Fuzzy Quasilinear Spaces

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Abstract

We introduce the concept of fuzzy quasilinear space and fuzzy quasilinear operator. Moreover we state some properties and give results which extend to the fuzzy context some results of linear functional analysis.

Keywords

Fuzzy sets, fuzzy-valued mappings, analysis.

1 Introduction

In [1] Assev has presented an abstract approach to the study of spaces of subsets and function spaces of multivalued mappings, by weakening the requirement of linearity in the constructions of linear functional analysis. Assev introduced the concepts of **quasilinear spaces** and **quasilinear operators** which enable us to consider both linear space and nonlinear spaces of subsets and multivalued mappings from a single point of view. He stated properties and theorems which are "quasilinear" counterparts of some results in linear functional analysis and differential calculus in Banach spaces. His work has motivated us to extend the notion of quasilinear spaces to the fuzzy context.

The purpose of this paper is to present the concept of fuzzy quasilinear spaces and to introduce a theory of fuzzy quasilinear operators. Motivated

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by the ideas introduced by Assev we give some properties about fuzzy quasilinear spaces and we state some propositions and theorems concerning a fuzzy “quasilinear” operator theory. For instance, we state a fuzzy version of the Banach-Steinhauss theorem and we introduce a duality theory for fuzzy quasilinear operators. We also present some results related to fuzzy differential inclusions. Others applications of this new theory in the direction of a fuzzy differential calculus have been develop in [4]. We believe that such results will play a important role in a construction of a consistence fuzzy analysis.

2 Preliminaries

A set X is called a **quasilinear space** if a partial order relation $\leqslant$, an algebraic sum operation $+$, and an operation of multiplication by real numbers $\cdot$, are defined in it and the following properties hold for any elements $x, y, z, v \in X$ and any real numbers $\alpha, \beta \in \mathbb{R}$:

$$
\begin{aligned}
x &\leqslant x, && (1)\\
x \leqslant y\ ,\ y \leqslant z &\Rightarrow x \leqslant z, && (2)\\
x \leqslant y\ ,\ y \leqslant x &\Rightarrow x = y, && (3)\\
x + y &= y + x, && (4)\\
x + (y + z) &= (x + y) + z, && (5)
\end{aligned}
$$

there exist an element $\theta \in X$, called neutral element, such that

$$
\begin{aligned}
x + \theta &= x, && (6)\\
\alpha\cdot(\beta\cdot x) &= (\alpha\cdot\beta)\cdot x, && (7)\\
\alpha\cdot(x + y) &= \alpha\cdot x + \alpha\cdot y, && (8)\\
1\cdot x &= x, && (9)\\
0\cdot x &= \theta, && (10)\\
(\alpha + \beta)\cdot x &\leqslant \alpha\cdot x + \beta\cdot x, && (11)\\
x \leqslant y \;\; e \;\; z \leqslant v &\Rightarrow x + z \leqslant y + v, && (12)\\
x \leqslant y &\Rightarrow \alpha\cdot x \leqslant \alpha\cdot y. && (13)
\end{aligned}
$$

An element $x' \in X$ is called **an inverse** of $x \in X$, if $x + x' = \theta$. If an element x' exist, then it is unique. If any element x in the quasilinear space

X has an inverse element $x^{'} \in X$ then the partial order on X is determined by equality and consequently X is a linear space with scalars in $\mathbb{R}$.

Let X be a quasilinear space. A real function $\|\cdot\|_X : X \to \mathbb{R}$ is called a **norm** if the following conditions hold:

$$\begin{aligned} if \;\; x \neq \theta \;\; &\Rightarrow \;\; \|x\|_X > 0, &(14)\\ \|x+y\|_X \;\; &\leqq \;\; \|x\|_X + \|y\|_X, &(15)\\ \|\alpha \cdot x\|_X \;\; &= \;\; |\alpha| \; \|x\|_X, &(16)\\ if \;\; x \leqslant y \;\; &\Rightarrow \;\; \|x\|_X \leqq \|y\|_X, &(17) \end{aligned}$$

if for any $\epsilon > 0$, exists an element $x_\epsilon \in X$ such that

$$x \leqslant y + x_\epsilon \;\; and \;\; \|x_\epsilon\| \leqq \epsilon \Rightarrow x \leqslant y.$$

Lemma 1 *A sum operation $+$, the multiplication by real numbers and the norm $\|.\|_X$ are continuous with respect to the Hausdorff metric H_X.*

A quasilinear space X with a norm defined on it is called a **normed quasilinear space**. If any $x \in X$ has an inverse element $x^{'} \in X$, then the concept of a normed quasilinear space coincides with the concept of a real normed linear space.

Let X be a normed quasilinear space. The Hausdorff metric on X is defined by

$$H_X(x, y) = \inf\{r \geq 0 / \; x \leqslant y + a_1^r, \;\; y \leqslant x + a_2^r, \;\; \|a_i^r\|_X \leq r\}.$$

Since that $x \leqslant y + (x - y)$ and $y \leqslant x + (y - x)$, the quantity $H_X(x, y)$ is defined for any elements $x, y \in X$, and $H_X(x, y) \leq \|x - y\|_X$. It is easy to see that the function $H_X(x, y)$ satisfies all the axioms of a metric.

Let X be a real normed linear space. Denote by $K(X)$ the space of nonempty closed and bounded subsets of X,i.e.,

$$K(X) = \{A \subset X / \; A \text{ is nonempty closed and bounded}\}$$

and denote by $K_C(X)$ its subspace of convex subsets of X:

$$K_C(X) = \{A \in K(X) / \; A \text{ is convex}\}.$$

The algebraic sum operation in $K(X)$ and multiplication by a real number $\alpha \in \mathbb{R}$ are defined by the expressions

$$A + B = \{a + b / \; a \in A, \; b \in B\}, \quad \alpha \cdot A = \{\alpha \cdot a / \; a \in A\}.$$

The space $K(X)$, with the partial order given by the inclusion, satisfies the conditions (1)-(13). A norm in $K(X)$ is given by $\|A\|_K = \sup_{a\in A}\|a\|$. Consequently $K(X)$ and $K_C(X)$ are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:

$$H(A, B) = \inf\{r \geq 0/\ A \subset B + rS_1(0),\ B \subset A + rS_1(0)\},$$

where $S_1(0)$ is the closed ball of radius r about $0 \in X$.

Definition 1 *Let X and Y be two quasilinear spaces. A application $\Gamma : X \to Y$ is called* ***quasilinear operator*** *if it satisfies the following conditions:*

$$\Gamma(\lambda x) = \lambda\Gamma(x) \ \ \forall\lambda \in \mathbb{R} \tag{18}$$

$$\Gamma(x + y) \leqslant \Gamma(x) + \Gamma(y) \ \ \forall x, y \in X \tag{19}$$

$$if \ \ x \leqslant y \ \Rightarrow \ \Gamma(x) \leqslant \Gamma(y) \tag{20}$$

A quasilinear operator $\Gamma : X \to Y$ is called ***bounded*** *if exists $k > 0$ such that*

$$\|\Gamma(x)\|_Y \leq k\|x\|_X \ \forall x \in X.$$

We denote by $L(X, Y)$ the space of all bounded quasilinear operator from X to Y. We write $\Gamma_1 \leqslant \Gamma_2$ if $\Gamma_1(x) \leqslant \Gamma_2(x) \ \ \forall x \in X$. Multiplication by real numbers is defined on $L(X, Y)$ by the equality $(\lambda\Gamma)(x) = \lambda\Gamma(x)$. Moreover, the algebraic sum on $L(X, Y)$ is defined by the equality $(\Gamma_1 + \Gamma_2)(x) = \Gamma_1(x) + \Gamma_2(x)$. Then $L(X, Y)$ is a quasilinear space.

A norm on $L(X, Y)$ is defined by

$$\|\Gamma\|_L = \sup_{\|x\|_X=1} \|\Gamma(x)\|_Y.$$

Consequently, $L(X, Y)$ is a normed quasilinear space.

3 Fuzzy Quasilinear Space

Let X be a Banach space. A fuzzy set on X is a function $u : X \to [0, 1]$. For $0 < \alpha \leq 1$, we denote by $L_\alpha u = \{x \in X/u(x) \geq \alpha\}$ the α-level of u, and $L_0 u = supp(u) = \overline{\{x \in X/u(x) > 0\}}$ is called the support of u.

A fuzzy set $u : X \to [0, 1]$ is called fuzzy compact set (fuzzy compact convex set, respectively) if $L_\alpha u$ is compact for all $\alpha \in [0, 1]$ (if $L_\alpha u$ is compact convex for all $\alpha \in [0, 1]$, respectively).

We denote by $\mathcal{F}(X)$ ($\mathcal{F}_C(X)$, respectively) the space of all fuzzy compact sets $u : X \to [0,1]$ (the space of all fuzzy compact convex sets $u : X \to [0,1]$, respectively).

Proposition 2 *If $u \in \mathcal{F}(X)$, then the family $\{L_\alpha u /\ \alpha \in [0,1]\}$ satisfies the following properties:*

(a) $L_0 u \supseteq L_\alpha u \supseteq L_\beta u \quad \forall\ 0 \le \alpha \le \beta.$

(b) *Se* $\alpha_n \uparrow \alpha \Rightarrow L_\alpha u = \bigcap_{n=1}^{\infty} L_{\alpha_n} u$

(i.e., the level-application is left-continuous),

(c) $u = v \Leftrightarrow L_\alpha u = L_\alpha v \quad \forall \alpha \in [0,1].$

(d) $L_\alpha u \neq \emptyset \quad \forall \alpha \in [0,1]$, *is equivalent to* $u(x) = 1$ *for some* $x \in X$.

If u *satisfies this condition we say that* u *is normal.*

(e) *We can to define a partial order* $\subseteq$ *on* $\mathcal{F}(X)$ *by setting*

$$u \subseteq v \Leftrightarrow u(x) \le v(x) \ \ \forall x \in X \Leftrightarrow L_\alpha u \subseteq L_\alpha v \quad \forall \alpha \in [0,1].$$

The algebraic sum operation and multiplication by a real number $\lambda \in \mathbb{R}$ on $\mathcal{F}(X)$ is defined by the expression

$$(u+v)(x) = \sup_{y \in Y} \min\{u(y), v(x-y)\} \quad \textit{and} \quad (\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases}$$

With these definitions we obtain $L_\alpha(u+v) = L_\alpha u + L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$, for all $u, v \in \mathcal{F}(X)$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$.

The space $\mathcal{F}(X)$ with the sum, multiplication by real numbers and the partial order, above defined, is a quasilinear space with neutral element $\chi_{\{0\}}$.

In $\mathcal{F}(X)$ we can define the norms

$$\|u\|_1 = \sup_{0 \le \alpha \le 1} \|L_\alpha u\|_K.$$

and

$$\|u\|_2 = \int_0^1 \|L_\alpha u\|_K d\alpha.$$

In the first case the Hausdorff metric is defined by

$$D_1(u,v) = \inf\{r \geq 0 / \ u \subseteq v + w_1^r, \ v \subseteq u + w_2^r, \ \ \|w_i^r\|_1 \leq r\},$$

or equivalently

$$D_1(u,v) = \inf\{r \geq 0 / \ u \subseteq v + r\omega, \ v \subseteq u + r\omega\},$$

where $\omega \in \mathcal{F}(X)$ is the fuzzy-compact set defined by

$$\omega(x) = \begin{cases} 1 & \text{if } x \in S_1(0) \\ 0 & \text{if } x \notin S_1(0) \end{cases}$$

Other form equivalent of write D_1, using the α-level, is given by

$$D_1(u,v) = \sup_{\alpha \in [0,1]} H(L_\alpha u, L_\alpha v).$$

In the second case, the Hausdorff metric is defined by

$$D_2(u,v) = \inf\{r \geq 0 / \ u \subseteq v + w_1^r, \ v \subseteq u + w_2^r, \ \ \|w_i^r\|_2 \leq r\}.$$

By using the α-level, D_2 is equivalent to

$$D_2(u,v) = \int_0^1 H(L_\alpha u, L_\alpha v) d\alpha.$$

The space $\mathcal{F}(X)$ extends $K(X)$ in the sense of, for each $A \in K(X)$, its characteristic functions χ_A belongs to $\mathcal{F}(X)$. Clearly if $A, B \in K(X)$, then

$$D_1(\chi_A, \chi_B) = D_2(\chi_A, \chi_B) = H(A,B).$$

It is well known that $(\mathcal{F}(X), D_1)$ is a complete metric space, but is not separable and $(\mathcal{F}(X), D_2)$ is a complete separable metric space [see [5],[6]].

Hereafter, the space $\mathcal{F}(X)$, with normed $\|.\|_{\mathcal{F}}$, will be called **fuzzy normed quasilinear space**. The Hausdorff metric deriving from $\|.\|_{\mathcal{F}}$ will be denote by $D(.,.)$.

Lemma 3 (a) *Suppose that $u_n \to u_0$ and $v_n \to v_0$, and that $u_n \subseteq v_n$ for any positive integer n. Then, $u_0 \subseteq v_0$.*

(b) *Suppose that $u_n \to u_0$ and $z_n \to u_0$. If $u_n \subseteq v_n \subseteq z_n$ for all n, then $v_n \to u_0$.*

(c) *Suppose that* $u_n + v_n \to u_0$ *and* $v_n \to \chi_{\{0\}}$*, then* $u_n \to u_0$.

Proof. (a) If $u_n \to u_0$ and $v_n \to v_0$, then for any $\epsilon > 0$ there exists N such that for any $n \geq N$ there exist elements $a_n^\epsilon, b_n^\epsilon \in \mathcal{F}(X)$ for which

$$u_0 \subseteq u_n + a_n^\epsilon \ , \ \|a_n^\epsilon\|_{\mathcal{F}} \leq \epsilon$$

and

$$v_n \subseteq v_0 + b_n^\epsilon \ , \ \|b_n^\epsilon\|_{\mathcal{F}} \leq \epsilon.$$

Thus,

$$u_0 \subseteq v_0 + a_n^\epsilon + b_n^\epsilon$$

for $n \geq N$. Since $\|a_n^\epsilon + b_n^\epsilon\|_{\mathcal{F}} \leq \|a_n^\epsilon\|_{\mathcal{F}} + \|b_n^\epsilon\|_{\mathcal{F}} \leq 2\epsilon$, follows from (18) that $u_0 \subseteq v_0$. The proofs of (b) and (c) are analogous. ■

Lemma 4 *Let* X*,* Y *be two Banach space. If* $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ *is a quasilinear operator, then* $\Psi(\chi_{\{0\}}) = \chi_{\{0\}}$.

Proof. The proof is easy. ■

Lemma 5 *A quasilinear operator* $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ *is bounded if and only if, is continuous at* $\chi_{\{0\}} \in \mathcal{F}(X)$.

Proof. Suppose that the operator Ψ is bounded. Then exist $k > 0$ such that

$$\|\Psi(u)\|_{\mathcal{F}} \leq k\|u\|_{\mathcal{F}} \ \forall u \in \mathcal{F}(X).$$

So, given $\epsilon > 0$ exists $\delta = \frac{\epsilon}{k}$ such that, if

$$D(u, \chi_{\{0\}}) = \|u\|_{\mathcal{F}} < \delta,$$

then

$$D(\Psi(u), \Psi(\chi_{\{0\}})) = \|\Psi(u)\|_{\mathcal{F}} \leq k\|u\|_{\mathcal{F}} < k\delta = \epsilon.$$

Thus, Ψ is continuous at $\chi_{\{0\}}$. Now suppose that Ψ is a quasilinear operator continuous in $\chi_{\{0\}} \in \mathcal{F}(X)$. Then, for any $\epsilon > 0$, exists $\delta > 0$ such that $D(u, \chi_{\{0\}}) = \|u\|_{\mathcal{F}} < \delta$, implies

$$D(\Psi(u), \Psi(\chi_{\{0\}})) < \epsilon \ \text{ or } \ \|\Psi(u)\|_{\mathcal{F}} < \epsilon.$$

So, for any $u \in \mathcal{F}(X)$

$$\|\Psi(\frac{\delta u}{2\|u\|_{\mathcal{F}}})\|_{\mathcal{F}} < \epsilon \ \ or \ \ \|\Psi(u)\|_{\mathcal{F}} < \frac{2\epsilon}{\delta}\|u\|_{\mathcal{F}}.$$

■

Lemma 6 *Let $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ be a quasilinear operator. If Ψ is continuous at $\chi_{\{0\}} \in \mathcal{F}(X)$, then Ψ is uniformly continuous on $\mathcal{F}(X)$.*

Proof. Suppose that Ψ is continuous at $\chi_{\{0\}}$. Then, for any $\epsilon > 0$, exists $\delta > 0$ such that

$$\|\Psi(u)\|_{\mathcal{F}} < \epsilon \text{ if } \|u\|_{\mathcal{F}} < \delta.$$

Let $u_0 \in \mathcal{F}(X)$ be given. If $D(u, u_0) < \delta$, then there exist w_1, w_2 such that

$$\|w_i\|_{\mathcal{F}} \leq \delta \text{ and } u \subseteq u_0 + w_1 \ , \ \ u_0 \subseteq u + w_2.$$

Since Ψ is a quasilinear operator, it follows that

$$\Psi(u) \subseteq \Psi(u_0) + \Psi(w_1) \text{ and } \Psi(u_0) \subseteq \Psi(u) + \Psi(w_2).$$

Since $\|w_i\|_{\mathcal{F}} < \delta$, then

$$\|\Psi(w_i)\|_{\mathcal{F}} < \epsilon.$$

Consequently, $D(\Psi(u), \Psi(u_0)) < \epsilon$. This completes the proof. ∎

Example 1 *If $f : X \to Y$ is a function, its Zadeh extension [see [5]], $\bar{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$, is defined by*

$$\bar{f}(u)(x) = \begin{cases} \sup_{y \in f^{-1}(x)} u(y) & \text{if } f^{-1}(x) \neq \phi, \\ 0 & \text{if } f^{-1}(x) = \phi \end{cases}$$

If f is continuous, Then $\bar{f}$ is a well defined function [see [5]] and

$$L_\alpha \bar{f}(u) = f(L_\alpha u), \ \ \forall \alpha \in [0,1] \ \forall u \in \mathcal{F}(X).$$

Now, suppose that f is linear. Then, $\bar{f}$ is a quasilinear operator, and from the continuity of f follows that $\bar{f}$ is bounded. Consequently, $\bar{f}$ is uniformly continuous.

The space $L(\mathcal{F}(X), \mathcal{F}(Y))$, of all the bounded quasilinear operator from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, is a normed quasilinear space, with norm defined by

$$\|\Psi\|_L = \sup_{\|u\|_{\mathcal{F}}=1} \|\Psi(u)\|_{\mathcal{F}}.$$

Lemma 7 *Suppose that the sequence* $\{\Psi_n\} \in L(\mathcal{F}(X), \mathcal{F}(Y))$ *converges at each point* $u \in \mathcal{F}(X)$*. Then the operator*

$$\Psi(u) = \lim_{n \to +\infty} \Psi_n(u)$$

is quasilinear.

Proof. It follows from limit properties. ■

We define the function $\varphi_\omega : [0, +\infty) \to \mathcal{F}(X)$ by $\varphi_\omega(t) = t\omega$. It satisfies the following conditions:

$$u \subseteq \varphi_\omega(\|u\|_{\mathcal{F}}) \tag{21}$$

$$if \;\; t \leq s \Rightarrow \varphi_\omega(t) \subseteq \varphi_\omega(s) \tag{22}$$

$$\varphi_\omega(t+s) = \varphi_\omega(t) + \varphi_\omega(s). \tag{23}$$

Lemma 8 *The operator* $\varphi : \mathcal{F}(X) \to \mathcal{F}(Y)$*, defined by*

$$\varphi(u) = \varphi_\omega(\|u\|_{\mathcal{F}})$$

belongs to $L(\mathcal{F}(X), \mathcal{F}(Y))$*.*

Proof. Let $u, v \in \mathcal{F}(X)$ be and suppose that $u \subseteq v$, then $\|u\|_{\mathcal{F}} \leq \|v\|_{\mathcal{F}}$. Consequently, $\|u\|_{\mathcal{F}}\,\omega \subseteq \|v\|_{\mathcal{F}}\,\omega$, so $\varphi_\omega(\|u\|_{\mathcal{F}}) \subseteq \varphi_\omega(\|v\|_{\mathcal{F}})$. Moreover,

$$\varphi_\omega(\|u+v\|_{\mathcal{F}}) \leq \varphi_\omega(\|u\|_{\mathcal{F}}) + \varphi_\omega(\|v\|_{\mathcal{F}}).$$

Now, $\varphi_\omega(\|\lambda u\|_{\mathcal{F}}) = |\lambda| \varphi_\omega(\|u\|_{\mathcal{F}})$. Since, $\omega = -\omega = (-1)\omega$, it follows that

$$\varphi_\omega(\|\lambda u\|_{\mathcal{F}}) = \lambda \varphi_\omega(\|u\|_{\mathcal{F}}).$$

Consequently, $\varphi(u)$ is a quasilinear operator from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Since,

$$\|\varphi(u)\|_{\mathcal{F}} = \|\varphi_\omega(\|u\|_{\mathcal{F}})\|_{\mathcal{F}} = \| \, \|u\|_{\mathcal{F}}\,\omega\|_{\mathcal{F}} = \|u\|_{\mathcal{F}} \|\omega\|_{\mathcal{F}},$$

we conclude that $\varphi \in L(\mathcal{F}(X), \mathcal{F}(Y))$. ■

Remark 1 (a) $\|\varphi\|_L = 1$.

(b) *If* $\|\Psi\|_L \leq \|\varphi\|_L$*, then* $\Psi \leqslant \varphi$*.*

(c) $\Psi \leqslant \|\Psi\|_L \varphi$*.*

The Hausdorff metric on $L(\mathcal{F}(X), \mathcal{F}(Y))$ is given by

$$H_L(\Psi_1, \Psi_2) = \inf\{r > 0/\ \Psi_1 \leqslant \Psi_2 + \Psi_1^r\ ,\ \Psi_2 \leqslant \Psi_1 + \Psi_2^r \quad \|\Psi_i^r\|_L \leqq r\},$$

or equivalently,

$$H_L(\Psi_1, \Psi_2) = \inf\{r > 0 : \Psi_1 \leqslant \Psi_2 + r\varphi\ ,\ \Psi_2 \leqslant \Psi_1 + r\varphi\}.$$

Theorem 9 *Suppose that $(\mathcal{F}(X), D)$ is a complete metric space. Then the normed quasilinear space $(L(\mathcal{F}(X), \mathcal{F}(Y)), H_L)$ is a complete metric space.*

Proof. Let $\{\Psi_n\}$ be a sequence of Cauchy on $L(\mathcal{F}(X), \mathcal{F}(Y))$. Then, for any $\epsilon > 0$, exists a N such that for all $n, m \geqq N$ there exist $\Psi_\epsilon^{n,m}, \Psi_\epsilon^{m,n}$ satisfying the conditions:

$$\Psi_n \leqslant \Psi_m + \Psi_\epsilon^{n,m},\quad \Psi_m \leqslant \Psi_n + \Psi_\epsilon^{m,n},\quad \|\Psi_\epsilon^{n,m}\|_L \leq \epsilon.$$

Consequently, $D(\Psi_n(u), \Psi_m(u)) \leq \epsilon\|u\|_{\mathcal{F}}$ for any $u \in \mathcal{F}(X)$. So, the sequence $\{\Psi_n(u)\}$ is Cauchy on $\mathcal{F}(Y)$. Since $\mathcal{F}(Y)$ is complete, there exists an element $\Psi(u) \in \mathcal{F}(Y)$ such that

$$\Psi(u) = \lim_{n\to\infty} \Psi_n(u).$$

Follows from Lema 7 that Ψ is a quasilinear operator from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Furthermore,

$$\|\Psi_n(u)\|_{\mathcal{F}} \leq \|\Psi_m(u)\|_{\mathcal{F}} + \|\Psi_\epsilon^{n,m}(u)\|_{\mathcal{F}} \leq (\|\Psi_m\|_L + \epsilon)\|u\|_{\mathcal{F}},$$

for any $n, m \geq N$. Fixing $m \geq N$ and taking the limit as $n \to \infty$ we get that

$$\|\Psi(u)\|_{\mathcal{F}} \leq (\|\Psi_m\|_L + \epsilon)\|u\|_{\mathcal{F}}.$$

Therefore, $\Psi \in L(\mathcal{F}(X), \mathcal{F}(Y))$. We now prove that $\{\Psi_n\}$ converges to Ψ in the quasilinear space $L(\mathcal{F}(X), \mathcal{F}(Y))$. Since

$$\begin{aligned} D(\Psi_n(u), \Psi(u)) &\leq D(\Psi_n(u), \Psi_m(u)) + D(\Psi_m(u), \Psi(u)) \\ &\leq \epsilon\|u\|_{\mathcal{F}} + D(\Psi_m(u), \Psi(u)) \end{aligned}$$

for $n, m \geqq N$, taking the limit as $m \to \infty$ we obtain that

$$D(\Psi_n(u), \Psi(u)) \leq \epsilon\|u\|_{\mathcal{F}} \ \ \forall n \geq N.$$

So, it follows that

$$\Psi_n \leqslant \Psi + \epsilon\varphi, \;\; \Psi \leqslant \Psi_n + \epsilon\varphi \;\; \forall n \geqq N.$$

■

The next result is an analogous to the Banach-Steinhaus theorem.

Theorem 10 *Suppose that $(\mathcal{F}(X), D)$ is a complete space metric. Let $\{\Psi_i\}$ be a family of elements in $L(\mathcal{F}(X), \mathcal{F}(Y))$ such that*

$$\sup_{i\in I} \|\Psi_i(u)\|_{\mathcal{F}} < \infty \;\; \forall u \in \mathcal{F}(X).$$

Then,

$$\sup_{i\in I} \|\Psi_i\|_L < \infty.$$

Equivalently, exists $c > 0$ such that

$$\|\Psi_i(u)\|_{\mathcal{F}} \leq c\|u\|_{\mathcal{F}} \;\; \forall u \in \mathcal{F}(X), \; \forall i \in I.$$

Proof. For each $n \geq 1$, we defined the sets

$$Z_n = \{u \in \mathcal{F}(X)/\, \forall n, \;\; \|\Psi_i(u)\|_{\mathcal{F}} \leq n\}.$$

Since Ψ is uniformly continuous, then Z_n is closed in $\mathcal{F}(X)$ for each $n \in \mathbb{N}$, and $\bigcup_{n=1}^{\infty} Z_n = X$. Consequently, using the Lema of Baire, $intZ_{n_0} \neq \emptyset$ for some $n_0 \geq 1$. So, there exist u_0 and $r > 0$ such that $S_r(u_0) \subset Z_{n_0}$. Then,

$$\|\Psi_i(v)\|_{\mathcal{F}} \leq n_0 \;\; \forall v \in S_r(u_0).$$

Now, If $\|u\|_{\mathcal{F}} \leq r$, then

$$u_0 \subseteq (u_0 + u) - u \;, \;\; (u_0 + u) \subseteq u_0 + u.$$

Consequently, $u_0 + u \in S_r(u_0)$. Since $u \subseteq (u_0 + u) - u_0$, it follows that

$$\Psi_i(u) \subseteq \Psi_i(u_0 + u) - \Psi_i(u_0).$$

Thus

$$\|\Psi_i(u)\|_{\mathcal{F}} \leq \|\Psi_i(u_0 + u)\|_{\mathcal{F}} + \|\Psi_i(u_0)\|_{\mathcal{F}} \leq 2n_0.$$

Now, for any $u \in \mathcal{F}(X)$,

$$\|\frac{ru}{\|u\|_{\mathcal{F}}}\|_{\mathcal{F}} = r$$

Therefore,

$$\|\Psi_i(u)\|_{\mathcal{F}} \leq \frac{2n_0}{r}\|u\|_{\mathcal{F}},$$

and the proof is complete. ■

Lemma 11 *Let $\{\Gamma_i\}_{i\in I}$, $\Gamma_i : \mathcal{F}(X) \to K(Y)$ be a family of bounded quasilinear operators such that the family $\{\Gamma_i(u)\}_{i\in I}$ is bounded in Y for each $u \in \mathcal{F}(X)$. Then, the application $R : \mathcal{F}(X) \to K(Y)$ defined by*

$$R(u) = \overline{\cup_{i\in I}\Gamma_i(u)},$$

is a bounded quasilinear operator.

Proof. Suppose that $u \subseteq v$, $u, v \in \mathcal{F}(X)$. Then,

$$R(u) = \overline{\cup_{i\in I}\Gamma_i(u)} \subseteq \overline{\cup_{i\in I}\Gamma_i(v)} = R(v).$$

Further, given $u, v \in \mathcal{F}(X)$,

$$\begin{aligned} R(u+v) &= \overline{\cup_{i\in I}\Gamma_i(u+v)} \\ &\subseteq \overline{\cup_{i\in I}(\Gamma_i(u)+\Gamma_i(v))} \\ &\subseteq \overline{\cup_{i\in I}\Gamma_i(u)} + \overline{\cup_{i\in I}\Gamma_i(v)} \\ &= R(u) + R(v). \end{aligned}$$

Moreover,

$$R(\lambda u) = \overline{\cup_{i\in I}\Gamma_i(\lambda u)} = \overline{\cup_{i\in I}\lambda\Gamma_i(u)} = \lambda\overline{\cup_{i\in I}\Gamma_i(u)} = \lambda R(u),$$

for any $u \in \mathcal{F}(X)$ and any $\lambda \in \mathbb{R}$. Consequently, $R : \mathcal{F}(X) \to K(Y)$ is a quasilinear operator. Now, we shall prove that R is bounded. We define $\Psi_i : \mathcal{F}(X) \to \mathcal{F}(Y)$ by $\Psi_i(u) = \chi_{\{\Gamma_i(u)\}}$. Since Γ_i is bounded quasilinear operator for each $i \in I$, it follows that Ψ_i is a quasilinear operator for each $i \in I$. Now,

$$\sup_{i\in I}\|\Psi_i(u)\|_{\mathcal{F}} = \sup_{i\in I}\|\chi_{\Gamma_i(u)}\|_{\mathcal{F}} = \sup_{i\in I}\|\Gamma_i(u)\|_K < \infty.$$

Then, by the Theorem 10, exists $c > 0$ such that

$$\|\Gamma_i(u)\|_K = \|\Psi_i(u)\|_{\mathcal{F}} \leq c\|u\|_{\mathcal{F}} \ \ \forall u \in \mathcal{F}(X),\ \forall i \in I.$$

So,

$$\|R(u)\|_K = \|\overline{\cup_{i\in I}\Gamma_i(u)}\|_K \leq c\|u\|_{\mathcal{F}} \ \ \forall u \in \mathcal{F}(X).$$

Consequently, R is bounded. ■

Lemma 12 *(a) An element $\Psi \in L(\mathcal{F}(X), \mathcal{F}(Y))$ satisfies the condition of Lipschitz with constant $\|\Psi\|_L$.*

(b) If $\Psi_1 \in L(\mathcal{F}(X), \mathcal{F}(Y))$ and $\Psi_2 \in L(\mathcal{F}(Y), \mathcal{F}(Z))$. Then, the operator $\Psi = \Psi_2 \circ \Psi_1$ is in the space $L(\mathcal{F}(X), \mathcal{F}(Z))$.

Proof. ■

Denote by $\mathcal{F}(X)^{\otimes}$ the space $L(\mathcal{F}(X), K(\mathbb{R}))$, and by $co\mathcal{F}(X)^{\otimes}$ the space $L(\mathcal{F}(X), K_C(\mathbb{R}))$. A quasilinear operator Γ from $\mathcal{F}(X)$ to $K(\mathbb{R})$ will be called **quasilinear functional**.

Let $\Psi \in L(\mathcal{F}(X), \mathcal{F}(Y))$. Then, for each $\Gamma \in \mathcal{F}(Y)^{\otimes}$ we can associate an element $\Phi \in \mathcal{F}(X)^{\otimes}$ in according to the rule $\Phi(u) = (\Gamma o \Psi)(u)$. Consequently, the operator $\Psi^{\otimes} : \mathcal{F}(Y)^{\otimes} \to \mathcal{F}(X)^{\otimes}$ given by $\Psi^{\otimes}(\Gamma) = \Gamma o \Psi$ is well defined.

Proposition 13 *Let $\Psi \in L(\mathcal{F}(X), \mathcal{F}(Y))$ be any. Then,*

(a) $\Psi^{\otimes} \in L(\mathcal{F}(Y)^{\otimes}, \mathcal{F}(X)^{\otimes})$.

(b) $\|\Psi^{\otimes}\|_L = \|\Psi\|_L$.

Proof. *(a)* Let $\Gamma_1, \Gamma_2 \in \mathcal{F}(Y)^{\otimes}$ and let $\alpha \in \mathbb{R}$ be given. Then,

$$\begin{aligned} \Psi^{\otimes}(\Gamma_1 + \Gamma_2)(u) &= (\Gamma_1 + \Gamma_2)(\Psi(u)) \\ &= \Gamma_1(\Psi(u)) + \Gamma_2(\Psi(u)) \\ &= (\Psi^{\otimes}(\Gamma_1) + \Psi^{\otimes}(\Gamma_2))(u). \end{aligned}$$

Consequently,

$$\Psi^{\otimes}(\Gamma_1 + \Gamma_2) \leq \Psi^{\otimes}(\Gamma_1) + \Psi^{\otimes}(\Gamma_2).$$

Moreover,

$$\Psi^{\otimes}(\alpha\Gamma)(u) = \alpha\Gamma(\Psi(u)) = \alpha\Psi^{\otimes}(\Gamma)(u).$$

Then, $\Psi^{\otimes}(\alpha\Gamma) = \alpha\Psi^{\otimes}(\Gamma)$. Now, suppose that $\Gamma_1 \leqslant \Gamma_2$. Then,

$$\Psi^{\otimes}(\Gamma_1)(u) = \Gamma_1(\Psi(u)) \leqslant \Gamma_2(\Psi(u)) = \Psi^{\otimes}(\Gamma_2)(u).$$

So, $\Psi^{\otimes}(\Gamma_1) \leqslant \Psi^{\otimes}(\Gamma_2)$. Consequently, the operator $\Psi^{\otimes} : \mathcal{F}(X)^{\otimes} \to \mathcal{F}(Y)^{\otimes}$ is quasilinear. Since

$$\|\Psi^{\otimes}(\Gamma)\|_L = \sup_{\|u\|_{\mathcal{F}}=1} \|\Gamma(\Psi(u))\|_{\mathcal{F}} \leq \|\Gamma\|_L \|\Psi\|_L,$$

the quasilinear operator $\Psi^{\otimes}$ is bounded, and $\|\Psi^{\otimes}\|_L \leq \|\Psi\|_L$.

(b) We shall prove that $\|\Psi^{\otimes}\|_L \geq \|\Psi\|_L$. The element $\Gamma_0(u) = [-\|u\|_{\mathcal{F}}, \|u\|_{\mathcal{F}}]$ is in the space $\mathcal{F}(Y)^{\otimes}$, and $\|\Gamma_0\|_L = 1$. Then,

$$\begin{aligned} \|\Psi^{\otimes}\|_L &= \sup_{\|\Gamma\|_L=1} \|\Psi^{\otimes}(\Gamma)\|_L \\ &\geq \|\Psi^{\otimes}(\Gamma_0)\|_L \\ &= \sup_{\|u\|_{\mathcal{F}}=1} \|\Gamma_0(\Psi(u))\|_{\mathcal{F}} \\ &= \sup_{\|u\|_{\mathcal{F}}=1} \|\Psi(u)\|_{\mathcal{F}} \\ &= \|\Psi\|_L. \end{aligned}$$

Therefore, $\|\Psi^{\otimes}\|_L \geqslant \|\Psi\|_L$. ■

4 Fuzzy quasilinear operator

The space $\mathcal{L}(X,Y)$ consisting of all bounded linear operators from X to Y is also a normed linear space. Consequently, the normed quasilinear space $K(\mathcal{L}(X,Y))$ is well defined (See Example 1). Each element $T \in K(\mathcal{L}(X,Y))$ defines a bounded set-valued quasilinear mapping $\Gamma(x) = Tx = \overline{\{Ax : \ A \in T\}}$ from X to $K(Y)$.

Definition 2 *We say that an element $\Gamma \in L(X, K(Y))$ has a* ***linear representation*** *if exist an element $T \in K(\mathcal{L}(X,Y))$ such that*

$$\Gamma(x) = Tx = \{Ax \ ; \ A \in T\}.$$

The next result is important and necessary to define the adjoint of a quasilinear operator (see [1]).

Theorem 14 *Suppose that $\Gamma \in L(X^*, K_C(\mathbb{R}))$. Then there exists a unique closed bounded convex subset $F \subset X$ such that for any $x^* \in X^*$*

$$\Gamma(x^*) = \langle F, x^* \rangle = \overline{\{\langle f, x^* \rangle \ ; \ f \in F\}}.$$

Corollary 15 *If X is a complete reflexive normed linear space, then any bounded quasilinear operator $\Gamma : X \to K_C(\mathbb{R})$ has a linear representation.*

Let X, Y be two normed linear spaces, and let $\Gamma : X \to K(Y)$ be a bounded quasilinear operator. Then there exists a unique bounded quasilinear operator $\Gamma^{\otimes} : Y^* \to K_C(X^*)$ such that $\langle \Gamma(x), y^* \rangle = \langle x, \Gamma^{\otimes}(y^*) \rangle$, for all $x \in X$ and $y^* \in Y^*$[see [1]]. The operator $\Gamma^{\otimes} \in L(Y^*, K(X^*))$ is called **the adjoint** of the operator $\Gamma \in L(X, K(Y))$.

An application $\Gamma : X \to \mathcal{F}(Y)$ is called **fuzzy quasilinear operator** if Γ satisfies the conditions (19), (20), i.e.,

$$\Gamma(\lambda x) = \lambda \Gamma(x) \quad \forall x \in X\ , \forall \lambda \in \mathbb{R}, \quad \Gamma(x_1 + x_2) \subseteq \Gamma(x_1) + \Gamma(x_2) \quad \forall x_1, x_2 \in X.$$

The condition (21) is automatically satisfied.

Let $\Gamma : X \to \mathcal{F}(Y)$ be a bounded fuzzy quasilinear operator. Then, for any $\alpha \in [0,1]$, the application of level $\Gamma_\alpha : X \to K(Y)$, defined by

$$\Gamma_\alpha(x) = L_\alpha \Gamma(x),$$

is a bounded quasilinear operator.

Definition 3 *A fuzzy valued mapping $\Gamma : X \to \mathcal{F}(Y)$ has a linear representation if, for each $\alpha \in [0,1]$, the application of level Γ_α has a linear representation.*

Proposition 16 *Let X be a complete reflexive normed linear space. Then each bounded fuzzy quasilinear operator $\Gamma : X \to \mathcal{F}_C(\mathbb{R})$ has a linear representation.*

Proof. Follows from corollary 15. ■

Given a bounded fuzzy quasilinear operator $\Gamma : X \to \mathcal{F}(Y)$, for each $\alpha \in [0,1]$ there exists a operator $\Gamma^{\otimes}_\alpha : Y^* \to K(X^*)$, the adjoint of Γ_α, such that $\langle \Gamma_\alpha(x), y^* \rangle = \langle x, \Gamma^{\otimes}_\alpha(y^*) \rangle$ for any $x \in X$ and $y^* \in Y^*$.

Consider the family $\{\Gamma^{\otimes}_\alpha(y^*)\}_{\alpha \in [0,1]}$. Such family satisfies the conditions of the theorem of representation due to Negoita and Ralescu [see [7]]. In fact,

i) By definition, it follows that $\Gamma^{\otimes}_\alpha(y^*) \in K(X^*)$ for all $\alpha \in [0,1]$.

ii) Let $\alpha \leq \beta$ be, we shall prove that $\Gamma^{\otimes}_\beta(y^*) \subseteq \Gamma^{\otimes}_\alpha(y^*)$, or equivalently $\langle x, \Gamma^{\otimes}_\beta(y^*) \rangle \subseteq \langle x, \Gamma^{\otimes}_\alpha(y^*) \rangle$ for all $x \in X$. Let $x \in X$ be given. Then

$$\langle x, \Gamma^{\otimes}_\beta(y^*) \rangle = \langle \Gamma_\beta(x), y^* \rangle \subseteq \langle \Gamma_\alpha(x), y^* \rangle = \langle x, \Gamma^{\otimes}_\alpha(y^*) \rangle.$$

iii) Consider the sequence $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq ...$, such that $\lim_{i \to \infty} \alpha_i = \alpha$. Then,

$$\cap_{i \geq 1} \Gamma_{\alpha_i}(x) = \Gamma_\alpha(x).$$

Consequently,

$$\begin{aligned}
\langle x, \cap_{i\geq 1}\Gamma^{\otimes}_{\alpha_i}(y^*)\rangle &= \langle x, (\cap_{i\geq 1}\Gamma_{\alpha_i})^{\otimes}(y^*)\rangle \\
&= \langle \cap_{i\geq 1}\Gamma_{\alpha_i}(x), y^*\rangle \\
&= \langle \Gamma_{\alpha}(x), y^*\rangle \\
&= \langle x, \Gamma^{\otimes}_{\alpha}(y^*)\rangle .
\end{aligned}$$

Follows from theorem of representation that there exists a unique fuzzy set $\Gamma^{\otimes}(y^*) : X^* \to [0,1]$ such that

$$L_\alpha \Gamma^{\otimes}(y^*) = \Gamma^{\otimes}_{\alpha}(y^*).$$

This result generalizes the concept of adjoint of a operator $\Gamma \in L(X, \mathcal{F}(Y))$.

Definition 4 *Let $\Gamma : X \to \mathcal{F}(Y)$ be a bounded fuzzy quasilinear operator. The adjoint of Γ is the operator $\Gamma^{\otimes} \in L(Y^*, \mathcal{F}(X^*)$ such that*

$$L_\alpha \Gamma^{\otimes}(y^*) = \Gamma^{\otimes}_{\alpha}(y^*).$$

for all $\alpha \in [0,1]$ and $y^* \in Y^*$.

5 Fuzzy Differential Inclusions

The following definition of fuzzy differential inclusion was introduced by Zhu and Rao [see [8]], where they obtained some results concerning existence of solution.

Definition 5 *Let $\Gamma : X \to \mathcal{F}(X)$ be a fuzzy valued mappings. Let $\alpha : X \to [0,1]$ be a function and let J be an interval in $\mathbb{R}$. We call **fuzzy differential inclusion** the following problem: to find $x \in C(J, X)$ such that*

$$\dot{x}(t) \in L_{\alpha(x(t))}\Gamma(x(t)). \tag{24}$$

If $\Gamma : X \to \mathcal{F}(X)$ is a fuzzy quasilinear operator, then the problem (24) is called a **fuzzy quasilinear differential inclusion**.

Suppose that the bounded fuzzy quasilinear operator $\Gamma : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ has linear representation, i.e., for each $\alpha \in [0,1]$, there exists a compact set T^α in the space of $n \times n$ matrices such that $\Gamma_\alpha(x) = T^\alpha x = \{Ax\ ;\ A \in T^\alpha\}$. Thus there exists a bounded fuzzy quasilinear operator $\Gamma^{\otimes} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$, the adjoint of Γ, and $\Gamma_\alpha^\otimes$ has linear representation $T^{\alpha\otimes} = \{A\,;\; A^* \in T^\alpha\}$ for each $\alpha \in [0,1]$.

Let $\Gamma : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy quasilinear operator, and let $\Gamma^\otimes : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be the adjoint of Γ. The fuzzy differential inclusion

$$\dot{x}(t) \in -L_{\alpha(x(t))}\Gamma^\otimes(x(t))$$

is called the adjoint of the fuzzy differential inclusions (24).

Proposition 17 *Suppose that the fuzzy quasilinear operator $\Gamma : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ has a linear representation. Then Γ is continuous.*

Proof. The proof is easy. ■

We denote by $\mathcal{F}^C(X)$ [see [7]] the subspace of $\mathcal{F}(X)$ for which the elements u are such that the mapping $\alpha \to L_\alpha u$ is H-continuous on $[0,1]$, i.e., given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\alpha - \beta| < \delta$ implies $H(L_\alpha u, L_\beta u) < \epsilon$. Since $[0,1]$ is a compact metric space, the application $\alpha \to L_\alpha u$ is, in fact, uniformly continuous.

Proposition 18 *Let $\Gamma : \mathbb{R}^n \to \mathcal{F}^C(\mathbb{R}^n)$ be a fuzzy quasilinear operator with a linear representation. Suppose that $\alpha : X \to [0,1]$ is continuous. Then the mapping $\overline{\Gamma} : X \to \mathcal{F}^C(X)$, defined by $\overline{\Gamma}(x) = L_{\alpha(x)}\Gamma(x)$, is continuous.*

Proof. Suppose that $x_n \to x$ in $\mathbb{R}^n$. Then

$$\begin{aligned} H(\overline{\Gamma}(x_n), \overline{\Gamma}(x)) &= H(L_{\alpha(x_n)}\Gamma(x_n), L_{\alpha(x)}\Gamma(x)) \\ &\leq H(L_{\alpha(x_n)}\Gamma(x_n), L_{\alpha(x_n)}\Gamma(x)) + H(L_{\alpha(x_n)}\Gamma(x), L_{\alpha(x)}\Gamma(x)) \\ &\leq D(\Gamma(x_n), \Gamma(x)) + H(L_{\alpha(x_n)}\Gamma(x), L_{\alpha(x)}\Gamma(x)) \to 0. \end{aligned}$$

■

Theorem 19 *Suppose that the fuzzy quasilinear operator $\Gamma : \mathbb{R}^n \to \mathcal{F}^C(\mathbb{R}^n)$ has a linear representation and $\alpha : \mathbb{R}^n \to [0,1]$ is continuous. Moreover, let $x(t)$ be a solution of the fuzzy differential inclusions (24). Then, there exists a measurable matrix-valued function $A : J \to \bigcup_{t\in J} T^{\alpha(x(t))}$ such that $x(t)$ is an absolutely continuous solution of the ordinary linear differential equation $\dot{x} = A(t)x$. Here J is the interval on which $x(t)$ is defined.*

Proof. We define the set valued mapping $\bar{\Gamma} : \mathbb{R}^n \to K(\mathbb{R}^n)$ by

$$\bar{\Gamma}(x) = L_{\alpha(x)} F(x).$$

Denote by $G = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ y \in \bar{F}(x)\}$ the graph of the set valued mapping $\bar{F}$. Then G is a closed set (Proposition 25). We define the set valued mapping $P : G \to K(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$ by

$$P(x, y) = \{A \in \cup_{t \in J} T^{\alpha(x(t))} \ / \ Ax = y\}.$$

We shall prove that P has closed graph. Suppose that $(x_n, y_n) \to (x, y)$ in G, $A_n \to A$ in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ with $A_n x_n = y_n$. Taking the limit as $n \to +\infty$ we have that $Ax = y$. Furthermore, $P(x, y)$ is bounded. Consequently P is upper semicontinuous. Let H be the set valued mapping given by $H(t) = P(x(t), \dot{x}(t))$. H is measurable, since for any open set $U \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, from the upper semicontinuous of P follows that $\{(x, y) \ : \ P(x, y) \subset U\}$ is an open set. Consequently, $\{t \in J \ ; \ H(t) \subset U\}$ is measurable. Therefore, it has a measurable single-valued branch $A : J \to \cup_{t \in J} T_{\alpha(x(t))}$, $A(t) \in H(t)$, such that $\dot{x}(t) = A(t)x(t)$ for a.e. $t \in J$. ■

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