condition for the class $W_1 \oplus W_3 \approx W_1 \oplus W_3 \oplus W_4$ is the anihilation of the tensor $\left(N\left(X, Y\right), X\right)_\Lambda$. We compute it by looking at the root vectors. Analogous to Lemma 7.1 we have $\left(N\left(X_\alpha, X_\beta\right), X_\gamma\right)_\Lambda = 0$ unless $\alpha + \beta + \gamma = 0$, and in this case

$$-\frac{1}{2}\left(N\left(X_\alpha, X_\beta\right), X_\gamma\right)_\Lambda = \lambda_\gamma m_{\alpha,\beta}\left(\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_\gamma + \varepsilon_\beta\varepsilon_\gamma + 1\right).$$

In particular, $\left(N\left(X_\alpha, X_\beta\right), X_\alpha\right)_\Lambda = 0$ for every root α. Hence, for $X = \sum_\alpha a_\alpha X_\alpha$ we get

$$\left(N\left(X, X_\beta\right), X\right)_\Lambda = \sum_{\alpha\neq\gamma}\left(\left(N\left(X_\alpha, X_\beta\right), X_\gamma\right)_\Lambda + \left(N\left(X_\gamma, X_\beta\right), X_\alpha\right)_\Lambda\right). \quad (12)$$

Now, $-\frac{1}{2}\left(\left(N\left(X_\alpha, X_\beta\right), X_\gamma\right)_\Lambda + \left(N\left(X_\gamma, X_\beta\right), X_\alpha\right)_\Lambda\right)$ is

$$m_{\alpha,\beta}\left(\left(\lambda_\alpha - \lambda_\beta\right)\left(\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_\gamma + \varepsilon_\beta\varepsilon_\gamma + 1\right)\right)$$

since $m_{\alpha,\beta} = m_{\beta,\gamma} = m_{\gamma,\alpha}$ if $\alpha + \beta + \gamma = 0$.

Lemma 7.6 *A necessary and sufficient condition for the invariant pair (J, Λ) to be in the class $W_1 \oplus W_3 \approx W_1 \oplus W_3 \oplus W_4$ is: $\lambda_\alpha = \lambda_\beta = \lambda_\gamma$ if $\{\alpha, \beta, \gamma\}$ is a $\{0, 3\}$ -triple.*

Proof: For roots α, β and γ with $\alpha+\beta+\gamma = 0$, the sum $\varepsilon_\alpha\varepsilon_\beta+\varepsilon_\alpha\varepsilon_\gamma+\varepsilon_\beta\varepsilon_\gamma+1$ is not zero if and only if $\{\alpha, \beta, \gamma\}$ is a $\{0, 3\}$-triple. Then the sufficiency of the condition is immediate from the identity (12). On the other hand, it is easy to see that the condition is necessary by computing $\left(N\left(X, Y\right), X\right)$ with X having the form $X = X_\alpha + X_\beta$, $\alpha, \beta \in \Pi$. □

The condition of this lemma implies the following existence of metrics.

Proposition 7.7 *Let $J = \{\varepsilon_\alpha\}$ be an* iacs *and denote by $C(J)$ the subset of roots α such that there exists a $\{0,3\}$ -triple $\{\alpha,\beta,\gamma\}$ containing α. Let $\Lambda = \{\lambda_\alpha\}$ be an invariant metric such that λ_α is constant on $C(J)$. Then the pair (J,Λ) is in the class $W_1 \oplus W_3 \approx W_1 \oplus W_3 \oplus W_4$.*

Proof: Follows immediately from the necessary and sufficient condition of the above lemma. □

Notice that the Cartan-Killing metric is a particular case of Λ in this proposition.

To complete our analysis of the invariant almost Hermitian structures it remains only to look at the near Kähler case W_1 (which is equivalent to $W_1 \oplus W_4$). The class of near Kähler structures (W_1) is the intersection of $W_1 \oplus W_2$ ((1,2) -symplectic) with $W_1 \oplus W_3$, which we have seen above. Hence, the condition of Lemma 7.6 is necessary for a pair (J,Λ) to be near Kähler. We use this condition together with the abelian ideal form for the $(1,2)$-symplectic strutures to show that any near Kähler is actually Kähler in most of the maximal flag manifolds. First we note that the condition of Lemma 7.6 can be restated by saying that if α and β are roots such that $\alpha+\beta$ is a root, and $\varepsilon_\alpha = \varepsilon_\beta = +1$, $\varepsilon_{\alpha+\beta} = -1$, then $\lambda_\alpha = \lambda_\beta = \lambda_{\alpha+\beta}$. (cf. [21], Theorem 9.17).

Let (J,Λ) be near Kähler. Then it is $(1,2)$-symplectic, so there are Σ and Π^+ a simple system of roots and positive roots where has the abelian ideal property with $M(J,\Sigma) = \{\alpha > 0 : \varepsilon_\alpha = -1\}$.

Lemma 7.8 *Suppose that there are $\alpha,\beta \in \Pi^+ \setminus M(J,\Sigma)$ such that $\alpha+\beta \in M(J,\Sigma)$ and $\beta = \beta_1+\beta_2$ with β_i, $i=1,2$, positive roots. Then (J,Λ) is not near Kähler.*

Proof: Suppose to the contrary that (J,Λ) is near Kähler. Then $\lambda_\alpha = \lambda_\beta$, by Lemma 7.6. On the other hand, either $\alpha+\beta_1$ or $\alpha+\beta_2$ is a root (see the proof of Lemma 4.11). Suppose, for instance, that $\alpha+\beta_1$ is a root. We have $\alpha+\beta = (\alpha+\beta_1)+\beta_2 \in M(J,\Sigma)$. Also, neither β_1 nor β_2 are in $M(J,\Sigma)$, because this set is an ideal and $\beta \notin M(J,\Sigma)$. Hence, $\{\beta_1,\beta_2,-\beta\}$ is a $\{1,2\}$-triple, so that $\lambda_\beta > \lambda_{\beta_1}, \lambda_{\beta_2}$. Also, another aplication of Lemma 7.6 implies that $\lambda_{\alpha+\beta_1} = \lambda_{\beta_2}$. Now, each possibility for $\varepsilon_{\alpha+\beta_1}$ lead to a contradiction. In fact, if $\varepsilon_{\alpha+\beta_1} = +1$ then $\{\alpha,\beta_1,-(\alpha+\beta_1)\}$ is a $\{1,2\}$-triple so that

$$\lambda_\alpha < \lambda_{\alpha+\beta_1} = \lambda_{\beta_2} < \lambda_\beta.$$

Otherwise, if $\varepsilon_{\alpha+\beta_1} = -1$, Lemma 7.6 applied to α and β_1 implies that $\lambda_\alpha = \lambda_{\beta_1} < \lambda_\beta$. Both cases contradict the fact that $\lambda_\alpha = \lambda_\beta$. □

Corollary 7.9 *Put*

$$M(J,\Sigma)_{\min} = \{\gamma \in M(J,\Sigma) : \exists \alpha \in \Sigma; \gamma - \alpha \in \Pi^+ \setminus M(J,\Sigma)\}.$$

Then (J,Λ) is not near Kähler if there exists $\gamma \in M(J,\Sigma)_{\min}$ having height $h(\gamma) > 2$.

Proof: Take $\gamma \in M(J,\Sigma)_{\min}$ with $h(\gamma) > 2$ and let $\alpha \in \Sigma$ be such that $\beta = \gamma - \alpha \in \Pi^+ \setminus M(J,\Sigma)$. Then $h(\beta) \geq 2$ so that $\beta = \beta_1 + \beta_2$ for some pair of positive roots. Therefore α and β are in the conditions of the lemma, showing that (J,Λ) cannot be near Kähler. □

Corollary 7.10 *If (J,Λ) is near Kähler and $M(J,\Sigma) \neq \emptyset$ then $M(J,\Sigma)$ contains every root α with $h(\alpha) = 2$, so that $M(J,\Sigma) = \{\alpha > 0 : h(\alpha) \geq 2\}$.*

The condition of this corollary is not satisfied in most of the root systems:

Lemma 7.11 *The set $I_2 = \{\alpha > 0 : h(\alpha) \geq 2\}$ is an abelian ideal only in the root systems A_l, $l \leq 3$, and B_2.*

Proof: Appart from A_l, $l \leq 3$, and B_2, every Dynkin diagram contains one of the root systems A_4, B_3, C_3, D_4 or G_2 as a subdiagram. It is easy to find in these low rank systems pairs of roots in I_2 whose sum is still a root. Hence, in these root systems I_2 is not abelian. Clearly, if a root system Π contains a subsystem such that the correponding I_2 is not abelian, then the same holds to Π. Finally, it is straighforward to check that I_2 is an abelian ideal in A_l, $l \leq 3$, and B_2, provig the lemma. □

Now we are able to prove that for most of the root systems every invariant near Kähler structure is Kähler.

Theorem 7.12 *Any invariant near Kähler structure is Kähler if $\mathfrak{g}$ is not A_2. In A_2 there exists one equivalence class of* iacs *admitting a 1-parameter family of near Kähler metrics.*

Proof: Take a near Kähler pair (J,Λ). Then it is $(1,2)$-symplectic, so that it can be put in canonical form. Clearly, (J,Λ) is Kähler if and only if $M(J,\Sigma)=\emptyset$. Hence, by Corollary 7.10 and the previous lemma it is enough to look at A_l, $l\leq 3$, and B_2. In the trivial case A_1, there are only Kähler structures. As to A_3, its positive roots are α_1, α_2, α_3, $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$ and $\alpha_1+\alpha_2+\alpha_3$. By Corollary 7.10, $\varepsilon_\alpha=+1$ if and only if α is a simple root. Now, by Lemma 7.6, the near Kähler condition implies that $\lambda_{\alpha_1}=\lambda_{\alpha_2}=\lambda_{\alpha_1+\alpha_2}$ and $\lambda_{\alpha_2}=\lambda_{\alpha_3}=\lambda_{\alpha_2+\alpha_3}$. However, by the $(1,2)$-symplectic property we must have $\lambda_{\alpha_1+\alpha_2}=\lambda_{\alpha_3}+\lambda_{\alpha_1+\alpha_2+\alpha_3}$, leading to $\lambda_{\alpha_1+\alpha_2+\alpha_3}=0$, a contradiction. Hence, there are no near Kähler structures on A_3 besides the Kähler one. Similarly, one checks in B_2 that for the highest root $\alpha_1+2\alpha_2$, $\lambda_{\alpha_1+2\alpha_2}=0$ if $M(J,\Sigma)\neq\emptyset$.

Finally, in A_2 we have $J=\{\varepsilon_\alpha\}$ with $\varepsilon_{\alpha_1}=\varepsilon_{\alpha_2}=+1$ and $\varepsilon_{\alpha_1+\alpha_2}=-1$, where α_1 and α_2 are the simple roots. This J together the one parameter family of metrics $\lambda_{\alpha_1}=\lambda_{\alpha_2}=\lambda_{\alpha_1+\alpha_2}$, give rise to near Kähler structures which are not Kähler. □

Remark: The above determination of the near Kähler structures on $\mathbb{F}$ gives a partial proof of the following conjecture stated Wolf and Gray in [21] : Let U/K be a homogeneous space of a compact Lie group U which is not Hermitian symmetric and such that the isotropy K has maximal rank. Then there are invariant almost Hermitian structures on U/K which are near Kähler but not Kähler if and only if the isotropy subalgebra is the fixed point set of an automorphism of order three. In fact, the unique flag manifold which is Hermitian symmetric is A_1, while A_1 and A_2 are the only flag manifolds having isotropy subalgebra as the fixed point set of an order three automorphism.

In summary we have the following classes of invariant almost Hermitian structures on $\mathbb{F}$:

1. **Kähler**: W_1 (near Kähler); W_2 (almost Kähler); W_3; W_4; $W_3\oplus W_4$ (integrable); $W_2\oplus W_4$ and $W_1\oplus W_4$.

2. $(1,2)$**-symplectic (quasi-Kähler)**: $W_1\oplus W_2$, $W_1\oplus W_2\oplus W_4$.

3. **Invariant**: $W_1\oplus W_2\oplus W_3$ (co-symplectic); $W_2\oplus W_3$; $W_2\oplus W_3\oplus W_4$; $W_1\oplus W_3$; $W_1\oplus W_3\oplus W_4$. (The last two for specific metrics and every *iacs*.)

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