Lyapunov graph continuation

M.A. BERTOLIM*
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo
São Carlos SP Brazil
bertolim@icmc.sc.usp.br

M.P. MELLO
Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas
Campinas SP Brazil
margarid@ime.unicamp.br

K.A. de REZENDE†
Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas
Campinas SP Brazil
ketty@ime.unicamp.br

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Abstract

In this paper the Poincaré-Hopf inequalities are shown to be necessary and sufficient conditions for an abstract Lyapunov graph L to be continued to an abstract Lyapunov graph of Morse type. The Lyapunov graphs considered represent smooth flows on closed orientable n-manifolds, $n \geq 2$. The continuation which is presented by means of a constructive algorithm, is shown to be unique in dimensions two and three. In all other dimensions, upper bounds on the number of possible continuations of L are presented.

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1 Introduction

In this article closed orientable n-manifolds M, $n \geq 2$ are considered. Given a smooth flow $\phi_t : M \to M$, there exists a smooth function $f : M \to \mathbb{R}$ associated to this flow with the property that it decreases along orbits outside the chain recurrent set $\mathcal{R}$, that is, if $x \notin \mathcal{R}$ then $f(\phi_t(x)) < f(\phi_s(x))$ given that $t > s$ and is constant on connected components of $\mathcal{R}$. This function is defined as a *Lyapunov function.* We refer to ϕ_t as a *gradient-like* flow with respect to f because of the properties above.

A set $S \subset M$ is invariant if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. A compact set $N \subset M$ is an *isolating neighborhood* if $\mathrm{inv(N,\phi) = \{x \in N : \phi_t(x) \subset N,\ \forall\ t \in \mathbb{R}\} \subset intN}$. A compact set N is an *isolating block* if $N^- = \{x \in N : \phi_{[0,t)}(x) \not\subset N, \forall t > 0\}$ is closed and $\mathrm{inv(N,\phi) \subset int\ N}$. An invariant set S is called an *isolated invariant set* if it is a maximal invariant set in some isolating neighborhood N, that is, $S = \mathrm{inv(N,\phi)}$.

A component R of $\mathcal{R}$ of the flow ϕ_t is an example of an invariant set. We will work under the hypothesis that $\mathcal{R}$ is the finite union of isolated invariant sets R_i. If f is a Lyapunov function associated to a flow and $c = f(R)$ then for $\varepsilon > 0$, the component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ that contains R is an isolating neighborhood for R. Take $(N, N^-) = (f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon))$ as an index pair for R. The Conley index is defined as the homotopy type of N/N^-. The Conley homology index is denoted by $CH_*(S)$ and its rank denoted by $h_* = \mathrm{rank\ CH_*(S)}$. For more details see [2].

Lyapunov graphs were first introduced by Franks [6] and have proven to be an excellent bookkeeping device of dynamical and topological information of the flow and the phase space. A *Lyapunov graph* is a finite, connected, oriented graph with no oriented cycles and labelled with vertices and edges.

A Lyapunov function $f : M \to \mathbb{R}$ associated to a flow, determines a Lyapunov graph by the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. Therefore, $M/\sim_f$ is a Lyapunov graph. It is possible to choose f so that each critical level contains a component R of $\mathcal{R}$. A point on $M/\sim_f$ is a *vertex point* if under the equivalence relation it corresponds to a level set containing a component R of $\mathcal{R}$. All other points are *edge points.* Each edge represents a codimension one submanifold Q of M times an open bounded interval I, $Q \times I$. In order to retain some topological information of $Q \times I$, the edges will be labelled with the Betti numbers of Q. All homology groups will be computed with coefficients $\mathbb{Z}_2$.

An *abstract Lyapunov graph* is an oriented graph with no oriented cycles such that each vertex

v is labelled with a list of non-negative integers $\{h_0(v) = k_0, \dots, h_n(v) = k_n\}$. Whenever $h_j(v) = 0$, it will be omitted from the list. Also, the labels on each edge $\{\beta_0 = 1, \beta_1, \dots, \beta_{n-2}, \beta_{n-1} = 1\}$ must be a collection of non-negative integers satisfying the Poincaré duality and if $n-1$ is even then $\beta_{\frac{n-1}{2}}$ is even.

An *abstract Lyapunov graph of Morse type* will be defined subsequently, but roughly speaking, it is an abstract Lyapunov graph with all vertices labelled with non-degenerate singularities, i. e., $\{h_j(v) = 1\}$ for some j.

Next, the notion of *vertex explosion* will be established in order to define continuation of abstract Lyapunov graphs. Let v be a vertex on an abstract Lyapunov graph labelled with $\{h_0(v), h_1(v) \dots, h_n(v)\}$. A vertex v can be exploded if v can be removed and substituted by an abstract Lyapunov graph I of Morse type with k vertices. The graph I must respect the orientations and labels of the incoming and outgoing edges of v. In other words, the new graph obtained must be oriented and with no oriented cycles. The incoming (outgoing) edges of v, must be incoming (outgoing) edges on vertices of I and all labels on the edges must respect the restrictions of the Morse type vertices. Moreover,

$$h_\lambda(v) = \sum_{j=1}^{k} h_\lambda(v_j),\ \forall\lambda.$$

An abstract Lyapunov graph admits a *continuation* to an abstract Lyapunov graph of Morse type if each vertex can be exploded.

The underlying idea in this paper is to describe under which conditions an abstract Lyapunov graph can be continued to an abstract Lyapunov graph of Morse type. These conditions are the Poincaré-Hopf inequalities which are shown to be necessary and sufficient for the continuation to occur. This is done by presenting an algorithm which not only constructs the continuation but also provides the number of possible continuations.

Theorem 1.1 *Every abstract Lyapunov graph that satisfies the Poincaré-Hopf inequalities at each vertex can be continued to an abstract Lyapunov graph of Morse type.*

This paper is organized as follows. Section (2) will introduce the techniques developed in [3]. The main idea is to develop a Morse theoretic approach of analyzing the changes to the manifold as handles are attached. For this purpose, a connection is drawn between the attachment of handles and the passage through a vertex on a Lyapunov graph L. The different type of attachments of handles will correspond to different labels on vertices of L.

In Section (3) a graph filtration is defined which corresponds to a filtration on the manifold and permits the proof of the generalized Morse inequalities using the ranks of the Conley homology index. Moreover, this filtration permits the translation of information of the flow to the graph and vice-versa. The Poincaré-Hopf inequalities are deduced from an analysis of long exact sequences of index pairs.

In Section (4) an algorithm for the explosion of a vertex is presented. The algorithm describes the number of different types of vertices that will replace v. This algorithm will differ slightly if the vertex is on a graph that represents a flow on a manifold of odd dimension, even dimension equal to $0 \mod 4$ or even dimension equal to $2 \mod 4$. Each step of the algorithm imposes a series of restrictions described by linear equations, thus forming three families of linear systems whose solution is the number of different types of vertices which will substitute v in the explosion.

In Section (5) the linear systems will be treated rigorously using integer linear programming techniques. In other words, it will be proved that the three different families of linear systems always have non-negative integer solutions provided certain inequalities, are satisfied. Moreover, an upper bound for the number of solutions is obtained.

In Section (6) it is shown that the inequalities in Section (5) required for the existence of non-negative integer solutions are precisely the Poincaré-Hopf inequalities.

2 Handle Theory

A description of how handles can be attached will be given in this section. Denote by D^j, the closed unit j-disk, i.e., the j-ball. Also, the boundary of D^j, ∂D^j is the $(j-1)$-sphere S^{j-1}.

Let $\overline{N} = N^- \times [0,1]$ be the n-manifold obtained as a collar of N^- and $H = D^\ell \times D^{n-\ell}$. Let $\theta : \partial D^\ell \times D^{n-\ell} \to \partial\overline{N}$ be an embedding which defines the new manifold $N' = \overline{N} \cup_\theta H$, which is the result of attaching an ℓ-handle to $\overline{N}$. The following loose notations are used $N' = \overline{N} \cup H^{(\ell)}$ or $N' = \overline{N} \cup H$ where $\partial N' = N^+ \cup N^-$.

In terms of a Lyapunov graph L, the attaching of a handle corresponds to the passage through a vertex in the opposite direction of the orientation of the graph. Hence a vertex of L represents the singularity. The vertex together with its labelled incident edges represents N'. The incoming edges represent $N^+ \times J$ and the outgoing edges represent $N^- \times J$ where J is an open interval.

The following definition distinguishes the effect on the Betti numbers of N^+ and N^- once the handle has been attached.

A handle containing a singularity of index ℓ or respectively, the corresponding vertex on L is called ℓ-disconnecting, in short ℓ-d, if this handle has the algebraic effect of increasing the ℓ-th Betti number of N^+ or respectively, the corresponding β_ℓ label on the incoming edge. A handle containing a singularity of index ℓ or the corresponding vertex on L is called $(\ell-1)$-*connecting*, in short $(\ell-1)$-c, if this handle has the algebraic effect of decreasing the $(\ell-1)$-th Betti number of N^+ or respectively, the corresponding $\beta_{\ell-1}$ label on the incoming edge. A handle containing a singularity of index ℓ or the corresponding vertex on L is called β-*invariant*, in short β-i, if all Betti numbers are kept constant. See Figure 1.

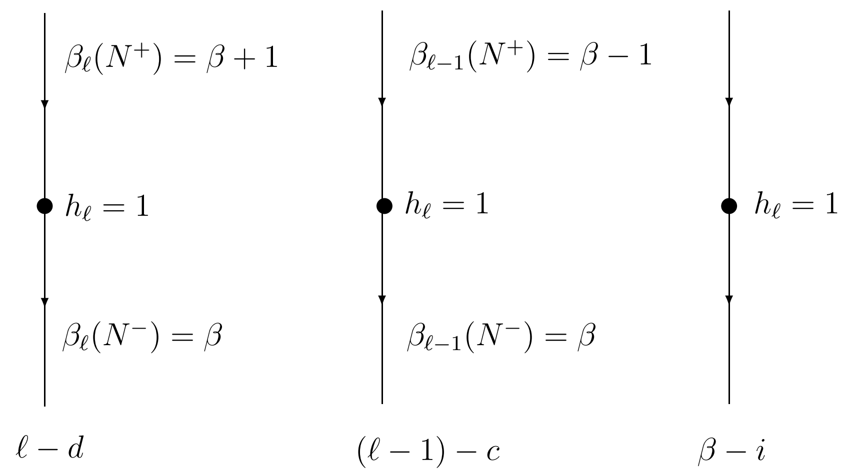

Figure 1: The three possible algebraic effects

The isolating blocks N' of a singularity of index $h_1 = 1$ in a manifold of dimension 2 of type $\beta - i$, $0 - c$ and $1 - d$ are shown in Figure 2. See [5]. Note the presence of an extra edge, in the $0 - c$ and $1 - d$ case precisely because the Betti numbers that decreased and increased respectively, control the number of connected components. It will be shown in Theorem 2.1 that on orientable even dimensional manifolds of dimension n, $\beta - i$ singularities can only occur if $n = 0 \mod 4$.

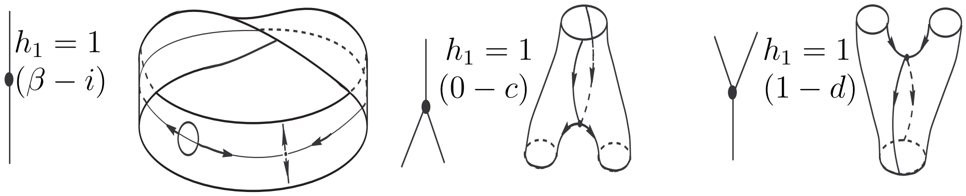

Figure 2: Isolating blocks for saddles in dimension 2

The isolating block N' of a singularity of index $h_1 = 1$ in an orientable manifold of dimension three corresponding to the $0 - c$ case and the $1 - d$ case are as in Figure 3. See [4].

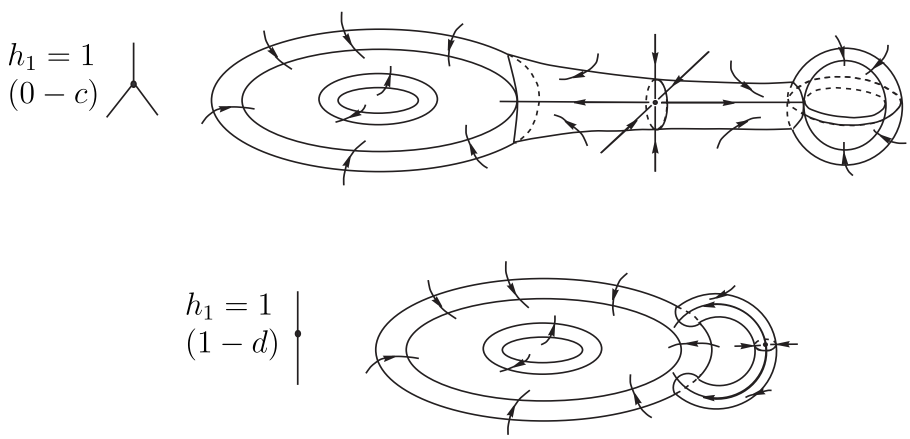

Figure 3: Isolating blocks for saddles in dimension 3

In this section part of the main result in [3] will be used.

Theorem 2.1 *Let $\phi_t : M \to M$ be a Morse-Smale flow with Lyapunov function $f : M \to \mathbb{R}$. Let v be a vertex of the associated Lyapunov graph L. Let N be a isolating block containing only one singularity that corresponds to v. Let N^-, N^+ be the exiting and entering components of N. If v is labelled as an index ℓ singularity then*

1. *v is ℓ-d, $(\ell - 1)$-c or β-i;*

2. *the sum of the labels on the incoming edges incident to v (i.e. the total Betti number of N^+) changes with respect to the sum of the labels on the outgoing edges incident to v (i.e. the total Betti number of N^-) by ± 2 or 0;*

3. *v cannot be β-i if $n \neq 0 \mod 4$.*

An *abstract Lyapunov graph of Morse type* is an abstract Lyapunov graph that satisfies the following:

1. every vertex is labelled with $h_j = 1$ for some $j = 0, \ldots, n$.

2. the number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:

   (a) if $h_j = 1$ for $j \neq 0, 1, n-1, n$ then $e^+ = 1$ and $e^- = 1$;

   (b) if $h_1 = 1$ then $e^+ = 1$ and $0 < e^- \leq 2$; if $h_{n-1} = 1$ then $e^- = 1$ and $0 < e^+ \leq 2$;

   (c) if $h_0 = 1$ then $e^- = 0$ and $e^+ = 1$; if $h_n = 1$ then $e^+ = 0$ and $e^- = 1$.

3. every vertex labelled with $h_\ell = 1$ must be of type ℓ-d or $(\ell-1)$-c. Furthermore if $n = 2i = 0 \mod 4$ and $h_i = 1$ then v may be labelled with β-i.

The specific effects of the ℓ-d, $(\ell-1)$-c or β-i vertices on the β_ℓ^+ labels on the incoming edge is specified in a series of Propositions in [3]. Let v be the vertex labelled with $h_\ell = 1$ on L, where L represents a flow whose phase space is an n dimensional manifold M. The dimension of M, n, is referred to as *ambient* dimension. Whenever M is an even dimensional manifold and ℓ coincides with half its dimension or whenever M is an odd dimensional manifold and ℓ coincides with either of the two middle dimensions then the analysis is slightly more elaborate. These cases will be referred to as the middle dimensional cases and the results are summarized in the following proposition.

Proposition 2.1 *1. Let $n = 2i$ and $\ell \neq i$ or $n = 2i+1$ and $\ell \neq i, i+1$. Then either*

(a) if v is ℓ-d then

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq \ell, n-\ell-1 \\ \beta_k(N^+) - 1 \text{ for } k = \ell, n-\ell-1 \end{cases}$$

or else,

(b) *if* v *is* $(\ell-1)$-c *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq \ell-1, n-\ell \\ \beta_k(N^+)+1 \text{ for } k = \ell-1, n-\ell \end{cases}$$

2. *if* $n = 2i$ *and* $\ell = i$ *then*

   (a) *if* v *is* $(\beta - i)$ *then*

$$\beta_k(N^-) = \beta_k(N^+) \text{ for all } k$$

   (b) *if* v *is* ℓ-d *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i, i-1 \\ \beta_k(N^+)-1 \text{ for } k = i, i-1 \end{cases}$$

   or else,

   (c) *if* v *is* $((\ell-1)-c)$ *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i, i-1 \\ \beta_k(N^+)+1 \text{ for } k = i, i-1 \end{cases}$$

3. *if* $n = 2i+1$ *and* $\ell = i$ *then*

   (a) *if* v *is* $(\ell - d)$ *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i \\ \beta_k(N^+)-2 \text{ for } k = i \end{cases}$$

   or else,

   (b) *if* v *is* $((\ell-1)-c)$ *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i+1, i-1 \\ \beta_k(N^+)+1 \text{ for } k = i+1, i-1 \end{cases}$$

4. *if* $n = 2i+1$ *and* $\ell = i+1$ *then*

   (a) *if* v *is* $((\ell-1)$-$c)$ *then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i \\ \beta_k(N^+)+2 \text{ for } k = i \end{cases}$$

   or else,

(b) if v is (ℓ-d) then

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) \text{ for all } k \neq i+1, i-1 \\ \beta_k(N^+) - 1 \text{ for } k = i+1, i-1 \end{cases}$$

To illustrate the use of $\ell - d$ and $(\ell - 1) - c$ vertices consider the following explosion of a vertex labelled with $\{h_1 = 3, h_2 = 3, h_3 = 2, h_4 = 2\}$ and edges labelled with non-negative integers as in Figure 4. The ambient dimension is $n = 5$.

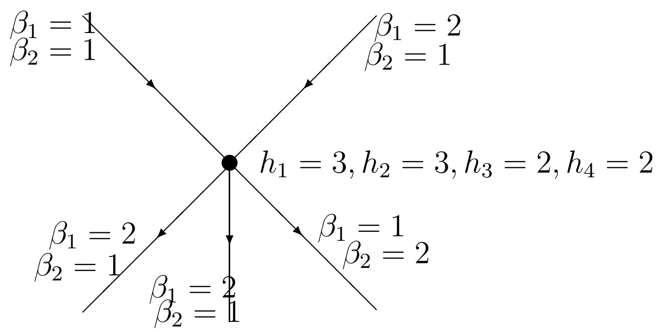

Figure 4: Vertex to be exploded

It will be shown that this vertex admits two different explosions as in Figure (5).

3 Poincaré-Hopf Inequalities

In Subsection (3.1) the classical Morse Inequalities are recalled. In the subsequent Subsection (3.2) a graph filtration is defined which corresponds to a flow-induced filtration on M. This makes it possible to transfer information from the flow on M to the Lyapunov graph and vice-versa. Finally, this filtration is used to prove the generalized Morse inequalities. In Subsection (3.3) the Poincaré-Hopf inequalities will be proven. These inequalities in contrast to the Morse inequalities also relate the homology indices of an isolated invariant set to the Betti numbers of the entering and exiting sets of an isolating block. Due to the graph filtration in Subsection (3.2) these results can be transferred to the graph.

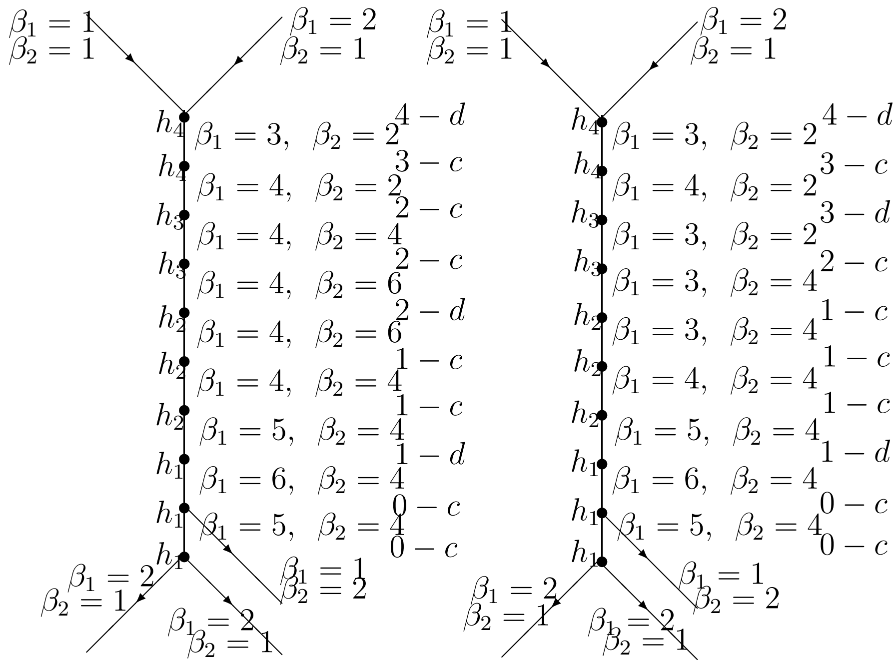

Figure 5: Vertex explosions

3.1 Morse Inequalities

The relationship between the topology of M and the critical points of a real valued function on M were described by the classical Morse inequalities. In this section basic results will be stated and further details can be found in [8].

Definition 3.1 *Let S be a function from certain pairs of spaces to the integers. S is subadditive if whenever $X \supset Y \supset Z$ we have $S(X, Z) \leq S(X, Y) + S(Y, Z)$. S is additive if equality holds.*

Considering $\beta_\lambda(X, Y)$ as the λ-th Betti number of (X, Y), that is, $\beta_\lambda(X, Y) = \text{rank}(H_\lambda(X, Y))$ and $\chi(X, Y) = \sum(-1)^\lambda \beta_\lambda(X, Y)$ as the Euler characteristic, it can be shown by analyzing the long exact sequence of (X, Y, Z) that β_λ is subadditive and χ is additive .

Lemma 3.1 *If S is subadditive and $X_0 \subset \ldots \subset X_n$, then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. If S is additive then equality holds. If $X_0 = \emptyset$ then $S(X_n) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$ and equality holds if S is additive.*

Proof: The proof is done by induction on n. ∎

Given a compact manifold M and a differentiable function on M with non-degenerate isolated critical points, define a filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \ldots \subset M^{a_k} = M$, where $a_1 < \ldots < a_k$. In this case, the sum of the ranks of $H_*(M_{a_i}, M_{a_{i-1}})$ is $\displaystyle\sum_{i=1}^{k} \beta_\lambda(M_{a_i}, M_{a_{i-1}}) = C_\lambda$.

Since β_λ is subadditive and χ is additive for the filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \ldots \subset M^{a_k} = M$, the following results are obtained:

1. *Weak Morse Inequalities*

$$\beta_\lambda(M) \leq C_\lambda.$$

2. *Poincaré-Hopf Equality*

$$\sum (-1)^\lambda \beta_\lambda(M) = \sum (-1)^\lambda C_\lambda,$$

where C_λ denotes the number of critical points of index λ.

Define the subadditive function $S_\lambda(X,Y) = \beta_\lambda(X,Y) - \beta_{\lambda-1}(X,Y) + - \ldots \pm \beta_0(X,Y)$. The Morse inequalities are obtained by using this function on the filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \ldots \subset M^{a_k} = M$.

$$S_\lambda(M) \leq \sum_{i=1}^{n} S_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda - C_{\lambda-1} + - \ldots \pm C_0$$

or in other words,

$$\beta_\lambda(M) - \beta_{\lambda-1}(M) + - \ldots \pm \beta_0(M) \leq C_\lambda - C_{\lambda-1} + - \ldots \pm C_0.$$

3.2 Graph Filtration

In this section we will describe an ordering procedure similar to topological ordering of a finite connected oriented graph with no oriented cycles.

Given a smooth flow on M with Lyapunov graph L, a graph filtration on L will be defined. This filtration induces a filtration on M.

Definition 3.2 *Let $v_1^1, v_1^2, \ldots, v_1^{k_1}$ be vertices of L of outdegree 0. Consider the incoming edges incident to these vertices. Denote by $N_1^1, N_1^2, \ldots, N_1^{k_1}$ the isolating blocks associated to these vertices and edges. Define $M_1 = \displaystyle\bigcup_{i=1}^{k_1} N_1^i$. Connect all vertices v_1^j, $j = 1, \ldots, k_1$ with the vertices*

$v_2^j,\ j=1,\dots,k_2$ such that the outgoing edges of these vertices are the incoming edges of the vertices $v_1^j,\ j=1,\dots,k_1$. Consider the outgoing and incoming edges of the vertices $v_2^j,\ j=1,\dots,k_2$. Denote by $N_2^1, N_2^2, \dots, N_2^{k_2}$ the isolating blocks associated to these vertices and their incident edges. Define $M_2 = M_1 \cup \bigcup_{i=1}^{k_2} N_2^i$. Repeat this process until all vertices of the graph are connected. This defines a filtration on M, $\emptyset = M_0 \subset M_1 \subset \dots \subset M_\ell = M$ which will be referred to as graph filtration,

$$M_\ell = M_{\ell-1} \cup \bigcup_{i=1}^{k_\ell} N_\ell^i.$$

The *homotopical Conley index* of the maximal invariant set represented by v_i^j will be denoted by $\mathcal{H}(v_i^j)$, and its homology $H_*(\mathcal{H}(v_i^j))$ is the Conley homology index and is denoted by $CH_*(v_i^j)$. Each vertex v_i^j of the graph is labelled with the ranks of the Conley homology indices, $h_*(v_i^j)$. If $CH_*(v_i^j)$ is trivial then for simplicity $h_*(v_i^j) = 0$ will be omitted.

Given a smooth flow on M with Lyapunov graph L, a flow line on M corresponds to an oriented path on L. Given that there is no oriented path between two vertices v and w in L then there is no connecting orbit between Λ_v and Λ_w.

It is important to note that the choice of the graph filtration $\emptyset = M_0 \subset M_1 \subset \dots \subset M_\ell = M$ implies that there is no oriented path connecting vertices that are at the same level $\{v_i\}_{j=1}^{k_i}$ and therefore the Λ_i^j associated to these vertices have no connecting orbits. Hence, this filtration defines an index pair (M_i, M_{i-1}) for the isolated invariant sets associated to $\bigcup_{j=1}^{k_i} v_i^j$, $\bigcup_{j=1}^{k_i} \Lambda_i^j$. This can easily be seen since (M_i, M_{i-1}) is a pair of compact spaces such that:

1. $cl(M_i, M_{i-1})$ is an isolating neighborhood for the disjoint union of the isolated invariant sets $\bigcup_{j=1}^{k_i} \Lambda_i^j$;
2. M_{i-1} is positively invariant in M_i, that is, if $x \in M_{i-1}$ and $\phi_{[0,T]}(x) \subset M_i$ then $\phi_{[0,T]}(x) \subset M_{i-1}$;
3. M_{i-1} is the exiting set for the flow, that is, if $x \in M_i$ and $\phi_{[0,\infty]}(x) \not\subset M_i$ then there is a $T > 0$ such that $\phi_{[0,T]}(x) \subset M_i$ and $\phi_T(x) \subset M_{i-1}$ (this last condition is vacuously satisfied).

Using the graph filtration and the subadditivity of β_λ and the additivity of χ as in the previous section, the following results are obtained:

1. *generalized weak Morse inequalities*

$$\beta_\lambda(M) \leq h_\lambda(M).$$

2. *generalized Poincaré- Hopf equality*

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^\lambda h_\lambda(M),$$

where $rank(CH_\lambda(M_i, M_{i-1})) = h_\lambda(M_i, M_{i-1})$ and $\sum_i h_\lambda(M_i, M_{i-1}) = h_\lambda(M)$.

Define the subadditive function S_λ as in the previous section and using the graph filtration the *generalized Morse inequalities* are obtained:

$$S_\lambda(M) \leq \sum_{i=1}^{l} S_\lambda(M_i, M_{i-1}) = h_\lambda(M) - h_{\lambda-1}(M) + - \ldots \pm h_0(M),$$

that is,

$$\beta_\lambda(M) - \beta_{\lambda-1}(M) + - \ldots \pm \beta_0(M) \leq h_\lambda(M) - h_{\lambda-1}(M) + - \ldots \pm h_0(M).$$

3.3 Poincaré-Hopf Inequalities

Given a flow φ_t and an isolated invariant set $\Lambda \subset N^n$ it will be assumed that the inverse flow φ_{-t} has an isolated invariant set Λ' with the property that

$$h_i = \dim CH_i(\Lambda) = \dim CH_{n-i}(\Lambda') = h_{n-i}(\Lambda').$$

There are many examples of isolated invariant sets that satisfy the above requirement. For instance if a flow φ_t has a singularity P of Conley index S^k ($h_k = 1$), then the singularity P for the reverse flow φ_{-t} has Conley index S^{n-k} ($h_{n-k} = 1$). The same holds for a periodic orbit γ of a flow φ_t with Conley index $S^k \vee S^{k+1}$ ($h_k = h_{k+1} = 1$). The reverse flow φ_{-t} has γ as a periodic orbit of Conley index $S^{n-k-1} \vee S^{n-k}$ ($h_{n-k-1} = h_{n-k} = 1$). Hence the dual of $h_k = 1$ is $h_{n-k} = 1$ and the dual of $h_{k+1} = 1$ is $h_{n-(k+1)} = 1$.

This can also be seen to hold for 3-dimensional flows with a suspension of the subshift of finite type σ with $n \times n$ matrix A, where $\overline{A}$ is equal to $A \mod 2$. The suspension will be denoted by $\tilde{\sigma}$. The homology Conley index of $\tilde{\sigma}$ is

$$CH_1(\tilde{\sigma}) = F^n/(I - \overline{A})F^n$$

$$CH_2(\tilde{\sigma}) = \ker(I - \overline{A}) \text{ on } F^n.$$

Hence,

$$h_1 = \dim CH_1(\tilde{\sigma}) = \dim F^n/(I - \overline{A})F^n = \dim \ker(I - \overline{A}) = \dim CH_2(\tilde{\sigma}) = h_2.$$

By considering the reverse flow, $\tilde{\sigma}^{-1}$ is now the suspension of the subshift of finite type σ^{-1} with the transpose of A, A^t as its matrix. Also, the suspension of the subshift of finite type for the inverse flow has the following homology indices:

$$CH_1(\tilde{\sigma}^{-1}) = F^n/(I - \overline{A}^t)F^n$$

$$CH_2(\tilde{\sigma}^{-1}) = \ker(I - \overline{A}^t) \text{ on } F^n$$

Hence, given that $\overline{h}_i$ represents the dimension of homology Conley index for the inverse flow,

$$\overline{h}_1 = \dim F^n/(I - \overline{A}^t)F^n = \dim \ker(I - \overline{A}^t) = \overline{h}_2.$$

Since $\ker(I - \overline{A}^t) \cong \ker(I - \overline{A})$, this implies $h_1 = h_2 = \overline{h}_1 = \overline{h}_2$. Therefore, h_1 is dual to $\overline{h}_2$.

Let (N, N^-) be an index pair for Λ and (N, N^+) an index pair for Λ'.

Consider the following long exact sequences for the pairs (N, N^-) and (N, N^+):

$$0 \to H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N, N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}}$$

$$\to H_{n-1}(N, N^-) \xrightarrow{\partial_{n-1}} H_{n-2}(N^-) \xrightarrow{i_{n-2}} H_{n-2}(N) \xrightarrow{p_{n-2}} H_{n-2}(N, N^-) \xrightarrow{\partial_{n-2}} \dots$$

$$\xrightarrow{\partial_4} H_3(N^-) \xrightarrow{i_3} H_3(N) \xrightarrow{p_3} H_3(N, N^-) \xrightarrow{\partial_3} H_2(N^-) \xrightarrow{i_2} H_2(N) \xrightarrow{p_2} H_2(N, N^-) \xrightarrow{\partial_2}$$

$$\to H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N, N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N, N^-) \to 0 \tag{1}$$

$$0 \to H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N, N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}}$$

$$\to H_{n-1}(N, N^+) \xrightarrow{\partial'_{n-1}} H_{n-2}(N^+) \xrightarrow{i'_{n-2}} H_{n-2}(N) \xrightarrow{p'_{n-2}} H_{n-2}(N, N^+) \xrightarrow{\partial'_{n-2}} \dots$$

$$\xrightarrow{\partial'_4} H_3(N^+) \xrightarrow{i'_3} H_3(N) \xrightarrow{p'_3} H_3(N, N^+) \xrightarrow{\partial'_3} H_2(N^+) \xrightarrow{i'_2} H_2(N) \xrightarrow{p'_2} H_2(N, N^+) \xrightarrow{\partial'_2}$$

$$H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N,N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N,N^+) \to 0 \tag{2}$$

It is an elementary result that given a long exact sequence of vector spaces,

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \ldots\ldots \to D \to 0$$

rank $\operatorname{Im} h +$ rank $\operatorname{Im} i =$ rank A. This follows from the fact that $\operatorname{rank} A = \operatorname{rank} \operatorname{Im} i + \operatorname{rank} \ker i$ and from the exactness of the sequence $\ker i = \operatorname{Im} h$. Hence,

$$\operatorname{rank} h = \operatorname{rank} A - \operatorname{rank} B + \operatorname{rank} C - + \ldots \pm \operatorname{rank} D \geq 0.$$

It will also be assumed that $h_i = 0$ for $i = 0, n$. This last assumption will be removed at the end of the section. Applying these arguments to the long exact sequences of the pairs (N, N^-) and (N, N^+), and considering that $H_n(N) = H_n(N^\pm) = 0$, $\operatorname{rank} H_i(N, N^-) = h_i$, $\operatorname{rank} H_i(N, N^+) = h_{n-i}$, $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$, $\operatorname{rank} H_0(N) = 1$ and $\operatorname{rank}(H_i(N^\pm)) = B_i^\pm$. Hence, the following equation is obtained:

$$\operatorname{rank} \partial_n = B_{n-1}^- - \operatorname{rank}(H_{n-1}(N)) + h_{n-1} - B_{n-2}^- + \operatorname{rank}(H_{n-2}(N)) - h_{n-2} \ldots$$

$$\pm B_3^- \pm \operatorname{rank}(H_3 N)) \pm h_3 \pm B_2^- \pm \operatorname{rank}(H_2 N)) \pm h_2$$

$$\pm B_1^- \pm \operatorname{rank}(H_1(N)) \pm h_1 \pm e^- \pm 1 \tag{3}$$

In the following subsection, the analysis of the long exact sequences of the pairs will be considered in two separate cases, namely when $\dim N$ is odd or even. In the former case, the classical Poincaré-Hopf equality is obtained.

3.3.1 Odd-dimensional Case, $n = 2i + 1$.

Poincaré duality on the boundary of N implies that $B_{(2i-j)}^- = B_j^-$. Also, since $\operatorname{rank} \partial_{(2i+1)} = 0$, it follows from (3) that:

$$2e^- - 2B_1^- + 2B_2^- - + \ldots \pm 2B_{i-1}^- \pm B_i^- + h_{2i} - h_{2i-1} + - \pm h_3 \pm h_2 \pm h_1 =$$

$$= \sum_{j=0}^{2i} (-1)^j \operatorname{rank}(H_j(N)) \tag{4}$$

Similarly, using the long exact sequence of the pair (N, N^{+}) and using the duality of the indices, $h_j = h_{[(2i+1)-j]}(\Lambda')$ the following equation holds:

$$2e^{+} - 2B_1^{+} + 2B_2^{+} - + \ldots \pm 2B_{i-1}^{+} \pm B_i^{+} + h_1 - h_2 + - \ldots \pm h_{2i-2} \pm h_{2i-1} \pm h_{2i} =$$

$$= \sum_{j=0}^{2i} (-1)^j \operatorname{rank}(H_j(N)) \tag{5}$$

Subtracting (4) from (5) and dividing by two, the following equation holds:

$$(e^{+} - e^{-}) - (B_1^{+} - B_1^{-}) + - \ldots \pm \left(B_{i-1}^{+} - B_{i-1}^{-}\right) \pm \left(B_i^{+} - B_i^{-}\right)$$

$$-h_{2i} + h_{2i-1} - + \ldots \pm h_i \pm \ldots - h_2 + h_1 = 0 \tag{6}$$

which can be represented in short by:

$$\mathcal{B}^{+} - \mathcal{B}^{-} = e^{-} - e^{+} + \sum_{j=1}^{2i} (-1)^j h_j \tag{7}$$

where,

$$\mathcal{B}^{+} = (-1)^i B_i^{+} \pm B_{i-1}^{+} \pm \ldots - B_1^{+}$$
$$\mathcal{B}^{-} = (-1)^i B_i^{-} \pm B_{i-1}^{-} \pm \ldots - B_1^{-}$$

This is an alternative form of the classical *Poincaré-Hopf Equality.* See [3] for more details.

Due to Poincaré duality, it is sufficient to analyze the long exact sequence of the pairs starting at mid-dimension.

Analyzing the ranks of p_i in (1), the following holds:

$$\mathrm{rank\ p_i = h_i - B_{i-1}^{-} + rank(H_{i-1}) - h_{i-1} + B_{i-2}^{-} - rank(H_{i-2}) + h_{i-2} - + \ldots}$$

$$\pm B_2^{-} \pm \mathrm{rank(H_2) \pm h_2 \pm B_1^{-} \pm rank(H_1) \pm h_1 \pm e^{-}} \pm 1 \geq 0$$

$$\Rightarrow h_i \geq B_{i-1}^{-} - B_{i-2}^{-} + - \ldots \pm B_2^{-} \pm B_1^{-} + h_{i-1} - h_{i-2} + - \ldots \pm h_2 \pm h_1 \pm e^{-} \pm 1$$

$$-\mathrm{rank}(\mathrm{H_{i-1}}) + \mathrm{rank}(\mathrm{H_{i-2}}) - + \ldots \pm \mathrm{rank}(\mathrm{H_2}) \pm \mathrm{rank}(\mathrm{H_1}) \tag{8}$$

Similarly, by considering rank ∂_i' in (2) and using the duality of the indices, the following inequality holds:

$$-\mathrm{rank}(\mathrm{H_{i-1}}) + \mathrm{rank}(\mathrm{H_{i-2}}) - + \ldots \pm \mathrm{rank}(\mathrm{H_2}) \pm \mathrm{rank}(\mathrm{H_1}) \geq -\mathrm{B^+_{i-1}} + \mathrm{B^+_{i-2}} - + \ldots \pm \mathrm{B^+_2} \pm \mathrm{B^+_1}$$

$$-h_{i+2} - + \ldots \pm h_{2i-1} \pm h_{2i} \pm e^+ \pm 1 \tag{9}$$

Substituting (9) in (8) the following inequality holds:

$$h_i \geq -\left(B^+_{i-1} - B^-_{i-1}\right) + \left(B^+_{i-2} - B^-_{i-2}\right) + - \ldots \pm \left(B^+_2 - B^-_2\right) \pm \left(B^+_1 - B^-_1\right)$$

$$-\left(h_{i+2} - h_{i-1}\right) + \left(h_{i+3} - h_{i-2}\right) + - \ldots \pm \left(h_{2i-1} - h_2\right) \pm \left(h_{2i} - h_1\right) \pm \left(e^+ - e^-\right) \tag{10}$$

Analogously analyzing p_i' and ∂_i as above and using the duality of the indices, the following inequality is obtained:

$$h_{i+1} \geq -\left[-\left(B^+_{i-1} - B^-_{i-1}\right) + \left(B^+_{i-2} - B^-_{i-2}\right) + - \ldots \pm \left(B^+_2 - B^-_2\right) \pm \left(B^+_1 - B^-_1\right)\right.$$

$$\left.-\left(h_{i+2} - h_{i-1}\right) + \left(h_{i+3} - h_{i-2}\right) + - \ldots \pm \left(h_{2i-1} - h_2\right) \pm \left(h_{2i} - h_1\right) \pm \left(e^+ - e^-\right)\right] \tag{11}$$

Note that in equation (10) h_i is greater than or equal to an integer number and in equation (11), h_{i+1} is greater than or equal to the opposite of that number. Obviously, since h_i and h_{i+1} are non-negative integers, one of the inequalities is redundant. This will continue to occur in the following inequalities.

The remaining inequalities will be obtained by analyzing two pairs of maps at a time,

$$\left\{\left[\left(p_{i-1}, \partial'_{i-1}\right), \left(p'_{i-1}, \partial_{i-1}\right)\right], \ldots \left[\left(p_2, \partial'_2\right), \left(p'_2, \partial_2\right)\right]\right\}.$$

Hence, the last pair to be analyzed is (p_2, ∂_2') and (p_2', ∂_2). The analysis of the former pair implies:

$$h_2 \geq -(B^+_1 - B^-_1) - (h_{2i} - h_1) + (e^+ - e^-) \tag{12}$$

while the analysis of the latter implies:

$$h_{2i-1} \geq -\left[-(B^+_1 - B^-_1) - (h_{2i} - h_1) + (e^+ - e^-)\right] \tag{13}$$

The last inequalities are obtained by analyzing p_1 and p_1':

$$\begin{aligned} e^- &\le h_1 + 1 \\ e^+ &\le h_{2i} + 1 \end{aligned} \tag{14}$$

The analysis of the long exact sequences of the isolating blocks has provided a collection of inequalities, equality (7) and inequalities (10), (11) and successively up to inequalities (12), (13) and (14) which will be called the *Poincaré-Hopf inequalities* for the odd-dimensional case.

3.3.2 Even-dimensional Case, $n = 2i$.

In this section, the inequalities are obtained in the same fashion, that is by analyzing the long exact sequences of the pairs $(N, N^{\pm})$ where $\dim N = 2i$.

The inequalities will be obtained by analyzing two pairs of maps at a time,

$$\left\{ \left[(p_i, \partial_i'), (p_i', \partial_i) \right], \dots \left[(p_2, \partial_2'), (p_2', \partial_2) \right] \right\}.$$

The last inequalities are obtained by analyzing p_1 and p_1'.

Therefore, the Poincaré-Hopf inequalities for the even dimensional case are identical to the odd dimensional case except for the middle dimension:

$$\begin{cases} h_i \ge -(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\ \qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \dots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (e^+ - e^-) \\ \\ h_i \ge -\left[-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \right. \\ \qquad \left. -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \dots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (e^+ - e^-) \right] \end{cases} \tag{15}$$

Moreover, in the case $n = 2 \mod 4$ it is necessary to require that

$$h_i - \sum_{j=1}^{i-1} (-1)^{j+1} (B_j^+ - B_j^-) - \sum_{j=1}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^- - e^+) \text{ be even.} \tag{16}$$

It is interesting to note that a Poincaré-Hopf equality does not exist in the even dimensional case.

By removing the restriction of $h_n = h_0 = 0$ we obtain the Poincaré-Hopf inequalities in all its generality. Recall that equality (7) must be included in the odd-dimensional case and condition

(16) must be included in the even-dimensional case. Essentially we add or subtract a term $(h_n - h_0)$ to the above inequalities.

$$
\begin{cases}
n = 2i+1 \begin{cases}
h_i \geq -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\
\qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \\
\qquad \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)] \\
\\
h_{i+1} \geq -\left[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \right. \\
\qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \\
\qquad \left. \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)] \right]
\end{cases} \\
\\
n = 2i \begin{cases}
h_i \geq -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \\
\qquad \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)] \\
\\
h_i \geq -\left[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \right. \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \\
\qquad \left. \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)] \right]
\end{cases} \\
\qquad \vdots \\
\begin{cases}
h_j \geq -(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \\
\\
h_{n-j} \geq -\left[-(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \right. \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\qquad \left. \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \right]
\end{cases} \\
\qquad \vdots \\
\begin{cases}
h_2 \geq -(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\
h_{n-2} \geq -\left[-(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \right]
\end{cases} \\
\\
\begin{cases}
h_1 \geq h_0 - 1 + e^- \\
h_{n-1} \geq h_n - 1 + e^+
\end{cases}
\end{cases}
\tag{17}
$$

4 Explosion Algorithms

In this section a vertex explosion algorithm will be presented for a *saddle type vertex* v, $h_0 = h_n = 0$ and $h_i,\ 0 < i < n$ non-negative integer. Recall that the vertex represents a singularity of a flow on a manifold of dimension n. This algorithm will be used in Subsection 4.2 to extend the result to generalized saddle type vertices, attractor and repeller vertices. The initial two steps of the algorithm do not depend on the parity of n. However, the third step will distinguish three cases: n odd, $n = 0 \mod 4$, $n = 2 \mod 4$, producing three different algorithms. Each algorithm will generate a linear system whose solutions are precisely the number of different $(\ell - d, \ell - c, \beta - i)$ vertices of Morse type used in the explosion of the vertex v. We end this section with the Continuation Theorem.

4.1 Vertex Explosion Algorithm

Let v be a saddle type vertex labelled vertex with $\{h_1, h_2, \ldots, h_{n-1}\}$ and incoming edges labelled with $\{(\beta_0^+, \ldots, \beta_{n-1}^+)_i\}_{i=1}^{e^+}$ and outgoing edges labelled with $\{(\beta_0^-, \ldots, \beta_{n-1}^-)_i\}_{i=1}^{e^-}$, where i denotes the edge. Let $B_j^+ = \displaystyle\sum_{i=1}^{e^+}(\beta_j^+)_i$ and $B_j^- = \displaystyle\sum_{i=1}^{e^-}(\beta_j^-)_i$. See Figure 6. Observe that $B_0^- = e^-$ e $B_0^+ = e^+$.

The algorithm will have three basic parts:

1. adjusting the incident edges;

2. the linear explosion without middle dimensions;

3. middle dimension explosion (n odd, $n = \mod 4$, $n = 2 \mod 4$).

The first part of the algorithm consists in adjusting the incident edges of the graph. For this purpose we will define two graphs G^+ and G^-. The graph G^+ has the property that it has e^+ incoming edges and one outgoing edge. It will be formed using only singularities of index $n-1$ of type $(n-1)$-d. The graph G^- has the property that it has e^- outgoing edges and one incoming edge. It will be formed using only singularities of index 1 of type 0-c.

The second part of the algorithm consists in joining the outgoing edge of G^+ to a linear graph L^+ formed by Morse type vertices labelled with $h_{n-j} = 1$ and $j <$ middle dimensions. Similarly, the incoming edge of G^- will be joined to a linear graph L^- formed by Morse type vertices labelled with $h_j = 1$ and $j <$ middle dimensions. As the singularities are inserted, recall that the variation of the labels on the edges is always considered in the opposite direction of the orientation of the graph.

The third part of the algorithm consists in inserting the middle dimensional singularities, if there are any, and joining L^+ to L^- if they have not already been joined in the previous part. Of course, all of this must be done, so that the weights on the last two edges to be joined coincide.

We will proceed to describe the algorithm.

Algorithm

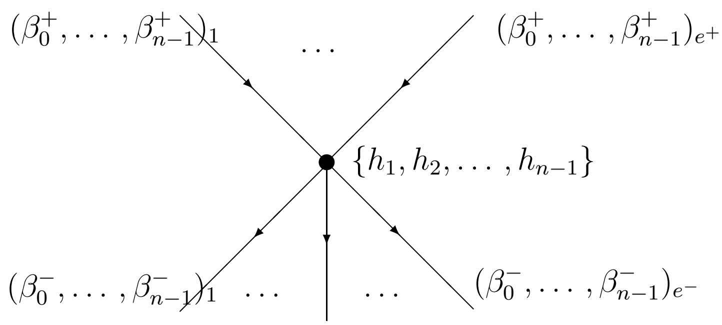

Figure 6: Vertex to be exploded

Step 0 - Adjusting the incident edges: G^- and G^+.

Let $h_1^c = e^- - 1$. By choosing this number of vertices labelled with 1-singularities, G^- is formed with e^- outgoing edges and one incoming edge. Singularities of type $0 - c$ do not alter the β_i with $0 < i < n-1$. This type of singularity decreases β_0 and by duality β_{n-1}. Hence, the incoming edge of G^- has $B_0^- = B_{n-1}^- = 1$ and $B_j^- = \displaystyle\sum_{i=1}^{e^-}(\beta_j^-)_i$ with $j = \{1, \dots, n-2\}$. See Figure 7.

Similarly, the graph G^+ is formed with $h_{n-1}^d = e^+ - 1$ vertices labelled with $(n-1)$-singularities. This graph will have e^+ incoming edges and one outgoing edge. These singularities do not alter β_i, $0 < i < n-1$. The outgoing edge of G^+ will be labelled with $B_0^+ = B_{n-1}^+ = 1$ and $B_j^+ = \displaystyle\sum_{i=1}^{e^+}(\beta_j^+)_i$ with $j = \{1, \dots, n-2\}$.

The linear explosion without middle dimensions (Step 1, ..., Step ℓ)

Step 1 - The adjusting of B_1 and by duality B_{n-2}.

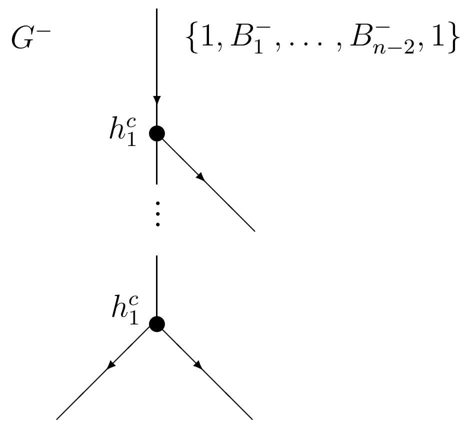

Figure 7: Outgoing edges exploded

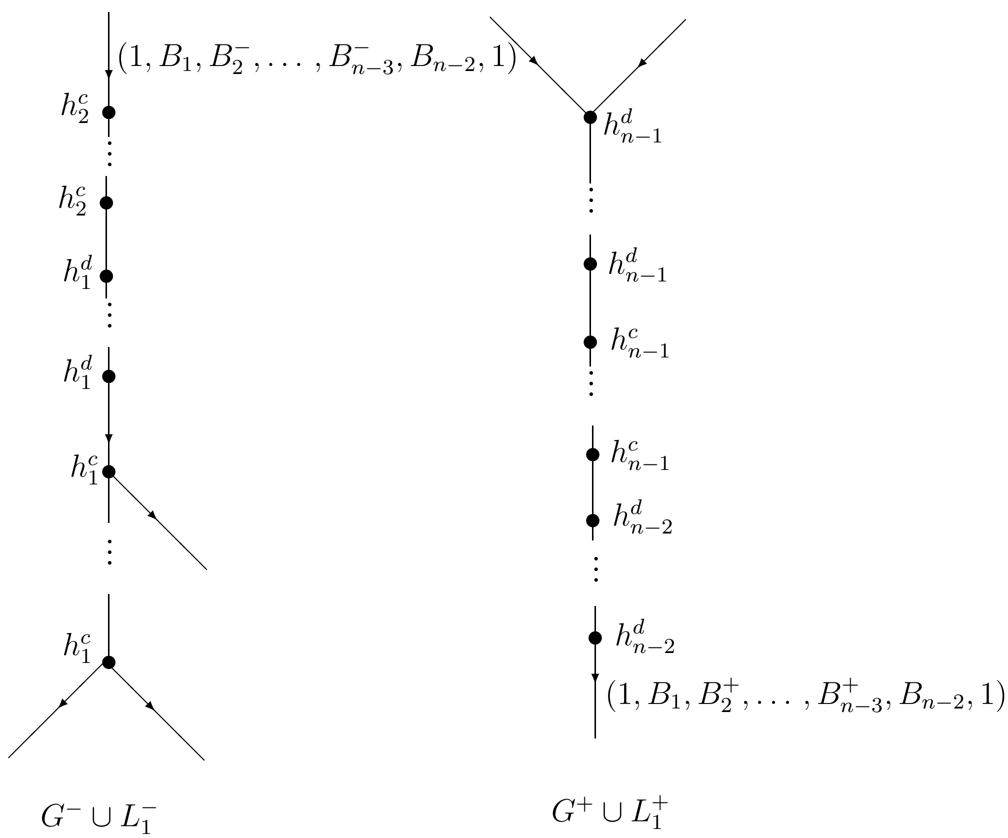

Figure 8: Linear explosion: L^- and L^+.

The label of the incoming edge of G^- is labelled with

$$\{1, B_1^-, B_2^-, \dots, B_{mid}^-, \dots, B_{n-2}^-, 1\}.$$

In order to adjust B_1^- and by duality B_{n-2}^- add a linear graph L_1^- to G^-. This is done by inserting h_1^d vertices $h_1 = 1$ of type $1-d$ to G^-. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B_1^- + h_1^d$. Observe that the insertion of vertices $h_2 = 1$ of type $1-c$ will decrease B_1^-. Thus, after the insertion of h_2^c vertices $h_2 = 1$ of type $1-c$, $G^- \cup L_1^-$ is formed. The label on the incoming edge of $G^- \cup L_1^-$ will be $B_1^- + h_1^d - h_2^c$. By duality the label B_{n-2}^- has been modified to $B_{n-2}^- + h_1^d - h_2^c$.

The label of the outgoing edge of G^+ is labelled with

$$\{1, B_1^+, B_2^+, \dots, B_{mid}^+, \dots, B_{n-2}^+, 1\}.$$

In order to adjust B_1^+ and by duality B_{n-2}^+ add a linear graph L_1^+ to G^+. This is done by inserting h_{n-1}^c vertices $h_{n-1} = 1$ of type $(n-2)-c$. After the insertion of these vertices the label on the outgoing edge of the last vertex of this type will be $B_{n-2}^+ + h_{n-1}^c$. Subsequently, add h_{n-2}^d vertices $h_{n-2} = 1$ of type $(n-2)-d$ forming $G^+ \cup L_1^+$. After the insertion of these vertices the label on the outgoing edge of $G^+ \cup L_1^+$ will be $B_{n-2}^+ - h_{n-2}^d + h_{n-1}^c$. By duality the label B_1^+ has been modified to $B_1^+ - h_{n-2}^d + h_{n-1}^c$.

Since the insertion of any other type of vertex will not alter the first and the $(n-2)$-th Betti number it is necessary that

$$B_1 = B_1^- + h_1^d - h_2^c = B_1^+ - h_{n-2}^d + h_{n-1}^c.$$

If the above equality is true then by duality the following equality holds:

$$B_{n-2} = B_{n-2}^- + h_1^d - h_2^c = B_{n-2}^+ + h_{n-2}^d - h_{n-1}^c.$$

Thus the labels of B_1 and B_{n-2} have been adjusted. The incoming edge of $G^- \cup L_1^-$ has the following labels:

$$\{1, B_1, B_2^-, \dots, B_\ell^-, \dots, B_{mid}^-, \dots, B_{n-\ell-1}^-, \dots, B_{n-3}^-, B_{n-2}, 1\}$$

and the outgoing edge of $G^+ \cup L_1^+$ has the following labels :

$$\{1, B_1, B_2^+, \dots, B_\ell^+, \dots, B_{mid}^+, \dots, B_{n-\ell-1}^+, \dots, B_{n-3}^+, B_{n-2}, 1\}$$

This process of adjusting B_ℓ^+ and $B_{n-\ell-1}^+$ will be repeated in increasing order except in the middle dimensions. The general procedure will be presented in the next step.

Step ℓ - The adjusting of B_ℓ and by duality $B_{n-\ell-1}$

At this point, it is assumed by induction that the adjustments of B_j for $j < \ell$ and by duality B_{n-j-1} for $j < \ell$ have been made in increasing order for j. Hence, several linear graphs have been added to G^+ forming at this point a graph $G^+ \cup \bigcup_{i=1}^{\ell-1} L_i^+$ whose outgoing edge is labelled with

$$\{1, B_1, \dots, B_{\ell-1}, B_\ell^+, \dots, B_{mid}^+, \dots, B_{n-\ell-1}^+, B_{n-\ell}, \dots, B_{n-2}, 1\}.$$

Similarly, several linear graphs have been added to G^- forming at this point a linear graph $G^- \cup \bigcup_{i=1}^{\ell-1} L_i^-$ whose incoming edge is labelled with

$$\{1, B_1, \dots, B_{\ell-1}, B_\ell^-, \dots, B_{mid}^-, \dots, B_{n-\ell-1}^+, B_{n-\ell}, \dots, B_{n-2}, 1\}.$$

In order to adjust B_ℓ and by duality $B_{n-\ell-1}$ insert h_ℓ^d vertices $h_\ell = 1$ of type $\ell - d$ to $G^- \cup \bigcup_{i=1}^{\ell-1} L_i^-$. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B_\ell^- + h_\ell^d$. Observe that the insertion of vertices of type $\ell - c$ will decrease B_ℓ^-. Thus, after the insertion of $h_{\ell+1}^c$ vertices $h_{\ell+1} = 1$ of type $\ell - c$ the label on the incoming edge of $G^- \cup \bigcup_{i=1}^{\ell} L_i^-$ will be $B_\ell^- + h_\ell^d - h_{\ell+1}^c$. By duality the label $B_{n-\ell-1}^-$ has been modified to $B_{n-\ell-1}^- + h_\ell^d - h_{\ell+1}^c$.

Similarly, the insertion of $h_{n-\ell}^c$ vertices $h_{n-\ell} = 1$ of type $(n-\ell-1) - c$ will produce an outgoing edge labelled with $B_{n-\ell-1}^+ + h_{n-\ell}^c$. Subsequently the insertion of $h_{n-\ell-1}^d$ vertices $h_{n-\ell-1} = 1$ of type $(n-\ell-1) - d$ will form $G^+ \cup \bigcup_{i=1}^{\ell} L_i^+$ and the labels on its outgoing edge is $B_{n-\ell-1}^+ - h_{n-\ell-1}^d + h_{n-\ell}^c$. By duality the label B_ℓ^+ has been modified to $B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c$.

Since the insertion of any other type of vertex will not alter the ℓ-th and the $(n-\ell-1)$-th Betti number it is necessary that

$$B_\ell = B_\ell^- + h_\ell^d - h_{\ell+1}^c = B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c$$

If the above equality is true then by duality the following equality holds:

$$B_{n-\ell-1} = B_{n-\ell-1}^- + h_\ell^d - h_{\ell+1}^c = B_{n-\ell-1}^+ - h_{n-\ell-1}^d + h_{n-\ell}^c.$$

The labels of B_ℓ and $B_{n-\ell-1}$ for $0 < \ell < mid$ have all been adjusted. It remains to adjust the middle dimensional labels. This is done in the next step.

Middle dimensional explosion

At this point the adjustments of the labels at the middle dimensions must be made. It is necessary to consider the case when $n-1$ is even and hence there is only one middle dimensional label $B_{\frac{n-1}{2}}$ and the case when $n-1$ is odd and there are two middle dimensional labels.

Case $n-1$ even

Let $n = 2i+1$. At this point there are two graphs $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$ with incoming edge labelled with $\{1, B_1, B_2, \dots, B_{i-1}, B_i^-, B_{i+1}, \dots, B_{2i-1}, 1\}$ and $G^+ \cup \bigcup_{j=1}^{i-1} L_j^+$ with outgoing edge labelled with $\{1, B_1, B_2, \dots, B_{i-1}, B_i^+, B_{i+1}, \dots, B_{2i-1}, 1\}$.

To adjust B_i^- insert h_i^d vertices $h_i = 1$ of type $i-d$ to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$. Hence, the label on the incoming edge of the last vertex inserted of this type is $B_i^- + 2h_i^d$.

Thus, after the insertion of h_{i+1}^c vertices $h_{i+1} = 1$ of type $(i+1)-c$ to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-1} L_j^+$. Thus, the label on the outgoing edge of last vertex inserted of this type is $B_i^+ + 2h_{i+1}^c$.

Moreover, it is necessary that

$$B_i = B_i^- + 2h_i^d = B_i^+ + 2h_{i+1}^c.$$

Since, the labels on the outgoing edge of one of the graphs now coincides entirely with the labels on the incoming edge of the other graph, they can be joined to form a connected graph.

Conclusion

Hence, at the end of this adjustment, we are left with the following linear system that must be solved for $\{h_1^c, h_1^d, \dots, h_{2i}^c, h_{2i}^d\}$, in order for the algorithm to work.

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \;\; j = 1, \ldots, 2i \\ e^+ - 1 - h_{2i}^d = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ \vdots \\ \frac{-(B_i^+ - B_i^-)}{2} + h_i^d - h_{i+1}^c = 0 \end{cases} \end{cases} \tag{18}$$

Case $n-1$ odd.

Let $n = 2i$. There are two cases to consider:

Case 1: $n = 0 \bmod 4$.

Observe that in this case the variation of the middle dimensional Betti number B_{i-1}^- and its dual B_i^- will vary by 0 if the vertex is of $\beta - i$ type or by 1 otherwise.

Insert h_{i-1}^d vertices $h_{i-1} = 1$ of type $(i-1) - d$ to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-2} L_j^-$. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B_{i-1}^- + h_{i-1}^d$. Subsequently, insert h_i^c vertices $h_i = 1$ of type $(i-1) - c$. The label on the incoming edge of $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$ will be $B_{i-1}^- + h_{i-1}^d - h_i^c$. By duality the label B_i^- will be modified to $B_i^- + h_{i-1}^d - h_i^c$.

Similarly, insert h_{i+1}^c vertices $h_{i+1} = 1$ of type $i - c$ to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-2} L_j^+$. Hence, the label on the outgoing edge of the last vertex of this type inserted will be $B_i^+ + h_{i+1}^c$. Subsequently, insert h_i^d vertices $h_i = 1$ of type $i - d$ forming $G^+ \cup \bigcup_{j=1}^{i} L_j^+$ and the label on the outgoing edge of this graph will be $B_i^+ - h_i^d + h_{i+1}^c$. By duality the label B_{i-1}^+ has been modified to $B_{i-1}^+ - h_i^d + h_{i+1}^c$.

Moreover, it is necessary that

$$B_{i-1}^- + h_{i-1}^d - h_i^c = B_{i-1}^+ - h_i^d + h_{i+1}^c.$$

By duality if the above equation holds then the following is true

$$B_i^- + h_{i-1}^d - h_i^c = B_i^+ - h_i^d + h_{i+1}^c.$$

Since, the labels on the outgoing edge of one of the graphs now coincides entirely with the labels on the incoming edge of the other graph, they can be joined to form a connected graph.

Conclusion

Hence, at the end of this adjustment, we are left with the following linear system that must be solved for $\{h_1^c, h_1^d, \ldots, \beta^i, \ldots, h_{2i-1}^c, h_{2i-1}^d\}$ in order for the algorithm to work.

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \ \ j = 1, \ldots, 2i-1, \beta^i = 0 \text{ if } j \neq i \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{cases} \end{cases} \tag{19}$$

Case 2: $n = 2 \bmod 4$ The adjustments in this case are identical to the previous case except for the fact that there are no vertices of type $\beta - i$.

In this case the linear system that must be solved for $\{h_1^c, h_1^d, \ldots, h_{2i-1}^c, h_{2i-1}^d\}$ is:

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \ \ j = 1, \ldots, 2i-1 \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{cases} \end{cases} \tag{20}$$

Under the hypothesis of Theorem 1.1, i.e., that the vertex satisfies the Poincaré-Hopf inequalities (10), (11) and successively up to inequalities (12), (13) and (14) in the odd case, and inequalities (15) in the even case, it will be shown in Section 5 that the above systems always have non negative integer solutions. In other words, the vertex can be exploded. Moreover, upper bound on the number of explosions a given vertex can have are also determined.

4.2 General Case

In this section the generalized saddle type vertex repeller and attractor vertices will be considered. For simplicity the particular case of a saddle type vertex with $h_0 = h_n = 0$ was considered in the previous section. However, the generalized saddle type vertex does not impose this restriction, that is, $h_0 \neq 0$ and $h_n \neq 0$ is permitted. Also, the attractor ($e^+ = 0$) and repeller vertices ($e^- = 0$) will be considered. The main body of the algorithm in these new cases is identical to the one in Subsection 4.1. Additional steps to adjust the incident edges to the graph must be performed before applying the algorithm 4.1.

4.2.1 Case 1 - Generalized saddle type vertex

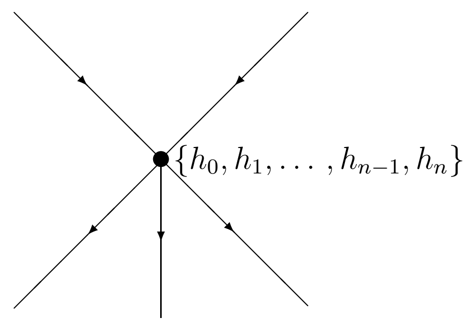

Figure 9: Generalized saddle vertex

Given a generalized saddle vertex labelled with $\{h_0, h_1, \ldots, h_n\}$, a partial explosion will be done in order to obtain a saddle vertex labelled with $\{h_1, \ldots, h_{n-1}\}$.

This partial explosion consists of a preliminary step to the algorithm in Section 4. Before defining G^+ and G^-, the graphs G_0^+ and G_0^- will be defined as follows

1. G_0^+ is formed with h_n vertices labelled with $h_n = 1$ and $h_n - 1$ vertices labelled with $h_{n-1} = 1$ of the type $(n-1) - d$ and has one outgoing edge which connects to v.

2. G_0^- is formed with h_0 vertices labelled with $h_0 = 1$ and $h_0 - 1$ vertices labelled with $h_1 = 1$ of the type $0 - c$ and has one incoming edge which comes from v.

Now, the vertex v has $e^+ + 1$ incoming edges and $e^- + 1$ outgoing edges. Moreover, the vertex is now labelled with

$$\{h_n = 0, h_{n-1} - (h_n - 1), h_{n-2}, \dots, h_3, h_2, h_1 - (h_0 - 1), h_0 = 0\}.$$

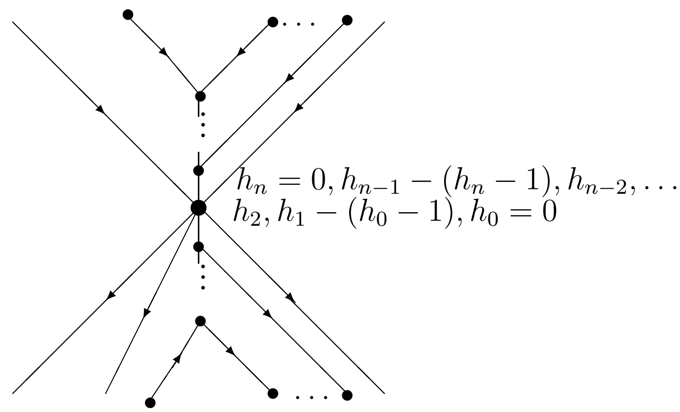

Figure 10: Partial explosion of a generalized saddle vertex

Observe that by substituting in the Poincaré-Hopf inequalities (17) in this section h_{n-1} by $h_{n-1} - (h_n - 1)$, h_1 by $h_1 - (h_0 - 1)$, e^+ by $e^+ + 1$ and e^- by $e^- + 1$, the Poincaré-Hopf inequalities for saddle type vertices (10), (11), and successively up to inequalities (12), (13) and (14), are satisfied for this partially exploded vertex. Hence, the algorithm in Section 4 may be applied in order to complete the explosion.

4.2.2 Case 2 - Attractor and Repeller Vertices.

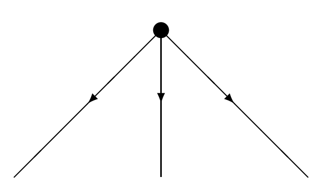

Figure 11: Repeller vertex

Consider a repeller vertex, $e^+ = 0$, labelled $\{h_0, h_1, \dots, h_n\}$, a partial explosion will be done in order to obtain a vertex labelled with $\{h_1, \dots, h_{n-1}\}$.

The graphs G_0^+ and G_0^- will be defined as in the previous case.

Now, vertex v has 1 incoming edge and $e^- + 1$ outgoing edges. Moreover, the vertex is now labelled with

$$\{h_n = 0, h_{n-1} - (h_n - 1), h_{n-2}, \dots, h_3, h_2, h_1 - (h_0 - 1), h_0 = 0\}.$$

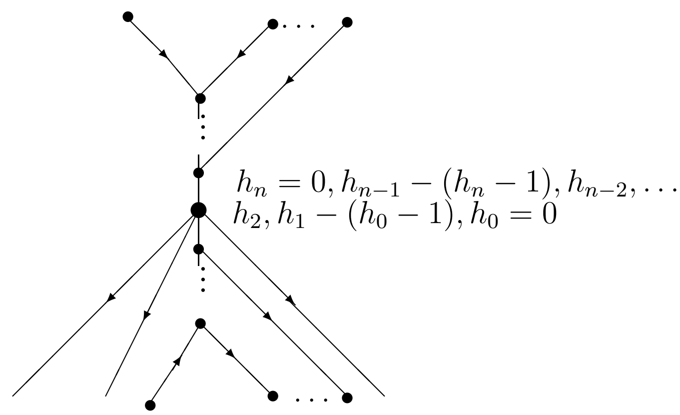

Figure 12: Partial explosion of a repeller vertex

Observe that by substituting in the Poincaré-Hopf inequalities (17) in this section h_{n-1} by $h_{n-1} - (h_n - 1)$, h_1 by $h_1 - (h_0 - 1)$, e^+ by 1 and e^- by $e^- + 1$, the Poincaré-Hopf inequalities for saddle type vertices (10), (11), and successively up to inequalities (12), (13) and (14), are satisfied for this partially exploded vertex. Hence, the algorithm in Subsection 4.1 may be applied in order to complete the explosion.

5 Solutions to the Linear Systems

There are four possibilities for n, namely $n = 0, 1, 2$ or $3 \mod 4$. The second and fourth possibilities correspond to the case n odd. This possibility is considered in Subsection 5.1. The first and third possibilities are analyzed in Subsections 5.2 and 5.3, respectively.

5.1 Case n odd

5.1.1 The linear system

Assume $n \geq 3$, $n = 2i+1$. We want to find a nonnegative integral solution of the linear system

$$
\begin{array}{r}
\text{1-st set } (2i+2) \text{ rows} \left\{ \vphantom{\begin{array}{c} 1\\1\\1\\1\\1\\1\\1 \end{array}} \right. \\
\text{2-nd set } i \text{ rows} \left\{ \vphantom{\begin{array}{c} 1\\1\\1\\1 \end{array}} \right.
\end{array}
\left(\begin{array}{cc|ccccccc|cc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots\ 1\ \ -1\ \cdots & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{4i-2} \\ x_{4i-1} \\ x_{4i} \end{pmatrix}
=
\begin{pmatrix} \theta_1 \\ d_2 \\ \theta_3 \\ d_4 \\ \vdots \\ \theta_4 \\ d_3 \\ \theta_2 \\ d_1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_i \end{pmatrix}
\tag{21}
$$

where θ_j and d_j, for $j = 1, \ldots, i+1$, are positive integers and β_j, for $j = 1, \ldots, i$, are integers. The equations in (21) are partitioned in two sets, containing the first $2i+2$ equations and the following i ones, respectively. The rows of the coefficient matrix $\overline{A}$ of (21) and its right-hand-side vector $\overline{b}$ inherit this partition. The formulas for the j-th row of the first group and the corresponding right-hand-side element are

$$
\overline{A}_j = \begin{cases} \boldsymbol{e}_1, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq 2i+1 \\ \boldsymbol{e}_{4i}, & \text{if } j = 2i+2 \end{cases}
$$

and

$$
\overline{b}_j = \begin{cases} \theta_j, & \text{if } 1 \leq j \leq i+1,\ j \text{ odd} \\ d_j, & \text{if } 1 \leq j \leq i+1,\ j \text{ even} \\ \theta_{2i+3-j}, & \text{if } i+1 < j \leq 2i+2,\ j \text{ odd} \\ d_{2i+3-j}, & \text{if } i+1 < j \leq 2i+2,\ j \text{ even} \end{cases}
$$

where $\boldsymbol{e}_j$ is the j-th vector of $\mathbb{R}^{4i}$'s canonical basis. In the second group, the j-th row and respective right-hand-side are as follows

$$\overline{A}_{2i+2+j} = \begin{cases} \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-2j} - \boldsymbol{e}_{4i-2j+1}, & \text{if } 1 \leq j \leq i-1 \\ \boldsymbol{e}_{2i} - \boldsymbol{e}_{2i+1}, & \text{if } j = i \end{cases}$$

and $\overline{b}_{2i+2+j} = \beta_j$, if $1 \leq j \leq i$.

Four variables in (21) have unique integer values that depend on certain elements of the right-hand-side. Hence the nonnegativity condition on the solution translates to a set of inequalities these values must satisfy. We are left with a reduced system in the remaining variables—the boxes depicted in (21) contain the coefficient matrix of this reduced system. In Subsection 5.1.2 we show that the coefficient matrix of this new system is totally unimodular, which implies that if the system has a nonnegative solution then it will have an nonnegative integral solution. A set of constraints that are equivalent to the existence of nonnegative solutions to the reduced system are constructed in Subsection 5.1.3.

The following equations may be used to eliminate some variables in (21):

1. The first equation implies $x_1 = \theta_1$.
2. The second equation implies $x_2 = d_2 - \theta_1$.
3. The $(2i+2)$-th equation implies $x_{4i} = d_1$.
4. The $(2i+1)$-th equation implies $x_{4i-1} = \theta_2 - d_1$.

Therefore, using the fact that the vectors θ, d and β are integral, the following are necessary and sufficient conditions for the partial solution $(x_1, x_2, x_{4i}, x_{4i-1})$ to be a nonnegative integral 4-tuple:

$$\begin{cases} d_2 - \theta_1 \geq 0 \\ \theta_2 - d_1 \geq 0 \end{cases} \tag{22}$$

It remains to obtain necessary and sufficient conditions for the existence of nonnegative integral solutions of the aforementioned reduced system.

5.1.2 Total unimodularity of the coefficient matrix of the reduced system

We arrive at the reduced system (23) below after eliminating equations 1, 2, $2i+1$ and $2i+2$ and variables x_1, x_2, x_{4i-1} and x_{4i} in (21):

$$\begin{array}{r} \text{1-st set} \\ (2i-2) \text{ rows} \\ \\ \text{2-nd set} \\ i \text{ rows} \end{array} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & -1 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots 1 \ -1 \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ \vdots \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} \theta_3 \\ d_4 \\ \vdots \\ \theta_4 \\ d_3 \\ \tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_i \end{pmatrix} \tag{23}$$

where $\tilde{\beta}_1 = \beta_1 - x_2 + x_{4i-1} = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-2$ equations in $4i-4$ unknowns.

The rows of $\overline{\overline{A}}$ and $\overline{\overline{b}}$, the coefficient matrix and right-hand-side vector of (23), inherit the partition established for their counterparts in (21). Hence the first set has $2i-2$ rows/elements (four equations were eliminated) and, the second, i. The formulas for the j-th row and right-hand-side element must be updated to account for the elimination of rows and columns (equations and variables). In the first group we have

$$\overline{\overline{A}}_j = \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, \text{ if } 1 \le j \le 2i-2$$

and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-1, \ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-1, \ j \text{ even} \\ \theta_{2i+1-j}, & \text{if } i-1 < j \le 2i-2, \ j \text{ odd} \\ d_{2i+1-j}, & \text{if } i-1 < j \le 2i-2, \ j \text{ even} \end{cases}$$

and, in the second set,

$$\overline{\overline{A}}_{2i-2+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}, & \text{if } j = 1 \\ \boldsymbol{e}_{2(j-1)} - \boldsymbol{e}_{2(j-1)+1} + \boldsymbol{e}_{4i-4-2(j-1)} - \boldsymbol{e}_{4i-4-2(j-1)+1}, & \text{if } 2 \le j \le i-1 \\ \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}, & \text{if } j = i \end{cases}$$

$\overline{\overline{b}}_{2i-1} = \tilde{\beta}_1$ and $\overline{\overline{b}}_{2i-2+j} = \beta_j$, for $j = 2, \ldots, i$. Note that $\boldsymbol{e}_\ell$ is now the ℓ-th vector of $\mathbb{R}^{4i-4}$'s canonical basis. The following block diagram helps in the understanding of $\overline{\overline{A}}$'s structure:

$$
\begin{array}{cc}
\left(\begin{array}{cccccccccccc}
1 & 1 & 0 & 0 & 0 & \cdots & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & \cdots & & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \ddots & & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & 0 & 1 & 1 \\
\hdashline
-1 & 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & \cdots & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & & & 1 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \iddots & & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\\vdots\\1\\1\end{array}}\right\}\begin{array}{l}\text{1-st set}\\(2i-2)\text{ rows}\end{array}\\
\left.\vphantom{\begin{array}{c}1\\1\\1\\\vdots\\1\end{array}}\right\}\begin{array}{l}\text{2-nd set}\\ i\text{ rows}\end{array}
\end{array}
\end{array}
$$

The first set of rows form a block diagonal matrix while in the second set there are two blocks per row, arranged symmetrically about the middle.

Lemma 5.1 *Each column of* $\overline{\overline{A}}$ *contains exactly two nonnull coefficients, one in the first set of rows and the other in the second set.*

Proof: Each entry in $\overline{\overline{A}}$ belongs to $\{0, 1, -1\}$. Thus it is equivalent to show that each column of $\tilde{A}$, the matrix obtained from $\overline{\overline{A}}$ by replacing each entry with its absolute value, contains two 1's, one in each set of rows.

In order to accomplish that we calculate the sum of the rows in the first and second sets of $\tilde{A}$:

1. Summing rows in the first set, that is, rows 1 through $2i-2$ of $\tilde{A}$:

$$\begin{aligned}
\sum_{j=1}^{2i-2} \tilde{A}_j = \sum_{j=1}^{2i-2} \overline{\overline{A}}_j &= \sum_{j=1}^{2i-2} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) \\
&= \sum_{\substack{\ell \in \{1,\dots,4i-4\} \\ \ell \text{ odd}}} \boldsymbol{e}_\ell + \sum_{\substack{\ell \in \{1,\dots,4i-4\} \\ \ell \text{ even}}} \boldsymbol{e}_\ell \\
&= \sum_{\ell=1}^{4i-4} \boldsymbol{e}_\ell = \mathbf{1}_{4i-4},
\end{aligned}$$

where $\mathbf{1}_m$ denotes the vector of all 1's in $\mathbb{R}^m$. Therefore each column of $\tilde{A}$ contains exactly one nonnull entry in the first set of rows, namely 1.

2. Summing the i rows in the second set:

$$\begin{aligned}
\sum_{j=1}^{i} \tilde{A}_{2i-2+j} &= \boldsymbol{e}_1 + \boldsymbol{e}_{4i-4} + \\
&\quad \sum_{j=2}^{i-1} (\boldsymbol{e}_{2(j-1)} + \boldsymbol{e}_{2(j-1)+1} + \boldsymbol{e}_{4i-4-2(j-1)} + \boldsymbol{e}_{4i-4-2(j-1)+1}) \\
&\quad + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} \\
&= \boldsymbol{e}_1 + \boldsymbol{e}_{4i-4} + \sum_{\substack{\ell \in \{2,\dots,2i-3\} \\ \ell \text{ even}}} \boldsymbol{e}_\ell + \sum_{\substack{\ell \in \{2,\dots,2i-3\} \\ \ell \text{ odd}}} \boldsymbol{e}_\ell \\
&\quad + \sum_{\substack{\ell \in \{1,\dots,2i-4\} \\ \ell \text{ even}}} \boldsymbol{e}_{4i-4-\ell} + \sum_{\substack{\ell \in \{1,\dots,2i-4\} \\ \ell \text{ odd}}} \boldsymbol{e}_{4i-4-\ell} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} \\
&= \boldsymbol{e}_1 + \boldsymbol{e}_{4i-4} + \sum_{\ell=2}^{2i-3} \boldsymbol{e}_\ell + \sum_{\ell=2i}^{4i-5} \boldsymbol{e}_\ell + \boldsymbol{e}_{2i-2}^T + \boldsymbol{e}_{2i-1} \\
&= \sum_{\ell=1}^{4i-4} \boldsymbol{e}_\ell = \mathbf{1}_{4i-4}
\end{aligned}$$

The above implies that each column of $\tilde{A}$ contains exactly one nonnull entry in the second set of rows, which is equal to 1.

The previous two items conclude the proof. ■

Multiplying by -1 the odd equations in (23), this system is transformed into the equivalent system (24) below:

$$\begin{array}{r} \text{1-st set} \\ (2i-2)\text{ rows} \\ \\ \text{2-nd set} \\ i \text{ rows} \end{array} \begin{pmatrix} -1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \cdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \cdots & & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & & \cdots & & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \cdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots (-1)^i & & (-1)^{i+1} \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ \vdots \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_4 \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ (-1)^i \beta_i \end{pmatrix} \tag{24}$$

The matrix of coefficients A of (24) and its right-hand-side vector b inherit the partition established for their predecessors in systems (21) and (23). The new formulas for the j-th row and right-hand-side element in the first set are:

$$A_j = (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), \text{ if } 1 \leq j \leq 2i-2$$

and

$$b_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \leq j \leq i-1,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \leq j \leq i-1,\ j \text{ even} \\ -\theta_{2i+1-j}, & \text{if } i-1 < j \leq 2i-2,\ j \text{ odd} \\ d_{2i+1-j}, & \text{if } i-1 < j \leq 2i-2,\ j \text{ even} \end{cases}$$

and in the second set we have:

$$A_{2i-2+j} = \begin{cases} -(-\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}), & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2(j-1)} - \boldsymbol{e}_{2(j-1)+1} + \boldsymbol{e}_{4i-4-2(j-1)} - \boldsymbol{e}_{4i-4-2(j-1)+1}), & \text{if } 2 \leq j \leq i-1 \\ (-1)^i(\boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}), & \text{if } j = i \end{cases}$$

$$b_{2i-2+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \leq j \leq i \end{cases}$$

Theorem 5.1 *Each column of A contains precisely two nonnull entries: 1 and -1.*

Proof: Lemma 5.1 implies that A has two nonnull entries of absolute value 1 per column. In order to conclude that this pair is 1, -1, it suffices to show that the sum of the rows produces the zero vector. We calculate separately the sums of the first and second set of rows:

$$\sum_{j=1}^{2i-2} A_j = \sum_{j=1}^{2i-2} (-1)^j (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) \tag{25}$$

and

$$\begin{aligned}
\sum_{j=1}^{i} A_{2i-2+j} &= \sum_{j=1}^{i} (-1)^j \overline{\overline{A}}_{2i-2+j} \\
&= -(-\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}) \\
&\quad + \sum_{j=2}^{i-1} (-1)^j (\boldsymbol{e}_{2(j-1)} - \boldsymbol{e}_{2(j-1)+1} + \boldsymbol{e}_{4i-4-2(j-1)} - \boldsymbol{e}_{4i-4-2(j-1)+1}) \\
&\quad + (-1)^i (\boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}) \\
&= \sum_{\ell=1}^{i-2} (-1)^{\ell+1} (\boldsymbol{e}_{2\ell} + \boldsymbol{e}_{4i-4-2\ell}) + (-1)^i \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{4i-4} \\
&\quad + \boldsymbol{e}_1 + (-1)^{i+1} \boldsymbol{e}_{2i-1} + \sum_{j=2}^{i-1} (-1)^{j+1} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2(j-1)+1}) \\
&= \sum_{\ell=1}^{i-2} (-1)^{\ell+1} \boldsymbol{e}_{2\ell} + (-1)^i \boldsymbol{e}_{2i-2} + \sum_{k=i}^{2i-3} (-1)^{k+1} \boldsymbol{e}_{2k} - \boldsymbol{e}_{4i-4} \\
&\quad + \boldsymbol{e}_1 + \sum_{j=2}^{i-1} (-1)^{j+1} \boldsymbol{e}_{2j-1} + (-1)^{i+1} \boldsymbol{e}_{2i-1} + \sum_{\ell=i+1}^{2i-2} (-1)^{\ell+1} \boldsymbol{e}_{2\ell-1} \\
&= \sum_{j=1}^{2i-2} (-1)^{j+1} \boldsymbol{e}_{2j} + \sum_{j=1}^{2i-2} (-1)^{j+1} \boldsymbol{e}_{2j-1}
\end{aligned} \tag{26}$$

Equations (25) e (26) imply that

$$\begin{aligned}
\sum_{j=1}^{3i-2} A_j &= \sum_{j=1}^{2i-2} A_j + \sum_{j=1}^{i} A_{2i-2+j} \\
&= \sum_{j=1}^{2i-2} (-1)^j (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) + \sum_{j=1}^{2i-2} (-1)^{j+1} (\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j-1}) \\
&= \mathbf{0}_{4i-4}
\end{aligned}$$

Hence the sum of rows in A is the zero vector in $\mathbb{R}^{4i-4}$, implying that the two nonzero entries in each column of A are 1 and -1. ■

Theorem 5.1 implies, see p. 274 of [10], that A is a totally unimodular matrix. Furthermore, it has the structure of the incidence matrix of a digraph, which can be used to our advantage in exploring the solution set.

5.1.3 Solutions of the reduced system

It follows from Theorem 5.1 that A (and thus also $\overline{\overline{A}}$) is totally unimodular and may be interpreted as the node-arc incidence matriz of a digraph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, with node set $\mathcal{N}$ and (directed) arc set $\mathcal{A}$. If the nonnull entries 1 and -1 of column j belong to rows k and ℓ, respectively, then the corresponding arc leaves node k and enters node ℓ. The variable x_{j+2} associated with column j is interpreted as the flow in arc j (leaving node k and entering node ℓ) and the i-th equation as

$$\text{flow that enters node } i - \text{flow that leaves node } i = b_i,$$

also called a flow balance constraint. Positive (resp., negative) b_i is interpreted as the demand (resp., $-$supply) of node i.

We seek a nonnegative integral flow that satisfies the balance equations. Since A is totally unimodular and b is integral, it is well known—see, for instance, p. 266–267 of [10]—that a nonnegative integral flow exists if a nonnegative one does.

The structure of $\mathcal{G}$ make it possible to translate the existence of nonnegative flow into a set of constraints involving the node constants b. The following necessary condition for the existence of solutions to (24) is a simple consequence of Theorem 5.1, and has a natural statement in terms of the flow model described: supply must equal demand. Since the rows of A add up to the zero vector in $\mathbb{R}^{4i-4}$, the sum of the elements of b must also be zero:

$$\begin{aligned}
0 &= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j + \sum_{j=1}^{i} b_{2i-2+j} \\
&= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j - \tilde{\beta}_1 + \sum_{j=2}^{i} (-1)^j \beta_j \\
&= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j - (\beta_1 + \theta_1 + \theta_2 - d_1 - d_2) + \sum_{j=2}^{i} (-1)^j \beta_j \\
&= -\sum_{j=1}^{i+1} \theta_j + \sum_{j=1}^{i+1} d_j + \sum_{j=1}^{i} (-1)^j \beta_j. \qquad (27)
\end{aligned}$$

Theorem 5.2 *The set $\mathcal{A}$ may be partitioned into $i-1$ subsets, each corresponding to a non-directed cycle of $\mathcal{G}$. The j-th cycle is given by the following sequence of nodes and arcs:*

If j is odd:

$$\{j, (j, 2i-2+j), 2i-2+j, (2i-2+j, 2i-1-j),$$
$$2i-1-j, (2i-1+j, 2i-1-j), 2i-1+j, (j, 2i-1+j), j\}$$

If j is even:

$$\{2i-1-j, (2i-1-j, 2i-2+j), 2i-2+j, (2i-2+j, j),$$
$$j, (2i-1+j, j), 2i-1+j, (2i-1-j), 2i-1+j), 2i-1-j\}$$

Proof: Since each cycle has four arcs, it suffices to show that the digraph contains the cycles described and that they are arc-disjoint, since then the union of these cycles will have $4(i-1)$ arcs, which is the cardinality of $\mathcal{A}$.

Denoting by $\mathcal{N}_j$ the node set $\{j, 2i-1-j, 2i-2+j, 2i-1+j\}$, the rows in the augmented matrix $A|b$ associated with nodes in $\mathcal{N}_j$ are:

$$\left(\begin{array}{c|c} A_j & b_j \\ A_{2i-1-j} & b_{2i-1-j} \\ A_{2i-2+j} & b_{2i-2+j} \\ A_{2i-1+j} & b_{2i-1+j} \end{array}\right) \tag{28}$$

where the elements b_j, b_{2i-1-j} are given by Table 1.

	$1 \le j \le i-1$	
	j odd	j even
b_j	$-\theta_{j+2}$	d_{j+2}
b_{2i-1-j}	d_{j+2}	$-\theta_{j+2}$
b_{2i-2+j}	$-\tilde{\beta}_1$, if $j=1$ $-\beta_j$, if $j \ge 3$	β_j
b_{2i-1+j}	β_{j+1}	$-\beta_{j+1}$

Table 1: Formulas for b_j and b_{2i-1-j}

Therefore, the elements of the right-hand-side associated with nodes in $\mathcal{N}_j$, independently of the parity of j, are

$$-\theta_{j+2},\ (-1)^j\beta_{2i-2+j},\ d_{j+2},\ (-1)^{j+1}\beta_{2i-1+j},$$

and the intersection of the subgraphs $\mathcal{G}_j$ and $\mathcal{G}_\ell$ induced respectively by $\mathcal{N}_j$ and $\mathcal{N}_\ell$, for $1 \leq j,\ \ell \leq i-1$, is:

$$\mathcal{N}_j \cap \mathcal{N}_\ell = \begin{cases} \{2i-1+j\}, & \text{if } \ell = j+1 \\ \{2i-2+j\}, & \text{if } \ell = j-1 \\ \emptyset, & \text{otherwise} \end{cases}$$

That is, the intersection contains at most one node, which means the subgraphs induced $\mathcal{G}_j$ and $\mathcal{G}_\ell$ are arc-disjoint. It remains to show that $\mathcal{G}_j$ consists of cycle $\mathcal{C}_j$ of Figure 13 below.

Obs.: the number written inside the node is the right-hand-side element of the associated equation.

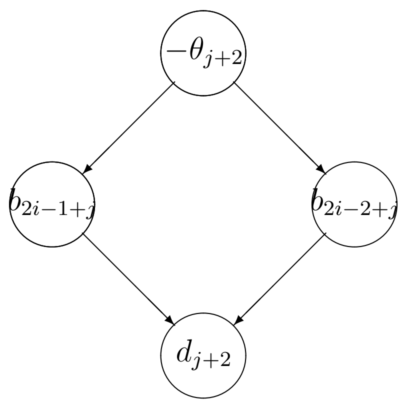

Figure 13: Cycle $\mathcal{C}_j$

For $j = 1$ the submatrix in (28) reduces to

$$\left(\begin{array}{c|c} A_1 & -\theta_3 \\ A_{2i-2} & d_3 \\ A_{2i-1} & -\tilde{\beta}_1 \\ A_{2i} & \beta_2 \end{array}\right) = \left(\begin{array}{c|c} -\boldsymbol{e}_1 - \boldsymbol{e}_2 & -\theta_3 \\ \boldsymbol{e}_{4i-5} + \boldsymbol{e}_{4i-4} & d_3 \\ \boldsymbol{e}_1 - \boldsymbol{e}_{4i-4} & -\tilde{\beta}_1 \\ \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_{4i-6} - \boldsymbol{e}_{4i-5} & \beta_2 \end{array}\right) \tag{29}$$

The arcs in $\mathcal{G}_1$ are those that have both end nodes in $\mathcal{N}_1$. Thus the incidence matrix of $\mathcal{G}_1$ contains the columns of (29) that have both their nonnull elements in the submatrix (29):

$$\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 4i\text{-}5 & 4i\text{-}4 \end{array} & \\ \left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -\theta_3 \\ 0 & 0 & 1 & 1 & d_3 \\ 1 & 0 & 0 & -1 & -\tilde{\beta}_1 \\ 0 & 1 & -1 & 0 & \beta_2 \end{array}\right) & \begin{array}{c} 1 \\ 2i-2 \\ 2i-1 \\ 2i \end{array} \end{array}$$

which implies $\mathcal{G}_1$ consists of the non-directed cycle $\mathcal{C}_1$

$$\{1, (1, 2i-1), 2i-1, (2i-1, 2i-2), 2i-2, (2i, 2i-2), 2i, (1, 2i), 1\},$$

in accordance with the theorem's statement.

If $2 \leq j < i-1$, the submatrix in (28) is, using the fact that $2i$ is even:

$$\left(\begin{array}{c|c} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) & b_j \\ (-1)^{2i-1-j}(\boldsymbol{e}_{4i-3-2j} + \boldsymbol{e}_{4i-2-2j}) & b_{2i-1-j} \\ (-1)^{2i-2+j}(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j}) & (-1)^{2i-2+j}\beta_j \\ (-1)^{2i-1+j}(\boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}) & (-1)^{2i-1+j}\beta_{j+1} \end{array}\right)$$
$$= \quad (-1)^j \left(\begin{array}{c|c} \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j} & (-1)^j b_j \\ -\boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j} & (-1)^j b_{2i-1-j} \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j} & \beta_j \\ -\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1} - \boldsymbol{e}_{4i-4-2j} + \boldsymbol{e}_{4i-3-2j} & -\beta_{j+1} \end{array}\right)$$

This implies that the incidence matrix of $\mathcal{G}_j$ is:

$$\begin{array}{cc} & \begin{array}{cccc} \text{2j-1} & \text{2j} & \text{4i-3-2j} & \text{4i-2-2j} \end{array} \\ (-1)^j & \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^j b_j \\ 0 & 0 & -1 & -1 & (-1)^j b_{2i-1-j} \\ -1 & 0 & 0 & 1 & \beta_j \\ 0 & -1 & 1 & 0 & -\beta_{j+1} \end{array}\right) \begin{array}{l} i \\ 2i-1-j \\ 2i-2+j \\ 2i-1+j \end{array} \end{array}$$

From this incidence matrix and the value of b_j and b_{2i-1-j} given in Table 1, we conclude that $\mathcal{G}_j$ is the cycle $\mathcal{C}_j$ depicted in Figure 13.

Finally, if $j = i-1$, the submatrix in (28) has the values

$$\left(\begin{array}{c|c} (-1)^{i-1}(\boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-1}) & b_{i-1} \\ (-1)^{i}(\boldsymbol{e}_{2i-1} + \boldsymbol{e}_{2i}) & b_{i} \\ (-1)^{3i-3}(\boldsymbol{e}_{2i-4} - \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i} - \boldsymbol{e}_{2i+1}) & (-1)^{3i-3}\beta_{i-1} \\ (-1)^{3i-2}(\boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}) & (-1)^{2i-1+j}\beta_{i} \end{array}\right)$$
$$= \quad (-1)^{i-1} \left(\begin{array}{c|c} \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} & (-1)^{i-1} b_{i-1} \\ -\boldsymbol{e}_{2i-1} - \boldsymbol{e}_{2i} & (-1)^{i-1} b_{i} \\ \boldsymbol{e}_{2i-4} - \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i} - \boldsymbol{e}_{2i+1} & \beta_{i-1} \\ -\boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} & -\beta_{i} \end{array}\right)$$

where the equality follows from the fact that $2i$ is even.

The subgraph $\mathcal{G}_{i-1}$ has, therefore, incidence matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{2i-3} & \text{2i-2} & \text{2i-1} & \text{2i} \end{array} \\
(-1)^{i-1} & \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^{i-1}b_{i-1} \\ 0 & 0 & \text{-1} & \text{-1} & (-1)^{i-1}b_i \\ \text{-1} & 0 & 0 & 1 & \beta_{i-1} \\ 0 & \text{-1} & 1 & 0 & -\beta_i \end{array}\right) \begin{array}{l} i-1 \\ i \\ 3i-3 \\ 3i-2 \end{array}
\end{array}
$$

This matrix and Table 1 imply that $\mathcal{G}_{i-1}$ consists of the $\mathcal{C}_{i-1}$, as desired. ■

Theorem 5.2 implies that the digraph $\mathcal{G}$ has the structure depicted in Figure 14. That is, the graph obtained by removing the orientation from $\mathcal{G}$'s arcs may be decomposed in $i-1$ biconnected components, each of them a cycle with four nodes and arcs. Nodes with indices $2i$, $2i+1$, ... , $3i-3$ are cutset nodes, since their removal disconnects the graph. Notice that these nodes correspond to rows in the second set of the partition. In fact all rows in the second set is associated with a cutset node, excepting the first and the last.

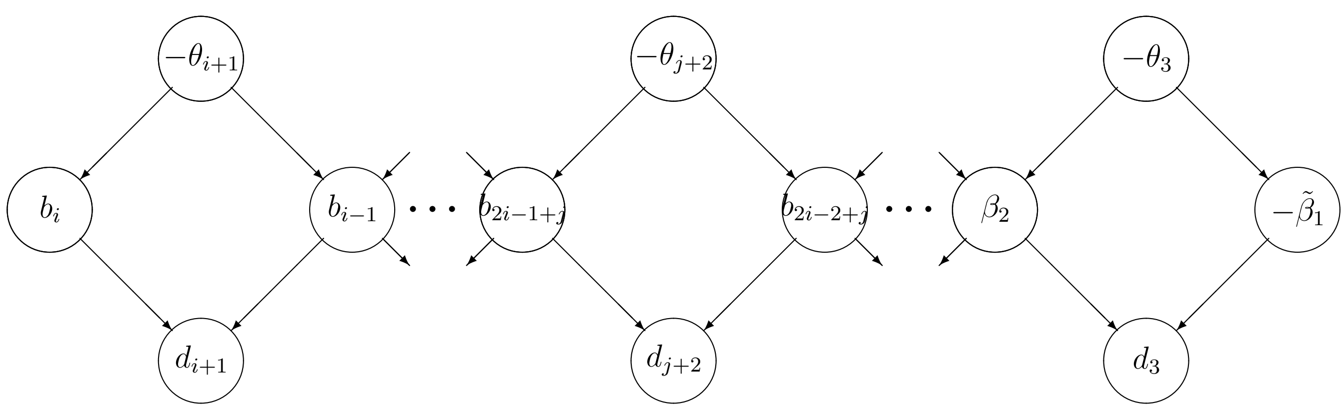

Figure 14: Structure of digraph $\mathcal{G}$.

Theorem 5.3 *If equation* (27) *holds, the system* (24) *is equivalent to a set of $i-1$ independent linear systems. The j-th system is given by*

$$
\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{30}
$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^{\ell} \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \qquad \textit{for } j = 1, \ldots, i-1, \tag{31}$$

and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-1-2j}, x_{4i-2j}), & \textit{if } 1 \le j \le i-1 \textit{ is odd} \\ (x_{4i-1-2j}, x_{4i-2j}, x_{2j+1}, x_{2j+2}), & \textit{if } 1 \le j \le i-1 \textit{ is even} \end{cases} \tag{32}$$

Proof: The decomposition of a flow problem in a graph into an equivalent set of problems in the biconnected components of the graph is a well known result, as mentioned in [7]. This decomposition involves the concept of the cut tree of the graph ([11], p. 90–92). In the present instance it is perhaps easier to verify this equivalence directly by means of a simple induction proof.

Consider equations 1, $2i-2$ and $2i-1$ of (24):

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ -\tilde{\beta}_1 \end{pmatrix}$$

Adding these equations to equation $2i$ we obtain

$$\begin{aligned} -x_5 + x_{4i-4} &= \beta_2 - \tilde{\beta}_1 - \theta_3 + d_3 \\ &= \beta_2 - (\beta_1 + \theta_1 + \theta_2 - d_1 - d_2) - \theta_3 + d_3 \\ &= \beta_2 - \beta_1 - \sum_{\ell=1}^{3} (\theta_\ell - d_\ell) = \hat{\beta}_2 \end{aligned}$$

Replacing equation $2i$ of (24) with the above equation we arrive at an equivalent system, since only elementary row operations were performed. In the new system, variables x_3, x_4, x_{4i-3} and x_{4i-2} show up only in equations 1, $2i-2$, $2i-1$ and these equations contain no other variables. This new system may therefore be decomposed in the pair of independent linear systems:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ -\tilde{\beta}_1 \end{pmatrix} \tag{33}$$

$$\begin{pmatrix} \tilde{A}_2 \\ \vdots \\ \tilde{A}_{2i-3} \\ \cdots\cdots \\ \tilde{A}_{2i} \\ \vdots \\ \tilde{A}_{3i-2} \end{pmatrix} \begin{pmatrix} x_5 \\ \vdots \\ x_{4i-4} \end{pmatrix} = \begin{pmatrix} d_4 \\ \vdots \\ -\theta_4 \\ \cdots\cdots \\ \hat{\beta}_2 \\ b_{2i} \\ \vdots \\ b_{3i-2} \end{pmatrix} \tag{34}$$

where $\tilde{A}_j$ is obtained from A_j eliminating four columns: the first and last two ones. Since A is the incidence matrix of the digraph depicted in Figure 14 and the columns and rows eliminated are associated with the arcs and node of biconnected component $\mathcal{C}_1$, with the exception of the cutset node, the matrix

$$\tilde{A} = \begin{pmatrix} \tilde{A}_2 \\ \vdots \\ \tilde{A}_{2i-3} \\ \tilde{A}_{2i} \\ \vdots \\ \tilde{A}_{3i-2} \end{pmatrix}$$

is the incidence matrix of the digraph given by the union of the remaining components: $\mathcal{C}_2, \ldots, \mathcal{C}_{i-1}$.

Furthermore, using the fact that $\tilde{\beta}_1 = \hat{\beta}_1$, we have that the system (35) below is equivalent to system (33), since it is obtained from the latter by appending to it a redundant equation:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ \hat{\beta}_1 \\ -\hat{\beta}_1 + \theta_3 - d_3 \end{pmatrix} \tag{35}$$

Note that (35) is the system (30), for $i = 1$, and the vector of unknowns is w^1.

Admit, by induction, that, after k analogous steps, a set of $k+1$ systems, equivalent to (24), were obtained, as follows: k systems of type (30) for $j = 1, \ldots, k$, and the system

$$\begin{pmatrix} \tilde{\tilde{A}}_{k+1} \\ \vdots \\ \tilde{\tilde{A}}_{2i-2-k} \\ \cdots\cdots \\ \tilde{\tilde{A}}_{2i-1+k} \\ \vdots \\ \tilde{\tilde{A}}_{3i-2} \end{pmatrix} \begin{pmatrix} x_{3+2k} \\ \vdots \\ x_{4i-2-2k} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ \vdots \\ b_{2i-1+k} \\ \cdots\cdots \\ \hat{\beta}_{k+1} \\ b_{2i-1+k} \\ \vdots \\ b_{3i-2} \end{pmatrix} \tag{36}$$

whose coefficient matrix $\tilde{\tilde{A}}$ is the incidence matrix of the digraph given by $\cup_{j=k+1}^{i-1}\mathcal{C}_j$, obtained eliminating from A the columns and rows associated with the arcs and nodes in $\cup_{j=1}^{k}\mathcal{C}_j$, with the exception of the cutset node $\mathcal{C}_k \cap \mathcal{C}_{k+1}$.

The equations associated with the nodes in $\mathcal{C}_{k+1}$, with the exception of the cutset node $\mathcal{C}_{k+1} \cap \mathcal{C}_{k+2}$, are

$$(-1)^k \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_{2k+3} \\ x_{2k+4} \\ x_{4i-3-2k} \\ x_{4i-2-2k} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ b_{2i-2-k} \\ \hat{\beta}_{k+1} \end{pmatrix} \tag{37}$$

The equation associated with node $\mathcal{C}_{k+1} \cap \mathcal{C}_{k+2}$ is

$$(-1)^k(x_{2k+4} - x_{2k+5} + x_{4i-4-2k} - x_{4i-3-2k}) = b_{2i-1+k} = (-1)^{k+2}\beta_{k+2} \tag{38}$$

Adding equations (37) to equation (38) and using Table 1 we have

$$\begin{aligned} (-1)^k(-x_{2k+5} + x_{4i-4-2k}) &= (-1)^{k+2}\beta_{k+2} - \theta_{k+3} + d_{k+3} + \hat{\beta}_{k+1} \\ &= \sum_{j=1}^{k+1}(-1)^j\beta_j - \sum_{j=1}^{k+3}(\theta_j - d_j) \\ &= \hat{\beta}_{k+2} \end{aligned} \tag{39}$$

Hence (36) is equivalent to the system obtained replacing equation (38) with equation (39). But this sustitution produces a system that may again be split in two: one of type (30) for $j = k+1$ (in the vector of unknowns w^{k+1}) and the other whose coefficient matrix is the incidence matrix of the digraph $\cup_{j=k+2}^{i-1}\mathcal{C}_j$ and whose right-hand-side vector is given by $b_{k+2}, \ldots, b_{2i-3-k}, \hat{\beta}_{k+2}, b_{2i+k}, \ldots, b_{3i-2}$. Therefore, by induction, this implies that (24) is equivalent to a set of $i-1$ systems of type (30) for $j = 1, \ldots, i-2$, where the $(i-1)$-th system is

$$(-1)^i \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ x_{2i} \\ x_{2i+1} \\ x_{2i+2} \end{pmatrix} = \begin{pmatrix} b_{i-1} \\ b_i \\ \hat{\beta}_{i-1} \\ (-1)^i\beta_i \end{pmatrix} \tag{40}$$

Using Table 1, adding the first three elements of the right-hand-side of (40) we obtain

$$\begin{aligned} b_{i-1} + b_i + \hat{\beta}_{i-1} &= -\theta_{i+1} + d_{i+1} + \sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i}(\theta_j - d_j) \\ &= \sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i+1}(\theta_j - d_j) \end{aligned}$$

Since we assume equation (27) holds, i.e., $(-1)^i\beta_i = -(\sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i+1}(\theta_j - d_j))$, Table 1 and equation (27) thus imply that (40) is also of the type (30) for $j = i-1$. ■

Theorem 5.3 implies that the solution to the original system (21) is the cartesian product of 4-tuples of the type $q_j = (x_{2j-1}, x_{2j}, x_{4i+1-2j}, x_{4i+2-2j})$, for $j = 1, \ldots, i$. In Subsection 5.1.1 we've seen that q_1 has a unique value, determined by equations 1, 2, $2i+2$ and $2i+1$ of (21). The 4-tuple q_j, for $j = 2, \ldots, i$, is a reordering of a vector w that must satisfy the linear system

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} -t \\ \delta \\ \xi \\ -\xi + t - \delta \end{pmatrix}. \tag{41}$$

Lemma 5.2 *The general solution to system* (41) *is given by*

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} \xi \\ t-\xi \\ \delta \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \tag{42}$$

where $\alpha \in \mathbb{R}$.

Proof: Substituting, it can be easily verified that $(\xi, t-\xi, \delta, 0)$ is a solution to (41) and that $(1, -1, -1, 1)$ is a solution to the homogeneous version of (41). Since the null space of the coefficient matrix of (41) has dimension 1 and the difference between two solutions is a vector of this null space, formula (42) encompasses all solutions to (41). ■

Lemma 5.3 *If t, ξ and δ are integers, and t, δ are nonnegative, the system* (41) *has a nonnegative integral solution if, and only if, the following inequalities are valid*

$$\begin{cases} \delta & \geq & -\xi \\ t & \geq & \xi \end{cases} \tag{43}$$

Furthermore, the number of nonnegative integral solutions to (41) *is equal to the number of integers in the interval*

$$[\max\{0, -\xi\}, \min\{t - \xi, \delta\}].$$

Proof: The matrix of coefficients of (41) is a submatrix of A and thus totally unimodular. Its right-hand-side is integral. These facts imply that system (41) has nonnegative integral solution if, and only if, it has a nonnegative solution. Lemma 5.2 implies that (41) has nonnegative solution if, and only if, the inequalities (44) are satisfied:

$$\begin{array}{rcl} \xi + \alpha & \geq & 0 \\ t - \xi - \alpha & \geq & 0 \\ \delta - \alpha & \geq & 0 \\ \alpha & \geq & 0 \end{array} \tag{44}$$

Applying Fourier-Motzkin (see [1], p. 57–62) to eliminate variable α in (44), the following inequalities are obtained:

$$\begin{array}{rcl} t - \xi & \geq & \alpha \\ \delta & \geq & \alpha \\ \alpha & \geq & -\xi \\ \alpha & \geq & 0 \end{array}$$

Lemma 5.2 establishes that the general solution to (41) is given by $(\xi, t - \xi, \delta, 0) + \alpha(1, -1, -1, 1)$. Therefore, for each integer value α in the interval $[\max\{0, -\xi\}, \min\{t - \xi, \delta\}]$ there is a nonnegative integral solution to (41).

Finally, the interval $[\max\{0, -\xi\}, \min\{t-\xi, \delta\}]$ is nonempty if, and only if, the inequalities below are satisfied:

$$
\begin{aligned}
t-\xi &\geq 0 && (45)\\
\delta &\geq 0 && (46)\\
t-\xi &\geq -\xi && (47)\\
\delta &\geq -\xi && (48)
\end{aligned}
$$

Inequalities (46) and (47) are satisfied, since δ and t are assumed nonnegative. The remaining two form system (43). ■

We collect below the necessary and sufficient constraints for the existence of nonnegative integral solutions to (21)

$$
\begin{cases}
d_2 \geq \theta_1 \\
\theta_2 \geq d_1 \\
\sum_{j=1}^{i} (-1)^j \beta_j - \sum_{j=1}^{i+1} (\theta_j - d_j) = 0 \\
\theta_{j+2} \geq \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \qquad \text{for } j = 1, \ldots, i-1 \\
d_{j+2} \geq - \left(\sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell) \right), \text{ for } j = 1, \ldots, i-1
\end{cases}
\tag{49}
$$

The number of solutions is given by the product $\prod_{j=1}^{i-1} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_j\}, \min\{\theta_{j+2} - \hat{\beta}_j, d_{j+2}\}]$ and $\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell)$.

5.2 Case $n = 0 \bmod 4$

5.2.1 The linear system

Let $4 \leq n = 2i$ be an integer multiple of 4 (or, equivalently, $2 \leq i$ is even). We want to find a nonnegative integral solution of the linear system

$$
\begin{array}{r}
\left.\begin{array}{r}\text{1-st set}\\ (2i+1)\text{ rows}\end{array}\right\}\\[2em]
\left.\begin{array}{r}\text{2-nd set}\\ i-1\text{ rows}\end{array}\right\}
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0\\
 & & & & & \ddots & & & & & & & \\
0 & & \cdots & & & & 0\;\;1\;\;1\;\;1\;\;0 & & & & \cdots & & 0\\
 & & & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 1\\
0 & 1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 1 & -1 & 0\\
0 & 0 & 0 & 1 & -1 & & \cdots & & 1 & -1 & 0 & 0 & 0\\
 & & & & & \ddots & & \iddots & & & & & \\
0 & & \cdots & & & & 1\;\;-1\;\;0\;\;1\;\;-1 & & & & \cdots & & 0
\end{pmatrix}
\begin{pmatrix}
x_1\\ x_2\\ x_3\\ \vdots\\ x_{2i-1}\\ y\\ x_{2i}\\ \vdots\\ x_{4i-4}\\ x_{4i-3}\\ x_{4i-2}
\end{pmatrix}
=
\begin{pmatrix}
\theta_1\\ d_2\\ \theta_3\\ d_4\\ \vdots\\ d_i\\ \Theta\\ \theta_i\\ \vdots\\ \theta_4\\ d_3\\ \theta_2\\ d_1\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_{i-1}
\end{pmatrix}
\tag{50}
$$

where Θ, θ_j and d_j, for $j = 1, \ldots, i$, are positive integers and β_j, for $j = 1, \ldots, i-1$, are integers. The system has $3i$ equations and $4i-1$ unknowns, coefficient matrix $\overline{A}$ and right-hand-side vector $\overline{b}$. The equations, rows of $\overline{A}$ and $\overline{b}$ are partitioned in two sets, containing the first $2i+1$ equations and the following $i-1$ ones, respectively. The formulas for the j-th row of the first group and the corresponding right-hand-side element are

$$
\overline{A}_j = \begin{cases}
\boldsymbol{e}_1, & \text{if } j = 1\\
\boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq i\\
\boldsymbol{e}_{2i-1} + \boldsymbol{e}_{2i} + \boldsymbol{e}_{2i+1}, & \text{if } j = i+1\\
\boldsymbol{e}_{2j-2} + \boldsymbol{e}_{2j-1}, & \text{if } i+2 \leq j \leq 2i\\
\boldsymbol{e}_{4i-1}, & \text{if } j = 2i+1
\end{cases}
$$

and

$$
\bar{b}_j = \begin{cases} \theta_j, & \text{if } 1 \leq j \leq i,\ j \text{ odd} \\ d_j, & \text{if } 1 \leq j \leq i,\ j \text{ even} \\ \Theta, & \text{if } j = i+1 \\ \theta_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1,\ j \text{ even} \\ d_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1,\ j \text{ odd} \end{cases}
$$

where $\boldsymbol{e}_j$ is the j-th vector of $\mathbb{R}^{4i-1}$'s canonical basis. In the second group, the j-th row and respective right-hand-side are as follows

$$
\overline{A}_{2i+1+j} \quad = \quad \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-1-2j} - \boldsymbol{e}_{4i-2j}, \quad \text{if } 1 \leq j \leq i-1
$$

and

$$
\bar{b}_{2i+1+j} \quad = \quad \beta_j, \quad \text{if } 1 \leq j \leq i-1 \ .
$$

As in the case of n odd, four variables in (50) are uniquely determined. The remaining variables must satisfy a reduced system. The coefficient matrix of the reduced system is not totally unimodular. Nevertheless, it is shown in Subsection 5.2.2 that, if we remove the column associated with variable x_{2i}, the remaining matrix is totally unimodular. We explore this fact in Subsection 5.2.3 to produce a set of conditions which is equivalent to the existence of a nonnegative integral solution to the reduced system.

The equations which may be used to eliminate variables in (50) are the same as in (21):

1. The first equation implies $x_1 = \theta_1$.
2. The second equation implies $x_2 = d_2 - \theta_1$.
3. The $(2i+1)$-th equation implies $x_{4i-2} = d_1$.
4. The $2i$-th equation implies $x_{4i-3} = \theta_2 - d_1$.

Thus the inequalities (22), already obtained in Subsection 5.1.1, are necessary and sufficient conditions for the partial solution established in 1–4 above to be a nonnegative integral vector.

The reduced system obtained with the removal of the 4-tuple and the equations used to determined their values is examined next.

5.2.2 The reduced system

We arrive at the reduced system (51) below after eliminating equations 1, 2, $2i$ and $2i+1$ and variables x_1, x_2, x_{4i-3} and x_{4i-2} in (50):

$$\begin{array}{r} \left.\begin{array}{r}\text{1-st set}\\ (2i-3)\text{ rows}\end{array}\right\} \\[6em] \left.\begin{array}{r}\text{2-nd set}\\ (i-1)\text{ rows}\end{array}\right\} \end{array}
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & \\
0 & & \cdots & & & 0\ 1 & 1\ 1 & 0 & & & \cdots & & 0 \\
 & & & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & & \cdots & & 1 & -1 & 0 & 0 & 0 \\
 & & & & & \ddots & & \iddots & & & & & \\
0 & & \cdots & & & 1\ -1 & 0 & 1\ -1 & & & \cdots & & 0
\end{pmatrix}
\begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix}
=
\begin{pmatrix} \theta_3 \\ d_4 \\ \vdots \\ d_i \\ \Theta \\ \theta_i \\ \vdots \\ \theta_4 \\ d_3 \\ \tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix} \tag{51}$$

where $\tilde{\beta}_1 = \beta_1 - x_2 + x_{4i-3} = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-4$ equations in $4i-5$ unknowns.

The rows of $\overline{\overline{A}}$ and $\overline{\overline{b}}$, the coefficient matrix and right-hand-side vector of (51), inherit the partition established for their counterparts in (50). Hence the sets have $2i-3$ and $i-1$ rows/elements, respectively. The formulas for the j-th row and right-hand-side element must be updated to account for the elimination of rows and columns (equations and variables). In the first group we have:

$$\overline{\overline{A}}_j = \begin{cases} \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, & \text{if } 1 \le j \le i-2 \\ \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1}, & \text{if } j = i-1 \\ \boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}, & \text{if } i \le j \le 2i-3 \end{cases}$$

and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ \Theta, & \text{if } j = i-1 \\ \theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd} \end{cases}$$

In the second set, the formulas are:

$$\overline{\overline{A}}_{2i-3+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-5}, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}, & \text{if } 2 \leq j \leq i-1 \end{cases}$$

and

$$\overline{\overline{b}}_{2i-3+j} = \begin{cases} \tilde{\beta}_1, & \text{if } j = 1 \\ \beta_j, & \text{if } 2 \leq j \leq i-1 \end{cases}$$

Note that $\boldsymbol{e}_j$ is now the i-th vector of $\mathbb{R}^{4i-5}$'s canonical basis.

Lemma 5.4 *Each column of $\overline{\overline{A}}$, except the (central) column $2i-2$, contains exactly two nonnull coefficients, one in the first set of rows and the other in the second set. Column $2i-2$ contains only one nonnull element, namely 1, in row $i-1$.*

Proof: Consider the matrix $\tilde{A}$ with each entry of $\overline{\overline{A}}$ replaced by its absolute value. Summing rows in the first set:

$$\begin{aligned} \sum_{j=1}^{2i-3} \tilde{A}_j &= \sum_{j=1}^{i-2} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} + \sum_{j=i}^{2i-3} (\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}) \\ &= \sum_{\substack{j\in\{1,\ldots,2i-4\} \\ j \text{ odd}}} \boldsymbol{e}_j + \sum_{\substack{j\in\{1,\ldots,2i-4\} \\ j \text{ even}}} \boldsymbol{e}_j + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} + \\ &\quad \sum_{\substack{j\in\{2i,\ldots,4i-5\} \\ j \text{ even}}} \boldsymbol{e}_j + \sum_{\substack{j\in\{2i,\ldots,4i-5\} \\ j \text{ odd}}} \boldsymbol{e}_j \\ &= \mathbf{1}_{4i-5}, \end{aligned}$$

Summing rows in the second set:

$$\begin{aligned} \sum_{j=1}^{i-1} \tilde{A}_{2i-3+j} &= \boldsymbol{e}_1 + \boldsymbol{e}_{4i-5} + \sum_{j=2}^{i-1} (\boldsymbol{e}_{2j-2} + \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} + \boldsymbol{e}_{4i-2-2j}) \\ &= \boldsymbol{e}_1 + \sum_{\substack{j\in\{2,\ldots,2i-3\} \\ j \text{ even}}} \boldsymbol{e}_j + \sum_{\substack{j\in\{2,\ldots,2i-3\} \\ j \text{ odd}}} \boldsymbol{e}_j + \\ &\quad \sum_{\substack{j\in\{2i-1,\ldots,4i-6\} \\ j \text{ odd}}} \boldsymbol{e}_j + \sum_{\substack{j\in\{2i-1,\ldots,4i-6\} \\ j \text{ even}}} \boldsymbol{e}_j + \boldsymbol{e}_{4i-5} \\ &= \mathbf{1}_{4i-5} - \boldsymbol{e}_{2i-2}. \end{aligned}$$

The first sum implies each column contains one nonnull element, namely 1 (since rows in the first set have nonnegative entries), in the first set of rows. Incidentally, row $i-1$ contributes with the nonnull entry in column $2i-2$. The second sum implies each column, with the execption of column $2i-2$ contains one nonnull element, with absolute value 1, in the second set of rows. The column $2i-2$ contains no nonnull elements in the second set. Thus the lemma is proved. ∎

Multiply by -1 the j-th equation, where we pick odd j, if $1 \le j \le i-1$, and even j, if $i \le j \le 3i-4$. The resulting system is

$$
\begin{matrix}
2i-3\left\{\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\\a\end{matrix}}\right. \\
i-1\left\{\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right.
\end{matrix}
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
 & & & & \ddots & & & & & & & & \\
0 & \cdots & 0 & -1 & -1 & -1 & 0 & & \cdots & & & & 0 \\
 & & & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & & \cdots & & -1 & 1 & 0 & 0 & 0 \\
 & & & & \ddots & & & \iddots & & & & & \\
0 & \cdots & -1 & 1 & 0 & -1 & 1 & & \cdots & & & & 0
\end{pmatrix}
\begin{pmatrix}
x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4}
\end{pmatrix}
=
\begin{pmatrix}
-\theta_3 \\ d_4 \\ \vdots \\ d_i \\ -\Theta \\ -\theta_i \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ -\beta_{i-1}
\end{pmatrix}
\tag{52}
$$

The new system (52) has coefficient matrix $\hat{A}$ and right-hand-side vector $\hat{b}$. As before they are partitioned in two sets. The j-th row and right-hand-side element in the first set are given by:

$$
\hat{A}_j = \begin{cases}
(-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } 1 \le j \le i-2 \\
-\boldsymbol{e}_{2i-3} - \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}, & \text{if } j = i-1 \\
(-1)^{j+1}(\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}), & \text{if } i \le j \le 2i-3
\end{cases}
$$

$$
\hat{b}_j = \begin{cases}
-\theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\
d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\
-\Theta, & \text{if } j = i-1 \\
-\theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\
d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd}
\end{cases}
$$

In the second set, the formulas are:

$$\hat{A}_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}), & \text{if } 2 \le j \le i-1 \end{cases}$$

$$\hat{b}_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \le j \le i-1 \end{cases}$$

Theorem 5.4 *The j-the column of $\hat{A}$ contains*

two nonnull entries, 1 *and* -1,	*if* $j \neq 2i-2, 2i-1$,
one nonnull entry, -1,	*if* $j = 2i-2$,
two nonnull entries, both -1,	*if* $j = 2i-1$.

Proof: By Lemma 5.4 and the fact that the entries of $\hat{A}$ belong to $\{0, -1, 1\}$, it suffices to show that the sum of the rows of $\hat{A}$ results in $-\boldsymbol{e}_{2i-2} - 2\boldsymbol{e}_{2i-1}$.

Adding the rows in the first set, we have:

$$\begin{aligned} \sum_{j=1}^{2i-3} \hat{A}_j &= \sum_{j=1}^{i-2} (-1)^j (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) - \boldsymbol{e}_{2i-3} - \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1} + \sum_{j=i}^{2i-3} (-1)^{j+1} (\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}) \\ &= \sum_{j=1}^{i-1} (-1)^j \boldsymbol{e}_{2j-1} + \sum_{j=1}^{i-2} (-1)^j \boldsymbol{e}_{2j} - \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1} + \\ &\quad \sum_{j=i}^{2i-3} (-1)^{j+1} \boldsymbol{e}_{2j} + \sum_{j=i+1}^{2i-2} (-1)^j \boldsymbol{e}_{2j-1} \end{aligned} \tag{53}$$

For the second set, we obtain:

$$\begin{aligned}
\sum_{j=1}^{i-1} \hat{A}_{2i-3+j} &= \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5} + \sum_{j=2}^{i-1} (-1)^j (\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}) \\
&= \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5} + \sum_{j=2}^{i-1} (-1)^j \boldsymbol{e}_{2j-2} + \sum_{j=2}^{i-1} (-1)^{j+1} \boldsymbol{e}_{2j-1} + \\
&\quad \sum_{j=2}^{i-1} (-1)^j \boldsymbol{e}_{4i-3-2j} + \sum_{j=2}^{i-1} (-1)^{j+1} \boldsymbol{e}_{4i-2-2j} \\
&= \sum_{j=1}^{i-2} (-1)^{j+1} \boldsymbol{e}_{2j} + \boldsymbol{e}_1 + \sum_{j=2}^{i-1} (-1)^{j+1} \boldsymbol{e}_{2j-1} + \\
&\quad \sum_{j=i}^{2i-3} (-1)^{j+1} \boldsymbol{e}_{2j-1} - \boldsymbol{e}_{4i-5} + \sum_{j=i}^{2i-3} (-1)^j \boldsymbol{e}_{2j} \\
&= \sum_{j=1}^{i-2} (-1)^{j+1} \boldsymbol{e}_{2j} + \sum_{j=1}^{i-1} (-1)^{j+1} \boldsymbol{e}_{2j-1} + \\
&\quad \sum_{j=i}^{2i-2} (-1)^{j+1} \boldsymbol{e}_{2j-1} + \sum_{j=i}^{2i-3} (-1)^j \boldsymbol{e}_{2j} \qquad (54)
\end{aligned}$$

Equations (53) and (54) imply the desired result. ■

Theorem 5.4 implies that the matrix obtained from $\hat{A}$ by removing column $2i-1$ is totally unimodular. In order to exploit this fact, we consider the parametric linear system obtained by interpreting x_{2i} (the variable associated to column $2i-1$) as a parameter. Thus, for each fixed value p of this parameter we have a linear system with right-hand-side equal to $\hat{b}$ minus p times column $2i-1$ of column $\hat{A}$. Furthermore, if we add the redundant equation $\boldsymbol{e}_{2i-2}x = -\sum_{j=1}^{3i-4} \hat{b}_j - 2p$, the coefficient matrix of the parametric system will be the incidence matrix of a digraph.

5.2.3 Solutions of the parametric system

The parametric system, corresponding to $x_{2i} = p$, described in the previous subsection has the following structure:

$$
\begin{array}{r}
2i-3\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \\
i\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right.
\end{array}
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & \\
0 & & \cdots & & & 0 & -1 & -1 & 0 & & \cdots & & 0 \\
 & & & & & & & & \ddots & & & & \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & & \cdots & & -1 & 1 & 0 & 0 & 0 \\
 & & & & & \ddots & & \iddots & & & & & \\
0 & & \cdots & & -1 & 1 & 0 & 1 & & & \cdots & & 0 \\
0 & & \cdots & & & & 1 & & & & \cdots & & 0
\end{pmatrix}
\begin{pmatrix}
x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i+1} \\ \vdots \\ x_{4i-5} \\ x_{4i-4}
\end{pmatrix}
=
\begin{pmatrix}
-\theta_3 \\ d_4 \\ \vdots \\ d_i \\ -\Theta + p \\ -\theta_i \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ -\beta_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p
\end{pmatrix}
\qquad (55)
$$

The coefficient matrix A of (55) is $(3i-3) \times (4i-6)$. The first set of the partition is the same as before. We have, for the j-th row of A and element of the right-hand-side b of (55) in the first set:

$$
A_j = \begin{cases} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } 1 \le j \le i-1 \\ (-1)^{j+1}(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } i \le j \le 2i-3 \end{cases}
$$

$$
b_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ -\Theta + p, & \text{if } j = i-1 \\ -\theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd} \end{cases}
$$

The redundant equation is added to the second set of the partition, thus we have:

$$A_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}), & \text{if } 2 \le j \le i-2 \\ -\boldsymbol{e}_{2i-4} + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-1}, & \text{if } j = i-1 \\ \boldsymbol{e}_{2i-2}, & \text{if } j = i \end{cases}$$

$$b_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \le j < i-1 \\ -\beta_{i-1} + p, & \text{if } j = i-1 \\ -\sum_{j=1}^{3i-4} b_j - 2p, & \text{if } j = i \end{cases}$$

Since the row vectors belong to $\mathbb{R}^{4i-6}$, we take $\boldsymbol{e}_k$ to be the k-th vector of the canonical basis of this space.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be the digraph whose node-arc incidence matrix is A. Equations (55) represent a parametric flow problem. As long as the parameter p is integer, the problem will have nonnegative integral solutions if it has nonnegative solutions. The structure of $\mathcal{G}$ is unraveled in the next theorem. It will help to break up (55) into a set of smaller systems, as done in Subsection 5.1.3.

Theorem 5.5 *The set $\mathcal{A}$ may be partitioned into $i-1$ subsets. Of those, $i-2$ correspond to non-directed cycles of length 4 in $\mathcal{G}$, and the remaining one corresponds to a non-directed path. The j-th subset is the set of arcs $\mathcal{A}_j$ in the subgraph $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{A}_j)$ induced by the nodes:*

$$\textit{If } j \le i-2: \qquad \mathcal{N}_j = \{j, 2i-2-j, 2i-3+j, 2i-2+j\}$$
$$\textit{If } j = i-1: \qquad \mathcal{N}_j = \{i-1, 3i-4, 3i-3\}$$

Proof: It suffices to show that the subgraphs $\mathcal{G}_j$ have the structure described and are arc-disjoint, since the sum of the cardinalities of $\mathcal{A}_j$, for $j = 1, \ldots, i-1$, is $4(i-2)+2 = 4i-6$, the cardinality of $\mathcal{A}$. The right-hand-side elements corresponding to nodes in $\mathcal{N}_j$ are given in Table 2.

From the table it is clear that the subgraphs $\mathcal{G}_j$ are arc-disjoint since

$$\mathcal{N}_\ell \cap \mathcal{N}_j = \begin{cases} \{2i-3+j\}, & \text{if } j = \ell+1 \\ \{2i-2+j\}, & \text{if } j = \ell-1 \\ \emptyset, & \text{otherwise} \end{cases}$$

	$1 \le j \le i-2$	
	j odd	j even
b_j	$-\theta_{j+2}$	d_{j+2}
b_{2i-2-j}	d_{j+2}	$-\theta_{j+2}$
b_{2i-3+j}	$-\tilde{\beta}_1$, if $j=1$ $-\beta_j$, if $j \ge 3$	β_j
b_{2i-2+j}	β_{j+1}	$-\beta_{j+1}$

Table 2: Node constants in $\mathcal{N}_j$

The submatrix of $A|b$ associated with nodes in $\mathcal{N}_j$ is

$$\left(\begin{array}{c|c} A_j & b_j \\ A_{2i-2-j} & b_{2i-2-j} \\ A_{2i-3+j} & b_{2i-3+j} \\ A_{2i-2+j} & b_{2i-2+j} \end{array}\right). \tag{56}$$

For $j = 1$ the submatrix (56) becomes:

$$\left(\begin{array}{c|c} -\boldsymbol{e}_1 - \boldsymbol{e}_2 & -\theta_3 \\ \boldsymbol{e}_{4i-7} + \boldsymbol{e}_{4i-6} & d_3 \\ \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6} & -\tilde{\beta}_1 \\ \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_{4i-8} - \boldsymbol{e}_{4i-7}) & \beta_2 \end{array}\right) \tag{57}$$

Therefore $\mathcal{G}_j$, the subgraph induced by $\mathcal{N}_j$ has incidence matrix

$$\begin{array}{cc} \begin{array}{cccc} 1 & 2 & \text{4i-7} & \text{4i-6} \end{array} & \\ \left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -\theta_3 \\ 0 & 0 & 1 & 1 & d_3 \\ 1 & 0 & 0 & -1 & -\tilde{\beta}_1 \\ 0 & 1 & -1 & 0 & \beta_2 \end{array}\right) & \begin{array}{c} 1 \\ 2i-3 \\ 2i-2 \\ 2i-1 \end{array} \end{array}$$

If we replace $j = 1$ in Figure 15, we obtain subgraph $\mathcal{G}_1$. Notice the similarity to Figure 13. We'll show that all subgraphs, except the last, have this structure.

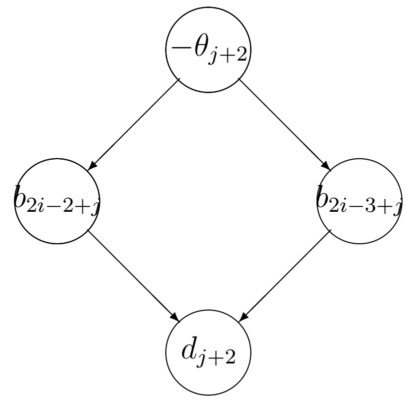

Figure 15: Subgraph $\mathcal{G}_j$ consists of non-directed cycle $\mathcal{C}_j$, for $j = 1, \ldots, i-2$.

The submatrix (56), for $2 \le j \le i-2$, is given by

$$\left(\begin{array}{c|c} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) & b_j \\ (-1)^{2i-1-j}(\boldsymbol{e}_{4i-5-2j} + \boldsymbol{e}_{4i-4-2j}) & b_{2i-2-j} \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}) & (-1)^j\beta_j \\ (-1)^{j+1}(\boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-6-2j} - \boldsymbol{e}_{4i-5-2j}) & (-1)^{j+1}\beta_{j+1} \end{array}\right)$$

$$= \quad (-1)^j \left(\begin{array}{c|c} \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j} & (-1)^j b_j \\ -\boldsymbol{e}_{4i-5-2j} - \boldsymbol{e}_{4i-4-2j} & (-1)^j b_{2i-2-j} \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j} & \beta_j \\ -\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1} - \boldsymbol{e}_{4i-6-2j} + \boldsymbol{e}_{4i-5-2j} & -\beta_{j+1} \end{array}\right)$$

using the fact that $2i$ is even.

Thus the incidence matrix of $\mathcal{G}_j$, for $j = 2, \ldots, i-2$, is

$$\begin{array}{cc} & \begin{array}{cccc} \text{2j-1} & \text{2j} & \text{4i-5-2j} & \text{4i-4-2j} \end{array} \\ (-1)^j & \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^j b_j \\ 0 & 0 & -1 & -1 & (-1)^j b_{2i-2-j} \\ -1 & 0 & 0 & 1 & \beta_j \\ 0 & -1 & 1 & 0 & -\beta_{j+1} \end{array}\right) \begin{array}{l} i \\ 2i-2-j \\ 2i-3+j \\ 2i-2+j \end{array} \end{array}$$

Using Table 2 and the above matrix we conclude that $\mathcal{G}_j$ has the desired structure.

Finally, we consider the submatrix of rows associated with nodes in $\mathcal{N}_{i-1}$:

$$\left(\begin{array}{c|c} -\boldsymbol{e}_{2i-3} - \boldsymbol{e}_{2i-2} & -\Theta + p \\ -\boldsymbol{e}_{2i-4} + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-1} & -\beta_{i-1} + p \\ \boldsymbol{e}_{2i-2} & -\sum_{j=1}^{3i-4} b_j - 2p \end{array}\right)$$

$$= \begin{array}{cc} & \begin{array}{cccc} \text{2i-4} & \text{2i-3} & \text{2i-2} & \text{2i-1} \end{array} \\ & \left(\begin{array}{cccc|c} 0 & -1 & -1 & 0 & -\Theta + p \\ -1 & 1 & 0 & 1 & -\beta_{i-1} + p \\ 0 & 0 & 1 & 0 & -\displaystyle\sum_{j=1}^{3i-4} b_j - 2p \end{array}\right) \begin{array}{c} i-1 \\ 3i-4 \\ 3i-3 \end{array} \end{array}$$

$$-s \xleftarrow{\;y\;} -\tilde{\Theta} \xrightarrow{\;x_{2i-1}\;} -\tilde{\beta}_{i-1}$$

Where $\tilde{\Theta} = \Theta - p$, $\tilde{\beta}_{i-1} = \beta_{i-1} - p$ and $s = \displaystyle\sum_{j=1}^{3i-4} b_j + 2p$

Figure 16: Subgraph $\mathcal{G}_{i-1}$.

Hence the subgraph $\mathcal{G}_{i-1}$, depicted in Figure 16 above, contains only two arcs, associated with columns $2i-3$ and $2i-2$ of matrix A given in (55). ■

The digraph $\mathcal{G}$ is the union of the subgraphs $\mathcal{G}_j$ for $j = 1, \ldots, i-1$. Figure 17 gives an idea of the whole digraph. The nodes associated with second set equations $2i-1, \ldots, 3i-4$, can be used to split the linear system (55) into an equivalent set of smaller linear systems, as done in Subsection 5.1.3.

Theorem 5.6 *The linear system* (55) *is equivalent to a set of* $i-1$ *systems. The* j*-th system in this set, for* $2 \le j \le i-2$*, is given by*

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{58}$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \qquad \textit{for } j = 1, \ldots, i-2, \tag{59}$$

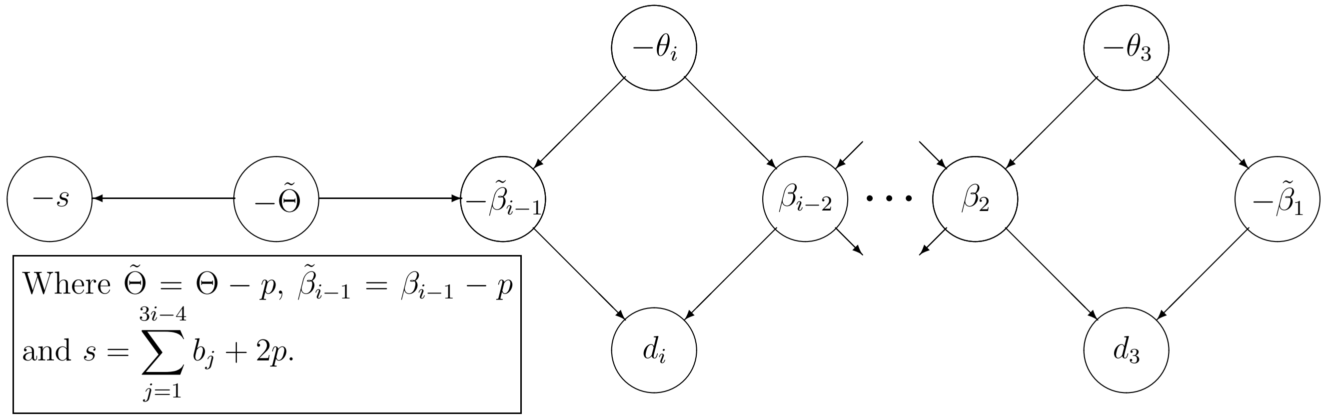

Figure 17: Structure of digraph $\mathcal{G}$.

and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-3-2j}, x_{4i-2-2j}), & \text{if } 1 \leq j \leq i-2 \text{ is odd} \\ (x_{4i-3-2j}, x_{4i-2-2j}, x_{2j+1}, x_{2j+2}), & \text{if } 1 \leq j \leq i-2 \text{ is even} \end{cases} \tag{60}$$

The $(i-1)$*-th system is*

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ y \end{pmatrix} = \begin{pmatrix} -\Theta + p \\ \hat{\beta}_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}, \tag{61}$$

where

$$\hat{\beta}_{i-1} = -\beta_{i-1} + \hat{\beta}_{i-2} - \theta_i + d_i. \tag{62}$$

Proof: Digraph $\mathcal{G}$ is the union of $\mathcal{G}_1, \ldots, \mathcal{G}_{i-2}, \mathcal{G}_{i-1}$, where the first $i-2$ are cycles like the one in Figure 15 and the last one is depicted in Figure 16. The procedure used in the proof given for Theorem 5.3 may be repeated, with analogous results, up to, and including, the $(i-3)$-th splitting.

That is, in the first step cutset node $2i-1$ (with node constant β_2) is used to "separate" subgraph $\mathcal{G}_1$ from the rest of $\mathcal{G}$. Algebraically speaking, the linear system (55) is split into a pair of systems, one associated with the subgraph $\mathcal{G}_1$ and the other with $\cup_{j=2}^{i-1}\mathcal{G}_j$. The system associated with $\mathcal{G}_1$ is given by (58), for $j = 1$. After $i-3$ such splittings, we've produce a set with $i-3$ linear systems given by (58) with $j = 1, \ldots, i-3$, plus one linear system associated with digraph $\mathcal{G}_{i-2} \cup \mathcal{G}_{i-1}$, where the constant associated with cutset node $\mathcal{G}_{i-3} \cap \mathcal{G}_{i-2}$ is $\hat{\beta}_{i-2}$, satisfying (59). Thus the remaining linear system is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{2i-3} \\ x_{2i-2} \\ x_{2i-1} \\ y \\ x_{2i+1} \\ x_{2i+2} \end{pmatrix} = \begin{pmatrix} d_i \\ -\Theta + p \\ -\theta_i \\ \hat{\beta}_{i-2} \\ -\beta_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix} \tag{63}$$

Adding the first, third and fourth equations of (63) to the fifth one, and rearranging rows and columns, we obtain the following equivalent system:

$$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x_{2i-3} \\ x_{2i-2} \\ x_{2i+1} \\ x_{2i+2} \\ x_{2i-1} \\ y \end{pmatrix} = \begin{pmatrix} d_i \\ -\theta_i \\ \hat{\beta}_{i-2} \\ -\Theta + p \\ -\beta_{i-1} + p + \hat{\beta}_{i-2} - \theta_i + d_i \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}$$

The system above is clearly decomposable along the lines shown into two equivalent ones: the first can be seen to be equivalent to (58) for $j = i-2$ (notice that the last equation in (58) is redundant) and the second is precisely (61), given the formulas (59) and (62) for $\hat{\beta}_{i-2}$ and $\hat{\beta}_{i-1}$, respectively. ■

It is important to notice that only in the vector of unknowns of the parametric system (55), only two components, namely x_{2i-1} and y, will depend on the parameter p (the value of x_{2i}), since this parameter shows up only in subsystem (61). Hence the solution to (50) has been broken into $i-1$ 4-tuples and one 3-tuple. The first 4-tuple, $q_1 = (x_1, x_2, x_{4i-3}, x_{4i-2})$ has the unique value $(\theta_1, d_2 - \theta_1, \theta_2 - d_1, d_1)$. The j-th 4-tuple, for $2 \le j \le i-1$, $q_j = (x_{2j-1}, x_{2j}, x_{4i-1-2j}, x_{4i-2j})$, is the appropriate reordering of the solution of the linear system

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^{j-1} \\ w_2^{j-1} \\ w_3^{j-1} \\ w_4^{j-1} \end{pmatrix} = \begin{pmatrix} -\theta_{j+1} \\ d_{j+1} \\ \hat{\beta}_{j-1} \\ -\hat{\beta}_{j-1} + \theta_{j+1} - d_{j+1} \end{pmatrix}$$

By Lemma 5.2, $w^{j-1} = (\hat{\beta}_{j-1}, \theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}, 0) + \alpha(1, -1, -1, 1)$, for $j = 2, \ldots, i-1$, and Lemma 5.3 gives necessary and sufficient conditions for such a solution to be nonnegative integral, as well as the number of solutions. The last piece of the puzzle is $q_i = (x_{2i-1}, y, x_{2i})$. System (61) has a solution if, and only, if

$$-\Theta + p + \hat{\beta}_{i-1} + p - \sum_{j=1}^{3i-4} b_j - 2p = 0.$$

But, as the above condition is, by construction, always satisfied, we conclude that $q_i = (\hat{\beta}_{i-1} + p, -\sum_{j=1}^{3i-4} b_j - 2p, p)$. This last vector must also satisfy nonnegativity and integrality conditions. Now q_i is integral if, and only if, p is integral, since all the data of the problem is integral. Nonnegativity conditions imply the three inequalities:

$$\begin{array}{rcl} p & \geq & -\hat{\beta}_{i-1} \\ \dfrac{-\sum_{j=1}^{3i-4} b_j}{2} = \dfrac{\Theta - \hat{\beta}_{i-1}}{2} & \geq & p \\ p & \geq & 0 \end{array}$$

Eliminating p with the Fourier-Motzkin we arrive at

$$\begin{array}{rcl} \Theta & \geq & -\hat{\beta}_{i-1} \\ \Theta & \geq & \hat{\beta}_{i-1} \end{array}$$

Thus (50) has a nonnegative integral solution if, and only if, the following conditions are satisfied:

$$\begin{cases} d_2 \geq \theta_1 \\ \theta_2 \geq d_1 \\ \theta_{j+1} \geq \hat{\beta}_{j-1} = \displaystyle\sum_{\ell=1}^{j-1} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j} (\theta_\ell - d_\ell), \qquad \text{for } j = 2, \ldots, i-1 \\ d_{j+1} \geq -\hat{\beta}_{j-1} = -\displaystyle\sum_{\ell=1}^{j-1} (-1)^\ell \beta_\ell + \sum_{\ell=1}^{j} (\theta_\ell - d_\ell), \text{ for } j = 2, \ldots, i-1 \\ \Theta \geq -\displaystyle\sum_{\ell=1}^{i-1} (-1)^\ell \beta_\ell + \sum_{\ell=1}^{i} (\theta_\ell - d_\ell) \\ \Theta \geq \displaystyle\sum_{\ell=1}^{i-1} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{i} (\theta_\ell - d_\ell) \end{cases} \tag{64}$$

The number of solutions is given by the product $\prod_{j=2}^{i} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_{j-1}\}, \min\{\theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}\}]$, for $j = 2, \ldots, i-1$, and n_i is the number of integers in the interval $[\max\{0, -\hat{\beta}_{i-1}\}, (\Theta - \hat{\beta}_{i-1})/2]$.

5.3 Case $n = 2 \bmod 4$

5.3.1 The linear system

Now we assume $n = 2i$, where $i \geq 3$ is odd. The problem is to find a nonnegative integral solution of the linear system

$$
\begin{array}{r}
\left.\begin{array}{r}\text{1-st set}\\ (2i+1)\text{ rows}\end{array}\right\{ \\[4em]
\left.\begin{array}{r}\text{2-nd set}\\ (i-1)\text{ rows}\end{array}\right\{
\end{array}
\begin{pmatrix}
1 & 0 & 0 & \cdots & & & & & \cdots & 0 & 0 & 0\\
1 & 1 & 0 & \cdots & & & & & \cdots & 0 & 0 & 0\\
 & & \ddots & & & & & & & & & \\
0 & \cdots & & & 1 & 0 & 0 & 0 & \cdots & & & 0\\
0 & \cdots & & & 0 & 1 & 1 & 0 & \cdots & & & 0\\
0 & \cdots & & & 0 & 0 & 0 & 1 & \cdots & & & 0\\
 & & & & & & & & \ddots & & & \\
0 & 0 & 0 & \cdots & & & & & \cdots & 0 & 1 & 1\\
0 & 0 & 0 & \cdots & & & & & \cdots & 0 & 0 & 1\\
0 & 1 & -1 & \cdots & & & & & \cdots & 1 & -1 & 0\\
 & & & \ddots & & & & & \iddots & & & \\
0 & 0 & 0 & \cdots & 1 & -1 & 1 & -1 & \cdots & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1\\ x_2\\ x_3\\ \vdots\\ x_{2i-1}\\ x_{2i}\\ \vdots\\ x_{4i-4}\\ x_{4i-3}\\ x_{4i-2}
\end{pmatrix}
=
\begin{pmatrix}
\theta_1\\ d_2\\ \theta_3\\ d_4\\ \vdots\\ \theta_i\\ \Theta\\ d_i\\ \vdots\\ \theta_4\\ d_3\\ \theta_2\\ d_1\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_{i-1}
\end{pmatrix}
\tag{65}
$$

where Θ, θ_j and d_j, for $j = 1, \ldots, i$, are positive integers and β_j, for $j = 1, \ldots, i-1$, are integers. The coefficient matrix $\overline{A}$ of (65) is $3i \times (4i-2)$, and its rows are partitioned in two sets, the first containing rows 1 through $2i+1$ and, the second, rows $2i+2$ through $3i$. The right-hand-side $\overline{b}$ and the equations are partitioned in the same way. The formulas for the j-th row of the first group and the corresponding right-hand-side element are

$$
\overline{A}_j = \begin{cases} \boldsymbol{e}_1, & \text{if } j = 1\\ \boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq 2i\\ \boldsymbol{e}_{2j-4}, & \text{if } j = 2i+1 \end{cases}
$$

and

$$
\overline{b}_j = \begin{cases} \theta_j, & \text{if } 1 \leq j \leq i,\ j \text{ odd} \\ d_j, & \text{if } 1 \leq j \leq i,\ j \text{ even} \\ \Theta, & \text{if } j = i+1 \\ \theta_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1,\ j \text{ even} \\ d_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1,\ j \text{ odd} \end{cases}
$$

where $\boldsymbol{e}_j$ is the j-th vector of $\mathbb{R}^{4i-2}$'s canonical basis. In the second group, the j-th row and respective right-hand-side are as follows

$$
\overline{A}_{2i+1+j} \;=\; \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j}, \quad \text{if } 1 \leq j \leq i-1
$$

and

$$
\overline{b}_{2i+1+j} \;=\; \beta_j, \quad \text{if } 1 \leq j \leq i-1 \ .
$$

We will follow the approach adopted in the previous cases studied. Our exposition will be succinct, since this is already the third time this approach is used. The first step is to eliminate variables x_1, x_2, x_{4i-2} and x_{4i-3}, in order to reduce the system. The equations used and values obtained are the same as in Subsection 5.2.1. They lead once more to the set of inequalities (22), obtained initially in Subsection 5.1.1, necessary and sufficient conditions for the partial solution $(x_1, x_2, x_{4i-2}, x_{4i-3}) = (\theta_1, d_2 - \theta_1, \theta_2 - d_1, d_1)$ to be nonnegative.

5.3.2 The reduced system and its solutions

Eliminating variables x_1, x_2, x_{4i-3} and x_{4i-2} and equations 1, 2, $2i+1$ and $2i$ the reduced system (51) is obtained from (65):

$$
\begin{array}{r}
\left.\begin{array}{r}\text{1-st set}\\ (2i-3)\text{ rows}\end{array}\right\{ \\[4em]
\left.\begin{array}{r}\text{2-nd set}\\ (i-1)\text{ rows}\end{array}\right\{
\end{array}
\begin{pmatrix}
1&1&0&0&0&\cdots&&&&&&0&0&0&0&0\\
0&0&1&1&0&\cdots&&&&&&0&0&0&0&0\\
&&&&\ddots&&&&&&&&&&&\\
0&\cdots&&&&&1&0&0&0&\cdots&&&&&0\\
0&\cdots&&&&&0&1&1&0&\cdots&&&&&0\\
0&\cdots&&&&&0&0&0&1&\cdots&&&&&0\\
&&&&&&&&&&\ddots&&&&&\\
0&0&0&0&0&\cdots&&&&&&0&1&1&0&0\\
0&0&0&0&0&\cdots&&&&&&0&0&0&1&1\\
-1&0&0&0&0&\cdots&&&&&&0&0&0&0&1\\
0&1&-1&0&0&\cdots&&&&&&0&0&1&-1&0\\
0&0&0&1&-1&\cdots&&&&&&1&-1&0&0&0\\
&&&&&\ddots&&&&&\iddots&&&&&\\
0&\cdots&&&&&1&-1&1&-1&\cdots&&&&&0
\end{pmatrix}
\begin{pmatrix}
x_3\\ x_4\\ \vdots\\ x_{2i-1}\\ x_{2i}\\ \vdots\\ x_{4i-5}\\ x_{4i-4}
\end{pmatrix}
=
\begin{pmatrix}
\theta_3\\ d_4\\ \vdots\\ \theta_i\\ \Theta\\ d_i\\ \vdots\\ \theta_4\\ d_3\\ \tilde{\beta}_1\\ \beta_2\\ \vdots\\ \beta_{i-1}
\end{pmatrix}
\qquad (66)
$$

where $\tilde{\beta}_1 = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-4$ equations in $4i-6$ unknowns.

System (66) has coefficient matrix $\overline{\overline{A}}$ and right-hand-side $\overline{\overline{b}}$, which inherit the two-set partition of $\overline{A}$ and $\overline{b}$. The j-th row and right-hand-side element in the first set is

$$\overline{\overline{A}}_j = \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, \quad \text{for } j = 1, \ldots, 2i-3$$

and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ \Theta, & \text{if } j = i-1 \\ \theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd} \end{cases}$$

In the second set, the formulas are:

$$\overline{\overline{A}}_{2i-3+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}, & \text{if } 2 \le j \le i-1 \end{cases}$$

and

$$\overline{\overline{b}}_{2i-3+j} = \begin{cases} \tilde{\beta}_1, & \text{if } j = 1 \\ \beta_j, & \text{if } 2 \leq j \leq i-1 \end{cases}$$

Note that $\boldsymbol{e}_j$ is now the i-th vector of $\mathbb{R}^{4i-6}$'s canonical basis.

Lemma 5.5 *Each column of $\overline{\overline{A}}$, contains exactly two nonnull coefficients, one in the first set of rows and the other in the second set.*

Proof: The proof is analogous to those of Lemmas 5.1 and 5.4. ∎

Multiplying by -1 the j-th equation of (66) where $1 \leq j \leq i-2$ is odd and $i-1 \leq j \leq 3i-4$ is even, we obtain the equivalent system

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & & & \\ 0 & \cdots\cdots & & -1 & 0 & 0 & 0 & \cdots\cdots & & & 0 \\ 0 & \cdots\cdots & & 0 & -1 & -1 & 0 & \cdots\cdots & & & 0 \\ 0 & \cdots\cdots & & 0 & 0 & 0 & 1 & \cdots\cdots & & & 0 \\ & & & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & \cdots\cdots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & \cdots\cdots & -1 & 1 & 0 & 0 & 0 \\ & & & & \ddots & & \cdots & & & & \\ 0 & \cdots\cdots & & 1 & -1 & 1 & -1 & \cdots\cdots & & & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_4 \\ \vdots \\ -\theta_i \\ -\Theta \\ d_i \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix} \qquad (67)$$

Formulas for the j-th row and right-hand-side element must be adapted accordingly. Mantaining the partition and letting A and b be the coefficient matrix and right-hand-side vector of (67), we have, for the first set:

$$A_j = \begin{cases} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } 1 \leq j \leq i-2 \\ (-1)^{j+1}(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } i-1 \leq j \leq 2i-3 \end{cases}$$

$$b_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \leq j \leq i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \leq j \leq i-2,\ j \text{ even} \\ -\Theta, & \text{if } j = i-1 \\ -\theta_{2i-j}, & \text{if } i \leq j \leq 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \leq j \leq 2i-3,\ j \text{ odd} \end{cases}$$

In the second set, the formulas are:

$$A_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}), & \text{if } 2 \leq j \leq i-1 \end{cases}$$

$$b_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j\beta_j, & \text{if } 2 \leq j \leq i-1 \end{cases}$$

Theorem 5.7 *System (67) is equivalent to a set of $i-1$ smaller systems. The j-th system, for $1 \leq j \leq i-2$, is given by*

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{68}$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \qquad \textit{for } j = 1, \ldots, i-2, \tag{69}$$

and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-3-2j}, x_{4i-2-2j}), & \textit{if } 1 \leq j \leq i-2 \textit{ is odd} \\ (x_{4i-3-2j}, x_{4i-2-2j}, x_{2j+1}, x_{2j+2}), & \textit{if } 1 \leq j \leq i-2 \textit{ is even} \end{cases} \tag{70}$$

System $i-1$, the last one, is given by

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} -\Theta \\ \hat{\beta}_{i-1} \end{pmatrix} \tag{71}$$

where

$$\hat{\beta}_{i-1} = \sum_{\ell=1}^{i-1} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{i} (\theta_\ell - d_\ell)$$

Proof: Applying precisely the same decomposition that was employed in the proofs of Theorems 5.3 and 5.6 to (67), we obtain $i-2$ flow problems in lenght 4 non-directed cycles represented by the systems (68). The remaining system is (71). ∎

System (71) has the unique solution $(x_{2i-1}, x_{2i}) = ((\Theta - \hat{\beta}_{i-1})/2, (\Theta + \hat{\beta}_{i-1})/2)$. This subvector of x is nonnegative and integral if, and only if, the following conditions hold:

$$\begin{array}{lll} \Theta - \hat{\beta}_{i-1} \text{ is even} & & \\ \Theta & \geq & \hat{\beta}_{i-1} \\ \Theta & \geq & -\hat{\beta}_{i-1} \end{array} \tag{72}$$

The decomposition afforded by Theorem 5.7 immediately leads to a set of conditions for the existence of nonnegative integral values for the remaining components of x. The set of all conditions results in system (64) together with the restriction on the parity of $\Theta - \hat{\beta}_{i-1}$. The number of solutions is given by the product $\prod_{j=2}^{i-1} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_{j-1}\}, \min\{\theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}\}]$, for $j = 2, \ldots, i-1$.

6 Conclusion

In this section the main theorem is proved, by establishing the correspondence between the linear systems obtained from the algorithms in Section 4 and the algebra in Section 5.

Proof of Theorem 1.1.

It is easy to see that the linear systems (18), (19) and (20) obtained from the algorithm in Section 4 are represented by the following matrices.

Odd dimensional case:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots 1 \; -1 \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_{2i}^c \\ h_{2i}^d
\end{pmatrix}
=
\begin{pmatrix}
e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{2i} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ \frac{B_i^+ - B_i^-}{2}
\end{pmatrix}
$$

Even dimensional case $n = 0 \mod 4$*:*

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & & & & & & \\
0 & & \cdots & & & & 0 & 1 & 1 & 1 & 0 & & & & & & \cdots & & & 0 \\
 & & & & & & & & & & & & & \ddots & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & \cdots & & & & & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & & & & & & \cdots & & & & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & & & & & & \cdots & & & & & 1 & -1 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & \iddots & & & & & & \\
0 & & \cdots & & & & 1 & -1 & 0 & 1 & -1 & & & & & & \cdots & & & 0
\end{pmatrix}
\begin{pmatrix}
h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_i^c \\ \beta^i \\ h_i^d \\ \vdots \\ h_{2i-1}^c \\ h_{2i-1}^d
\end{pmatrix}
=
\begin{pmatrix}
e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_i \\ \vdots \\ h_{2i-1} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ B_{i-1}^+ - B_{i-1}^-
\end{pmatrix}
$$

Even dimensional case $n = \ 2 \mod 4$*:*

$$\begin{pmatrix}
1 & 0 & 0 & & \cdots\cdots\cdots & & & & 0 & 0 & 0 \\
1 & 1 & 0 & & \cdots\cdots\cdots & & & & 0 & 0 & 0 \\
 & & \ddots & & & & & & & & \\
0 & \cdots\cdots & & 1 & 0 & 0 & 0 & \cdots\cdots & & & 0 \\
0 & \cdots\cdots & & 0 & 1 & 1 & 0 & \cdots\cdots & & & 0 \\
0 & \cdots\cdots & & 0 & 0 & 0 & 1 & \cdots\cdots & & & 0 \\
 & & & & & & & \ddots & & & \\
0 & 0 & 0 & & \cdots\cdots\cdots & & & & 0 & 1 & 1 \\
0 & 0 & 0 & & \cdots\cdots\cdots & & & & 0 & 0 & 1 \\
0 & 1 & -1 & & \cdots\cdots\cdots & & & & 1 & -1 & 0 \\
 & & & \ddots & & & & \iddots & & & \\
0 & 0 & 0 & \cdots\ 1 & -1 & 1 & -1 & \cdots & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_i^c \\ h_i^d \\ \vdots \\ h_{2i-1}^c \\ h_{2i-1}^d
\end{pmatrix}
=
\begin{pmatrix}
e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_i \\ \vdots \\ h_{2i-1} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ B_{i-1}^+ - B_{i-1}^-
\end{pmatrix}$$

These matrices correspond to (21), (50) and (65) respectively, in Section 5. In that section necessary and sufficient conditions were proved for the existence of non-negative integer solutions for these systems. These conditions are summarized in Section 5 in the odd dimensional case in (49), in the case $0 \mod 4$ in (64) and in the case $2 \mod 4$ in (64) and (72). These conditions correspond to the Poincaré-Hopf inequalities (17) where h_n and h_0 are taken to be zero together with (7) in the odd dimensional case and (16) in the $2 \mod 4$ dimensional case.

Hence the Poincaré-Hopf inequalities are necessary and sufficient conditions in order for a vertex to explode. The continuation of the abstract Lyapunov graph will occur if this condition is verified for each vertex.

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