Existence of weak solutions of micropolar fluid equations in a time dependent domain

A. Jaime Resendiz B.[1,2]
DIM-Universidad de Chile, Casilla17013 Correo 3, Santiago, Chile
and
M.A. Rojas-Medar[3]
DMA-IMECC-UNICAMP, CP 6065, 13081-970, Campinas-SP, Brazil

Abstract: In this paper we study the existence of weak solutions of the initial-boundary value problem for micropolar equations in domains with smoothly moving boundaries.

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1. Introduction

Many problems in fluid mechanics occur in time-varying regions. Such situations arise for instance in the case

· A fluid in a vessel with moving boundaries.

· A fluid in a vessel containing rigid bodies moving through it.

Partial differential equations governing such phenomena are defined in a non-cylindrical domain. This leads to theorical as well as numerical difficulties.

Is well known that the micropolar fluids model is an essential generalization of the well established Navier-Stokes model in the sense that it takes into account the microstructure of the fluid. It may represent fluids consisting of randomly oriented (or spherical) particles suspended in a viscous medium, whom the deformation of

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[1]Ph-D Student, supported by FONDAP Programme in Mathematical-Mechanics (CONICYT-Chile).

[2]Permanent Adress: Dpto. De Ciencias Básicas, Instituto Tecnológico de Querétaro, Av. Tecnológico esq. M. Escobedo, Querétaro Qro. México.

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fluid particles is ignored. Micropolar fluid were introduced in [4]. They are non-Newtonian fluids with nonsymmetric stress tensor.

The purpose of this paper is to show the existence and periodicity of weak solutions of the initial-boundary value problem for the micropolar equations in domains with smoothly moving boundaries.

The micropolar fluids equations in $Q_\infty = \cup_{t\in I\!R}\Omega(t) \times \{t\}$ is the following:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\nu+\nu_r)\Delta\mathbf{u} + \nabla p = 2\nu_r \text{ rot } \mathbf{w} + \mathbf{f}, \text{ in } \mathbf{x}\in\Omega(t), \\ \text{div } \mathbf{u} = 0 \text{ in } \mathbf{x}\in\Omega(t),, \\ \dfrac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w} \\ \qquad +4\nu_r\mathbf{w} = 2\nu_r \text{ rot } \mathbf{u} + \mathbf{g}, \text{ in } \mathbf{x}\in\Omega(t), \\ \mathbf{u} = \beta, \ \ \mathbf{w} = \eta, \text{ on } \quad \mathbf{x}\in\partial\Omega(t), \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \ \ \mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega(0). \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, and $\mathbf{w} = (w_1, w_2, w_3)$ is the microrotational interpreted as the angular velocity field of rotational of particles. The fields $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_1, g_3)$ are external forces and moments respectively. Positive constants ν, ν_r, c_0, c_a, c_d represent viscosity coefficients, ν is the usual Newtonian viscosity and ν_r is called the microrotational viscosity. And $\Omega(t)$ is a bounded domain in $I\!R^3$, with smooth boundary $\partial\Omega(t)$. $\beta = (\beta_1, \beta_2, \beta_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ are given on the boundary $\partial Q_\infty = \cup_{t\in I\!R}\partial\Omega(t) \times \{t\}$.

It has pointed out that similar time-dependent problem but for the Navier-Stokes equations have been studied by many different authors. This is the case, for instance, of the works by J.L. Lions [8], [9], H. Fujita and N. Sauer [5], H. Morimoto [13], T. Miyakawa and Y. Teramoto [14], R. Salvi [20], etc. In particular, we would like to emphasize that the arguments in [9] requires $\Omega(t)$ to be nondecreasing with respect to t (see problem 11.9, p. 426 of this book). Our paper, other generalize these previous works in the sense that problem (1.1) includes the microrotational velocity, does not assume this nondecreasing condition on $\Omega(t)$.

We observe that the model (1.1) was early studied in [19], [16] for a class special of domains (see also [2]). In this work, we use the approach given in [14].

The class of domains considered in [14] is very general and include the domains used in [19], [16].

Over the past years, many existence (weak and strong), uniqueness results has been done for micropolar fluids. In a fixed domain see [10], [11], [12], [17], [18] and the references therein.

2. Statements and notations

In addition to the conditions (A.1) and (A.2) given for $\Omega(t)$ and β respectively in [14], we give the condition (A.3):

(A.1) There exist a cylindrical domain $\tilde{Q}_\infty = \tilde{\Omega} \times I\!R$ and a level-preserving C^∞ diffeomorphism $\Phi : \overline{Q}_\infty \to \overline{\tilde{Q}}_\infty$,

$$(\mathbf{y}, s) = \Phi(\mathbf{x}, t) = (\phi_1(\mathbf{x}, t), \phi_2(x, t), \phi_3(\mathbf{x}, t),\ t)$$

such that

$$\det\,[\partial\phi_i(\mathbf{x}, t)/\partial x_j] \equiv J(t)^{-1} > 0 \text{ for } (\mathbf{x}, t) \in \overline{Q}_\infty.$$

(A.2) β is the restriction to ∂Q_∞ of a C^2 vector field ψ, which is divergence-free on each $\Omega(t)$ and bounded on $\overline{Q}_\infty$ together with its first and second derivatives.

(A.3) η is the restriction to ∂Q_∞ of a C^2 vector field φ.

Then, by (A.2) and (A.3) the micropolar fluids equations can be reduced to the case zero boundary values.

Putting $\mathbf{u} = \psi + \mathbf{v}$ and $\mathbf{w} = \varphi + \mathbf{z}$ in the above equations, we obtain,

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - c\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\psi + (\psi\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \\ 2\nu_r \text{ rot } \mathbf{z} + \mathbf{F}, \quad \mathbf{x} \in \Omega(t), \\ \text{div } \mathbf{u} = 0 \quad \mathbf{x} \in \Omega(t),, \\ \dfrac{\partial \mathbf{z}}{\partial t} - c_1\Delta\mathbf{z} - c_2\nabla \text{ div } \mathbf{z} + (\mathbf{v}\cdot\nabla)\varphi + (\psi\cdot\nabla)\mathbf{z} + (\mathbf{v}\cdot\nabla)\mathbf{z} \\ \qquad +4\nu_r\mathbf{z} = 2\nu_r \text{ rot } \mathbf{v} + \mathbf{G}, \quad \mathbf{x} \in \Omega(t), \\ \mathbf{v} = 0, \ \ \mathbf{z} = 0, \quad \mathbf{x} \in \partial\Omega(t), \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x}), \ \ \mathbf{z}(\mathbf{x}, 0) = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \Omega(0). \end{cases} \tag{2.1}$$

where $c = (\nu+\nu_r)$, $c_1 = (c_a + c_d)$, $c_2 = (c_0 + c_d - c_a)$, $\mathbf{F} = -\psi_t + (\nu+\nu_r)\Delta\psi - (\psi\cdot\nabla)\psi + 2\nu_r \text{ rot } \psi$, $\mathbf{G} = -\varphi_t + C_1\Delta\varphi + C_2\nabla \text{ div } \varphi - 4\nu_r\varphi - (\varphi\cdot\nabla)\varphi + 2\nu_r \text{ rot } \psi$, $\mathbf{a}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) - \psi(\mathbf{x},0)$, and $\mathbf{b}(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) - \varphi(\mathbf{x},0)$.

Here for every vectorial field, we denote

$$\widetilde{u}_j(\mathbf{y},s) = \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} u_k(\Phi_1^{-1}(\mathbf{y},s))$$

analogously for every scalar field

$$\widetilde{q}(\mathbf{y},s) = q(\Phi_1^{-1}(\mathbf{y},s))$$

Using these transformations, the original system of equations (1)-(3) in Q, becomes:

$$\begin{cases} \frac{\partial}{\partial s}\tilde{\mathbf{v}} - O\tilde{\mathbf{v}} + M\tilde{\mathbf{v}} + N_1\tilde{\mathbf{v}} + N_2\tilde{\mathbf{v}} = \text{ rot }_g\widetilde{\mathbf{z}} + \widetilde{\mathbf{F}} - \nabla_g\widetilde{p} \\ \quad \mathbf{y}\in\tilde{\Omega}(s), \quad s>0, \\ \text{div } \tilde{\mathbf{u}} = 0 \quad \mathbf{y}\in\Omega(s), \\ \frac{\partial}{\partial s}\widetilde{\mathbf{z}} - L\widetilde{\mathbf{z}} + T\widetilde{\mathbf{z}} + X_1\widetilde{\mathbf{z}} + X_2\widetilde{\mathbf{z}} + 4\nu_r\widetilde{\mathbf{z}} = \text{ rot }_g\tilde{\mathbf{u}} + \widetilde{\mathbf{G}}, \\ \quad \mathbf{y}\in\tilde{\Omega}(s), \quad s>0, \\ \tilde{\mathbf{v}} = 0, \ \ \tilde{\mathbf{z}} = 0, \quad \mathbf{y}\in\partial\tilde{\Omega}(s), \quad s>0, \\ \mathbf{v}(\mathbf{y},0) = 0\,, \quad \mathbf{z}(\mathbf{y},0) = 0, \quad \mathbf{y}\in\partial\tilde{\Omega}(0), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} (O\tilde{\mathbf{v}})_i &= cg^{jk}\nabla_j\nabla_k\tilde{\mathbf{v}}_i, \qquad (L\tilde{\mathbf{v}})_i = c_1 g^{jk}\nabla_j\nabla_k\tilde{\mathbf{v}}_i + c_2\frac{\partial y_i}{\partial x_j}(\frac{\partial^2}{\partial x_j\partial y_l})\tilde{\mathbf{v}}_l \\ (M\tilde{\mathbf{v}})_i &= \frac{\partial y_j}{\partial t}\nabla_j\tilde{\mathbf{v}}_i + \frac{\partial y_j}{\partial x_k}(\frac{\partial^2 x_k}{\partial s\partial y_j})\tilde{\mathbf{v}}_j, \qquad (T\widetilde{\mathbf{z}})_i = \frac{\partial y_j}{\partial t}\nabla_j\widetilde{\mathbf{z}}_i + \frac{\partial y_i}{\partial x_k}(\frac{\partial^2 x_k}{\partial s\partial y_j})\widetilde{\mathbf{z}}_j, \\ (N_1\tilde{\mathbf{v}})_i &= \psi_j\nabla_j\tilde{\mathbf{v}}_i + \tilde{\mathbf{v}}_j\nabla_j\psi_i, \qquad (N_2\tilde{\mathbf{v}})_i = \tilde{\mathbf{v}}_j\nabla_j\tilde{\mathbf{v}}_i, \\ (X_1\widetilde{\mathbf{z}})_i &= \psi_j\nabla_j\widetilde{\mathbf{z}}_i + \tilde{\mathbf{v}}_j\nabla_j\varphi_i, \qquad (X_2\widetilde{\mathbf{z}})_i = \tilde{\mathbf{v}}_j\nabla_j\widetilde{\mathbf{z}}_i, \\ (\nabla_g\tilde{p})_i &= g^{jk}\frac{\partial\tilde{p}}{\partial y_j}, \end{aligned}$$

$$
\begin{aligned}
(\text{ rot }_g\widetilde{\mathbf{z}})_i &= (\frac{\partial y_i}{\partial x_j}\text{ rot }\mathbf{z})_j = \frac{\partial y_i}{\partial x_j}(\xi_{jkl}\frac{\partial}{\partial x_k}\mathbf{z}_l) \\
&= \frac{\partial y_i}{\partial x_j}\xi_{jkl}(\frac{\partial^2 x_l}{\partial x_k \partial y_r}\widetilde{\mathbf{z}}_r + \frac{\partial x_l}{\partial y_r}\frac{\partial}{\partial x_k}\widetilde{\mathbf{z}}_r).
\end{aligned}
$$

and

$$
\begin{aligned}
g^{jk} &= \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}, \qquad g_{i,j} = \frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j}, \\
\nabla_j \widetilde{\mathbf{v}}_i &= \frac{\partial \widetilde{\mathbf{v}}_i}{\partial y_j} + \Gamma^l_{jk}\widetilde{\mathbf{v}}_k, \\
\nabla_k \nabla_j \widetilde{\mathbf{v}}_i &= \frac{\partial(\nabla_j \widetilde{\mathbf{v}}_i)}{\partial y_k} + \Gamma^i_{kl}\nabla_j \widetilde{\mathbf{v}}_l - \Gamma^l_{kj}\nabla_l \widetilde{\mathbf{v}}_i, \\
\Gamma^k_{ij} &= \frac{\partial y_k}{\partial x_l}\frac{\partial^2 x_l}{\partial y_i \partial y_j}.
\end{aligned}
$$

We are using summation convention, i.e.e take sum over repeated indices. For details about above, see [14].

3. Existence of weak solutions

We will denote by C a generic constant. This will appear in most of the estimates to the be obtained. When for any reason we want to emphasize the dependence of a certain constant on a given parameter we will denote this constant with a subscript. Throughout the paper we need the following function spaces $C_0^\infty(\tilde{\Omega})^n, L^2(\tilde{\Omega})^n, H^1(\tilde{\Omega})^3$ and

$$
\begin{aligned}
C^\infty_{0,\sigma}(\tilde{\Omega})^n &= \{\mathbf{v} \in C_0^\infty(\tilde{\Omega})^n \ / \ \text{div } = 0 \text{ in } \tilde{\Omega}\}, \\
H &= \text{closure of } C^\infty_{0,\sigma}(\tilde{\Omega})^n \text{ in } L^2(\tilde{\Omega})^n, \\
V &= \text{closure of } C^\infty_{0,\sigma}(\tilde{\Omega})^n \text{ in } H^1(\tilde{\Omega})^n, \\
U &= \text{closure of } C^\infty_{0}(\tilde{\Omega})^n \text{ in } L^2(\tilde{\Omega})^n, \\
S &= \text{closure of } C^\infty_{0}(\tilde{\Omega})^n \text{ in } H^1(\tilde{\Omega})^n,
\end{aligned}
$$

and similarly are defined the spaces H_t, V_t, U_t and S_t on Ω_t, with their inner products and norms following:

For H and U

$$\begin{aligned}\langle \tilde{\mathbf{v}}, \tilde{\mathbf{u}}\rangle_t &= \int_{\tilde{\Omega}} g_{ij}(\mathbf{y},t)\tilde{\mathbf{v}}_i(\mathbf{y})\tilde{\mathbf{u}}_j(\mathbf{y})\mathbf{J}(t)d\mathbf{y};\\ |\tilde{\mathbf{v}}|_t &= \langle \tilde{\mathbf{v}}, \tilde{\mathbf{v}}\rangle_t^{1/2},\end{aligned}$$

for V and S

$$\begin{aligned}\langle \nabla_g\tilde{\mathbf{v}}, \nabla_g\tilde{\mathbf{u}}\rangle_t &= \int_{\tilde{\Omega}} g_{ij}(\mathbf{y},t)g^{kl}(\mathbf{y},t)\nabla_k\tilde{\mathbf{v}}_i(\mathbf{y})\nabla_l\tilde{\mathbf{u}}_j(\mathbf{y})\mathbf{J}(t)d\mathbf{y};\\ |\nabla_g\tilde{\mathbf{v}}|_t &= \langle \nabla_g\tilde{\mathbf{v}}, \nabla_g\tilde{\mathbf{v}}\rangle_t^{1/2},\end{aligned}$$

for H_t and U_t

$$\begin{aligned}(\mathbf{v},\mathbf{u})_t &= \int_{\Omega(t)} \mathbf{v}_i(\mathbf{x})\mathbf{u}_i(\mathbf{x})d\mathbf{x};\\ \|\mathbf{v}\|_t &= (\mathbf{v},\mathbf{v})_t^{1/2},\end{aligned}$$

for V_t and S_t

$$\begin{aligned}(\nabla\mathbf{v},\nabla\mathbf{u})_t &= \int_{\Omega(t)} \nabla_k\mathbf{v}_i(\mathbf{x})\nabla_l\mathbf{u}_j(\mathbf{x})d\mathbf{x};\\ \|\nabla\mathbf{v}\|_t &= (\nabla\mathbf{v},\nabla\mathbf{v})_t^{1/2}.\end{aligned}$$

We denote also, for each t, V_t^* the dual space of V_t. The norm of $f \in V_t^*$ is defined by

$$\|f\|_t^* = \sup_{\|\nabla\mathbf{v}\|_t\le 1} \langle f,\mathbf{v}\rangle,$$

analogously is defined the norm on V^* and denoted it by $|\cdot|_t^*$.

Now we define a weak solution of the problem (2.1).

Definition 3.1. *Given $\mathbf{a} \in H_0$ and $\mathbf{b} \in U_0$, and $\mathbf{F} \in L^2(0,T;V_t^*)$, and $\mathbf{G} \in L^2(0,T;S_t^*)$, with $T > 0$. We say that $\mathbf{v} \in L^2(0,T;V_t) \cap L^\infty(0,T;H_t)$ and $\mathbf{z} \in L^2(0,T;S_t) \cap L^\infty(0,T;U_t)$ are a weak solution of problem (2.1), if and only if the following identity is satisfied:*

$$-\int_0^T \langle \tilde{\mathbf{v}}, \tilde{\mathbf{u}}_t\rangle_t - \int_0^T \langle \tilde{\mathbf{v}}, M\tilde{\mathbf{u}}\rangle_t + \int_0^T \langle \nabla_g\tilde{\mathbf{v}}, \nabla_g\tilde{\mathbf{u}}\rangle_t + \int_0^T \langle N_1\tilde{\mathbf{v}} + N_2\tilde{\mathbf{v}}, \tilde{\mathbf{u}}\rangle_t$$
$$= \langle \tilde{a}, \tilde{\mathbf{u}}(0)\rangle_0 + \int_0^T \langle \tilde{\mathbf{F}}, \tilde{\mathbf{u}}\rangle_t + \int_0^T \langle \ rot\ _g\tilde{\mathbf{z}}, \tilde{\mathbf{u}}\rangle_t.$$

$$-\int_0^T \langle \widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}_t\rangle_t - \int_0^T \langle \widetilde{\mathbf{z}}, T\widetilde{\mathbf{w}}\rangle_t + \int_0^T \langle \nabla_g \widetilde{\mathbf{z}}, \nabla_g \widetilde{\mathbf{w}}\rangle_t + \int_0^T \langle div\ \widetilde{\mathbf{z}},\ div\ (\nabla \mathbf{y} \widetilde{\mathbf{w}})\rangle_t$$
$$+ \int_0^T \langle X_1 \widetilde{\mathbf{z}} + X_2 \widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}\rangle_t + 4\nu_r \int_0^T \langle \widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}\rangle_t$$
$$= \langle \widetilde{b}, \widetilde{\mathbf{w}}(0)\rangle_0 + \int_0^T \langle \widetilde{\mathbf{G}}, \widetilde{\mathbf{w}}\rangle_t + \int_0^T \langle\ rot\ _g \widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}\rangle_t.$$

for any $\widetilde{\mathbf{u}} = l(t)\widetilde{\mathbf{d}}$ *and* $\widetilde{\mathbf{w}} = h(t)\widetilde{\mathbf{e}}$ *such that* $\widetilde{\mathbf{u}} \in V$, $\widetilde{\mathbf{w}} \in U$, *and* $l, h \in C^1([0,T]; I\!R)$ *with* $l(T) = h(T) = 0$.

The results that we will prove are

Theorem 3.2. *Given* $\mathbf{a} \in H_0$ *and* $\mathbf{b} \in U_0$, *and* $\mathbf{F} \in L^2(0,T;V_t^*)$, *and* $\mathbf{G} \in L^2(0,T;S_t^*)$, *with* $T > 0$. *Then exist a weak solution of problem (2.1), on* $[0,T]$.

Theorem 3.3. *When* $n = 2$, *the solution given in Theorem is unique.*

4. Auxiliar problems

In order to prove our results, we established some preliminaries results. We use Galerkin approximation , then we define approximate solution $\widetilde{\mathbf{v}}_m(t)$, $m \geq 1$ as follows,

$$\begin{aligned}
\widetilde{\mathbf{v}}_m(\mathbf{y},t) &= l_{jm}(t)\widetilde{\mathbf{d}}_j(\mathbf{y},t),\\
\widetilde{\mathbf{v}}_m(\mathbf{y},0) &= l^0_{jm}\widetilde{\mathbf{d}}_j(\mathbf{y},0) \text{ and } l^0_{jm} = \langle \widetilde{a}, \widetilde{\mathbf{d}}_j\rangle_0\\
\widetilde{\mathbf{z}}_m(\mathbf{y},t) &= h_{jm}(t)\widetilde{\mathbf{e}}_j(\mathbf{y},t),\\
\widetilde{\mathbf{z}}_m(\mathbf{y},0) &= h^0_{jm}\widetilde{\mathbf{e}}_j(\mathbf{y},0) \text{ and } h^0_{jm} = \langle \widetilde{b}, \widetilde{\mathbf{e}}_j\rangle_0
\end{aligned}$$

where $\left\{\widetilde{\mathbf{d}}_j(\mathbf{y},t)\right\}$ and $\{\widetilde{\mathbf{e}}_j(\mathbf{y},t)\}$ are the Schmidt orthogonalization with respect to the inner product of H and V, of the sequences $\{\tilde{\alpha}_j\}$ and $\left\{\tilde{\beta}_j\right\}$ of linearly independent vectors in $C^\infty_{0,\sigma}(\tilde{\Omega})$, and in $C^\infty_0(\tilde{\Omega})$, respectively.

$\{l_{jm}(t)\}$ and $\{h_{jm}(t)\}$ are defined as solution of problem following:

$$\begin{aligned}
&\langle \widetilde{\mathbf{v}}'_m, \widetilde{\mathbf{d}}_j\rangle_t + \langle M\widetilde{\mathbf{v}}_m, \widetilde{\mathbf{d}}_j\rangle_t - \langle 0\widetilde{\mathbf{v}}_m, \widetilde{\mathbf{d}}_j\rangle_t + \langle N_1\widetilde{\mathbf{v}} + N_2\widetilde{\mathbf{v}}_m, \widetilde{\mathbf{d}}_j\rangle_t\\
= \ &\langle \widetilde{\mathbf{F}}, \widetilde{\mathbf{d}}_j\rangle_t + \langle \text{ rot } _g\widetilde{\mathbf{z}}_m, \widetilde{\mathbf{d}}_j\rangle_t, \qquad (4.1)
\end{aligned}$$

and

$$\begin{aligned} & \langle \widetilde{\mathbf{z}}'_m, \widetilde{\mathbf{e}}_j \rangle_t + \langle T\widetilde{\mathbf{z}}_m, \widetilde{\mathbf{e}}_j \rangle_t + \langle L\widetilde{\mathbf{z}}_m, \widetilde{\mathbf{e}}_j \rangle_t + \langle X_1\widetilde{\mathbf{z}}_m + X_2\widetilde{\mathbf{z}}_m, \widetilde{\mathbf{e}}_j \rangle_t + 4\nu_r \langle \widetilde{\mathbf{z}}_m, \widetilde{\mathbf{e}}_j \rangle_t \\ = \; & \langle \widetilde{\mathbf{G}}, \widetilde{\mathbf{e}}_j \rangle_t + \langle \text{ rot }_g \tilde{\mathbf{v}}_m, \widetilde{\mathbf{e}}_j \rangle_t. \end{aligned} \tag{4.2}$$

Analogously to [14], [17], [19], it is easy to see that $(\tilde{\mathbf{v}}_m, \widetilde{\mathbf{z}}_m)$ is determined uniquely by the above relation, in a neighborhood of $t = 0$. The proof of the next lemma guarantees that $(\tilde{\mathbf{v}}_m, \widetilde{\mathbf{z}}_m)$ is defined on the whole interval $[0, T]$.

Lemma 4.1. *The solution* $(\mathbf{v}_m, \mathbf{z}_m)$, *is bounded in* $[L^2(0,T;V_t) \cap L^\infty(0,T;H_t)] \times [L^2(0,T;S_t) \cap L^\infty(0,T;U_t)\}$.

Proof. Multiplying (4.2) and (4.3) by $l_{jm}(t)$ and $h_{jm}(t)$, respectively and then summing in j, after returning to Q_∞, we have

$$\begin{aligned} & \frac{d}{dt}\|\mathbf{v}_m(t)\|_t^2 + 2c\|\nabla \mathbf{v}_m(t)\|_t^2 + \\ \leq \; & 2\,|(\mathbf{v}_m \nabla\psi, \mathbf{v}_m)_t| + 2(F(t), \mathbf{v}_m(t))_t + +4\nu_r(\text{ rot } \mathbf{z}_m(t), \mathbf{v}_m(t))_t, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & \frac{d}{dt}\|\mathbf{z}_m(t)\|_t^2 + 2\|L^{\frac{1}{2}}\mathbf{z}_m(t)\|_t^2 + 8\nu_r\|\mathbf{z}_m(t)\|_t^2 ds \\ \leq \; & 2\,|(\mathbf{v}_m \nabla\varphi, \mathbf{z}_m)_t| + 2(G(t), \mathbf{z}_m(t))_t + 4\nu_r(\text{ rot } \mathbf{v}_m(t), \mathbf{z}_m(t))_t \end{aligned} \tag{4.4}$$

where the operator L is defined by

$$L\mathbf{z} = -(c_a + c_d)\Delta \mathbf{z} - (c_0 + c_d - c_a)\nabla \text{ div } \mathbf{z},$$

with domain $D(L) = H_0^1 \cap H^2$.

By using the condition (A.2), we have

$$|(\mathbf{v}_m \cdot \nabla\psi, \mathbf{v}_m)_t| \leq \sup_{\Omega(t)} |\psi| \, \|\mathbf{v}_m(t)\|_t^2 \leq C\|\mathbf{v}_m(t)\|_t^2.$$

Also, by using the condition (A.3), we obtain

$$\begin{aligned} |(\mathbf{v}_m \cdot \nabla\varphi, \mathbf{z}_m)_t| & \leq \sup_{\Omega(t)} |\varphi| \, \|\mathbf{v}_m(t)\|_t \|\mathbf{z}_m(t)\|_t \\ & \leq C\|\mathbf{v}_m(t)\|_t \|\mathbf{z}_m(t)\|_t \\ & \leq c_\varepsilon \|\mathbf{v}_m(t)\|_t^2 + \varepsilon\|\mathbf{z}_m(t)\|_t^2. \end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities, we have

$$(F(t), \mathbf{v}_m(t))_t \leq c_\delta \|F(t)\|_t^{*2} + \delta \|\nabla \mathbf{v}_m(t)\|_t^2,$$

analogously

$$(G(t), \mathbf{z}_m(t))_t \leq c_\eta \|G(t)\|_t^{*2} + \eta \|L^{1/2} \mathbf{z}_m(t)\|_t^2,$$

here was used the equivalence de norms between ∇ and $L^{1/2}$.

Then taking $\delta = \frac{c}{2}$ and $\eta = \frac{1}{2}$ in the above estimates and replacing in (4.3) and (4.4), after addying and integrating in t we obtain

$$\begin{aligned} & \|\mathbf{v}_m(t)\|_t^2 + \|\mathbf{z}_m(t)\|_t^2 + c \int_0^t \|\nabla \mathbf{v}_m(\tau)\|_\tau^2 + \int_0^t \|L^{\frac{1}{2}} \mathbf{z}_m(\tau)\|_\tau^2 \\ \leq \; & \|\mathbf{v}_m(0)\|_0^2 + \|\mathbf{z}_m(0)\|_0^2 + c(\int_0^t \|\mathbf{v}_m(\tau)\|_\tau^2 + \|\mathbf{z}_m(\tau)\|_\tau^2) \\ & +2c_\delta \int_0^t \|F(\tau)\|_\tau^{*2} + 2c_\eta \int_0^t \|G(\tau)\|_\tau^{*2} \end{aligned}$$

from the Gronwall' inequality we conclude that $\{\mathbf{v}_m(t)\}$ is bounded in $L^\infty(0,T;H_t) \cap L^2(0,T;V_t)$ and $\{\mathbf{z}_m(t)\}$ is bounded in $L^\infty(0,T;U_t) \cap L^2(0,T;S_t)$. ■

In the proof of our result we need the Lemma 2.5 of [14], this is

Lemma 4.2. *For each $\varepsilon > 0$ there exist a positive integer $N = N_\varepsilon$ independent of $t \in [0,T]$ such that for any $\mathbf{v} \in V_t$ we have*

$$\|\mathbf{v}\|_t^2 \leq \sum_{j=1}^N (\mathbf{v}, \mathbf{d}_j)_t^2 + \varepsilon \|\nabla \mathbf{v}\|_t^2.$$

Analogous result is true for $\mathbf{z} \in S_t$. With this estimates we prove

Lemma 4.3. *The solution $(\tilde{\mathbf{v}}_m, \tilde{\mathbf{z}}_m)$, is precompact in $[L^2(0,T;V_t) \cap L^\infty(0,T;H_t)] \times [L^2(0,T;S_t) \cap L^\infty(0,T;U_t)]$.*

Proof. *Put $\rho_{mj}(t) = (\mathbf{v}_m, \mathbf{d}_j)_t$ for $m \geq j$. we shall show that $\left\{\rho_{mj}(t)\right\}_{m \geq j}$ is uniformly bounded and equicontinuous on $[0,T]$ for each fixed j. In fact, since there exists for each j a constant M_j such that*

$$\begin{aligned} |\mathbf{d}_j(\mathbf{x},t)| \;&\leq\; M_j, \;\; |\nabla \mathbf{d}_j(\mathbf{x},t)| \leq M_j \text{ and } \left|\mathbf{d}_j'(\mathbf{x},t)\right| \leq M_j, \\ \text{for all } \mathbf{x} \;&\in\; \Omega(t), \;\; t \in [0,T], \end{aligned}$$

it follows from Lemma 3.3 that

$$\left|\rho_{mj}(t)\right| \leq M_j \left|\Omega(t)\right|^{1/2} \|\mathbf{v}_m(t)\|_t \leq M_j'.$$

Furthermore, for $t \in [0,T]$ *and* $s > 0$

$$\begin{aligned}
\left|\rho_{mj}(t+s) - \rho_{mj}(t)\right| &= \left|\int_t^{t+s} \frac{d}{d\tau}(\mathbf{v}_m, \mathbf{d}_j)_\tau\right| \\
&\leq \left|\int_t^{t+s} (\mathbf{v}_m', \mathbf{d}_j)_{t\tau}\right| + \left|\int_t^{t+s} (\mathbf{v}_m, \mathbf{d}_j')_\tau\right| \\
&\leq c\int_t^{t+s} \left|(\nabla \mathbf{v}_m, \boldsymbol{\nabla} \mathbf{d}_j)_\tau\right| + \int_t^{t+s} \left|(\mathbf{v}_m \nabla \mathbf{v}_m, \mathbf{d}_j)_\tau\right| + \int_t^{t+s} \left|(\psi \nabla \mathbf{v}_m, \mathbf{d}_j)_\tau\right| \\
&\quad + \int_t^{t+s} \left|(\mathbf{v}_m \nabla \psi, \mathbf{d}_j)_\tau\right| + \int_t^{t+s} \left|(F, \mathbf{d}_j)_\tau\right| + \\
&\quad 2\nu_r \int_t^{t+s} |\ rot\ \mathbf{z}_m, \mathbf{d}_j)_t| + \left|\int_t^{t+s} (\mathbf{v}_m, \mathbf{d}_j')_t\right| \\
&\leq c_j(\int_t^{t+s} \|\nabla \mathbf{v}_m\|_\tau + \int_t^{t+s} \|\nabla \mathbf{v}_m\|_\tau \|\mathbf{v}_m\|_\tau + \int_t^{t+s} \|\mathbf{v}_m\|_\tau \\
&\quad + \int_t^{t+s} \|F\|_\tau^* + \int_t^{t+s} \|L^{1/2}\mathbf{z}_m\|_\tau) \\
&\leq c_j(s^{1/2}\left\{1 + \sup \|\mathbf{v}_m\|_t\right\}\left\{\int_0^T \|\nabla \mathbf{v}_m\|_\tau^2\right\}^{1/2} + s \times \sup \|\mathbf{v}_m\|_t \\
&\quad + s^{1/2}\left\{\int_0^T \|L^{1/2}\mathbf{z}_m\|_\tau\right\}^{1/2} + s^{1/2}\left\{\int_0^T \|F\|_\tau^*\right\}^{1/2}),
\end{aligned}$$

where C_j *is a constant depending only on* n *and* M_j*. So the equicontinuity is obtained. Therefore, applying the diagonal argument we can choose a sequence* $\{m_k\}$ *of positive integers such that* $\left\{\rho_{m_k j}(t)\right\}_{m_k \geq j}$ *converge uniformly on* $[0,T]$ *for each fixed* j*. Considering* $\mathbf{v} = \mathbf{v}_{m_k} - \mathbf{v}_{m_l}$ *in the Lemma 4.4 and integrating in* t*, we obtain*

$$\int_0^T \|\mathbf{v}_{m_k} - \mathbf{v}_{m_l}\|_\tau^2 \leq \sum_{j=1}^{k} \int_0^T \left|\rho_{m_k j} - \rho_{m_l j}\right|^2 + 2\varepsilon \sup_m \int_0^T \|\nabla \mathbf{v}_m\|_\tau^2$$

letting $k, l \to \infty$,

$$\limsup \int_0^T \|\mathbf{v}_{m_k} - \mathbf{v}_{m_l}\|_\tau^2 \leq 2\varepsilon \sup_m \int_0^T \|\nabla \mathbf{v}_m\|_\tau^2,$$

since $\varepsilon > 0$ *is arbitrary and* $\{\mathbf{v}_m\}$ *is bounded in* $L^2(0,T;V_t)$ *the proof is completed, for* $\mathbf{v}$ *in the first equation of (2.1). Analogously we can conclude for* $\mathbf{z}$.

5. Proof of the Theorems

In this Section, we prove the main results. In first time, we prove the Theorem 3.2.

By Lemmas 3.3 we may assume that there exist $(\mathbf{v}_m, \mathbf{z}_m) \in [L^2(0,T;V_t) \cap L^\infty(0,T;H_t)] \times [L^2(0,T;S_t) \cap L^\infty(0,T;U_t)]$, such that

$$\begin{array}{rcl} \mathbf{v}_m & \rightharpoonup & \mathbf{v}, \text{ in } L^\infty(0,T;H_t) \text{ weak-start }, \\ \mathbf{v}_m & \rightharpoonup & \mathbf{v}, \text{ in } L^2(0,T;V_t) \text{ weak }, \\ \mathbf{z}_m & \rightharpoonup & \mathbf{z}, \text{ in } L^\infty(0,T;U_t) \text{ weak-start }, \\ \mathbf{z}_m & \rightharpoonup & \mathbf{z} \text{ in } L^2(0,T;S_t) \text{ weak }, \end{array}$$

and the Lemma 4.3 imply

$$\begin{array}{rcl} \mathbf{v}_m & \longrightarrow & \mathbf{v} \text{ in } L^2(0,T;H_t) \text{ strong }, \\ \mathbf{z}_m & \longrightarrow & \mathbf{z} \text{ in } L^2(0,T;U_t) \text{ strong }. \end{array}$$

Taking $h, l \in C^1([0,T];R)$ with $h(T) = L(T) = 0$, and setting $\tilde{\mathbf{u}} = \mathbf{d}_j l(t)$ and $\tilde{\mathbf{w}} = \mathbf{e}_j l(t)$. Multiplying the equalities (4.1) and (4.2) by l and h respectively, and returning to Q_∞ we obtain by integration by parts

$$(\mathbf{v}'_m, \mathbf{u})_t + c(\nabla \mathbf{v}_m, \nabla \mathbf{u})_t - ((\mathbf{v}_m \cdot \nabla)\mathbf{v}_m, \mathbf{u})_t - ((\mathbf{v}_m \cdot \nabla)\psi, \mathbf{u})_t - ((\psi \cdot \nabla)\mathbf{v}_m, \mathbf{u})_t$$

$$= (2\nu_r \text{ rot } \mathbf{z}_m + \mathbf{f}, \mathbf{u})_t,$$

$$(\mathbf{z}'_m, \mathbf{w})_t + c_1(\nabla \mathbf{z}_m, \nabla \mathbf{w})_t + c_2(\text{div } \mathbf{z}_m, \text{div } \mathbf{w})_t - ((\mathbf{v}_m \cdot \nabla)\mathbf{z}_m, \mathbf{w})_t$$

$$-((\mathbf{v}_m \cdot \nabla)\varphi, \mathbf{w})_t - (\psi \cdot \nabla)\mathbf{z}_m, \mathbf{w})_t + 4\nu_r(\mathbf{z}_m, \mathbf{w})_t$$

$$= (2\nu_r \text{ rot } \mathbf{u}_m + \mathbf{g}, \mathbf{w})_t,$$

integrating by parts in $[0,T]$, we obtain

$$-\int_0^T (\mathbf{v}_m, \mathbf{u}')_t + c\int_0^T (\nabla \mathbf{v}_m, \nabla \mathbf{u})_t - \int_0^T ((\mathbf{v}_m \cdot \nabla)\mathbf{v}_m, \mathbf{u})_t - \int_0^T ((\mathbf{v}_m \cdot \nabla)\psi, \mathbf{u})_t$$

$$-\int_0^T ((\psi \cdot \nabla)\mathbf{v}_m, \mathbf{u})_t = (\mathbf{v}_m(0), \mathbf{u}(0))_0 + \int_0^T (2\nu_r \text{ rot } \mathbf{z}_m + \mathbf{f}, \mathbf{u})_t,$$

$$-\int_0^T (\mathbf{z}_m, \mathbf{w}')_t + c_1\int_0^T (\nabla \mathbf{z}_m, \nabla \mathbf{w})_t + c_2\int_0^T (\text{div } \mathbf{z}_m, \text{div } \mathbf{w})_t - \int_0^T ((\mathbf{v}_m \cdot \nabla)\mathbf{z}_m, \mathbf{w})_t$$

$$-\int_0^T ((\mathbf{v}_m\cdot\nabla)\varphi,\mathbf{w})_t - \int_0^T (\psi\cdot\nabla)\mathbf{z}_m,\mathbf{w})_t + 4\nu_r\int_0^T (\mathbf{z}_m,\mathbf{w})_t$$

$$= (\mathbf{z}_m(0),\mathbf{w}(0))_0 + \int (2\nu_r \text{ rot } \mathbf{u}_m + \mathbf{g},\mathbf{w})_t.$$

Since $\mathbf{v}_m(0)\longrightarrow \mathbf{a}$ in H_0 and $\mathbf{z}_m(0)\longrightarrow \mathbf{b}$ in U_0, as $m\longrightarrow\infty$, we have

$$-\int_0^T (\mathbf{v},\mathbf{u}')_t + c\int_0^T (\nabla\mathbf{v},\nabla\mathbf{u})_t - \int_0^T ((\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{u})_t - \int_0^T ((\mathbf{v}\cdot\nabla)\psi,\mathbf{u})_t - \int_0^T ((\psi\cdot\nabla)\mathbf{v},\mathbf{u})_t$$

$$= (\mathbf{a},\mathbf{u}(0))_0 + \int_0^T (2\nu_r \text{ rot } \mathbf{z} + \mathbf{f},\mathbf{u})_t,$$

$$-\int_0^T (\mathbf{z},\mathbf{w}')_t + c_1\int_0^T (\nabla\mathbf{z},\nabla\mathbf{w})_t + c_2\int_0^T (\text{div } \mathbf{z},\text{div } \mathbf{w})_t - \int_0^T ((\mathbf{v}\cdot\nabla)\mathbf{z},\mathbf{w})_t$$

$$-\int_0^T ((\mathbf{v}\cdot\nabla)\varphi,\mathbf{w})_t - \int_0^T (\psi\cdot\nabla)\mathbf{z},\mathbf{w})_t + 4\nu_r\int_0^T (\mathbf{z},\mathbf{w})_t$$

$$= (\mathbf{b},\mathbf{w}(0))_0 + \int (2\nu_r \text{ rot } \mathbf{u} + \mathbf{g},\mathbf{w})_t.$$

The convergence in the terms non lineares is warranted by the fact that $\mathbf{u}$ and $\mathbf{w}$ are a linear combinations of functions $C_{0,\sigma}^\infty$ and C_0^∞, which are denses in V_t, and S_t, respectively.

Expressing the above equality in $\tilde{Q}_\infty$, we have

$$-\int_0^T \langle\tilde{\mathbf{v}},\tilde{\mathbf{u}}_t\rangle_t - \int_0^T \langle\tilde{\mathbf{v}},M\tilde{\mathbf{u}}\rangle_t + c\int_0^T \langle\nabla_g\tilde{\mathbf{v}},\nabla_g\tilde{\mathbf{u}}\rangle_t + \int_0^T \langle N_1\tilde{\mathbf{v}} + N_2\tilde{\mathbf{v}},\tilde{\mathbf{u}}\rangle_t$$

$$= \langle\tilde{\mathbf{a}},\tilde{\mathbf{u}}(0)\rangle_0 + \int_0^T \langle\widetilde{\mathbf{F}},\tilde{\mathbf{u}}\rangle_t + \int_0^T \langle \text{ rot }_g\tilde{\mathbf{z}},\tilde{\mathbf{u}}\rangle_t.$$

$$-\int_0^T \langle\tilde{\mathbf{z}},\tilde{\mathbf{w}}_t\rangle_t - \int_0^T \langle\tilde{\mathbf{z}},T\tilde{\mathbf{w}}\rangle_t + c_1\int_0^T \langle\nabla_g\tilde{\mathbf{z}},\nabla_g\tilde{\mathbf{w}}\rangle_t + c_2\int_0^T \langle\text{div } \tilde{\mathbf{z}},\ \text{div } \tilde{\mathbf{w}}\rangle_t \quad (5.1)$$

$$+\int_0^T \langle X_1\tilde{\mathbf{z}} + X_2\tilde{\mathbf{z}},\tilde{\mathbf{w}}\rangle_t + 4\nu_r\int_0^T \langle\tilde{\mathbf{z}},\tilde{\mathbf{w}}\rangle_t$$

$$= \langle\tilde{\mathbf{b}},\tilde{\mathbf{w}}(0)\rangle_0 + \int_0^T \langle\widetilde{\mathbf{G}},\tilde{\mathbf{w}}\rangle_t + \int_0^T \langle \text{ rot }_g\tilde{\mathbf{v}},\tilde{\mathbf{w}}\rangle_t.$$

Remark 1. *By taking* $\widetilde{\mathbf{u}} = l(t)\widetilde{\mathbf{d}}$ *and* $\widetilde{\mathbf{w}} = h(t)\widetilde{\mathbf{e}}$ *such that* $\widetilde{\mathbf{u}} \in C^{\infty}_{0,\sigma}(\widetilde{\Omega}), \in C^{\infty}_{0}(\widetilde{\Omega})$, *and* $l, h \in C^{\infty}_{0}([0,T]; I\!R)$ *in (5.1), we see that in sense of distribution*

$$\begin{aligned}
\frac{d}{dt}\langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{u}}\rangle_t &= \langle \widetilde{\mathbf{v}}, M\widetilde{\mathbf{u}}\rangle_t - c\langle \nabla_g \widetilde{\mathbf{v}}, \nabla_g \widetilde{\mathbf{u}}\rangle_t - \langle N_1\widetilde{\mathbf{v}} + N_2\widetilde{\mathbf{v}}, \widetilde{\mathbf{u}}\rangle_t \\
&\quad \langle \widetilde{\mathbf{F}}, \widetilde{\mathbf{u}}\rangle_t + \langle \ rot\ _g\widetilde{\mathbf{z}}, \widetilde{\mathbf{u}}\rangle_t \\
\frac{d}{dt}\langle \widetilde{\mathbf{z}}, \widetilde{\mathbf{e}}_j\rangle_t &= \langle \widetilde{\mathbf{z}}, T\widetilde{\mathbf{w}}\rangle_t - c_1\langle \nabla_g \widetilde{\mathbf{z}}, \nabla_g \widetilde{\mathbf{w}}\rangle_t - c_2\langle div\ \widetilde{\mathbf{z}},\ div\ \widetilde{\mathbf{w}}\rangle_t \\
&\quad -\langle X_1\widetilde{\mathbf{z}} + X_2\widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}\rangle_t - 4\nu_r\langle \widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}\rangle_t + \langle \widetilde{\mathbf{G}}, \widetilde{\mathbf{w}}\rangle_t + \langle \ rot\ _g\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}\rangle_t.
\end{aligned}$$

by definition of M, N_1, T, *and* X_1, *and the estimates obtain in the above Lemmas, we see that the right-hand side defines an elements of* $L^1(0,T;V^*)$ *and* $L^1(0,T;S^*)$, *respectively. By applying Lemma in ([21], chap. 3 §1), it follow that* $\widetilde{\mathbf{v}}', \widetilde{\mathbf{z}}'$, *exist as elements of* $L^1(0,T;V^*)$ *and* $L^1(0,T;S^*)$, *respectively, and so* $\widetilde{\mathbf{v}}, \widetilde{\mathbf{z}}$ *are weakly continuous on* $[0,T]$ *with values in* H *and* U, *respectively, since* $\widetilde{\mathbf{v}} \in L^{\infty}(0,T;H)$, *and* $\widetilde{\mathbf{z}} \in L^{\infty}(0,T;U)$. *Hence we have* $\widetilde{\mathbf{v}}(0) = \widetilde{\mathbf{a}}$, *and* $\widetilde{\mathbf{z}}(0) = \widetilde{\mathbf{b}}$.

The following result is necessary to prove the Theorem 3.3.

Lemma 5.1. *If* $\widetilde{\mathbf{v}} \in L^2([0,T);V)$, $\widetilde{\mathbf{z}} \in L^2([0,T);S)$, *and* $\widetilde{\mathbf{v}}' \in L^2([0,T);V^*)$, $\widetilde{\mathbf{z}} \in L^2([0,T);S^*)$, *then* $\widetilde{\mathbf{v}}$, *is continuous on* $[0,T]$ *with values in* $\tilde{H}$. *Furthermore, we have*

$$\frac{d}{dt}\,|\widetilde{\mathbf{v}}|_t^2 = 2\langle \widetilde{\mathbf{v}}' + M\widetilde{\mathbf{v}}, \widetilde{\mathbf{v}}\rangle_t.$$

Analogous results are true for $\widetilde{\mathbf{z}}$.

Now, we prove the Theorem 3.3. To do it we consider that $(\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{z}}_1)$ and $(\widetilde{\mathbf{v}}_2, \widetilde{\mathbf{z}}_2)$ are two solution of problem (2.1) corresponding to the same $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\ \widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$. Define differences

$$\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2,\ \ \widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}_1 - \widetilde{\mathbf{z}}_2.$$

They satisfy

$$\begin{aligned}
&\langle \widetilde{\mathbf{v}}', \widetilde{\mathbf{u}}\rangle_t = \langle \widetilde{\mathbf{v}}, M\widetilde{\mathbf{u}}\rangle_t - c\langle \nabla_g \widetilde{\mathbf{v}}, \nabla_g \widetilde{\mathbf{u}}\rangle_t - \langle N_1\widetilde{\mathbf{v}} + N_2\widetilde{\mathbf{v}}_1 - N_2\widetilde{\mathbf{v}}_2, \widetilde{\mathbf{u}}\rangle_t \\
&2\nu_r\langle \text{ rot }_g\widetilde{\mathbf{z}}, \widetilde{\mathbf{u}}\rangle_t,
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
&\langle \widetilde{\mathbf{z}}', \widetilde{\mathbf{e}}_j\rangle_t = \langle \widetilde{\mathbf{z}}, T\widetilde{\mathbf{w}}\rangle_t - c_1\langle \nabla_g \widetilde{\mathbf{z}}, \nabla_g \widetilde{\mathbf{w}}\rangle_t - c_2\langle \text{div } \widetilde{\mathbf{z}},\ \text{div } \widetilde{\mathbf{w}}\rangle_t \\
&-\langle X_1\widetilde{\mathbf{z}} + X_2\widetilde{\mathbf{z}}_1 - X_2\widetilde{\mathbf{z}}_2, \widetilde{\mathbf{w}}\rangle_t - 4\nu_r\langle \widetilde{\mathbf{z}}, \widetilde{\mathbf{w}}\rangle_t + 2\nu_r\langle \text{ rot }_g\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}\rangle_t.
\end{aligned} \tag{5.3}$$

for every $\widetilde{\mathbf{u}}\in V$ and $\widetilde{\mathbf{w}}\in S$.

On the other hand,

$$N_2\widetilde{\mathbf{v}}_1 - N_2\widetilde{\mathbf{v}}_2 = \mathbf{v}_1\cdot\nabla\mathbf{v}_1 - \mathbf{v}_2\cdot\nabla\mathbf{v}_2 = \mathbf{v}_1\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v}_2.$$

Since

$$|(\mathbf{v}\cdot\nabla\mathbf{u},\mathbf{w})_t| = |(\mathbf{v}\cdot\nabla\mathbf{w},\mathbf{u})_t| \le 2^{\frac{1}{2}}\left\{\|\mathbf{v}\|_t\|\nabla\mathbf{v}\|_t\right\}^{1/2}\left\{\|\mathbf{u}\|_t\|\nabla\mathbf{u}\|_t\right\}^{1/2}\|\mathbf{w}\|_t$$

for every $\mathbf{v},\mathbf{u},\mathbf{w}\in V$ or S

Then

$$|\langle N_2\widetilde{\mathbf{v}}_1 - N_2\widetilde{\mathbf{v}}_2,\widetilde{\mathbf{v}}\rangle_t| = |((\mathbf{v}\cdot\nabla)\mathbf{v}_2,\mathbf{v})_t| \le c\|\widetilde{\mathbf{v}}\|_t\|\nabla_g\widetilde{\mathbf{v}}\|_t\|\nabla_g\widetilde{\mathbf{v}}_2\|_t$$

analogously

$$X_2\widetilde{\mathbf{z}}_1 - X_2\widetilde{\mathbf{z}}_2 = \mathbf{v}_1\cdot\nabla\mathbf{z}_1 - \mathbf{v}_2\cdot\nabla\mathbf{z}_2 = \mathbf{v}_1\cdot\nabla\mathbf{z} - \mathbf{v}\cdot\nabla\mathbf{z}_2$$

and

$$|\langle X_2\widetilde{\mathbf{z}}_1 - X_2\widetilde{\mathbf{z}}_2,\widetilde{\mathbf{z}}\rangle_t| = |((\mathbf{v}\cdot\nabla)\mathbf{z}_2,\mathbf{z})_t| \le c\|\widetilde{\mathbf{z}}\|_t\|\nabla_g\widetilde{\mathbf{v}}\|_t\|\nabla_g\widetilde{\mathbf{z}}_2\|_t.$$

Since $\widetilde{\mathbf{v}}' \in L^2([0,T);V^*)$, and $\widetilde{\mathbf{z}} \in L^2([0,T);S^*)$, it follows from (5.2) and (5.3), in the above Lemmas and Young Inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{v}}\|_t^2 + c\|\nabla_g\widetilde{\mathbf{v}}\|_t^2\\
\le\ & c_\varepsilon\|\widetilde{\mathbf{v}}\|_t^2 + \varepsilon\|\nabla_g\widetilde{\mathbf{v}}\|_t^2 + c_\delta\|\widetilde{\mathbf{v}}\|_t^2\|\nabla_g\widetilde{\mathbf{v}}_2\|_t^2 + \delta\|\nabla_g\widetilde{\mathbf{v}}\|_t^2 + 2\nu_r\|\widetilde{\mathbf{z}}\|_t^2 + \frac{\nu_r}{2}\|\nabla_g\widetilde{\mathbf{v}}\|_t^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{z}}\|_t^2 + c_1\|\nabla_g\widetilde{\mathbf{z}}\|_t^2 + c_2\|\mathrm{div}\ \widetilde{\mathbf{z}}\|_t^2 + 4\nu_r\|\widetilde{\mathbf{z}}\|_t^2\\
\le\ & c_{\varepsilon'}\|\widetilde{\mathbf{v}}\|_t^2 + \varepsilon'\|\nabla_g\widetilde{\mathbf{z}}\|_t^2 + c_{\delta'}\|\widetilde{\mathbf{z}}\|_t^2\|\nabla_g\widetilde{\mathbf{z}}_2\|_t^2 + \delta'\|\nabla_g\widetilde{\mathbf{v}}\|_t^2 + 2\nu_r\|\widetilde{\mathbf{z}}\|_t^2 + \frac{\nu_r}{2}\|\nabla_g\widetilde{\mathbf{v}}\|_t^2
\end{aligned}$$

taking appropiate $\varepsilon,\varepsilon',.\delta$, and δ', then additying, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|_t^2 + \|\widetilde{\mathbf{z}}\|_t^2) &\le c(1+\|\nabla_g\widetilde{\mathbf{v}}_2\|_t^2)\|\widetilde{\mathbf{v}}\|_t^2 + c_{\delta'}\|\widetilde{\mathbf{z}}\|_t^2\|\nabla_g\widetilde{\mathbf{z}}_2\|_t^2\\
&\le c(\|\widetilde{\mathbf{v}}\|_t^2 + \|\widetilde{\mathbf{z}}\|_t^2)
\end{aligned}$$

integrating in t, we obtain

$$\|\widetilde{\mathbf{v}}\|_t^2 + \|\widetilde{\mathbf{z}}\|_t^2 \le \int_0^t c(\|\widetilde{\mathbf{v}}\|_\tau^2 + \|\widetilde{\mathbf{z}}\|_\tau^2).$$

Applying Gronwall's inequality we obtain $\widetilde{\mathbf{v}} = 0$ and $\widetilde{\mathbf{z}} = 0$. ■

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