Vector fields on manifolds with boundary, discontinuities and reversibility

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(*Dedicated to Mauricio Matos Peixoto: teacher, colleague, on the occasion of his* 80^{th} *birthday)*

1 Introduction

The geometric-qualitative study of flows and general dynamical systems on surfaces has been during many decades object of a growing interest in many branches of pure and applied mathematics. After the works of Poincaré, Lyapunov and Bendixson this has become a well-established subject in mathematics and focus of considerable attention. Moreover, nowadays it is fairly accessible for a broad scientific audience. From various sides, attention has been paid to the structural stability concept and specially to the results of Peixoto (mainly those published in An. Ac. Bras. Sci , 1959 and Topology, 1962) and higher dimensional extensions (due mainly to Smale and Anosov).

A brief historical outline follows: in 1937 Andronov and Pontrjagin [AP] announced the characterization of the structural stability of a class of vector fields defined on a compact region in the plane. In 1959 , Peixoto & Peixoto [PP] generalized this result to a larger class of systems still defined on a planar region. This last theorem was extended by Peixoto [P] in 1962 to 2-dimensional manifolds. A bibliographical guide of this matter is contained in many expository works (for example in [AZ] or in [MP]).

Here we present an elementary discussion of three aspects of this theory: classification problems arising in bifurcation of vector fields defined in manifolds with boundary, ordinary differential equations with discontinuous second members and reversible systems. All of them strongly depend on results and techniques appearing for the first time in the work of M. Peixoto on structural stability. We focus the discussion on two-dimensional systems.

The main point treated here concerns the contact between a general vector field and the boundary of a manifold. More specifically, a tangency point between the vector field and the boundary is a distinguished singularity- an important object to be analyzed when one studies discontinuous or reversible systems. We observe also that in [A], Arnold observed the importance of such singularities in the oblique-derivative problem. We still point out that there is

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a natural mathematical approach to studying such a phenomenon by means of singularity mappings theory; see for instance [S1], [ST1], [T1] and [V].

2 Historical remarks

In 1937, Andronov & Pontrjagin (in [AP] introduced the concept of structural stability (via $C^0 - \textit{orbital equivalence}$) for C^r $(r \geq 1)$ planar vector fields X defined in a neighborhood of a compact region M in R^2 bounded by a Jordan curve ∂M . They considered the set χ^* of all such vector fields which are transverse to ∂M. The following result was stated:

Theorem 1- $X \in \chi^*$ **is structurally stable (in** χ^***) if and only if,**
(1i) all its critical points and periodic orbits are hyperbolic;
(1ii) there is no saddle connection.

In 1959, Peixoto & Peixoto (in [PP]) generalized Theorem 1, by considering the set χ of all C^r $(r \geq 1)$ vector fields X defined in a neighborhood of a compact region M in R^2. That is:

Theorem 2- $X \in \chi$ **is structurally stable (in** χ**) if and only if,**
(1i) all its critical points and periodic orbits are hyperbolic;
(1ii) there is no saddle connection;
(2i) all critical points and periodic orbits are in the interior of M;
(2ii) any trajectory of X has at most one point of tangency with ∂M;
(2iii) any saddle separatrix is transverse to ∂M;
(2iv) if a trajectory of X is tangent to ∂M **at p, then this contact is quadratic.**

It is convenient to observe that the $C^0 - \textit{equivalence}$ introduced by Andronov-Pontryagin was made via an $\varepsilon -$homeomorphism whereas in the Peixoto sense the equivalence is considered just as an homeomorphism.

Let M be a 2-dimensional manifold and $\chi = \chi^r(M)$ be the space of all $C^r - \textit{vector fields}$ on M with the C^r-topology. We denote by $\Sigma_0 = \Sigma_0(M)$ the set of all structurally stable vector fields in χ. For simplicity, we may call any element of Σ_0 a codimension$-\textit{zero}$ vector field of χ.

In 1962 , M. Peixoto (in [P]) proved the following result:

Theorem 3- Let $X \in \chi^r(M)$ $(r \geq 1)$ where M is a compact orientable surface or compact non-orientable surface of genus $1 \leq g \leq 3$. Then $X \in \Sigma_0(M)$ if and only if it is a Morse-Smale vector field.

By means of a result of C. Pugh (in [Pu]) in the C^1-topology we may add to the last result that Σ_0 is open and dense in χ. In this direction many results were obtained (also in higher dimension), for example by Gutierrez, deMelo/Gutierrez, Palis, Smale, Mañe, Robbin etc..This subject is so rich in many aspects which makes to be inevitable that a number of distinguished mathematicians, topics and results deriving from the last theorem, receive no mention here.

In our approach we have to mention the work of Sotomayor in [S1] which generalizes (or continues) the results of Andronov- Pontrjagin-Peixoto for the so called codimension$-one$ element of χ or in Sotomayor's nomenclature: *"first order structurally stable vector fields of "* χ. In [T1] this result was generalized for 2-manifolds with boundary in which the techniques and results of Theorem 2 were fundamental. These two results are summarized as follows:

Theorem 4: Call $\chi_1^r = \chi^r - \Sigma_0$ $(r \geq 3)$ the bifurcation set of χ. There exists a $C^{r-1} - immersed$ codimension-one submanifold Σ_1 of χ such that:

(i) Σ_1 is dense in χ_1;

(ii) for any X in Σ_1, there exists a neighborhood B in the intrinsic topology of Σ_1 such that any Y in B is $C^0 - equivalent$ to X;

(iii) Σ_1^r, as well as the part of Σ_1^r imbedded in χ , are characterized.

Following the last theorem we may of course classify the stable one-parameter families X_λ of vector fields in χ by means of the concept of transversality. It is usual to say that X_λ presents a codimension-one bifurcation at $\lambda = 0$ if $X_0 \in \Sigma_1$. This research program attempts the classification of the codimension - k bifurcations in $\chi^r(M)$. It should be mentioned that [T2] contains results concerning codimension 2 bifurcations of vector fields defined on manifolds with boundary. Again the main ideas and techniques come from the former results of Peixoto. When, throughout the paper, the treatment is local we use the germ terminology.

3 Vector fields in manifolds with boundary

In this section we discuss some results concerning the problem of classification of dynamical systems defined on manifolds with boundary under $C^0 - orbital$ equivalence. The techniques introduced in above results on the contact between vector fields and S are used frequently, but details are omitted. We also recall that, tools in singularity of mappings are fundamental in this approach.

3.1 In dimension 2

We present here the terminology, concepts and some results introduced in [T1].

3.1.1 Structural stability in manifolds with boundary

For simplicity we assume in this subsection that there exists $f: M \to R$, a C^∞ function having 0 as regular value with S = $\{f^{-1}(0)\}$ and $f(q) \geq 0$ for all q in M.

Let $X \in \chi$ be as above. Call $S = \partial M$.

Definition 1: We say that $p \in S$ is an $S-singularity$ of X if either $X(p) = 0$ or $X(p) \neq 0$ and $Xf(p) = 0$.

Definition 2: We say that $p \in S$ is a $fold$ singularity of X if $X(p) \neq 0$, $Xf(p) = 0$ and $XXf(p) \neq 0$. In this case we say that the contact between the orbit of X and S at p is *quadratic*.

A separatrix of X is an orbit which connects either two saddle critical points or two tangency points between the vector field and S or a tangency point and a saddle critical point. Any equivalence between two vector fields in χ must preserve such objects.

3.1.2 Generic bifurcation in manifolds with boundary (local setting)

In this subsection we comment briefly the boundary codimension-one singularities. They play an important role in characterizing the set Σ_1 presented in Theorem 4.

Let $p \in S$ and $\chi(p)$ be the space of all germs of $C^r - vector\ fields$ at p. The sets $\Sigma_0(p)$

and $\chi_1(p)$ are defined as above. Assume that $X \in \chi_1(p)$.

Definition 3: We say that $p \in S$ is a *cusp* singularity of X if $X(p) \neq 0$, $Xf(p) = XXf(p) = 0$ and $XXXf(p) \neq 0$.

Definition 4: A codimension-one $S-singularity$ of X is either a *cusp* singularity or an $S-hyperbolic$ **critical point** p in S of the vector field. In the second case this means that p is a hyperbolic critical point of X. Some generic extra assumptions are usually assumed.

The set of elements $X \in \chi_1(p)$ such that p is an $S-singularity$ of X will be denoted by $\Sigma_1(p)$.

We recall that given $X \in \chi_1(p)$, the following orbits have to be distinguished: a) an invariant manifold of a saddle critical point $p \in S$; b) a strong invariant manifold of a nodal critical point $p \in S$; c) an orbit of X tangent to S at p. Any C^0 equivalence between two elements of χ must necessarily preserve such objects. We may refer to them as $S-separatrices$ of X.

The next result is in proved in [T1] will be used in the sequel

Proposition 1 Let $X \in \chi_1(p)$ and $p \in S$. The vector field X is structurally stable (at $p \in S$) relative to $\chi_1(p)$ if and only if $X \in \Sigma_1(p)$. Moreover, $\Sigma_1(p)$ is an embedded codimension-one submanifold and dense in $\chi_1(p)$.

The following result is also in [T1].

Proposition 2(Normal forms) (1) $X \in \Sigma_0(p)$ iff X is equivalent to one of the following normal forms:

$(0.i) : X(x,y) = (0,1)$ (regular case);

$(0.ii) : X(x,y) = (1, \delta x)$ with $\delta = \pm 1$ (fold singularity).

(2) Any one-parameter family X_λ , $(\lambda \in (-\varepsilon, \varepsilon))$ in χ transverse to $\Sigma_1(p)$ at X_0, has one of the following normal forms:

(1.1) $X_\lambda(x,y) = (1, \lambda + x^2)$ (cusp singularity);

(1.2) $X_\lambda(x,y) = (ax, x + by + \lambda), a = \pm 1, b = \pm 2$;

(1.3) $X_\lambda(x,y) = (x, x - y + \lambda)$;

(1.4) $X_\lambda(x,y) = (x + y, -x + y + \lambda)$.

3.2 In dimension 3:

We discuss here the results in [ST1], where is studied the local behavior of a a vector field **near the boundary** of a 3-manifold. We use the same notations as in 3.1 (in 2D); that means M and $S = \partial M$. In this way let $f : M, S \to R, 0$ be a germ representation of the **boundary** of M around p.

Theorem 5:

(i) $X \in \Sigma_0(p)$ **if and only if**

a) $X(p) \neq 0$;

b) either (b$_1$) $Xf(p) \neq 0$,**(b$_2$)** $Xf(p) = 0$ **and** $X^2 f(p) \neq 0$ **or (b$_3$)** $Xf(p) = X^2 f(p) = 0$ **and** $\{df(p), dXf(p), dX^2 f(p)\}$ **are linearly independent;**

(ii) $\Sigma_0(p)$ **is open and dense in** $\chi(p)$.

The points in S at which $Xf \neq 0$ (resp. $Xf = 0$) are called $S - regular$ (resp. $S - singular$) points of X. The points of S where (b$_2$) is satisfied are called *fold* singularities; they form smooth curves in S, along which X has quadratic contact with S. The set where (b$_3$) is satisfied is the union of isolated points of cubic contact between X and S , located at the extremes of the curves of fold singularities, called *cusp* singularities.

At this point we observe that $\chi_1(p)$ splits as $\chi_1(p) = A \cup B$ such that $X \in A$ (resp. $X \in B$) provided $X(p) = 0$ (resp. $X(p) \neq 0$).

Definition 5: An *S − hyperbolic critical point* of X is a hyperbolic critical point $p \in S$ of X such that:

(i) the eigenvalues of $DX(p)$ are pairwise distinct and the corresponding eigenspaces are transverse to S at p;

(ii) each pair of non complex conjugate eigenvalues have distinct real parts.

Denote by $\Sigma_1(a)$ the the collection of X in A such that p is an $S-hyperbolic$ *critical point* of X.

Definition 6: Call $\Sigma_1(b)$ the set of vector fields X in B such that $X(p) \neq 0, Xf(p) = 0, X^2f(p) = 0$ and one of the following conditions hold:

$Q_1 : X^3f(p) \neq 0$, and $rank\{Df(p), DXf(p), DX^2f(p)\} = 2$ and the function $Xf\mid_S$ has a non-degenerate critical point at p;

$Q_2 : X^3f(p) = 0, X^4f(p) \neq 0$ and p is a regular point of $Xf\mid_S$.

The following result is proved in [ST1].

Theorem 6:(i) $\Sigma_1(p) = \Sigma_1(a) \cup \Sigma_1(b)$;(ii) $\Sigma_1(p)$ is a codimension-one submanifold of $\chi(p)$;(iii) $\Sigma_1(p)$ is open and dense in $\chi_1(p)$;(iv) the normal forms of the stable one-parameter families of vector fields in $\chi(p)$ transversal to $\Sigma_1(p)$ are exhibited.

3.3 In dimension n:

3.3.1 A theorem of Sotomayor:

Consider now χ be the space of C^∞ vector fields defined on a compact C^∞ *n-dimensional* manifold with boundary. Endow χ with the $C^r -$ *topology* with $r > n$. In [S2] is stated the following result:

Theorem 7: There is an open generic set $\Sigma \subset \chi$ such that: (i) for any $\xi : [0,1] \to \Sigma$, continuous, whose the evaluation $\xi(\lambda)(x)$ is C^∞ on $[0,1] \times M$, there is an isotopy $h(\lambda)$ (between $h(1)$ and $h(0) = Id$) of M such that $h(\lambda)$ maps orbits of $\xi(\lambda)$ onto orbits of $\xi(0)$, for every $\lambda \in [0,1]$; (ii) Any $X \in \Sigma$ is isotopically C^r structurally stable.

3.3.2 A theorem of Percell:

In [Pe], Percell presents the normal forms for codimension 0 tangential singularities of $X \in \chi$. Also he classifies the class of transient vector fields generically in χ. That means those vector fields where each integral curve leaves M in finite positive and negative time. The result is:

Theorem 8: (i) The set $\Upsilon \subset \chi$ of transient vector fields is non-empty and open in χ; (ii) $X \in \Upsilon$ iff it is a gradient field (for some metric) with no critical points; (iii) The set of structurally stable vector fields is open and dense in Υ.

Finally we recall that Vishik in [V] has presented a very nice normal form of a generic vector field near p, obtained from a smooth change of coordinates. It is:

Theorem 9: (Vishik's normal form): Assume that $X \in \chi$, with $p \in \partial M$ and $X(p) \neq 0$. There exists a coordinate systems around p such that:

$$X(x_1, x_2, ..., x_n) = (x_2, x_3, ..., x_{k+1}, 1, 0, ..., 0) \text{ with } 0 \leq k \leq m-1;$$

and the boundary is represented by the the equation $\{x_1 = 0\}$.

4 Discontinuous vector fields

Some problems in control theory and nonlinear oscillations lead to differential equations whose right hand terms are defined by discontinuous vector fields.

Let N be an $n-$dimensional compact manifold and $f : N \to R$ be a C^∞ function having 0 as regular value. Denote $S = \{f^{-1}(0)\}$, $N^+ = f^{-1}(0,\infty)$ and $N^- = f^{-1}(-\infty, 0)$.

Denote the space of C^r vector fields on M $(r > 1)$ by χ.

Let $\Omega = \Omega(N, f)$ be the space of vector fields Z on N defined by:

$$Z(q) = \begin{cases} X(q) & \text{if} \quad f(q) > 0. \\ Y(q) & \text{if} \quad f(q) < 0. \end{cases}$$

where $X, Y \in \chi^r$. To point out the dependence on X and Y we write $Z = (X, Y)$.

On S the solution curves of Z are given by the rules of Gantmaher and Filippov (see [F]) which are given in what follows

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Given any $Z = (X, Y)$ in Ω we distinguish the following regions in S :

- *Sewing* Region (SW), characterized by $(Xf)(Yf) > 0$.

- *Escaping* Region (ES) , given by the inequalities $Xf > 0$ and $Yf < 0$.

- *Sliding* Region (SL), given by the inequalities XXf$Xf < 0$ and $Yf > 0$. On this region we define a vector field $F^+ = F^+(X, Y)$ (called the SL-vector field associated to $Z = (X, Y)$) as follows. If $p \in SL$, then $F^+(p)$ denotes the vector in the cone spanned by $X(p)$ and $Y(p)$ tangent to S.

Observe that on ES we define another vector field F^- by $F^-(p) = (-F^+(-X, -Y))(p)$. We refer to either F^+ or F^- as $F(Z)$.

4.1 Regularization

In what follows we are going to discuss some results on regularization of discontinuous planar vector fields contained in [ST2]. We restrict ourselves to the local theory. We just mention that in [LT] similar results were also obtained in 3D.

Let N be the standard 2-sphere in R^3 and $f : N \rightarrow R$ be a C^∞ function having 0 as regular value. We assume for simplicity that $S = \{f^{-1}(0)\}$ has a single connected component in such a way that $N \setminus S$ has two connected components, that are two discs denoted by $N^+ = f^{-1}(0, \infty)$ and $N^- = f^{-1}(-\infty, 0)$.

By a *transition function* we mean a C^∞ function
$\varphi : \rightarrow \Re$ such that: $\varphi(t) = 0$ if $t \leq -1$, $\varphi(t) = 1$ if $t \geq 1$ and $\varphi'(t) > 0$ if $t \in (-1, 1)$.

Definition 8: The φ_ε-regularization of $Z = (X, Y) \in \Omega$ is the one parameter family of vector fields Z_ε in χ^r given by

$$Z_\varepsilon(q) = (1 - \varphi_\varepsilon(f(q))Y(q) + \varphi_\varepsilon(f(q))X(q) \text{ where } \varphi_\varepsilon(t) = \varphi(\tfrac{t}{\varepsilon}) \text{ .}$$

In that paper we gave conditions on $Z = (X, Y)$ which determine the global phase portrait of its regularization and guarantee the structural stability of Z_ε, for any transition function and small ε. This is achieved by using the characterization of the class Σ_0 of the structurally stable vector fields on smooth submanifolds of N, due to Andronov-Pontryaguin and Peixoto (given above). In our approach we restricted ourselves to local settings.

4.1.1 Local Settings

Let $p \in S$ and $Z = (X, Y)$.
Definition 9: A point $p \in S$ is an **S-regular point** of Z if one of the following conditions is satisfied:
Definition 10: $p \in S$ is an **elementary S-singular point** of $Z = (X, Y)$ if one of the following conditions is satisfied:

(i) p is a *fold point* of $Z = (X, Y)$. This means that:
either p is a "*fold point of* Y": $Xf(p) \neq 0, Yf(p) = 0$ and $YYf(p) \neq 0$;
or "*p is a fold point of* X": $Yf(p) \neq 0, Xf(p) = 0$ and $XXf(p) \neq 0$;

(ii) $Xf(p).Yf(p) < 0$, $Det[X, Y](p) = 0$ but $d(Det[X, Y]_{|S})(p) \neq 0$. A simple calculation shows that this condition is equivalent to: " p is a *hyperbolic critical point* of $F(Z)$".

Theorem 10: Let p be an elementary $S-$singularity of $Z = (X, Y)$. Then there exists a positive number ε_0 such that for any $\varepsilon < \varepsilon_0$, Z_ε is in $\Sigma_0(p)$.

4.2 Stability

It should be mentioned that the structural stability inside the class of discontinuous systems has been studied by Kozlova [K] with no appeal to regularization methods. In [T4] and [T7] aspects of the structural stability and asymptotic stability of discontinuous vector fields in 3D are analyzed.

5 Reversible vector fields

It is generally acknowledged that time-reversal symmetry is one of the fundamental symmetries discussed in many branches of physics. Time-reversible systems share many properties of Hamiltonian systems. In [LR] an interesting survey on reversibility in dynamical systems is presented.

Let M be a C^∞ compact orientable two-dimensional manifold and $f : M \to R$ be a C^∞ function having 0 as regular value. Call $S = \{f^{-1}(0)\}$, $M^+ = f^{-1}[0, \infty)$, $M^- = f^{-1}(-\infty, 0]$.

Let $\varphi : M \to M$ be a C$^\infty$ diffeomorphism (an involution) from M onto M such that $\varphi \circ \varphi = Id$ (φ is an *involution*) and $Fix\varphi = S$.

We say that a vector field X on M is $\varphi - reversible$ (or simply reversible) if

$$\varphi * X = -X \circ \varphi.$$

Let Φ^r be the space of the C^r $\varphi-$reversible vector fields on M endowed with the C^r - topology ($r > 2$).

Any critical point of $X \in \Phi^r$ contained in S is called a *symmetric* singularity of the vector field.

The main result in[T6] has a close connection with the results in [A], [P], [PP], [S1] and [T1]. It says that:

Theorem 11: *The set Σ_0 of all vector fields in M which are structurally stable is open and dense in Φ^r. Moreover $X \in \Sigma_0$ if and only if the following conditions are satisfied:*

- *(0) X does not have nontrivial recurrent trajectories;*
- *(i) all asymmetric critical points of X are hyperbolic;*
- *(ii) all asymmetric periodic orbits of X are hyperbolic;*
- *(iii) X does not have saddle connections on M^+;*
- *(iv) all symmetric singularities of X are of codimension 0.*

Call $\Phi_1 = \Phi^r - \Sigma_0$ the bifurcation set of Φ^r. There exists a C^{r-1} immersed codimension-one submanifold Σ_1 of Φ^r such that:

(i) Σ_1 is dense in Φ_1 (both with the relative topology);

(ii) for any X in Σ_1, there exists a neighborhood B_1 in the intrinsic topology of Σ_1 such that any Y in B_1 is topologically equivalent to X;

(iii) the part of Σ_1 imbedded in Φ^r is also characterized.

In [T5], we classified all the symmetric singularities of codimension 0, 1 and 2 of $X \in \Phi^r$. There we presented a technique which enabled us to classify in a simple manner those singularities. In that paper the treatment is local and the technique consists in making a special change of coordinates around the point and then address the analysis to the **study of the contact between a general system and S**. We followed those ideas and use extensively the tools of the Singularity Theory and the results contained in [A], [P], [PP] and [T1]. In our setting, the strategy is to establish a connection between a reversible system on M and a vector defined on M^+. Roughly speaking, having reduced the system to the study of vector fields defined in manifolds with boundary, the next step is to employ known results.

In the class of reversible vector fields some persistent phenomena occur which cannot be destroyed by perturbations in Φ. Examples ares periodic orbits and saddle connections which meet the submanifold S. However, concerning non trivial recurrences no surprises arise at all. As a matter of fact, this point becomes in some sense simpler in this class. We mention for example that such reversible systems on the torus do not admit an irrational flow.

5.1 Local Settings

Let Ω_0 be the space of the germs of C^r reversible vector fields at 0 on R^2 endowed with the C^r topology, $r > 3$.

Theorem 9: (i) The normal forms of a codimension 0 singularity in Ω_0 are: (0) $X_0(x,y) = (0, 1/2)$, (i) $X_{01}(x,y) = (y, \frac{x}{2})$ and (ii) $X_{02}(x,y) = (-y, \frac{x}{2})$.

(b) *(codimension one singularity classification)* - In the space of one-parameter families of vector fields in Ω_0, an everywhere dense set is formed by generic families such that their (C^0-) normal forms are:

(1.0) The codimension 0 normal forms in Ω_0;

(1.1) $X_\lambda(x,y) = (y, \frac{\lambda + x^2}{2})$;

(1.2) $X_\lambda(x,y) = (\varepsilon xy, \frac{2\varepsilon y^2 + x + \lambda}{2})$ with $\varepsilon = \pm 1$;

(1.3) $X_\lambda(x,y) = (xy, \frac{-y^2 + x + \lambda}{2})$;

(1.4) $X_\lambda(x,y) = (xy + y^3, \frac{-x + y^2 + \lambda}{2})$.

5.1.1 Basic concepts and definitions

We shall deal with those involutions which are germs of C^∞ diffeomorphisms (at 0) $\phi : \mathbb{R}^2, 0 \Rightarrow \mathbb{R}^2, 0$, satisfying $(\phi \circ \phi) = \mathrm{Id}$ and $\mathrm{Det}(\mathrm{D}\phi(0)) = -1$. The set $S = \mathrm{Fix}\{\phi\}$ is a smooth curve in $\mathbb{R}^2, 0$. It is well known (Montgomery-Bochner Theorem in [MZ]) that such an involution is C^∞ conjugated to $\phi(x, y) = (x, -y)$.

Let X be a (germ of) C^∞ vector field on$R^2, 0$ and ϕ be an involution.

We fix coordinates in$R^2, 0$ in such a way that $\phi(x, y) = (x, -y)$ and denote by Ω the set of all ϕ-reversible (or just reversible) vector fields on$R^2, 0$. In these coordinates we have that $S = \{y = 0\}$.

Endow Ω_0 with the C^r-topology with $r > 3$.

Any critical point of $X \in \Omega_0$ on S (fixed set of ϕ) is called a *symmetric singularity* (or simply singularity) of X; otherwise it is an asymmetric singularity. Any other point in$R^2, 0$ is a regular point of X.

5.1.2 A construction

The coming construction will be useful in the sequel.

Let X be in Ω_0. In the coordinates (x, y) given above, the definition 2.1.1 implies the following general form for X:

$$X(x, y) = (yf(x, y^2), \tfrac{g(x,y^2)}{2}) \tag{III.1}$$

In the half-plane $y > 0$, consider

$$u = x \quad \text{and} \quad \mathrm{v} = \mathrm{y}^2.$$

A simple calculation shows that in these coordinates X is transformed into:

$$X'(u, v) = (\sqrt{v} f(u, v), \sqrt{v} g(u, v)) \quad \text{in} \quad \mathrm{v} > 0.$$

It follows that in $y > 0, X$ is topologically equivalent to $F = F(X)$, where

$$F(u, v) = (f(u, v), g(u, v)) \quad \text{for} \quad \mathrm{v} > 0.$$

Observe now that F can be C^r extended to a full neighborhood of 0. Due to the symmetry properties of X (with respect to the canonical involution) we deduce that the behavior of $F(X)$ at $M, 0$ determines completely the behavior of X at 0. So the problem now is carried out to analyze the phase portrait of

F in M. We make no distinction between F and any one of its extensions.

Recall that at a regular point the trajectory of X is always orthogonal to S. At a critical point of X, the contact between an invariant manifold and S decays by a factor of $1/2$ in comparison with the orbit or invariant manifold of $F(X)$ passing through the same point. We illustrate this fact by assuming that $\{v = u^k, k > 0\}$ is an invariant manifold of $F(X)$ on the region $v \geq 0$. This implies that the curve $y = x^{k/2}$ is a an invariant manifold of X on $y \geq 0$.

5.1.3 Symmetric singularities

Now we present some examples where a key connection between vector fields defined in manifolds with boundary and reversible systems is established
Example 1: As $X(0) = 0$ we have that $g(0) = 0$. So we may write $f(u, v) = a_0 + a_{10}u + a_{01}v+$ h.o.t and $g(u, v) = b_{01}u + b_{10}v+$ h.o.t.

Observe that $Fh(u, v) = g(u, v)$.

Assume that 0 is a codimension zero singularity of $F(X)$. Then the origin is either a saddle critical point (in the case $F^2h(0) > 0$) or a elliptic critical point (in the case $F^2h(0) < 0$).

By means of the results in [T1] and using the same technique of the lemma we may classify the *codimension-k* symmetric singularities in Ω_0. We mention that in [MT] similar result as Theorem 8 was obtained in $3D$.
Example 2: Consider now X a germ of a C^∞ vector field on $\mathbb{R}^2, 0$ with $X(0) = 0$ which is $C^0 - equivalent$ to:

$$X_{k,m}(x, y) = (y^{2k-1}(a + f(x, y^{2k})), bx^m + g(x, y^{2k}))$$

with $k, m \in N$, $k \geq 1$, $f(0, 0) = 0$, $g(0, 0) = 0$ and $g(x, 0) = O(m + 1)$.

For each $X_{\alpha\beta}$ given above we consider the associated linear vector field $T_{\alpha\beta}$ given by:

$$T_{\alpha\beta}(u, v) = (\alpha(au + bv), 2\beta(cu + bv)).$$

Let λ_1 and λ_2 be the eigenvalues of $T_{\alpha\beta}$ and denote T_1, T_2 the respective eigenspaces. Denote $Trace(T_{\alpha\beta}) = Tr_{\alpha\beta}$, $D_{\alpha\beta} = Det(T_{\alpha\beta})$ and $\Delta_{\alpha\beta} = (Tr_{\alpha\beta})^2 - 4D_{\alpha\beta}$.

As a matter of fact, we can classify generically this class of vector fields by means of such $T_{\alpha\beta}$. Below the conditions imposed on $T_{\alpha\beta}$ are easy to be expressed from the parameters a, b, c and d.

In [T8] it is proved the following result:

"(i) 0 is a singularity of center type of X provided that m is odd and $ab<0$.
(ii) 0 is a cusp provided m is even (the degeneracy of such cusp depends on m and k).
(iii) 0 is a saddle point provided m is odd and $ab>0$ (the degeneracy of such saddle depends on m and k).
(iv) $X_{k,m}(x,y)=(ay^{2k-1},bx^m)$ is in fact a C^0-normal form for this class of vector fields."

We give a brief idea of the proof of the above case (i); the other cases are treated similarly.

First of all, observe that the vector field X is $\varphi-reversible$, with $\varphi(x,y)=(x,-y)$.

On the region $K=\{y>0\}$, we define the following coordinates:

$$u=x \qquad \text{and} \qquad v=y^{2k}.$$

So

$$X(u,v)=y^{2k-1}((a+f(u,v)),2ky^{2k-1}(bu^m+g(u,v))) \text{ on } \{v\geq 0\}$$

In the same spirit as the above construction we define the associated vector field :

$$H(u,v)=(a+f(u,v),2kbu^m+2kg(u,v)).$$

The contact between H and $v=0$ at 0 is even. Due to the relation $ab<0$, we easily deduce that 0 is a center for X.

6 Open problems on regularization of discontinuous vector fields

6.1 Introduction

Let M be a n-dimensional compact, connected and orientable C^∞ manifold .

Let $f: M\to I\!R$ be a C^∞ function having 0 as regular value. Call $S=\{f^{-1}(0)\}$. Denote by χ^r the space of C^r vector fields on M with $r>1$ and by χ the space of Lipschitz vector fields on M.

Denote $M^+=\{f^{-1}[0,\infty)\}$ and $M^-=\{f^{-1}(-\infty,0]\}$.

Let Ω^r be the space of vector fields Z on M defined by

$$Z(q)=\begin{cases} X(q) & \text{if} \quad f(q)>0,\\ Y(q) & \text{if} \quad f(q)<0,\end{cases}$$

where $X, Y \in \chi^r$. This vector field is denoted by $Z = (X, Y)$.

Let $Z = (X, Y)$ be in Ω^r. Given a positive number ε we consider the C^r function $\varphi_\varepsilon : I\!R \to I\!R$ such that

- $\varphi_\varepsilon(t) = 0$ if $t \leq -\epsilon$
- $\varphi_\varepsilon(t) = 1$ if $t \geq \epsilon$
- $\varphi'_\varepsilon(t) > 0$ if $t \in (-\varepsilon, \varepsilon)$

Recal that an $\varepsilon - regularization$ of $Z = (X, Y) \in \Omega^r$ is a vector field Z_ε in χ^r defined by

$$Z_\varepsilon(q) = (1 - \varphi_\varepsilon(f(q)))Y(q) + \varphi_\varepsilon(f(q))X(q).$$

We also consider the *piecewise linear* function $\phi_\varepsilon : I\!R \to I\!R$ such that

$$\phi_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad if\ t \leq -\varepsilon, \\ \frac{t+\varepsilon}{2\varepsilon} & \text{if} \quad if\ t \in (-\varepsilon, \varepsilon), \\ 1 & \text{if} \quad if\ t \geq \varepsilon. \end{cases}$$

An $\varepsilon - L - regularization$ of $Z = (X, Y) \in \Omega^r$ is a vector field Z_ε in χ defined by

$$Z_\varepsilon(q) = (1 - \phi_\varepsilon(f(q)))Y(q) + \phi_\varepsilon(f(q))X(q).$$

This kind of regularization of Z unfolds the discontinuous surface $S = \{f^{-1}(0)\}$, and we are able to discuss whether the $\lim_{\varepsilon \to 0} Z_\varepsilon$ agrees with the Filippov's convention (in [F]) about the extension of orbit solutions of the vector field Z through the discontinuous surface.

Recall the definitions contained in Section 4.

6.2 Problems

I - CASE $\partial M = \emptyset$

1. Local Settings

In [ST2] and [LT] the extension of orbit solutions through the discontinuity surface of $Z = (X, Y)$ in Ω^r and its relationship with the Filippov's convention in dimension two and three respectively, was analyzed using this regularization technique. Moreover the qualitative behavior of the $\varepsilon - regularizated$ and $\varepsilon - L - regularized$ vector fields Z_ε around generic singularities were also studied there.

The problems in question are:

Question (i): *Classify the generic S-singularities of* $Z \in \Omega^r$ *and exhibit their* C^0- *normal forms* (Papers related with this question: [F], [K], [LT], [ST1], [ST2], [T3], [T4] and [V]).

Question (ii): *Study the behavior of the* $\varepsilon-$ *regularized and* $\varepsilon - L-$ *regularized vector fields* Z_ε *derived from the normal forms found above* (Papers related with this question: [LT] and [ST2]).

Question (iii): *Prove or disprove the following statement:* $x : [0,T] \to I\!R$ *is a (Filippov) solution of the discontinuous differential equation* $x'(t) = Z(x(t))$, $x(0) \in S$ *if and only if there is a sequence of smooth solutions* $x_\varepsilon : [0,T] \to I\!R$ *of the regularized differential system* $x'_\varepsilon(t) = Z_\varepsilon(x(t)$ *uniformly converging to* $x(t)$ *as* $\varepsilon \to 0^+$ (Papers related with this question: [F], [LT] and [ST2]).

2. Periodic Orbits

A closed curve γ formed by pieces of regular orbits of X in M^+ and regular orbits of Y in M^- is an $S-$*periodic orbit* of $Z = (X,Y)$ if γ meets S only in SW.

Question (iv): *Under which conditions on* Z *there exists* $\varepsilon_0 > 0$ *such that for every positive* $\varepsilon < \varepsilon_0$, Z_ε *has a periodic orbit* γ_ε *nearby* γ *in* M? *How about its hyperbolicity?* (Paper related with this question:[ST2]).

3. Invariant manifolds

Assume that S is connected.

Question (v): Suppose that every point p in S satisfies $Xf(p).Yf(p) < 0$. This phenomenon is called a *simple graph*.

Prove or disprove: There exists $\varepsilon_0 > 0$ *such that for every positive* $\varepsilon < \varepsilon_0$, Z_ε *has an invariant manifold* S_ε , *diffeomorphic to S. If the last assertion is true then under which conditions is* $Z_\varepsilon |_{S_\varepsilon}$ *a Morse-Smale vector field?* (Paper related with this question: [ST2]).

Question (vi): For simplicity assume here that $n = 3$. We assume that there are an annulus A contained in S and V a small neighborhood of A in S, such that:

(a) $\partial A = A_1 \cup A_2$ (union of two circles);

(b) $A_1 = \{p \in R; Xf(p) = 0\}$. Moreover any $p \in A_1$satisfies $XXf(p) > 0$.

(c) A_1 is a separating curve in V, of the regions $A^+ = \{Xf > 0\}$ and $A^- = \{Xf < 0\}$. We are imposing that A_2 is contained in A^-.

d) Any forward (resp. backward) orbit of X passing through any point $q_1 \in A_1$ (resp. $q_2 \in A_2$) meets S at $p_2 \in A_2$ (resp. $p_1 \in A_1$).

(d) All points in A outside A_1 is a regular point of Z.

(e) $Yf(q) > 0$ for every $q \in V$.

A *singular graphic* of Z is the topological surface T formed by A and the saturated of A_1 , by the flow of X, bounded by A_1 and A_2 in M^+.

Prove or disprove: There exists $\varepsilon_0 > 0$ such that for every positive $\varepsilon < \varepsilon_0$, Z_ε has an invariant torus T_ε. If the last assertion is true then under which conditions is $Z_\varepsilon |_{T_\varepsilon}$ a Morse-Smale vector field? (Paper related with this question: [ST2]).

We remark that similar question can be stated in higher dimension.

4. General Problem

Give necessary and sufficient conditions on Z for the existence of an $\varepsilon_0 > 0$ such that for every positive $\varepsilon < \varepsilon_0$, Z_ε is a Morse-Smale vector field.

II- Case $\partial M \neq \emptyset$

5. Transient vector fields

The problem contained in this section was communicated to me by J. Sotomayor.

Problem: *Give necessary and sufficient conditions on Z for the existence of an $\varepsilon_0 > 0$ such that for every positive $\varepsilon < \varepsilon_0$, Z_ε is a transient vector field.* (Papers related with this question: [Pe] and [ST2]).

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