Weak Solutions of a Phase-Field Model with Convection for Solidification of an Alloy

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Abstract

In recent years, the phase-field methodology has achieved considerable importance in modeling and numerically simulating a range of phase transitions and complex growth structures that occur during solidification processes. In attempt to understand the mathematical aspects of such methodology, in this article we consider a simplified model of this sort for a nonstationary process of solidification/melting of a binary alloy with thermal properties. The model includes the possibility of occurrence of natural convection in non-solidified regions and, therefore, leads to a free-boundary value problem for a highly non-linear system of partial differential equations consisting of a phase-field equation, a heat equation, a concentration equation and a modified Navier-Stokes equations by a penalization term of Carman-Kozeny type, which accounts for the mushy effects, and Boussinesq terms to take in consideration the effects of variations of temperature and concentration in the flow.

A proof of existence of weak solutions for the system is given. The problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder's fixed point theorem. A solution of the original problem is then found by using compactness arguments.

1 Introduction

In recent years, the phase-field methodology, which is an alternative formulation both to the sharp-interface methodology (Stefan type approach) or to the enthalpy methodology, has achieved considerable importance in modeling and numerically simulating a range of phase transitions and complex growth structures that occur during solidification. This has spurred many articles using this approach and proposing several mathematical models consisting of highly nonlinear systems of partial differential equations.

Rigorous mathematical analysis is in general difficult, but for pure materials undergoing phase change, several authors have undertaking the task. Examples of this sort of analysis are [4, 14, 16, 20], where existence and uniqueness results are investigated for various types of non-linearities.

Several phase-field models have also been developed for binary alloys. One of the first works in this direction was due to Wheeler et al. [30] and was concerned with isothermal solidification. Warren and Boettinger [29] extended this model, while recently Rappaz and Scheid [22] investigated the well-posedness under suitable assumptions for the non-linearities. Caginalp et al. [6, 5] extended this kind of model by including temperatures changes. For such model, the governing equations for the phase-field and the concentration are derived from a free energy functional; then an appropriate balance equation for the temperature, accounting for the liberation of latent heat by addition of a term proportional to the time derivative of the phase-field, is added to complete the model. The existence of weak solutions for this model was recently studied in [3].

The previously mentioned phase-field models do not consider the possibility of flow of the non-solidified material. However, there are many cases where such flows do occur and are significant, having important effects on the outcome of the solidification process. From the mathematical point of view, the inclusion of such effects in the model brings another order of difficulty to the analysis, whatever the approach used for modeling phase change. For instance [7, 8, 9, 10, 21, 24] consider several mathematical aspects of the interplay between fluid motion and phase change for pure material; the first four of these papers used the Stefan approach, while the last two used the enthalpy approach.

Voller et al. [26, 27] proposed models using the enthalpy technique for a convection/diffusion phase change process by including in the model a modification of the Navier-Stokes equations by the inclusion of a certain term that takes in consideration the flow in mushy regions. Particular expressions for this term may be obtained by modeling the mushy region as porous medium. Another model of this type was proposed by Voss and Tsai [28]. In Blanc et al. [1] performed a rigorous mathematical analysis of a stationary model for the solidification process with convection of a binary alloy. The model in [1] used an enthalpy approach and, as suggested in Voller et al. [27], a Carman-Kozeny penalization term was added to the Navier-Stokes equations to model the flow in mushy regions. Other authors have proposed models using the phase-field method for solidification process of binary alloys in presence of convection. For instance, Beckermann et al. [2] and Diepers et al. [11] proposed models of this sort using arguments of mixture theory. They also presented numerical simulations to validate their models.

In this paper we are interested in the rigorous mathematical analysis of a phase-field type model for a non-stationary solidification process of a binary alloy, with the possibility of flow of the non-solid phase. Differently of models in [2] and [11], the model we consider here combines ideas of Voller et al. [27] and of Blanc et al. [1] to model the possibility of flow with those of Caginalp et al. [6] for the phase-field and the thermal properties of the alloy. Our system of equations will described in detail in the next section; here we just observe that our system includes the Navier-Stokes equations with a Carman-Kozeny type term as described above, and also a Boussinesq type term to take in consideration buoyancy forces due to thermal and concentration effects. Since these equations for the flow only hold in an a priori unknown non-solid region, the model corresponds to a free boundary value problem. Moreover, since the Carman-Kozeny term is dependent on the local solid fraction, this is assumed to be functionally related to the the phase-field.

Our objective is to present a result on existence of weak solutions for this mathematical model. The proof will be based on a regularization technique that combines ideas already used in [1] and [3]: an auxiliary positive parameter will be introduced in the equations in such way that the original free boundary value problem will be transformed in a more standard (penalized) problem. We say that this transformed problem is the regularized problem. By solving it, one hopes to recover a solution of the original problem as the parameter approaches zero. To accomplish such program, we will firstly solve the regularized problem by using the Faedo-Galerkin method, just in the modified Navier-Stokes equations, and the Leray-Schauder fixed point theorem. Then, by taking a sequence of values of the parameter approaching zero, we will have a sequence of approximate solutions. By obtaining suitable estimates for this sequence, we will then be able to take the limit along a subsequence and, by compactness arguments, to show that we have a solution of the original problem.

The paper is organized as follows. In Section 2 we describe the mathematical model and its variables; we fix the notation and describe the functional spaces to be used; we also state our technical hypotheses and main result. In Section 3 we introduce and analyze the regularized problem. Section 4 is dedicated to the proof of the existence of weak solutions of the original free boundary value problem.

2 The model and main result

Let $0 < T < +\infty$ and Ω be an open bounded domain in $I\!R^N$, $N = 2$ or 3, with smooth boundary $\partial\Omega$ (of class C^3 will be enough for our purposes). Being $Q = \Omega \times (0, T)$, we will consider the following system of equations:

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \quad \text{in } Q, \tag{1}$$

$$v_t - \nu\Delta v + \nabla p + v\cdot\nabla v + k(f_s(\phi))v = \mathcal{F}(c,\theta) \quad \text{in } Q_{ml}, \tag{2}$$

$$\text{div } v = 0 \quad \text{in } Q_{ml}, \tag{3}$$

$$v = 0 \quad \text{in } Q_s, \tag{4}$$

$$C_{\rm v}\theta_t + C_{\rm v} v\cdot\nabla\theta = \nabla\cdot[K_1(\phi)\nabla\theta] + \frac{l}{2}f_s(\phi)_t \quad \text{in } Q, \tag{5}$$

$$c_t + v.\nabla c = K_2\left(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]\right) \quad \text{in } Q, \tag{6}$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial\theta}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad v = 0 \quad \text{on } \partial Q_{ml}, \tag{7}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \quad \text{in } \Omega, \quad v(0) = v_0 \quad \text{in } \Omega_{ml}(0), \tag{8}$$

Here, ϕ is the phase-field variable (sometimes called order parameter), which is the state variable characterizing the different phases; v is the velocity field; p is the associated hydrostatic pressure; $f_s \in [0,1]$ is the solid fraction; θ is the temperature; $c \in [0,1]$ is the concentration of the solute (i.e., the fraction of one of the two materials in the mixture).

We recall that the phase-field methodology in its simplest approach assumes the existence of two real numbers $\phi_s < \phi_l$ and a order parameter (phase-field) $\phi(x,t)$, depending on the spatial variable x and time t, such that if $\phi(x,t) \leq \phi_s$ then the material at point x at time t is in solid state; if $\phi_l \leq \phi(x,t)$ the material at point x at time t is in liquid state; if $\phi_s < \phi(x,t) < \phi_l$ then, at time t the point x is in the mushy region (a region of microscopic mixture of solid and liquid). This setting must be physically coherent with the concept of solid fraction, which we assume to be functionally dependent on the phase-field. This requires that $f_s(z)$ be a function such that $f_s(z) = 1$ for $z \leq \phi_s$, $f_s(z) = 0$ for $z \geq \phi_l$, and $0 < f_s(z) < 1$ for $\phi_s < z < \phi_l$. The required regularity assumptions on f_s will be described later on.

In the first of the previous equations (the phase-field equation), $\alpha > 0$ is the relaxation scaling; $\beta = \epsilon[s]/3\sigma$ where $\epsilon > 0$ is a measure of the interface width; σ the surface tension, and $[s]$ is the entropy density difference between phases. θ_A and θ_B are the melting temperatures of the two materials composing the binary alloy.

In the second of the previous equations, $\nu > 0$ is the viscosity, assumed to be constant. The penalization term $k(f_s)$ accounts for the mushy effect in the flow. The original Carman-Kozeny expression for it is $k(x) = C_0x^2/(1-x)^3$; however, we will consider more general expressions for this term. The term $\mathcal{F}(c,\theta)$ is the buoyancy force, which by using Boussinesq approximation can be expressed as $\mathcal{F}(c,\theta) = \rho\mathbf{g}\left(c_1(\theta-\theta_r) + c_2(c-c_r)\right) + F$, where ρ is the mean value of the density (which for simplicity we will assume to be a positive constant); $\mathbf{g}$ is the acceleration of gravity (for simplicity also assumed to be constant); c_1 and c_2 are two constants; θ_r, c_r are respectively the reference temperature and concentration (again for simplicity of exposition, both will be assumed to be zero), and F is an external force field.

In the equation for the temperature, $C_{\mathrm{v}} > 0$ is the specific heat (constant); l is a positive constant associated to the latent heat. We also observe that this equation comes from the balance of the internal energy that in this case has the form $e = C_{\mathrm{v}}\theta + \dfrac{l}{2}(1-f_s)$, where $1 - f_s$ is the liquid fraction. The thermal conductivity $K_1 > 0$ is assumed to depend on the phase-field.

In the last equation, $K_2 > 0$ is the solute diffusivity and M is a constant related to the slopes of solidus and liquidus lines.

The domain Q is composed of three regions, Q_s, Q_m and Q_l. The first one corresponds to the fully solid region; the second one corresponds to the mushy region, while the third is fully liquid region. They are defined by

$$\begin{array}{rcl} Q_s & = & \{(x,t) \in Q \ : \ f_s(\phi(x,t)) = 1\}, \\ Q_m & = & \{(x,t) \in Q \ : \ 0 < f_s(\phi(x,t)) < 1\}, \\ Q_l & = & \{(x,t) \in Q \ : \ f_s(\phi(x,t)) = 0\}. \end{array} \tag{9}$$

Q_{ml} will refer to the non-solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{(x,t) \in Q \ : \ 0 \leq f_s(\phi(x,t)) < 1\}. \tag{10}$$

We also define the subsets of Ω associated respectively to the solid and non-solid regions at time $t \in (0,T]$

$$\begin{array}{r} \Omega_s(t) = \{x \in \Omega \ : \ f_s(\phi(x,t)) = 1\}, \\ \Omega_{ml}(t) = \{x \in \Omega \ : \ 0 \leq f_s(\phi(x,t)) < 1\}. \end{array} \tag{11}$$

Observe that as we said above, all these previously described regions are a priori unknown, the model corresponds to a free boundary value problem.

Throughout this paper we will assume the following assumptions:

(H1) k is nondecreasing function of class $C^1[0,1)$ satisfying $k(0) = 0$ and $\lim_{x \to 1^-} k(x) = +\infty$,

(H2) f_s depends only on the phase field and is a Lipschitz continuous function defined on $I\!R$ and satisfying $0 \leq f_s(r) \leq 1$ for $r \in I\!R$ with f_s' measurable,

(H3) K_1 depends only on the phase-field and is a Lipschitz continuous function defined on $I\!R$; moreover, there exist $a > 0$ and $b > 0$ such that

$$0 < a \leq K_1(r) \leq b \qquad \text{for all } r \in I\!R,$$

(H4) F is a given function in $L^2(Q)$.

We remark that the concentration equation as it is written in [6] (up to addition of a proper convection term) is the following:

$$c_t + v.\nabla c = K_2 \nabla \cdot \left[c(1-c) \nabla \left(M\phi + \ \ln \frac{c}{1-c} \right) \right] \ \text{ in } Q.$$

This form of the equation forces $c \in (0,1)$ and is equivalent to equation (6) in this case. Thus, (6) is more general than this last form since it allows c to assume the values 0 and 1, which are associated to regions of pure materials.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq I\!R^N$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0,T)$ Then,

$$
\begin{aligned}
\mathcal{V}(G) &= \left\{ w \in \left(C_0^\infty(G)\right)^N,\ \mathrm{div}\ w = 0 \right\}, \\
H(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^N, \\
V(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^N, \\
H^{\tau,\tau/2}(\overline{Q}_G) &= \text{Hölder continuous functions of exponent } \tau \text{ in } x \\
&\quad\ \text{and exponent } \tau/2 \text{ in } t, \\
W_q^{2,1}(Q_G) &= \{ w \in L^q(Q_G) /\, D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \}.
\end{aligned}
$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$. Properties of these functional spaces can be found for instance in [15, 25]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^N$.

The main result of this paper is the following.

Theorem 1 *Let be given $T > 0$, $\Omega \subset I\!R^N$, $N = 2$, or 3, a bounded open domain of class C^3, and assume that* **(H1)-(H4)** *hold. Let also be given $(N+2)/2 < q \leq 2(N+2)/N$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, satisfying the compatibility condition $\dfrac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$, $v_0 \in H(\Omega_{ml}(0))$, $\theta_0 \in L^2(\Omega)$, and $c_0 \in L^2(\Omega)$ satisfying $0 \leq c_0 \leq 1$ a.e. in Ω. Then, there exist functions (ϕ, v, θ, c, J) satisfying:*

(i) $\phi \in W_q^{2,1}(Q)$, $\phi(0) = \phi_0$,

(ii) $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $v = 0$ *a.e. in* $\overset{o}{Q}_s$, $v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where Q_s is defined by (9) and $\Omega_{ml}(0)$ by (11),*

(iii) $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta(0) = \theta_0$,

(iv) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q

Moreover, they satisfy

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \quad a.e.\ in\ Q, \tag{12}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad a.e.\ on\ \partial\Omega \times (0,T), \tag{13}$$

$$
\begin{aligned}
(v(t),\eta(t))_{\Omega_{ml}(t)} - \int_0^t (v,\eta_t)_{\Omega_{ml}(s)}ds + \nu \int_0^t (\nabla v, \nabla\eta)_{\Omega_{ml}(s)}ds \\
+ \int_0^t (v\cdot\nabla v,\eta)_{\Omega_{ml}(s)}ds + \int_0^t (k(f_s(\phi))v,\eta)_{\Omega_{ml}(s)}ds \\
= \int_0^t (\mathcal{F}(c,\theta),\eta)_{\Omega_{ml}(s)}ds + (v_0,\eta(0))_{\Omega_{ml}(0)},
\end{aligned} \tag{14}
$$

for $t \in (0,T]$ *and any* $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ *with compact support contained in* $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ *and* $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$ *where* Q_{ml} *is defined by (10) and* $\Omega_{ml}(t)$ *by (11),*

$$
\begin{aligned}
-C_{\mathrm{v}}\int_\Omega \theta_0\xi(0)dx - C_{\mathrm{v}}\int_0^T\int_\Omega \theta\xi_t dxdt - C_{\mathrm{v}}\int_0^T\int_\Omega v\theta\cdot\nabla\xi\, dxdt \\
+ \int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\xi\, dxdt = \frac{l}{2}\int_0^T\int_\Omega f_s(\phi)_t\xi\, dxdt
\end{aligned} \tag{15}
$$

for any $\xi \in L^4(0,T;H^1(\Omega))$ *with* $\xi_t \in L^2(Q)$ *and* $\xi(T) = 0$ *in* Ω,

$$
\begin{aligned}
-\int_0^T\int_\Omega c\zeta_t dxdt - \int_0^T\int_\Omega vc\cdot\nabla\zeta\, dxdt + K_2\int_0^T\int_\Omega \nabla c\cdot\nabla\zeta\, dxdt \\
+K_2 M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta\, dxdt = \int_\Omega c_0\zeta(0)dx,
\end{aligned} \tag{16}
$$

for any $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\zeta_t \in L^2(Q)$ *and* $\zeta(T) = 0$ *in* Ω.

Remarks:

1. The restriction $q > N + 2/2$ ensures the continuity of phase-field; in fact, in this case $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$, for $\tau = 2 - (N+2)/q$ ([15] p. 80). Therefore, the set Q_{ml} is open, and we have a suitable interpretation for the equations of velocity field. The restriction $q \leq 2(N+2)/N$ is consequence of the obtained regularity of the temperature. (iii) implies that $\theta \in L^{2(N+2)/N}(Q)$, and then, from the existence theorem for the phase-field equation given in ([14] Thm 2.1), we know that $\phi \in W^{2,1}_{2(N+2)/N}(Q)$.

2. We observe that the phase-field models without convection studied in [3] or [16] allow the thermal conductivity K_1 to vanish. In the presence of convection, we were not able to prove the existence of global weak solutions in this degenerate case; thus, we had to assume the more restrictive assumption **(H3)**. It is possible, however, to prove the existence of a slightly different local weak solution of (1)-(8) in the degenerate case. This will be done elsewhere.

3 A regularized problem

In this section we introduce an auxiliary regularized problem by performing suitable modifications of the original equations. The first objective of these modifications is to introduce coefficients ensuring enough regularity for the arguments to be used. The second objective, as in Blanc [1], is to change the modified Navier-Stokes equations in such way that it holds in the whole domain instead of holding just in an a priori unknown region.

The proof of existence of solutions for such regularized problem will be done by using Faedo-Galerkin method, with the help of the Leray-Schauder Fixed Point Theorem as stated in ([12], p. 189):

Theorem (Leray-Schauder): *Consider a transformation $y = \mathcal{T}_\lambda(x)$ where x, y belong to a Banach space B and λ is a real parameter which varies in a bounded interval, say $0 \leq \lambda \leq 1$. Assume:*

(a) $\mathcal{T}_\lambda(x)$ is defined for all $x \in B$, $0 \leq \lambda \leq 1$,

(b) for any fixed λ, $\mathcal{T}_\lambda(x)$ is continuous in B,

(c) for x in bounded sets of B, $\mathcal{T}_\lambda(x)$ is uniformly continuous in λ,

(d) for any fixed λ, $\mathcal{T}_\lambda(x)$ is a compact transformation,

(e) there exists a (finite) constant M such that every possible solution x of $\mathcal{T}_\lambda(x) = x$ satisfies: $\|x\|_B \leq M$,

(f) the equation $\mathcal{T}_0(x) = x$ has a unique solution in B.

Under the assumptions (a)-(f), there exists a solution of the equation $x - \mathcal{T}_1(x) = 0$.

Now, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^{N+1})$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^{N+1})} \leq C\, \|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [19] p. 157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^{N+1})$ be a family of symmetric positive mollifier functions with compact support converging to the Dirac delta function (we can take the support of ρ_δ contained in the ball of radius δ), and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^{N+1})$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $I\!R^{N+1}\backslash Q$. Then, as in [19] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $I\!R^{N+1}$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(I\!R^{N+1})$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in I\!R^N : d(x,\Omega) < \delta\}. \tag{17}$$

We also define the associated space-time cylinder

$$Q^\delta = \Omega^\delta \times (0,T). \tag{18}$$

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$, we conclude that $\rho_\delta(v) \in L^\infty(0,T;H(\Omega^\delta)) \cap L^2(0,T;V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \leq \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \leq \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. Let $\delta(\Omega)$ be as described after (17); for each $\delta \in (0, \delta(\Omega)]$, we consider the system

$$\alpha\epsilon^2\phi^\delta_t - \epsilon^2\Delta\phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) = \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta, \quad (19)$$

$$\begin{aligned} \frac{d}{dt}(v^\delta, u) \;&+\; \nu(\nabla v^\delta, \nabla u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f^\delta_s(\phi^\delta) - \delta)v^\delta, u) \\ &=\; (\mathcal{F}(c^\delta, \theta^\delta), u) \quad \text{for all } u \in V(\Omega^\delta),\ t \in (0,T), \end{aligned} \quad (20)$$

$$C_{\rm v}\theta^\delta_t + C_{\rm v}\rho_\delta(v^\delta)\cdot\nabla\theta^\delta = \nabla\cdot\left(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\right) + \frac{l}{2}f^\delta_s(\phi^\delta)_t \quad \text{in } Q^\delta, \quad (21)$$

$$c^\delta_t - K_2\Delta c^\delta + \rho_\delta(v^\delta)\cdot\nabla c^\delta = K_2 M\nabla\cdot\left(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\right) \quad \text{in } Q^\delta, \quad (22)$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \qquad \frac{\partial\theta^\delta}{\partial n} = 0, \qquad \frac{\partial c^\delta}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \quad (23)$$

$$\phi^\delta(0) = \phi^\delta_0, \qquad v^\delta(0) = v^\delta_0, \qquad \theta^\delta(0) = \theta^\delta_0, \qquad c^\delta(0) = c^\delta_0 \qquad \text{in } \Omega^\delta. \quad (24)$$

Concerning this system we will prove the following existence result.

Proposition 1 *Let $T > 0$, $\delta(\Omega) > 0$ be as described following (17), and $1/2 < \gamma \leq 1$. For each $\delta \in (0, \delta(\Omega)]$, consider $\phi^\delta_0 \in H^{1+\gamma}(\Omega^\delta)$, $v^\delta_0 \in H(\Omega^\delta)$, $\theta^\delta_0 \in H^{1+\gamma}(\Omega^\delta)$ and $c^\delta_0 \in C^1(\overline{\Omega^\delta})$ satisfying the compatibility conditions $\frac{\partial\phi^\delta_0}{\partial n} = \frac{\partial\theta^\delta_0}{\partial n} = \frac{\partial c^\delta_0}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 \leq c^\delta_0 \leq 1$ in $\overline{\Omega^\delta}$. Assume also that* **(H1)-(H4)** *hold. Then, there exist a solution $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ of (19)-(24) satisfying*

i) $\phi^\delta \in L^2(0,T; H^2(\Omega^\delta))$, $\phi^\delta_t \in L^2(Q^\delta)$,

ii) $v^\delta \in L^2(0,T; V(\Omega^\delta)) \cap L^\infty(0,T; H(\Omega^\delta))$, $v^\delta_t \in L^2(0,T; V(\Omega^\delta)')$,

iii) $\theta^\delta \in L^2(0,T; H^2(\Omega^\delta))$, $\theta^\delta_t \in L^2(Q^\delta)$,

iv) $c^\delta \in C^{2,1}(Q^\delta)$, $\quad 0 \leq c^\delta \leq 1$.

The proof of this proposition will depend on an another existence result for other approximate problem, obtained from (19)-(24) by discretizing just the modified Navier-Stokes equations using Faedo-Galekin method. By solving this approximate problem, we will recover the solution of the regularized problem as the discretization dimension m increases to $+\infty$.

For this purpose, first we introduce the spaces $V_s(\Omega^\delta)$,

$$V_s(\Omega^\delta) = \text{ the closure of } \mathcal{V}(\Omega^\delta) \text{ in } (H^s(\Omega^\delta))^N, \ s \geq 1,$$

endowed with the usual Hilbert scalar product

$$((u,v))_s = \sum_{i=1}^{N} (u_i, v_i)_{H^s(\Omega^\delta)}.$$

We also consider the spectral problem:

$$((u,v))_s = \lambda(u,v) \quad \text{ for all } v \in V_s(\Omega^\delta) \text{ and } s = \frac{N}{2},$$

which admits a sequence of solutions w_j corresponding to the sequence of eigenvalues $\lambda_j > 0$.

With the help of these eigenfunctions, we define the following approximate problem of order m: find $(\phi_m^\delta, v_m^\delta, \theta_m^\delta, c_m^\delta)$, with

$$v_m^\delta(t) = \sum_{j=1}^{m} g_{jm}^\delta(t) w_j \in V_m = \text{span}\{w_1, \ldots, w_m\},$$

such that

$$\alpha\epsilon^2 \phi_{mt}^\delta - \epsilon^2 \Delta\phi_m^\delta - \frac{1}{2}(\phi_m^\delta - ({\phi_m^\delta}^3) = \beta\left(\theta_m^\delta + (\theta_B - \theta_A)c_m^\delta - \theta_B\right) \text{ in } Q^\delta, \quad (25)$$

$$\begin{aligned} \frac{d}{dt}(v_m^\delta, w_j) \quad &+ \quad \nu(\nabla v_m^\delta, \nabla w_j) + (v_m^\delta \cdot \nabla v_m^\delta, w_j) + (k(f_s^\delta(\phi_m^\delta) - \delta)v_m^\delta, w_j) \\ &= \quad (\mathcal{F}(c_m^\delta, \theta_m^\delta), w_j) \quad 1 \leq j \leq m, \ t \in (0,T), \end{aligned} \quad (26)$$

$$C_{\text{v}}\theta_{mt}^\delta + C_{\text{v}}\rho_\delta(v_m^\delta) \cdot \nabla\theta_m^\delta = \nabla \cdot \left(K_1(\rho_\delta(\phi_m^\delta))\nabla\theta_m^\delta\right) + \frac{l}{2} f_s^\delta(\phi_m^\delta)_t \quad \text{in } Q^\delta, \quad (27)$$

$$c_{mt}^\delta - K_2\Delta c_m^\delta + \rho_\delta(v_m^\delta) \cdot \nabla c_m^\delta = K_2 M \nabla \cdot \left(c_m^\delta(1 - c_m^\delta)\nabla\rho_\delta(\phi_m^\delta)\right) \quad \text{in } Q^\delta, \quad (28)$$

$$\frac{\partial\phi_m^\delta}{\partial n} = 0, \quad \frac{\partial\theta_m^\delta}{\partial n} = 0, \quad \frac{\partial c_m^\delta}{\partial n} = 0 \quad \text{ on } \partial\Omega^\delta \times (0,T), \quad (29)$$

$$\phi_m^\delta(0) = \phi_{0m}^\delta, \quad v_m^\delta(0) = v_{0m}^\delta, \quad \theta_m^\delta(0) = \theta_{0m}^\delta, \quad c_m^\delta(0) = c_{0m}^\delta \quad \text{ in } \Omega^\delta. \quad (30)$$

We then have the following existence result.

Proposition 2 *Let $T > 0$, $\delta(\Omega)$ be as described after (17), and $1/2 < \gamma \leq 1$. Fix $\delta \in (0, \delta(\Omega)]$ and $m \in I\!N$; let $\phi_{0m}^\delta \in H^{1+\gamma}(\Omega^\delta)$, $v_{0m}^\delta \in H(\Omega^\delta)$, $\theta_{0m}^\delta \in H^{1+\gamma}(\Omega^\delta)$ and $c_{0m}^\delta \in C^1(\overline{\Omega^\delta})$ satisfying the compatibility conditions $\dfrac{\partial \phi_{0m}^\delta}{\partial n} = \dfrac{\partial \theta_{0m}^\delta}{\partial n} = \dfrac{\partial c_{0m}^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 < c_{0m}^\delta < 1$ in $\overline{\Omega^\delta}$. Assume also that* **(H1)-(H4)** *hold. Then, there exist a solution $(\phi_m^\delta, v_m^\delta, \theta_m^\delta, c_m^\delta)$ satisfying (25)-(30) and*

i) $\phi_m^\delta \in L^2(0, T; H^2(\Omega^\delta))$, $\phi_{mt}^\delta \in L^2(Q^\delta)$,

ii) $v_m^\delta \in C^1([0, T]; V_m)$,

iii) $\theta_m^\delta \in L^2(0, T; H^2(\Omega^\delta))$, $\theta_{mt}^\delta \in L^2(Q^\delta)$,

iv) $c_m^\delta \in C^{2,1}(Q^\delta)$, $0 < c_m^\delta < 1$.

Proof: For simplicity of notation, in this proof we shall omit the index δ used in ϕ_m^δ, v_m^δ, θ_m^δ, c_m^δ.

We consider the family of operators, for $0 \leq \lambda \leq 1$,

$$T_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(Q^\delta) \times L^2(0, T; H(\Omega^\delta)) \times L^2(Q^\delta) \times L^2(Q^\delta),$$

which maps $(\hat{\phi}_m, \hat{v}_m, \hat{\theta}_m, \hat{c}_m) \in B$ into $(\phi_m, v_m, \theta_m, c_m)$, with $v_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \in V_m$, obtained by solving the problem

$$\alpha\epsilon^2 \phi_{mt} - \epsilon^2 \Delta\phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda\beta \left(\hat{\theta}_m + (\theta_B - \theta_A)\hat{c}_m - \theta_B\right) \text{ in } Q^\delta, \quad (31)$$

$$\begin{aligned} \frac{d}{dt}(v_m, w_j) \quad &+ \quad \nu(\nabla v_m, \nabla w_j) + (v_m \cdot \nabla v_m, w_j) + (k(f_s^\delta(\phi_m) - \delta)v_m, w_j) \\ &= \quad \lambda(\mathcal{F}(\hat{c}_m, \hat{\theta}_m), w_j) \quad 1 \leq j \leq m, \ t \in (0, T), \qquad (32) \end{aligned}$$

$$C_{\rm v}\theta_{mt} + C_{\rm v}\rho_\delta(v_m) \cdot \nabla\theta_m = \nabla \cdot (K_1(\rho_\delta(\phi_m))\nabla\theta_m) + \frac{l}{2} f_s^\delta(\phi_m)_t \quad \text{in } Q^\delta, \quad (33)$$

$$c_{mt} - K_2\Delta c_m + \rho_\delta(v_m) \cdot \nabla c_m = K_2 M \nabla \cdot (c_m(1 - c_m)\nabla\rho_\delta(\phi_m)) \quad \text{in } Q^\delta, \quad (34)$$

$$\frac{\partial \phi_m}{\partial n} = \frac{\partial \theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \qquad \text{on } \partial\Omega^\delta \times (0,T), \tag{35}$$

$$\phi_m(0) = \phi^\delta_{0m}, \quad v_m(0) = v^\delta_{0m}, \quad \theta_m(0) = \theta^\delta_{0m}, \quad c_m(0) = c^\delta_{0m} \text{ in } \Omega^\delta. \tag{36}$$

Clearly $(\phi_m, v_m, \theta_m, c_m)$ is a solution of (25)-(30) if and only if it is a fixed point of the operator T_1. In the following, we prove that T_1 has at least one fixed point using the Leray-Schauder Fixed Point Theorem.

To begin with, observe that since $\hat{\theta}_m$, $\hat{c}_m \in L^2(Q^\delta)$ we infer from Theorem 2.1 [14] that there is a unique solution ϕ_m of equation (31) with $\phi_m \in W^{2,1}_2(Q^\delta)$.

Now, (32) is a nonlinear system of ordinary differential equations for the functions $g_{1m}, \ldots, g_{mm}$. This problem has an unique maximal solution defined on same interval $[0, t_m)$ and $v_m \in C^1([0,t_m); V_m)$. The *a priori* estimates we shall prove later will show in particular that $t_m = T$.

Observe that since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi_m) \in C^\infty(I\!R^{N+1})$, we have that $K_1(\rho_\delta(\phi_m)) \in W^{1,1}_r(Q^\delta)$, $1 \le r \le \infty$, and since $\rho_\delta(v_m) \in L^{N+2}(Q^\delta)$ and $f^\delta_s(\phi_m)_t = f^{\delta'}_s(\phi_m)\phi_{mt} \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([15], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ_m of equation (33) with $\theta_m \in W^{2,1}_2(Q^\delta)$.

We observe that equation (34) is a semi-linear parabolic equation with smooth coefficients and growth conditions on the nonlinear forcing terms as the ones required for a semigroup result on global existence result given in [13], p. 75. Thus, there is a unique classical global solution c_m. In addition, note that equation (34) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principle together with the conditions $0 < c^\delta_{0m} < 1$ and $\dfrac{c_m}{\partial n} = 0$ on $\partial\Omega^\delta$, we can deduce that

$$0 < c_m(x,t) < 1, \qquad \forall\, (x,t) \in Q^\delta. \tag{37}$$

Therefore, the mapping T_λ is well defined from B into B.

To prove the continuity of T_λ, let $(\hat{\phi}^k_m, \hat{v}^k_m, \hat{\theta}^k_m, \hat{c}^k_m)$, $k \in I\!N$ be a sequence in B strongly converging to $(\hat{\phi}_m, \hat{v}_m, \hat{\theta}_m, \hat{c}_m) \in B$ and for each k, let $(\phi^k_m, v^k_m, \theta^k_m, c^k_m)$, the solution of the problem:

$$\alpha\epsilon^2\phi^k_{mt} - \epsilon^2\Delta\phi^k_m - \frac{1}{2}(\phi^k_m - (\phi^k_m)^3) = \lambda\beta\left(\hat{\theta}^k_m + (\theta_B - \theta_A)\hat{c}^k_m - \theta_B\right) \text{ in } Q^\delta, \tag{38}$$

$$
\begin{aligned}
v_m^k(t) &= \sum_{j=1}^{m} g_{jm}^k(t) w_j \in V_m, \\
\frac{d}{dt}(v_m^k, w_j) &+ \nu(\nabla v_m^k, \nabla w_j) + (v_m^k \cdot \nabla v_m^k, w_j) + (k(f_s^\delta(\phi_m^k) - \delta) v_m^k, w_j) \\
&= \lambda(\mathcal{F}(\hat{c}_m^k, \hat{\theta}_m^k), w_j),\ 1 \le j \le m,\ t \in (0,T), \qquad (39)
\end{aligned}
$$

$$
C_{\rm v}\theta_{mt}^k + C_{\rm v}\rho_\delta(v_m^k) \cdot \nabla\theta_m^k = \nabla \cdot \Big(K_1(\rho_\delta(\phi_m^k))\nabla\theta_m^k\Big) + \frac{l}{2} f_s^\delta(\phi_m^k)_t \text{ in } Q^\delta, \quad (40)
$$

$$
c_{mt}^k - K_2 \Delta c_m^k + \rho_\delta(v_m^k) \cdot \nabla c_m^k = K_2 M \nabla \cdot \Big(c_m^k(1 - c_m^k)\nabla\rho_\delta(\phi_m^k)\Big) \text{ in } Q^\delta, \quad (41)
$$

$$
\frac{\partial\phi_m^k}{\partial n} = \frac{\partial\theta_m^k}{\partial n} = \frac{\partial c_m^k}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \qquad (42)
$$

$$
\phi_m^k(0) = \phi_{0m}^\delta, \quad v_m^k(0) = v_{0m}^\delta, \quad \theta_m^k(0) = \theta_{0m}^\delta, \quad c_m^k(0) = c_{0m}^\delta \qquad \text{in } \Omega^\delta. \quad (43)
$$

We show that the sequence $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$ converges strongly in B to $(\phi_m, v_m, \theta_m, c_m) = T_\lambda(\hat{\phi}_m, \hat{v}_m\hat{\theta}_m, \hat{c}_m)$. For that purpose, we will obtain estimates to $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$ independent of k. As usual, we will denote by C_i, with a proper indexes i, positive constants independent of k.

We multiply (38) by ϕ_m^k, ϕ_{mt}^k and $-\Delta\phi_m^k$, we integrate over $\Omega^\delta \times (0,t)$ and by parts, and we use the Hölder's and Young's inequalities to obtain the following three estimates:

$$
\begin{aligned}
&\frac{\alpha\epsilon^2}{2}\int_{\Omega^\delta} |\phi_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} \Big(\epsilon^2 |\nabla\phi_m^k|^2 + \frac{1}{4}(\phi_m^k)^4\Big) dxdt \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \Big(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |\phi_m^k|^2\Big) dxdt, \qquad (44) \\
&\frac{\alpha\epsilon^2}{2}\int_0^t \int_{\Omega^\delta} |\phi_{mt}^k|^2 dxdt + \int_{\Omega^\delta} \left(\frac{\epsilon}{2}|\nabla\phi_m^k|^2 + \frac{(\phi_m^k)^4}{8} - \frac{(\phi_m^k)^2}{4}\right) dx \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \Big(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2\Big) dxdt, \qquad (45) \\
&\frac{\alpha\epsilon^2}{2}\int_{\Omega^\delta} |\nabla\phi_m^k|^2 dx + \frac{\epsilon^2}{2}\int_0^t \int_{\Omega^\delta} |\Delta\phi_m^k|^2 dxdt \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \Big(|\nabla\phi_m^k|^2 + |\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2\Big) dxdt. \qquad (46)
\end{aligned}
$$

Multiplying (45) by $\alpha\epsilon^2$ and adding the result to (44), we find

$$
\begin{aligned}
\int_{\Omega^\delta} |\phi_m^k|^2 + |\nabla\phi_m^k|^2 &+ (\phi_m^k)^4 dx \\
\le C_1 &+ C_2 \int_0^t \int_{\Omega^\delta} \Big(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |\phi_m^k|^2\Big) dxdt. \qquad (47)
\end{aligned}
$$

Since $\|\hat{\theta}_m^k\|_{L^2(Q^\delta)}$ and $\|\hat{c}_m^k\|_{L^2(Q^\delta)}$ are bounded independent of k, we infer from (47) and Gronwall's inequality that

$$\|\phi_m^k\|_{L^\infty(0,T;H^1(\Omega^\delta))} \leq C_1. \tag{48}$$

Then, thanks to (44)-(46) we have

$$\|\phi_m^k\|_{L^2(0,T;H^2(\Omega^\delta))} + \|\phi_{mt}^k\|_{L^2(Q^\delta)} \leq C_1. \tag{49}$$

We multiply (39) by $g_{jm}^k(t)$ and add these equations for $j = 1, \dots, m$. Using that $(u \cdot \nabla v, v) = 0$, $u \in V(\Omega^\delta)$, $v \in (H^1(\Omega^\delta))^N$ we get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega^\delta} |v_m^k|^2 dx \quad &+ \quad \int_{\Omega^\delta} \left(\nu|\nabla v_m^k|^2 + k(f_s^\delta(\phi_m^k) - \delta)|v_m^k|^2\right) dx \\ &\leq \quad C_1 \int_{\Omega^\delta} \left(|F|^2 + |\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |v_m^k|^2\right) dx. \end{aligned}$$

By using Gronwall's inequality, we obtain

$$\|v_m^k\|_{L^\infty(0,T;H(\Omega^\delta))\cap L^2(0,T;V(\Omega^\delta))} \leq C_1. \tag{50}$$

Let now P_m be the projector of $H(\Omega^\delta)$ on the space V_m. Note that P_m is a $V_s(\Omega^\delta)$-orthogonal projector on V_m and thus $\|P_m\|_{\mathcal{L}(V_s(\Omega^\delta),V_s(\Omega^\delta))} \leq 1$. Therefore, from equation (39), we infer that

$$\begin{aligned} \|v_{mt}^k\|_{V_s(\Omega^\delta)'} \quad &\leq \quad C_1 \Big(\|v_m^k\|_{V(\Omega^\delta)} + \|v_m^k\|^2_{L^{\frac{2N}{N-1}}(\Omega^\delta)} + \|F\|_{L^2(\Omega^\delta)} \\ &+ \quad \|\hat{\theta}_m^k\|_{L^2(\Omega^\delta)} + \|\hat{c}_m^k\|_{L^2(\Omega^\delta)}\Big). \end{aligned}$$

Then, by using (50) and interpolation ([17] p.73), we obtain

$$\|v_{mt}^k\|_{L^2(0,T;V_s(\Omega^\delta)')} \leq C_1. \tag{51}$$

Now, by multiplying (40) by θ_m^k, one obtains similarly that

$$\int_{\Omega^\delta} |\theta_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla\theta_m^k|^2 dxdt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\phi_{mt}^k|^2 + |\theta_m^k|^2\right) dxdt, \tag{52}$$

and we infer from (49) and Gronwall's inequality that

$$\|\theta_m^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{53}$$

Hence, it follows from (52) that

$$\|\theta_m^k\|_{L^2(0,T;H^1(\Omega^\delta))} \le C_1. \tag{54}$$

Now we take the scalar product of (40) with $\eta \in H^1(\Omega^\delta)$ and integrate by parts using Hölder's and Young's inequalities to obtain

$$\|\theta_{mt}^k\|_{H^1(\Omega^\delta)'} \le C_1 \left(\|\nabla\theta_m^k\|_{L^2(\Omega^\delta)} + \|v_m^k\|_{L^4(\Omega^\delta)}\|\theta_m^k\|_{L^4(\Omega^\delta)} + \|\phi_{mt}^k\|_{L^2(\Omega^\delta)} \right)$$

and we infer from (49),(50) and (54) that

$$\|\theta_{mt}^k\|_{L^{4/3}(0,T;H^1(\Omega^\delta)')} \le C_1. \tag{55}$$

Next, multiplying (41) by c_m^k and reasoning as before with the help of (37), we conclude that

$$\int_{\Omega^\delta} |c_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c_m^k|^2 dxdt \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla \phi_m^k|^2 dxdt.$$

Hence, from (49), we obtain

$$\|c_m^k\|_{L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta))} \le C_1. \tag{56}$$

In order to get an estimate for (c_{mt}^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we go back to equation (41) and proceed similarly as before to obtain

$$\|c_{mt}^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \le C_1. \tag{57}$$

We now infer from (48)-(57) that the sequence (ϕ_m^k) is uniformly bounded with respect to k in

$$W_1 = \left\{ w \in L^2(0,T;H^2(\Omega^\delta)),\ w_t \in L^2(0,T;L^2(\Omega^\delta)) \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H^1(\Omega^\delta)),\ w_t \in L^2(0,T;L^2(\Omega^\delta)) \right\};$$

the sequence (v_m^k) is bounded in

$$W_3 = \left\{ w \in L^2(0,T;V(\Omega^\delta)),\ w_t \in L^2(0,T;V_s(\Omega^\delta)') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;H(\Omega^\delta)),\ w_t \in L^2(0,T;V_s(\Omega^\delta)') \right\};$$

the sequence (θ_m^k) is bounded in

$$W_5 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\; w_t \in L^{4/3}(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_6 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\; w_t \in L^{4/3}(0,T;H^1(\Omega^\delta)') \right\};$$

and the sequence (c_m^k) is bounded in

$$W_7 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\; w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_8 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\; w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Now we observe that W_1 is compactly embedded into $L^2(0,T;H^1(\Omega^\delta))$, and the same holds for W_2 into $C([0,T];L^2(\Omega^\delta))$; for W_3, W_5 and W_7 into $L^2(Q^\delta)$; with W_4 into $C([0,T];V_s(\Omega^\delta)')$, and with W_6 and W_8 into $C([0,T];H^1(\Omega^\delta)')$ ([23] Cor.4).

It follows that there exist $(\phi_m,\, v_m,\, \theta_m,\, c_m)$ satisfying:

$$\begin{array}{rcl}
\phi_m & \in & L^2(0,T;H^2(\Omega^\delta)) \cap L^\infty(0,T;H^1(\Omega^\delta)), \text{ with } \phi_{mt} \in L^2(Q^\delta), \\
v_m & \in & L^2(0,T;V(\Omega^\delta)) \cap L^\infty(0,T;H(\Omega^\delta)), \text{ with } v_{mt} \in L^2(0,T;V_s(\Omega^\delta)'), \\
\theta_m & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)), \text{ with } \theta_{mt} \in L^{4/3}(0,T;H^1(\Omega^\delta)'), \\
c_m & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)), \text{ with } c_{mt} \in L^2(0,T;H^1(\Omega^\delta)'),
\end{array}$$

and a subsequence of $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$, which for simplicity of notation we keep denoting $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$, such that as $k \to +\infty$ we have

$$\begin{array}{rcl}
\phi_m^k & \to & \phi_m \text{ strongly in } L^2(0,T;H^1(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)), \\
\phi_m^k & \rightharpoonup & \phi_m \text{ weakly in } L^2(0,T;H^2(\Omega^\delta)), \\
v_m^k & \to & v_m \text{ strongly in } L^2(Q^\delta) \cap C([0,T];V_s(\Omega^\delta)'), \\
v_m^k & \rightharpoonup & v_m \text{ weakly in } L^2(0,T;V(\Omega^\delta)), \\
\theta_m^k & \to & \theta_m \text{ strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\
\theta_m^k & \rightharpoonup & \theta_m \text{ weakly in } L^2(0,T;H^1(\Omega^\delta)), \\
c_m^k & \to & c_m \text{ strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\
c_m^k & \rightharpoonup & c_m \text{ weakly in } L^2(0,T;H^1(\Omega^\delta)).
\end{array} \tag{58}$$

It now remains to pass to the limit as k tends to $+\infty$ in (38)-(43).

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^9(Q^\delta)$ is compact ([18] p.15), and (ϕ_m^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi_m^k)^3$ converges to ϕ_m^3 in $L^2(Q^\delta)$. We then pass to the limit as k tends to $+\infty$ in (38) and get

$$\alpha\epsilon^2\phi_{mt} - \epsilon^2\Delta\phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda\beta\left(\hat{\theta}_m + (\theta_B - \theta_A)\hat{c}_m - \theta_B\right) \text{ a.e. in } Q^\delta.$$

Now we observe that for fixed $\delta > 0$, $k(f_s^\delta(\cdot) - \delta)$ is a bounded Lipschitz continuous function from $I\!R$ in $I\!R$; therefore, $k(f_s^\delta(\phi_m^k) - \delta)$ converges to $k(f_s^\delta(\phi_m) - \delta)$ in $L^p(Q^\delta)$ for any $1 \le p < +\infty$. Since the passing to the limit of the other terms of (39) can be done in standard ways, we get

$$\begin{aligned}\frac{d}{dt}(v_m, w_j) + \nu(\nabla v_m, \nabla w_j) \quad & +(v_m \cdot \nabla v_m, w_j) + (k(f_s^\delta(\phi_m) - \delta)v_m, w_j) \\ & = \lambda(\mathcal{F}(\hat{c}_m, \hat{\theta}_m), w_j) \quad 1 \le j \le m,\ t \in (0,T).\end{aligned}$$

Also, since V_m is a closed subspace, we have that $v_m(t) = \sum_{j=1}^m g_{jm}(t)w_j \in V_m$.

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions and ϕ_m^k converges to ϕ_m in $L^2(Q^\delta)$, we have that $K_1(\rho_\delta(\phi_m^k))$ converges to $K_1(\rho_\delta(\phi_m))$ and $f_s^{\delta\prime}(\phi_m^k)$ converges to $f_s^{\delta\prime}(\phi_m)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. These facts and (58) yield the weak convergence of $K_1(\rho_\delta(\phi_m^k))\nabla\theta_m^k$ to $K_1(\rho_\delta(\phi_m))\nabla\theta_m$ and $f_s^{\delta\prime}(\phi_m^k)\phi_{mt}^k$ to $f_s^{\delta\prime}(\phi_m)\phi_{mt}$ in $L^{3/2}(Q^\delta)$. Now, multiplying (40) by $\eta \in \mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta \times (0,T)$ and by parts, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega^\delta} C_{\rm v}\left(\theta_{mt}^k + \rho_\delta(v_m^k)\cdot\nabla\theta_m^k\right)\eta \quad & + \quad K_1(\rho_\delta(\phi_m^k))\nabla\theta_m^k \cdot \nabla\eta\, dxdt \\ & = \quad \int_0^T\int_{\Omega^\delta} \frac{l}{2} f_s^{\delta\prime}(\phi_m^k)\phi_{mt}^k\eta\, dxdt.\end{aligned}$$

Then, we may pass to the limit and find that

$$C_{\rm v}\theta_{mt} + C_{\rm v}\rho_\delta(v_m)\cdot\nabla\theta_m = \nabla\cdot(K_1(\rho_\delta(\phi_m))\nabla\theta_m) + \frac{l}{2}f_s^{\delta\prime}(\phi_m)\phi_{mt} \text{ in } \mathcal{D}'(Q^\delta). \tag{59}$$

Now, by using the L^p-theory of parabolic equations, we conclude that (59) holds almost everywhere in Q^δ.

It remains to pass to the limit in (41). We infer from (58) that $\nabla\rho_\delta(\phi_m^k)$ converges to $\nabla\rho_\delta(\phi_m)$ in $L^2(Q^\delta)$. Also, since $\|c_m^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c_m^k(1 - c_m^k)$ converges to $c_m(1 - c_m)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. Thus, we may pass to the limit in (41) to obtain

$$c_{mt} - K_2\Delta c_m + \rho_\delta(v_m)\cdot\nabla c_m = K_2 M\nabla\cdot(c_m(1 - c_m)\nabla\rho_\delta(\phi_m)) \text{ in } Q^\delta.$$

Therefore, T_λ is continuous for each $0 \le \lambda \le 1$.

At the same time, T_λ is bounded in $W_1 \times W_3 \times W_5 \times W_7$, and the embedding of this space in B is compact. We conclude that T_λ is a compact operator.

To prove that for $(\hat{\phi}_m, \hat{v}_m, \hat{\theta}_m, \hat{c}_m)$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $(\phi_{mi}, v_{mi}, \theta_{mi}, c_{mi})$, $(i = 1, 2)$ the corresponding solutions of (31)-(36). We observe that $\phi_m = \phi_{m1} - \phi_{m2}$, $v_m = v_{m1} - v_{m2}$ $\quad (v_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \in V_m)$, $\theta_m = \theta_{m1} - \theta_{m2}$ and $c_m = c_{m1} - c_{m2}$ satisfy the following problem:

$$\begin{aligned}\alpha\epsilon^2 \phi_{mt} - \epsilon^2 \Delta\phi_m = \; & \frac{1}{2}\phi_m \left(1 - (\phi_{m1}^2 + \phi_{m1}\phi_{m2} + \phi_{m2}^2)\right) \\ & + (\lambda_1 - \lambda_2)\beta \left(\hat{\theta}_m + (\theta_B - \theta_A)\hat{c}_m - \theta_B\right) \quad \text{in } Q^\delta,\end{aligned} \tag{60}$$

$$\begin{aligned}&\frac{d}{dt}(v_m, w_j) + \nu(\nabla v_m, \nabla w_j) + (v_m \cdot \nabla v_{m1}, w_j) - (v_{m2} \cdot \nabla v_m, w_j) \\ &\quad + (k(f_s^\delta(\phi_{m1}) - \delta) v_m, w_j) + \left(\left[k(f_s^\delta(\phi_{m1}) - \delta) - k(f_s^\delta(\phi_{m2}) - \delta)\right] v_{m2}, w_j\right) \\ &\quad = (\lambda_1 - \lambda_2)(\mathcal{F}(\hat{c}_m, \hat{\theta}_m), w_j), \quad 1 \le j \le m,\end{aligned} \tag{61}$$

$$\begin{aligned}&C_{\mathrm{v}}\theta_{mt} - \nabla \cdot (K_1(\rho_\delta(\phi_{m1}))\nabla\theta_m) - \nabla \cdot \left[K_1(\rho_\delta(\phi_{m1})) - K_1(\rho_\delta(\phi_{m2}))\right] \nabla\theta_{m2} \\ &\quad + \; C_{\mathrm{v}}\rho_\delta(v_m) \cdot \nabla\theta_{m1} + C_{\mathrm{v}}\rho_\delta(v_{m2}) \cdot \nabla\theta_m \\ &\quad = \; \frac{l}{2} f_s^{\delta\prime}(\phi_{m1})\phi_{mt} + \frac{l}{2}\left[f_s^{\delta\prime}(\phi_{m1}) - f_s^{\delta\prime}(\phi_{m2})\right]\phi_{m2t} \quad \text{in } Q^\delta,\end{aligned} \tag{62}$$

$$\begin{aligned}c_{mt} - K_2\Delta c_m \;=\; & K_2 M \nabla \cdot \left(c_{m1}(1 - c_{m1})\left[\nabla\rho_\delta(\phi_{m1}) - \nabla\rho_\delta(\phi_{m2})\right]\right) \\ + \; & \rho_\delta(v_m) \cdot \nabla c_{m1} + \rho_\delta(v_{m2}) \cdot \nabla c_m \\ + \; & K_2 M \nabla \cdot (c_m(1 - (c_{m1} + c_{m2}))\nabla\rho_\delta(\phi_{m2})) \quad \text{in } Q^\delta,\end{aligned} \tag{63}$$

$$\frac{\partial\phi_m}{\partial n} = \frac{\partial\theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0, T), \tag{64}$$

$$\phi_m(0) = 0, \quad v_m(0) = 0, \quad \theta_m(0) = 0, \quad c_m(0) = 0 \quad \text{in} \quad \Omega^\delta. \tag{65}$$

We remark that $d := \phi_{m1}^2 + \phi_{m1}\phi_{m2} + \phi_{m2}^2 = (\phi_{m1}/\sqrt{2} + \phi_{m2}/\sqrt{2})^2 + \phi_{m1}^2/2 + \phi_{m2}^2/2 \ge 0$. Now, by multiplying equation (60) by ϕ_m, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\begin{aligned}\int_{\Omega^\delta} |\phi_m|^2 dx \; + \; & \int_0^t \int_{\Omega^\delta} |\nabla\phi_m|^2 dx dt \le C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 (1 - d) dx dt \\ + \; & C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2\right) dx dt.\end{aligned}$$

Applying Gronwall's inequality, we get

$$\|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} + \|\phi_m\|^2_{L^2(0,T;H^1(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{66}$$

Now, by multiplying (60) by ϕ_{m_t} and using Hölder's inequality, we conclude

$$\begin{aligned}
\alpha\epsilon^2 \int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dxdt \;&+\; \frac{\epsilon^2}{2}\int_{\Omega^\delta} |\nabla\phi_m|^2 dx \\
&\leq\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 dxdt + \frac{\alpha\epsilon^2}{2}\int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dxdt \\
&+\; C_2 \left(\int_0^t \int_{\Omega^\delta} |\phi_m|^{10/3} dxdt\right)^{3/5} \left(\int_0^t \int_{\Omega^\delta} |d|^5 dxdt\right)^{2/5} \\
&+\; C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2\right) dxdt.
\end{aligned}$$

Since $W_2^{2,1}(Q^\delta) \hookrightarrow L^{10}(Q^\delta)$, the following interpolation inequality holds

$$\|\phi_m\|^2_{L^{10/3}(Q^\delta)} \leq \eta\,\|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} + \tilde{C}\,\|\phi_m\|^2_{L^2(Q^\delta)} \text{ for all } \eta > 0,$$

and since $\|d\|_{L^5(Q^\delta)} \leq C$, depending on $\|\phi_{m1}\|_{L^{10}(Q^\delta)}$ and $\|\phi_{m2}\|_{L^{10}(Q^\delta)}$, rearranging the different terms, we obtain

$$\begin{aligned}
\int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dxdt \;&+\; \int_{\Omega^\delta} |\nabla\phi_m|^2 dx \\
&\leq\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 dxdt + C_2\,\eta\,\|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} \\
&+\; C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2\right) dxdt.
\end{aligned} \tag{67}$$

Multiplying (60) by $-\Delta\phi_m$, and proceeding similarly as before, we infer that

$$\begin{aligned}
\int_{\Omega^\delta} |\nabla\phi_m|^2 dx \;&+\; \int_0^t \int_{\Omega^\delta} |\Delta\phi_m|^2 dxdt \\
&\leq\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 + |\nabla\phi_m|^2 dxdt + C_2\,\eta\,\|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} \\
&+\; C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2\right) dxdt.
\end{aligned} \tag{68}$$

Taking $\eta > 0$ small enough and considering (66), we conclude from (67) and (68) that

$$\|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} + \|\phi_m\|^2_{L^\infty(0,T;H^1(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{69}$$

Multiplying (61) by $g_{jm}(t)$ and adding these equations for $j = 1, \cdots, m$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta} |v_m|^2 dx + \int_{\Omega^\delta} \nu|\nabla v_m|^2 + k(f_s^\delta(\phi_{m1}) - \delta)|v_m|^2 dx \\
&\quad \le \int_{\Omega^\delta} |k(f_s^\delta(\phi_{m1}) - \delta) - k(f_s^\delta(\phi_{m2}) - \delta)||v_{m2}||v_m| dx \\
&\qquad + \int_{\Omega^\delta} |v_m||\nabla v_{m1}||v_m| dx + |\lambda_1 - \lambda_2| \int_{\Omega^\delta} \mathcal{F}(\hat{c}_m, \hat{\theta}_m) v_m dx \\
&\quad \le C_1 \Big(\|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \|v_{m2}\|^2_{V(\Omega^\delta)} \\
&\qquad + \|v_{m1}\|^2_{L^\infty(0,T;V_s(\Omega^\delta))} \|v_m\|^2_{L^2(\Omega^\delta)} \Big) + \frac{\nu}{2}\int_{\Omega^\delta} |\nabla v_m|^2 dx \\
&\qquad + C_2 |\lambda_1 - \lambda_2|^2 \int_{\Omega^\delta} |F|^2 + |\hat{\theta}_m|^2 + |\hat{c}_m|^2 dx + C_3 \int_{\Omega^\delta} |v_m|^2 dx.
\end{aligned}
$$

By integrating this last inequality with respect to t and using our previous estimates and Gronwall's inequality, we obtain

$$
\|v_m\|^2_{L^\infty(0,T;H(\Omega^\delta)) \cap L^2(0,T;V(\Omega^\delta))} \le C_1 \, |\lambda_1 - \lambda_2|^2. \tag{70}
$$

Multiplying (62) by θ_m, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^\delta} |\theta_m|^2 dx + a \int_{\Omega^\delta} |\nabla \theta_m|^2 dx \\
&\quad \le C_1 \int_{\Omega^\delta} |\rho_\delta(\phi_m)||\nabla \theta_{m2}||\nabla \theta_m| + |\rho_\delta(v_m)||\nabla \theta_{m1}||\theta_m| dx \\
&\qquad + C_2 \int_{\Omega^\delta} |\phi_{mt}||\theta_m| + |\phi_m||\phi_{m2t}| \, |\theta_m| dx \\
&\quad \le C_1 \|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \|\nabla \theta_{m2}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta} |\nabla \theta_m|^2 dx \\
&\qquad + C_2 \|v_m\|^2_{L^\infty(0,T;H(\Omega^\delta))} \|\nabla \theta_{m1}\|^2_{L^2(\Omega^\delta)} \\
&\qquad + C_4 \|\phi_m\|^2_{L^\infty(0,T;H^1(\Omega^\delta))} \|\phi_{m2t}\|^2_{L^2(\Omega^\delta)} + C_3 \int_{\Omega^\delta} \left(|\phi_{mt}|^2 + |\theta_m|^2 \right) dx.
\end{aligned}
$$

Integration with respect to t and the use of Gronwall's Lemma and (69)-(70) lead to the estimate

$$
\|\theta_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1 \, |\lambda_1 - \lambda_2|^2. \tag{71}
$$

We multiply (63) by c_m, integrate over $\Omega^\delta \times (0, t)$ and by parts, and we use

Hölder's and Young's inequalities and (37) to obtain

$$\begin{aligned}\int_{\Omega^\delta} |c_m|^2 dx + & \int_0^t \int_{\Omega^\delta} |\nabla c_m|^2 dx dt \\ \leq & \ C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla \rho_\delta(\phi_{m1}) - \nabla \rho_\delta(\phi_{m2})|^2 + |v_m|^2 + |c_m|^2 \right) dx dt \\ \leq & \ C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla \phi_m|^2 + |v_m|^2 + |c_m|^2 \right) dx dt.\end{aligned}$$

Applying Gronwall's inequality and using (69)-(70) we arrive at

$$\|c_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{72}$$

Therefore, it follows from (69)-(72) that T_λ is uniformly continuous with respect to λ on bounded sets of B.

To estimate the set of all fixed points of T_λ, let $(\phi_m, v_m, \theta_m, c_m) \in B$ be such any given fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2 \phi_{mt} - \epsilon^2 \Delta \phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda\beta \left(\theta_m + (\theta_B - \theta_A) c_m - \theta_B\right) \text{ in } Q^\delta, \tag{73}$$

$$\begin{aligned} v_m(t) &= \sum_{j=1}^m g_{jm}(t) w_j \in V_m = \text{span}\{w_1, \dots, w_m\}, \\ \frac{d}{dt}(v_m, w_j) &+ \nu(\nabla v_m, \nabla w_j) + (v_m \cdot \nabla v_m, w_j) + (k(f_s^\delta(\phi_m) - \delta) v_m, w_j) \\ &= \lambda(\mathcal{F}(c_m, \theta_m), w_j) \quad 1 \leq j \leq m, \ t \in (0,T), \end{aligned} \tag{74}$$

$$C_{\rm v} \theta_{mt} + C_{\rm v} \rho_\delta(v_m) \cdot \nabla \theta_m = \nabla \cdot (K_1(\rho_\delta(\phi_m)) \nabla \theta_m) + \frac{l}{2} f_s^\delta(\phi_m)_t \text{ in } Q^\delta, \tag{75}$$

$$c_{mt} - K_2 \Delta c_m + \rho_\delta(v_m) \cdot \nabla c_m = K_2 M \nabla \cdot (c_m(1 - c_m) \nabla (\rho_\delta(\phi_m))) \text{ in } Q^\delta, \tag{76}$$

$$\frac{\partial \phi_m}{\partial n} = \frac{\partial \theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{77}$$

$$\phi_m(0) = \phi_{0m}^\delta, \quad v_m(0) = v_{0m}^\delta, \quad \theta_m(0) = \theta_{0m}^\delta, \quad c_m(0) = c_{0m}^\delta \quad \text{in} \quad \Omega^\delta. \tag{78}$$

Multiplying the first equation (73) by ϕ_m, ϕ_{mt} and $-\Delta\phi_m$, respectively, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities, we obtain

$$\frac{\alpha\epsilon^2}{2} \frac{d}{dt} \int_{\Omega^\delta} |\phi_m|^2 dx \ + \ \int_{\Omega^\delta} \left(\epsilon^2 |\nabla \phi_m|^2 + \frac{1}{4} \phi_m^4 \right) dx$$

$$\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\phi_m|^2 \right) dx, \quad (79)$$

$$\frac{\alpha\epsilon^2}{2} \int_{\Omega^\delta} |\phi_{mt}|^2 dx + \frac{d}{dt} \int_{\Omega^\delta} \left(\frac{\epsilon^2}{2} |\nabla \phi_m|^2 + \frac{1}{8}\phi_m^4 - \frac{1}{4}|\phi_m|^2 \right) dx$$
$$\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 \right) dx, \quad (80)$$

$$\frac{\alpha\epsilon^2}{2} \frac{d}{dt} \int_{\Omega^\delta} |\nabla \phi_m|^2 dx + \int_{\Omega^\delta} \frac{\epsilon^2}{2} |\Delta \phi_m|^2 dx$$
$$\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\nabla \phi_m|^2 \right) dx. \quad (81)$$

Now, for each $j = 1, \dots, m$, we multiply (74) by $g_{jm}(t)$ and add the resulting equations to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\Omega^\delta} |v_m|^2 dx + \int_{\Omega^\delta} \left(\nu |\nabla v_m|^2 + k(f_s^\delta(\phi_m) - \delta)|v_m|^2 \right) dx \\ \leq C_1 \int_{\Omega^\delta} |F|^2 + |\theta_m|^2 + |c_m|^2 + |v_m|^2 dx. \end{aligned} \quad (82)$$

By multiplying (75) by θ_m and (76) by c_m and proceeding similarly as above lead us to the following inequalities

$$\frac{d}{dt} \int_{\Omega^\delta} \frac{C_{\rm v}}{2} |\theta_m|^2 dx + a \int_{\Omega^\delta} |\nabla \theta_m|^2 dx \leq \frac{\alpha^2 \epsilon^4}{4} \int_{\Omega^\delta} |\phi_{mt}|^2 dx + C_1 \int_{\Omega^\delta} |\theta_m|^2 dx, \quad (83)$$

$$\frac{d}{dt} \int_{\Omega^\delta} |c_m|^2 dx + K_2 \int_{\Omega^\delta} |\nabla c_m|^2 dx \leq C_1 \int_{\Omega^\delta} |\nabla \phi_m|^2 dx, \quad (84)$$

where we used (37) to obtain the last inequality.

Now, by multiplying (80) by $\alpha\epsilon^2$ and adding the result to (79),(81)-(84), we obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega^\delta} \left(\frac{\alpha\epsilon^2}{4} |\phi_m|^2 + \left(\frac{\alpha\epsilon^2}{2} + \frac{\alpha\epsilon^4}{2} \right) |\nabla \phi_m|^2 + \frac{\alpha\epsilon^2}{8} \phi_m^4 + \frac{1}{2}|v_m|^2 \right. \\ &\quad \left. + \frac{C_{\rm v}}{2} |\theta_m|^2 + |c_m|^2 \right) dx + \int_{\Omega^\delta} \left(\epsilon^2 |\nabla \phi_m|^2 + \frac{1}{4}\phi_m^4 + \frac{\alpha^2\epsilon^4}{4} |\phi_{mt}|^2 \right. \\ &\quad \left. + \frac{\epsilon^2}{2} |\Delta \phi_m|^2 + \nu |\nabla v_m|^2 + k(f_s^\delta(\phi_m) - \delta)|v_m|^2 + a|\nabla \theta_m|^2 + K_2 |\nabla c_m|^2 \right) dx \\ &\quad \leq C_1 + C_1 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\phi_m|^2 + |\nabla \phi_m|^2 + |v_m|^2 \right) dx, \end{aligned} \quad (85)$$

where C_1 is independent of λ, m and δ.

Hence, integrating (85) with respect t and using Gronwall's inequality, we obtain

$$\|\phi_m\|_{L^\infty(0,T;H^1(\Omega^\delta))} + \|v_m\|_{L^\infty(0,T;H(\Omega^\delta))} + \|\theta_m\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c_m\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1,$$

where C_1 is independent of λ. Therefore, we have a bound for all fixed points of T_λ in B independent of λ.

Finally, proceeding exactly as we did to prove that T_λ is well defined, we conclude that for $\lambda = 0$, problem (31)-(36) has a unique solution.

Thus, we can apply Leray-Schauder theorem and conclude that there is a fixed point $(\phi_m, v_m, \theta_m, c_m) \in B \cap W_2^{2,1}(Q^\delta) \times C^1(0,T;V_m) \times W_2^{2,1}(Q^\delta) \times C^{2,1}(Q^\delta)$ of the operator T_1, that is, $(\phi_m, v_m, \theta_m, c_m) = T_1(\phi_m, v_m, \theta_m, c_m)$. This is a solution of problem (25)-(30), and the proof of Proposition 2 is complete. ∎

We now proceed with the

Proof of Proposition 1: We choose $\phi^\delta_{0m} = \phi^\delta_0$, $\theta^\delta_{0m} = \theta^\delta_0$, $c^\delta_{0m} \in C^1(\bar\Omega)$ with $0 < c^\delta_{0m} < 1$, and $v^\delta_{0m} \in V_m$ such that $c^\delta_{0m} \to c^\delta_0$ and $v^\delta_{0m} \to v^\delta_0$ in the norm of $H(\Omega^\delta)$ as $m \to +\infty$. We then infer from Proposition 2 that, for each $\delta \in (0, \delta(\Omega)]$ and $m \in I\!N$, there exist functions $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$ satisfying the system (25)-(30). We will derive bounds, independent of m, for this solution and then pass to the limit in the approximate problem as m tends to $+\infty$ by using compactness arguments.

Lemma 1 *There exists a constant C_1 independent of $m \in I\!N$ such that*

$$\|\phi^\delta_m\|_{L^\infty(0,T;H^1(\Omega^\delta))\cap L^2(0,T;H^2(\Omega^\delta))} + \|\phi^\delta_{mt}\|_{L^2(Q^\delta)} \le C_1, \tag{86}$$
$$\|v^\delta_m\|_{L^\infty(0,T;H(\Omega^\delta))\cap L^2(0,T;V(\Omega^\delta))} \le C_1, \tag{87}$$
$$\|\theta^\delta_m\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \tag{88}$$
$$\|c^\delta_m\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1. \tag{89}$$

Proof: It follows from the inequality (85). ∎

Lemma 2 *There exists a constant C_1 independent of $m \in I\!N$ such that*

$$\|v^\delta_{mt}\|_{L^2(0,T;V_s(\Omega^\delta)')} \le C_1, \tag{90}$$
$$\|\theta^\delta_{mt}\|_{L^{4/3}(0,T;H^1(\Omega^\delta)')} \le C_1, \tag{91}$$
$$\|c^\delta_{mt}\|_{L^2(0,T;H^1(\Omega^\delta)')} \le C_1. \tag{92}$$

Proof: From the equation (26), we infer that

$$\begin{aligned}\|v^\delta_{mt}\|_{V_s(\Omega^\delta)'} \leq\ & C_1 \Big(\|v^\delta_m\|_{V(\Omega^\delta)} + \|v^\delta_m\|^2_{L^{\frac{2N}{N-1}}(\Omega^\delta)} \\ & + \|F\|_{L^2(\Omega^\delta)} + \|\theta^\delta_m\|_{L^2(\Omega^\delta)} + \|c^\delta_m\|_{L^2(\Omega^\delta)}\Big) .\end{aligned}$$

Then, by using (87)-(89) and interpolation ([17] p.73), we obtain (90). By taking the scalar product of (27) with $\eta \in H^1(\Omega)$ and using Hölder's inequality, we find

$$\|\theta^\delta_{mt}\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla \theta^\delta_m\|_{L^2(\Omega^\delta)} + \|\phi^\delta_{mt}\|_{L^2(\Omega^\delta)} + \|v^\delta_m\|_{L^4(\Omega^\delta)} \|\theta^\delta_m\|_{L^4(\Omega^\delta)}\right) .$$

Then, (91) follows from (86)-(88). (92) can be obtained similarly by using Lemma 1. ∎

We infer from Lemma 1 and 2 using the compact embedding ([23] Cor.4) that there exist

$$\begin{array}{lcl} \phi^\delta & \in & L^2(0,T;H^2(\Omega^\delta)) \cap L^\infty(0,T;H^1(\Omega^\delta)) \text{ with } \phi^\delta_t \in L^2(Q), \\ v^\delta & \in & L^2(0,T;V(\Omega^\delta)) \cap L^\infty(0,T;H(\Omega^\delta)) \text{ with } v^\delta_t \in L^2(0,T;V_s(\Omega^\delta)'), \\ \theta^\delta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } \theta^\delta_t \in L^{4/3}(0,T;H^1(\Omega^\delta)'), \\ c^\delta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } c^\delta_t \in L^2(0,T;H^1(\Omega^\delta)'), \end{array}$$

and a subsequence of $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$, which we keep calling $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$ to ease the notation, such that, as $m \to +\infty$,

$$\begin{array}{lcll} \phi^\delta_m & \to & \phi^\delta & \text{strongly in } L^2(0,T;H^{2-\overline{\gamma}}(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)), \\ & & & 0 < \overline{\gamma} \leq 1/2 \\ \phi^\delta_{mt} & \rightharpoonup & \phi^\delta_t & \text{weakly in } L^2(Q^\delta), \\ v^\delta_m & \to & v^\delta & \text{stronly in } L^2(Q^\delta) \cap C([0,T];V_s(\Omega^\delta)'), \\ v^\delta_{mt} & \rightharpoonup & v^\delta_t & \text{weakly in } L^2(0,T;V_s(\Omega^\delta)'), \\ \theta^\delta_m & \to & \theta^\delta & \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\ \theta^\delta_m & \rightharpoonup & \theta^\delta & \text{weakly in } L^2(0,T;H^1(\Omega^\delta)), \\ c^\delta_m & \to & c^\delta & \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\ c^\delta_m & \rightharpoonup & c^\delta & \text{weakly in } L^2(0,T;H^1(\Omega^\delta)). \end{array} \tag{93}$$

Thus, letting $m \to +\infty$ in (25), we get

$$\alpha\epsilon^2 \phi^\delta_t - \epsilon^2 \Delta\phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) = \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ a.e. in } Q^\delta.$$

Since $k(f_s^\delta(\cdot)-\delta)$ is a bounded Lipschitz continuous function we have that $k(f_s^\delta(\phi_m^\delta)-\delta)$ converges to $k(f_s^\delta(\phi^\delta)-\delta)$ in $L^p(Q^\delta)$, for $p\in[1,\infty)$; then $k(f_s^\delta(\phi_m^\delta)-\delta)v_m^\delta$ converges to $k(f_s^\delta(\phi^\delta)-\delta)v^\delta$ in $L^{3/2}(Q^\delta)$ as m tends to $+\infty$. As usual ([17] p.76) we may pass to the limit in the other terms in (26) and get

$$\begin{aligned}\frac{d}{dt}(v^\delta,w_j)+\nu(\nabla v^\delta,\nabla w_j) \quad &+(v^\delta\cdot\nabla v^\delta,w_j)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,w_j)\\ &=(\mathcal{F}(c^\delta,\theta^\delta),w_j)\quad \text{for all } j\in I\!N.\end{aligned}$$

We conclude that

$$\frac{d}{dt}(v^\delta,u)+\nu(\nabla v^\delta,\nabla u)+(v^\delta\cdot\nabla v^\delta,u)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,u)=(\mathcal{F}(c^\delta,\theta^\delta),u),$$

for all $u\in V_s(\Omega^\delta)$, and then for all $u\in V(\Omega^\delta)$.

Since $K_1(\rho_\delta)$ and ${f_s^\delta}'$ are bounded Lipschitz continuous functions we have that $K_1(\rho_\delta(\phi_m^\delta))$ converges to $K_1(\rho_\delta(\phi^\delta))$ and ${f_s^\delta}'(\phi_m^\delta)$ to ${f_s^\delta}'(\phi^\delta)$ in $L^p(Q^\delta)$ for any $p\in[1,\infty)$ as m tends to $+\infty$. Using these facts and (93) we pass to the limit in (27) and obtain

$$C_{\rm v}\theta_t^\delta+C_{\rm v}\rho_\delta(v^\delta)\cdot\nabla\theta^\delta=\nabla\cdot\Big(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\Big)+\frac{l}{2}{f_s^\delta}'(\phi^\delta)\phi_t^\delta \text{ in } \mathcal{D}'(Q^\delta).$$

Applying L^p-theory of parabolic equations, we have that $\theta^\delta\in W_2^{2,1}(Q^\delta)$.

Similarly we pass to the limit in (28) and obtain

$$c_t^\delta-K_2\Delta c^\delta+\rho_\delta(v^\delta)\cdot\nabla c^\delta=K_2M\nabla\cdot\Big(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\Big) \text{ in } Q^\delta.$$

Observe that c^δ is a classical solution and satisfies $0\le c^\delta\le 1$. Finally, it follows from (93) that $\dfrac{\partial\phi^\delta}{\partial n}=\dfrac{\partial\theta^\delta}{\partial n}=\dfrac{\partial c^\delta}{\partial n}=0$, $\phi^\delta(0)=\phi_0^\delta$, $v^\delta(0)=v_0^\delta$, $\theta^\delta(0)=\theta_0^\delta$ and $c^\delta(0)=c_0^\delta$. Therefore, the proof of Proposition 1 is complete. ∎

4 Proof of Theorem 1

In this section we prove the existence Theorem 1. For $0<\delta\le\delta(\Omega)$ as in the statement of Theorem 1, we choose $\phi_0^\delta\in W^{2-2/q,q}(\Omega^\delta)\cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta\in H(\Omega^\delta)$, $\theta_0^\delta\in H^{1+\gamma}(\Omega)$, $1/2<\gamma\le 1$, $c_0^\delta\in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial\phi_0^\delta}{\partial n}=$

$\dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$, $\|\theta_0^\delta\|_{L^2(\Omega^\delta)} \le C$, and $0 \le c_0^\delta \le 1$ in $\overline{\Omega^\delta}$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0))$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (19)-(24). We will derive bounds, independent of δ, for this solution and then use compactness arguments and passage to the limit procedure for δ tends to 0 to establish the desired existence result. They are stated in following in a sequence of lemmas; however, most of them are ease consequence of the previous estimates (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 3 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\phi_t^\delta\|_{L^2(Q)} \le \|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega^\delta))\cap L^2(0,T;H^2(\Omega^\delta))} + \|\phi_t^\delta\|_{L^2(Q^\delta)} \le C_1, \tag{94}$$

$$\|v^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \tag{95}$$
$$\le \|v^\delta\|_{L^\infty(0,T;H(\Omega^\delta))\cap L^2(0,T;V(\Omega^\delta))} + \int_0^T \int_{\Omega^\delta} k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \le C_1,$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1, \tag{96}$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \le C_1. \tag{97}$$

Proof: It follows from the inequality (85). ∎

Lemma 4 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} \le C_1, \tag{98}$$

$$\|\theta_t^\delta\|_{L^{4/3}(0,T;H_o^1(\Omega)')} \le C_1, \tag{99}$$

$$\|\phi^\delta\|_{W_q^{2,1}(Q)} \le C_1, \quad \textit{for any } 2 \le q \le 2(N+2)/N. \tag{100}$$

Proof: Using that $0 \le c^\delta \le 1$ in Q, we infer from (22) that,

$$\|c_t^\delta\|_{H_o^1(\Omega)'} \le C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla \phi^\delta\|_{L^2(\Omega)} \right).$$

Then, (98) follows from Lemma 3.

Now, we take the scalar product of (21) with $\eta \in H^1_o(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_{\rm v}\|\theta^\delta_t\|_{H^1_o(\Omega)'} \leq C_1 \left(\|\nabla\theta^\delta\|^2_{L^2(\Omega)} + \|\theta^\delta\|_{L^4(\Omega)}\|v^\delta\|_{L^4(\Omega)} + \|\phi^\delta_t\|_{L^2(\Omega)}\right).$$

Then, (99) follows from Lemma 3.

Now, from a result of Hoffman and Jiang ([14] Thm 2.1), we conclude that ϕ^δ satisfies the following inequality, for any $2 \leq q < \infty$,

$$\|\phi^\delta\|_{W^{2,1}_q(Q^\delta)} \leq C_1 \left(\|\theta^\delta\|_{L^q(Q^\delta)} + \|c^\delta\|_{L^q(Q^\delta)} + \|\phi^\delta_0\|_{W^{2,\infty}(\Omega^\delta)} + C_1\right). \quad (101)$$

Then, (100) holds due to $\|c^\delta\|_{L^\infty(Q^\delta)}$ and by interpolation $\|\theta^\delta\|_{L^{2(N+2)/N}(\Omega^\delta)}$ are bounded independent of δ. ∎

Lemma 5 *There exist a constant C_1 and $\delta_0 \in (0, \delta(\Omega))$ such that, for any $\delta < \delta_0$,*

$$\|v^\delta_t\|_{L^{4/3}(t_1,t_2;V(U)')} \leq C_1 \quad (102)$$

where $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ and such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (20) that for a.e. $t \in (t_1, t_2)$,

$$(v^\delta_t, u) = -\nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx - \int_U k(f^\delta_s(\phi^\delta) - \delta) v^\delta u dx + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx \quad \text{for u } \in V(U).$$

In order to estimate $\|v^\delta_t\|_{V(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W^{2,1}_q(Q)$, for $2 \leq q \leq 2(N+2)/N$, in particular, for $q > (N+2)/2$ we have that $W^{2,1}_q(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - (N+2)/q$ ([15] p. 80). Due to theorem of Arzela-Ascoli, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q : 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega : 0 \leq f_s(\phi(x,t)) < 1\}$. Note that there is $\overline{\gamma} \in (0,1)$ such that for any $(x,t) \in [t_1, t_2] \times \bar{U}$, we have

$$f_s(\phi(x,t)) < 1 - \overline{\gamma}.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0, \delta_0)$ and for all $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \overline{\gamma}/2.$$

By assumption **(H1)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \overline{\gamma}/2) \quad \text{for } (x,t) \in [t_1, t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned} \|v_t^\delta\|_{V(U)'} \quad \leq \quad & C_1\Big(\|v^\delta\|_V + \|v^\delta\|^2_{L^4(\Omega)} + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^2(\Omega)} \\ & + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)}\|v^\delta\|_{L^2(\Omega)}\Big). \end{aligned}$$

Hence, (102) follows from Lemma 3. ∎

¿From (95), we conclude that the sequence (v^δ) is also uniformly bounded in $L^2(t_1, t_2; H^1(U))$. Then, by the compact embedding ([23] Cor. 4) , there exist v and a subsequence of (v^δ) (which we still denote by v^δ), such that

$$v^\delta \to v \quad \text{strongly in } L^2((t_1, t_2) \times U).$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i, t_{i+1}) \times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{strongly in } L^2_{loc}(Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)). \tag{103}$$

Moreover, from (95) and the fact that $v^\delta \in L^2(0,T; V(\Omega^\delta))$ we have that $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{aligned} v^\delta \quad & \rightharpoonup \quad v \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \\ v^\delta \quad & \overset{*}{\rightharpoonup} \quad v \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{104}$$

Now, from Lemma 3 and Lemma 4, by using compact embedding ([23] Cor.4), we infer that there exist

$$\begin{aligned} \phi \quad & \in \quad W_q^{2,1}(Q) \text{ for } 2 \leq q \leq 2(N+2)/N, \\ \theta \quad & \in \quad L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \\ c \quad & \in \quad L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \end{aligned}$$

and a subsequence of $(\phi^\delta, \theta^\delta, c^\delta)$ (which we still denote by $(\phi^\delta, \theta^\delta, c^\delta)$) such that, as $\delta \to 0$,

$$\begin{array}{llll} \phi^\delta & \to & \phi & \text{uniformly in } Q, \\ \phi^\delta & \to & \phi & \text{strongly in } L^q(0,T;W^{2-\overline{\gamma},q}(\Omega)),\ 0<\overline{\gamma}<1/2, \\ \phi_t^\delta & \rightharpoonup & \phi_t & \text{weakly in } L^q(Q), \\ \theta^\delta & \to & \theta & \text{strongly in } L^2(Q)\cap C([0,T];H_o^1(\Omega)'), \\ \theta^\delta & \rightharpoonup & \theta & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ c^\delta & \to & c & \text{strongly in } L^2(Q)\cap C([0,T];H_o^1(\Omega)'), \\ c^\delta & \rightharpoonup & c & \text{weakly in } L^2(0,T;H^1(\Omega)). \end{array} \tag{105}$$

It now remains to pass to the limit as δ decreases to zero in (19)-(24).

It follows from (105) that we may pass to the limit in (19), and find that (12) holds almost everywhere.

Now, we take $u=\eta(t)$ in (20) where $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ with compact support contained in $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ and $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$; after integration over $(0,t)$, we find

$$\begin{aligned} \int_0^t \Big((v_t^\delta,\eta)+(\nabla v^\delta,\nabla\eta)+(v^\delta\cdot\nabla v^\delta,\eta)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,\eta)\Big)\,ds \\ =\int_0^t(\mathcal{F}(c^\delta,\theta^\delta),\eta)ds. \end{aligned} \tag{106}$$

Since supp $\eta\subseteq Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ we have that supp $\eta(t)\subseteq\Omega_{ml}(t)$ a.e. $t\in[0,T]$. Moreover, we observe that

$$\int_0^t(v_t^\delta,\eta)ds=-\int_0^t(v^\delta,\eta_t)_{\Omega_{ml}(s)}ds+(v^\delta(t),\eta(t))_{\Omega_{ml}(t)}-(v_0^\delta,\eta(0))_{\Omega_{ml}(0)}.$$

Because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H1)**, it follows that $k(f_s^\delta(\phi^\delta)-\delta)$ converges to $k(f_s(\phi))$ uniformly on compact subsets of $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$. These facts, together with (103)-(105), ensure that we may pass to the limit in (106) and get (14).

To check that $v=0$ a.e. in $\mathring{Q}_s$, take a compact set $K\subseteq\mathring{Q}_s$. Then there is an $\delta_K\in(0,1)$ such that

$$f_s^\delta(\phi^\delta(x,t))=1 \quad \text{in } K \text{ for } \delta<\delta_K,$$

hence, $k(f_s^\delta(\phi^\delta(x,t)-\delta)=k(1-\delta)$ in K for $\delta<\delta_K$. From (95) we infer that

$$k(1-\delta)\|v^\delta\|_{L^2(K)}^2\le C_1 \quad \text{for } \delta<\delta_K$$

where C_1 is independent of δ. As δ tends to 0, by assumption **(H1)**, $k(1-\delta)$ blows up and consequently $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K. Since K is an arbitrary subset, we conclude that $v = 0$ a.e. in $\overset{o}{Q}_s$.

In order to pass to the limit in (21), we notice that given $\xi \in L^4(0,T;H^1(\Omega))$ with $\xi_t \in L^2(0,T;L^2(\Omega))$ satisfying $\xi(T) = 0$, we can consider an extension of ξ such that $\xi^\delta \in L^4(0,T;H^1(\Omega^\delta))$ with $\xi^\delta_t \in L^2(0,T;L^2(\Omega^\delta))$ satisfying $\xi^\delta(T) = 0$. Now, we take the scalar product of (21) with ξ^δ,

$$-C_{\rm v}\int_{\Omega^\delta}\theta_0^\delta\xi^\delta(0)dx - C_{\rm v}\int_0^T\int_{\Omega^\delta}\theta^\delta\xi_t^\delta dxdt - C_{\rm v}\int_0^T\int_{\Omega^\delta}\rho_\delta(v^\delta)\theta^\delta\cdot\nabla\xi^\delta dxdt + \int_0^T\int_{\Omega^\delta}K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\cdot\nabla\xi^\delta dxdt = \frac{l}{2}\int_0^T\int_{\Omega^\delta}f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta\xi^\delta dxdt. \tag{107}$$

Observe that, since K_1 is a bounded Lipschitz continuous function, $K_1(\rho_\delta(\phi^\delta))$ converges to $K_1(\phi)$ in $L^p(Q)$ for $p \in [1,\infty)$. We notice that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0,T;H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0,T];H^1_o(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$. Observe also that $f_s^{\delta\prime} \to f_s'$ in $L^q(I\!R)$ for $2 \le q < \infty$, then from (105) we infer that $f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 3 the integrals over $\Omega^\delta\backslash\Omega$ are bounded independent of δ and since $|\Omega^\delta\backslash\Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (107) and obtain

$$-C_{\rm v}\int_\Omega\theta_0\xi(0)dx - C_{\rm v}\int_0^T\int_\Omega\theta\xi_t dxdt - C_{\rm v}\int_0^T\int_\Omega v\,\theta\cdot\nabla\xi\,dxdt + \int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\xi\,dxdt = \frac{l}{2}\int_0^T\int_\Omega f_s'(\phi)\phi_t\xi\,dxdt$$

for all $\xi \in L^4(0,T;H^1(\Omega))$ with $\xi \in L^2(0,T;L^2(\Omega))$ and $\xi(T) = 0$.

It remains to pass to the limit in (22). For that purpose, we proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta^\delta_t \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$-\int_0^T\int_{\Omega^\delta}c^\delta\zeta_t^\delta dxdt - \int_0^T\int_{\Omega^\delta}\rho_\delta(v^\delta)c^\delta\cdot\nabla\zeta^\delta dxdt + K_2\int_0^T\int_{\Omega^\delta}\nabla c^\delta\cdot\nabla\zeta^\delta dxdt + K_2M\int_0^T\int_{\Omega^\delta}c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\cdot\nabla\zeta^\delta\,dxdt = \int_{\Omega^\delta}c_0^\delta\zeta^\delta(0)dx.$$

Then, from (104),(105), and using the fact that sequence (c^δ) is bounded in

$L^\infty(Q)$, we may pass to the limit as $\delta \to 0$ to obtain

$$-\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega vc \cdot \nabla\zeta \, dxdt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla\zeta \, dxdt$$
$$+K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\zeta \, dxdt = \int_\Omega c_0 \zeta(0) dx,$$

which holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ satisfying $\zeta(T) = 0$. Observe that since $0 \leq c^\delta \leq 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \leq c \leq 1$ a.e. in Q.

Finally, it follows from (105) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$. Furthermore, $v(0) = v_0$ in $\Omega_{ml}(0)$ because $v^\delta(0) \to v(0)$ in $V'(U)$ for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$. The proof of Theorem 1 is then complete. ∎

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