On the differentiability of fuzzy-valued mappings and the stability of a fuzzy differential inclusion

Yurilev Chalco-Cano[1,2], Marko A. Rojas-Medar[3,4]

IMECC-UNICAMP, CP 6065, 13081-970, Campinas-SP, Brazil

and

Adilson J. V. Brandão[5]

DEMAT-ICEB-UFOP, Ouro Preto-MG, Brazil

Abstract

We introduce a new concept of differentiability for fuzzy-valued mapping and we study some of its properties. Using this concept, we give a result on stability of the Lyapunov type for fuzzy differential inclusions.

Keywords

Fuzzy sets, differentiability, fuzzy-valued mappings, stability of fuzzy differential inclusions.

1 Introduction

The concept of differentiability for fuzzy valued mappings has been considered by many authors from different points of view. For instance, the concept of $H-$differentiability due to Puri and Ralescu[14] has been studied and applied by several mathematicians in the context of fuzzy differential equations, including Ding and Kandel [5], Kaleva[9, 10] and Seikkala[18]. Goetschel and Voxman [7] have introduced the notion of a derivative for fuzzy mappings of

[1]Ph. student, supported by FAPESP-Brazil through project 00/00055-0

[2]e-mail: katary@ime.unicamp.br

[3]This work was partially support by CNPq-Brazil through project 300116/93(RN)

[4]e-mail marko@ime.unicamp.br

[5]adilson@iceb.ufop.br

one variable. Basically, they viewed fuzzy numbers in a topological vector space and then they defined differentiation of fuzzy mappings of one variable in ways paralleling the definition of real-valued functions. Syau [19] extends such definition for fuzzy mappings of several variables. Others concepts of differentiability were introduced by Diamond and Kloeden[6] and Román-Flores and Rojas-Medar[17], which extend to the fuzzy context, the concepts of Fréchet differentiability (see De Blasi[4]) and Gâteaux differentiability (see Ibrahim[8]) for set-valued mappings respectively.

As we know, the main idea of the classic differential calculus consists in local approximation of a mapping by a linear operator. In this article we propose a new notion of differentiability for fuzzy mappings, where the role of linear operators is played, in the fuzzy context, by fuzzy quasilinear operators. The theory of fuzzy quasilinear spaces and fuzzy quasilinear operators have been introduced by the authors in [3], inspired in the concept of quasilinear spaces and quasilinear operators given by Assev in [1].

This new concept of differentiability is followed by some properties, examples and rules of calculus. As an application of our results we prove a theorem on stability of a fuzzy differential inclusion. Zhu and Rao [20] have introduced a notion of fuzzy differential inclusion and stated some results on existence of solution. This work has motivated us to develop some ideas concerning stability of fuzzy differential inclusion by using our new notion of differentiability of fuzzy mappings.

The structure of this paper is as follows. In section 2 we give the definitions and previous results that will be used in this article. In section 3 we introduce the concepts of Fréchet and Gâteaux differentiability for fuzzy valued mappings and we give some properties. In section 4 we present some rules of calculus and in the last section we give some applications on stability of fuzzy differential inclusions.

2 Preliminaries

Let Y be a real separable Banach space with norm $\|\cdot\|$ and dual Y^*. Let $K(Y)$ and $K_C(Y)$ be respectively, the class of all nonempty and compact subsets of Y and the class of all nonempty compact and convex subsets of Y.

The Hausdorff metric H on $K(Y)$ is defined by

$$H(A,B) = \inf\{r \geq 0 : \ A \subset B + rS_1(\theta), \ B \subset A + rS_1(\theta)\},$$

where $S_1(\theta)$ is the closed ball of radius 1 about $\theta \in Y$. It is known that $(K(Y), H)$ is a complete and separable metric space and $K_C(Y)$ is a closed subspace of $K(Y)$ [see [2], [6], [16]]. Also, the algebraic sum operation and multiplication by a real number $\lambda \in \mathbb{R}$ on $K(Y)$ is defined by

$$A + B = \{a + b / \quad a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a / a \in A\},$$

for all $A, B \in K(Y)$ and $\lambda \in \mathbb{R}$.

For any $A \in K(Y)$, the support function s_A of A is defined on Y^* as

$$s_A(x^*) = \sup_{a \in A} x^*, \; \forall x^* \in Y^*.$$

Given $A, B \in K(Y)$ we have that

$$s_{A+B}(x^*) = s_A(x^*) + s_B(x^*).$$

A fuzzy set of Y is a function $u : Y \to [0,1]$ and, for $0 < \alpha \leq 1$, we denote by $L_\alpha u = \{y \in Y / u(y) \geq \alpha\}$ the α-level of u, and $L_0 u = supp(u) = \{y \in Y / u(y) > 0\}$ is called the support of u.

A fuzzy set $u : Y \to [0,1]$ is called fuzzy compact set (fuzzy compact convex set, respectively) if $L_\alpha u$ is compact for all $\alpha \in [0,1]$ (if $L_\alpha u$ is compact convex for all $\alpha \in [0,1]$, respectively).

We denote by $\mathcal{F}(Y)$ ($\mathcal{F}_C(Y)$, respectively) the space of all fuzzy compact sets $u : Y \to [0,1]$ (the space of all fuzzy compact convex sets $u : Y \to [0,1]$, respectively).

Remark 1 *If $u \in \mathcal{F}(Y)$, then the family $\{L_\alpha u : \; \alpha \in [0,1]\}$ satisfies the following properties:*

(*a*) $L_0 u \supseteq L_\alpha u \supseteq L_\beta u \quad \forall \; 0 \leq \alpha \leq \beta$.

(*b*) *Se* $\alpha_n \uparrow \alpha \Rightarrow L_\alpha u = \bigcap_{n=1}^{\infty} L_{\alpha_n} u$
(i.e., the level-application is left-continuous).

(*c*) $u = v \Leftrightarrow L_\alpha u = L_\alpha v \quad \forall \alpha \in [0,1]$.

(*d*) $L_\alpha u \neq \emptyset \quad \forall \alpha \in [0,1]$, *is equivalent to* $u(y) = 1$ *for some* $x \in Y$.
If u satisfies this condition we say that u is normal.

(*e*) *We can define a partial order* $\subseteq$ *on* $\mathcal{F}(Y)$ *by setting*

$$u \subseteq v \Leftrightarrow u(y) \leq v(y) \ \ \forall y \in Y \Leftrightarrow L_\alpha u \subseteq L_\alpha v \ \ \forall \alpha \in [0,1].$$

The algebraic sum operation and multiplication by a real number $\lambda \in \mathbb{R}$ on $\mathcal{F}(Y)$ is defined by

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\} \quad and \quad (\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & \text{if } \lambda \neq 0, \\ \chi_{\{\theta\}}(x) & \text{if } \lambda = 0. \end{cases}$$

With these definitions we obtain $L_\alpha(u+v) = L_\alpha u + L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$, for all $u, v \in \mathcal{F}(Y)$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$[see [2], [6], [16]].

The quasinorm on $\mathcal{F}(Y)$ is defined by

$$\|u\|_{\mathcal{F}} = \sup_{0 \leqq \alpha \leqq 1} \|L_\alpha u\| = \|L_0 u\|,$$

where $\|L_\alpha u\| = \sup_{y \in L_\alpha u} \|y\|$. Then, $\mathcal{F}(Y)$ and $\mathcal{F}_C(Y)$, with the operation algebraic and partial order $\subseteq$ defined above, are normed quasilinear spaces [see [1],[3]].

Also, we can define a metric on $\mathcal{F}(Y)$ as follows:

$$D(u,v) = \sup_{\alpha \in [0,1]} H(L_\alpha u, L_\alpha v),$$

where H define the Hausdorff metric [see [1], [2], [3], [6]].

It is known that $(\mathcal{F}(Y), D)$ is a complete but non-separable metric space [see [16]].

Proposition 1 *(see [2], [3], [6], [16]) If* $u, v, w, u_1, v_1 \in \mathcal{F}(Y)$. *Then*

(*a*) $D(\lambda u, \lambda v) = \lambda D(u,v)$, *for all* $\lambda \geq 0$.

(*b*) $D(u+v, u_1+v_1) \leq D(u+u_1, v+v_1)$.

If $u, v \in \mathcal{F}_C(Y)$ *we have*

(*c*) $D(u+w, v+w) = D(u,v)$.

An application $\Gamma : X \to \mathcal{F}(Y)$ is called a fuzzy valued mapping.

Definition 1 *A fuzzy valued mapping* $\Gamma : X \to \mathcal{F}(Y)$ *will be called a* ***quasilinear operator*** *if it satisfies the following conditions:*

$$F(\lambda x) = \lambda F(x) \quad \forall x \in X, \forall \lambda \in \mathbb{R} \tag{1}$$
$$F(x_1 + x_2) \subseteq F(x_1) + F(x_2) \quad \forall x_1, x_2 \in X. \tag{2}$$

A fuzzy valued mapping $\Gamma : X \to \mathcal{F}(Y)$ is said to be **bounded** if there exists a number $k > 0$ such that $\|\Gamma(x)\|_{\mathcal{F}} \leq k\|x\|$ for any $x \in X$.

Theorem 2 *([1], [3])The quasilinear operator* $\Gamma : X \to \mathcal{F}(Y)$ *is bounded if and only if it is continuous at the point* $\theta \in X$*. The continuity of* Γ *at* θ *implies that it is uniformly continuous on* X*.*

Denote by $L(X, \mathcal{F}(Y))$ the space of all bounded quasilinear operators from X to $\mathcal{F}(Y)$. We write $\Gamma_1 \leq \Gamma_2$ if $\Gamma_1(x) \leq \Gamma_2(x)$ for any $x \in X$. Multiplication by real numbers is defined on $L(X, \mathcal{F}(Y))$ by the equality $(\lambda\Gamma)(x) = \lambda\Gamma(x)$. Moreover, it is assumed that the operation of algebraic sum is defined on $L(X, \mathcal{F}(Y))$ by the equality $(\Gamma_1 + \Gamma_2)(x) = \Gamma_1(x) + \Gamma_2(x)$. The space $L(X, \mathcal{F}(Y))$ is closed under these operation of algebraic sum and multiplication by real numbers. Then $L(X, \mathcal{F}(Y))$ is a quasilinear space.

The quasinorm on $L(X, \mathcal{F}(Y))$ is defined by

$$\|\Gamma\|_L = \sup_{\|x\|=1} \|\Gamma(x)\|_{\mathcal{F}}.$$

Then $L(X, \mathcal{F}(Y))$ is a normed quasilinear space.

3 Differentiability of fuzzy mappings

In this section we extend the notion of Fréchet differentiability to the fuzzy-valued context, by using the concept of bounded quasilinear operator.

Definition 2 *A fuzzy valued mapping* $\Gamma : X \to \mathcal{F}(Y)$*is said to be* ***Fréchet differentiable*** *at* $x_0 \in X$ *if exists a bounded quasilinear operator* $\mathcal{D}^F_{x_0}(\Gamma) : X \to \mathcal{F}_C(Y)$ *such that*

$$D(\Gamma(x), \Gamma(x_0) + \mathcal{D}^F_{x_0}(\Gamma)(x - x_0)) = o(\|x - x_0\|).$$

The quasilinear operator $\mathcal{D}^F_{x_0}(\Gamma)$ *is called the Fréchet differential of* Γ *at* x_0*.*

Theorem 3 *The Fréchet differential $\mathcal{D}^F_{x_0}(\Gamma)$ is unique if it exists.*

Proof: Let $\mathcal{D}^F_{x_0}(\Gamma)$ and $\overline{\mathcal{D}^F_{x_0}(\Gamma)}$ be two differentials of Γ at x_0. Then, by Proposition 1, we have that

$$\begin{aligned} & D(\mathcal{D}^F_{x_0}(\Gamma)(x-x_0), \overline{\mathcal{D}^F_{x_0}(\Gamma)}(x-x_0)) \\ = \;& D(\Gamma(x_0)+\mathcal{D}^F_{x_0}(\Gamma)(x-x_0), \Gamma(x_0)+\overline{\mathcal{D}^F_{x_0}(\Gamma)}(x-x_0)) \\ \leq \;& D(\Gamma(x), \Gamma(x_0)+\mathcal{D}^F_{x_0}(\Gamma)(x-x_0)) \\ & +D(\Gamma(x), \Gamma(x_0)+\overline{\mathcal{D}^F_{x_0}(\Gamma)}(x-x_0)) \\ = \;& o(\|x-x_0\|). \end{aligned}$$

Thus, $D(\mathcal{D}^F_{x_0}(\Gamma)(x-x_0), \overline{\mathcal{D}^F_{x_0}(\Gamma)}(x-x_0)) = o(\|x-x_0\|)$ for all $x \in X$. This prove que $\mathcal{D}^F_{x_0}(\Gamma) = \overline{\mathcal{D}^F_{x_0}(\Gamma)}$. ■

Proposition 4 *A fuzzy valued application $\Gamma : X \to \mathcal{F}(Y)$ is constant if and only if, for every $x_0 \in X$, $\mathcal{D}^F_{x_0}(\Gamma)(x) = \chi_{\{\theta\}}$ $\forall x \in X$.*

Proposition 5 *Let $\Gamma : X \to \mathcal{F}(Y)$ be a bounded quasilinear operator. Then Γ is Fréchet differentiable at $\theta \in X$ and $\mathcal{D}^F_\theta(\Gamma) = \Gamma$.*

Theorem 6 *If $\Gamma : X \to \mathcal{F}(Y)$ is differentiable at x_0, then Γ is continuous at x_0.*

Proof: Suppose that $x_i \to x_0$, then

$$\begin{aligned} D(\Gamma(x_i), \Gamma(x_0)) \;\leq\; & D(\Gamma(x_i), \Gamma(x_0)+\mathcal{D}^F_{x_0}(\Gamma)(x_i-x_0)) \\ & +D(\Gamma(x_0), \Gamma(x_0)+\mathcal{D}^F_{x_0}(\Gamma)(x_i-x_0)) \\ = \;& o(\|x_i-x_0\|) + \|\mathcal{D}^F_{x_0}(\Gamma)\|_{\mathcal{F}}\|x_i-x_0\| \to 0 \end{aligned}$$

as $i \to \infty$. The Theorem is proved.■

Let $\Gamma : X \to \mathcal{F}(Y)$ be a fuzzy-valued mapping. The level set-valued mapping $\Gamma_\alpha : X \to \mathcal{F}(Y)$, with $\alpha \in [0,1]$, is defined by

$$\Gamma_\alpha(x) = L_\alpha \Gamma(x).$$

Proposition 7 *If Γ is differentiable at x_0, then the level set-valued mapping Γ_α is differential at x_0 for each $\alpha \in [0,1]$ and*

$$\mathcal{D}^F_{x_0}(\Gamma_\alpha) = L_\alpha \mathcal{D}^F_{x_0}(\Gamma).$$

Proof.

Let $\alpha \in [0,1]$ be arbitrary. Then

$$\begin{aligned} & H(L_\alpha\Gamma(x), L_\alpha\Gamma(x_0) + L_\alpha\mathcal{D}^F_{x_0}(\Gamma)(x - x_0)) \\ = \ & H(L_\alpha\Gamma(x), L_\alpha(\Gamma(x_0) + \mathcal{D}^F_{x_0}(\Gamma)(x - x_0))) \\ \leq \ & D(\Gamma(x), \Gamma(x_0) + \mathcal{D}^F_{x_0}(\Gamma)(x - x_0)) \\ = \ & o(\|x - x_0\|) \end{aligned}$$

Consequently, the proposition is proved. ■

Example 1 *Let $X = [0,1]$ be given. Consider the fuzzy valued mapping $\Gamma : [0,1] \to \mathcal{F}([0,1])$ defined by*

$$\Gamma(t)(x) = \begin{cases} \frac{x}{t} & \text{if } x \leq t \\ 0 & \text{otherwise,} \end{cases}$$

if $t \neq 0$, and $\Gamma(0) = \chi_{[0,1]}$. It is easily seen that $L_\alpha\Gamma(t) = [\alpha t, 1]$ for each $\alpha \in [0,1]$. Now,

$$\begin{aligned} D(\Gamma(t), \Gamma(0) + \chi_{\{0\}}) &= \sup_{\alpha\in[0,1]} H([\alpha t, 1], [0,1] + \{0\}) \\ &= \sup_{\alpha\in[0,1]} H([\alpha t, 1], [0,1]) \\ &= \sup_{\alpha\in[0,1]} |\alpha t| = |t|. \end{aligned}$$

It follows that Γ is Fréchet differentiable at $t = 0$ and $\mathcal{D}^F_0(\Gamma)(x) = \chi_{\{0\}}$. It can be easily checked that for each $\alpha \in [0,1]$ the level set valued mapping Γ_α is not differentiable at $t \neq 0$. Consequently, Γ is differentiable only at $t = 0$.

Definition 3 *A fuzzy valued mapping $\Gamma : X \to \mathcal{F}(Y)$ is **Gâteaux differentiable** at $x_0 \in X$ if exists an bounded quasilinear operator $\mathcal{D}^G_{x_0}(\Gamma) : X \to \mathcal{F}_C(Y)$ such that, for all $z \in X$*

$$D(\Gamma(x_0 + tz), \Gamma(x_0) + t\mathcal{D}^G_{x_0}(\Gamma)(z)) = o(t) \ \text{ as } \ t \to +0.$$

*$\mathcal{D}^G_{x_0}(\Gamma)(z)$ is called the **Gâteaux derivative** of Γ at x_0.*

Theorem 8 *The Gâteaux derivative is unique if it exists.*

Proof: Let $\mathcal{D}_{x_0}^G(\Gamma)$ and $\overline{\mathcal{D}_{x_0}^G(\Gamma)}$ two derivatives of Γ at x_0. Then, from Proposition 1 follows that

$$\begin{aligned} D(\mathcal{D}_{x_0}^G(\Gamma)(tz), \overline{\mathcal{D}_{x_0}^G(\Gamma)}(tz)) &= D(\Gamma(x_0) + \mathcal{D}_{x_0}^G(\Gamma)(tz), \Gamma(x_0) + \overline{\mathcal{D}_{x_0}^G(\Gamma)}(tz)) \\ &\leq D(\Gamma(x_0 + tz), \Gamma(x_0) + \mathcal{D}_{x_0}^G(\Gamma)(tz)) \\ &\quad + D(\Gamma(x_0 + tz), \Gamma(x_0) + \overline{\mathcal{D}_{x_0}^G(\Gamma)}(tz)) \\ &= o(t) \text{ as } t \to +0 \end{aligned}$$

Consequently,

$$D(\mathcal{D}_{x_0}^G(\Gamma)(z), \overline{\mathcal{D}_{x_0}^G(\Gamma)}(z)) = \frac{o(t)}{t} \text{ as } t \to +0.$$

Therefore, $\mathcal{D}_{x_0}^G(\Gamma)(z) = \overline{\mathcal{D}_{x_0}^G(\Gamma)}(z)$ for all $z \in X$. ■

Theorem 9 *Suppose that a fuzzy valued mapping $\Gamma : X \to \mathcal{F}(Y)$ is Fréchet differentiability at $x_0 \in X$. Then, Γ is Gâteaux differentiable and*

$$\mathcal{D}_{x_0}^G(\Gamma) = \mathcal{D}_{x_0}^F(\Gamma).$$

Proof:

$$\begin{aligned} & D(\Gamma(x_0 + tz), \Gamma(x_0) + t\mathcal{D}_{x_0}^F(\Gamma)(z)) \\ = \; & D(\Gamma(x_0 + tz), \Gamma(x_0) + \mathcal{D}_{x_0}^F(\Gamma)(tz)) \\ = \; & o(t\|z\|) = o(t) \text{ as } t \to +\infty. \quad ■ \end{aligned}$$

Definition 4 *A fuzzy-valued mapping $\Gamma : X \to \mathcal{F}(Y)$ is De Blasi differentiable at $x_0 \in X$ if exists an upper semicontinuous, positive homogeneous mapping $\mathbb{D}_{x_0}^F(\Gamma) : X \to \mathcal{F}_C(Y)$ such that*

$$D(\Gamma(x_0 + x), \Gamma(x_0) + \mathbb{D}_{x_0}^F(\Gamma)(x)) = o(\|x\|).$$

The mapping $\mathbb{D}_{x_0}^F(\Gamma)$ is called the De Blasi differential of Γ at x_0.

Definition 5 *A fuzzy-valued mapping* $\Gamma : X \to \mathcal{F}(Y)$ *is Ibrahim-Gâteaux differentiable at* $x_0 \in X$ *if exists an upper semicontinuous, positive homogeneous mapping* $\mathbb{D}^G_{x_0}(\Gamma) : X \to \mathcal{F}_C(Y)$ *such that, for all* $z \in X$

$$D(\Gamma(x_0 + tz), \Gamma(x_0) + t\mathbb{D}^G_{x_0}(\Gamma)(z)) = o(t) \ \ as \ \ t \to +0.$$

$\mathbb{D}^F_{x_0}(\Gamma)$ *is called the Ibrahim-Gâteaux differential of* Γ *at* x_0*.*

It is clear that if $\Gamma : X \to \mathcal{F}(Y)$ is Fréchet differentiable at x_0 (Gâteaux differential) then Γ is De Blasi differentiable at x_0 (Ibrahim-Gâteaux differential, respectively) and $\mathcal{D}^F_{x_0}(\Gamma)(x) = \mathbb{D}^F_{x_0}(\Gamma)(x)$ ($\mathcal{D}^G_{x_0}(\Gamma)(x) = \mathbb{D}^G_{x_0}(\Gamma)(x)$ respectively).

4 Rules of calculus

Theorem 10 *Let* Γ_1 *and* Γ_2 *be two fuzzy valued mapping from* X *to* $\mathcal{F}(Y)$*. If* Γ_1 *and* Γ_2 *are Fréchet differentiable at* $x_0 \in X$*, then the mapping* $\Gamma = \lambda\Gamma_1 + \beta\Gamma_2$ *with* $\lambda, \beta \in \mathbb{R}$*, is Fréchet differentiable at* x_0*, and*

$$\mathcal{D}^F_{x_0}(\lambda\Gamma_1 + \beta\Gamma_2) = \lambda\mathcal{D}^F_{x_0}(\Gamma_1) + \beta\mathcal{D}^F_{x_0}(\Gamma_2).$$

Proof:

$$\begin{aligned} & D(\Gamma(x), \Gamma(x_0) + (\lambda\mathcal{D}^F_{x_0}(\Gamma_1) + \beta\mathcal{D}^F_{x_0}(\Gamma_2))(x - x_0)) \\ \leq \ & D(\lambda\Gamma_1(x), \lambda\Gamma_1(x_0) + \lambda\mathcal{D}^F_{x_0}(\Gamma_1)(x - x_0)) \\ & + D(\lambda\Gamma_2(x), \lambda\Gamma_2(x_0) + \lambda\mathcal{D}^F_{x_0}(\Gamma_2)(x - x_0)) \\ \leq \ & |\lambda| o(\|x - x_0\|) + |\beta| o(\|x - x_0\|) \\ = \ & o(\|x - x_0\|). \end{aligned}$$

This prove the Theorem. ■

Remark 2 *Theorem 12 still holds if we suppose that* Γ_1 *and* Γ_2*are Gâteaux differentiable at* x_0*.*

Theorem 11 *A fuzzy valued mapping* $\Gamma : X \to \mathcal{F}_C(Y)$ *is Gâteaux differentiable at* x_0 *if and only if, the support function* $S_{\Gamma(x)}(\alpha, \psi)$ *is Gâteaux differentiable at* x_0 *and* $\mathcal{D}^G_{x_0}(S_{\Gamma(x)})$ *is a support function. Moreover, in this case*

$$\mathcal{D}^G_{x_0}(S_{\Gamma(x)})(\alpha, \psi) = S_{\mathcal{D}^G_{x_0}(\Gamma)(x)}(\alpha, \psi).$$

Proof: Suppose that Γ is differentiable at x_0 and $z \in X$. Then

$$\begin{aligned}
& \frac{1}{t}\|S_{\Gamma(x_0+t.z)}(\alpha,\psi) - S_{\Gamma(x_0)} - t.S_{\mathcal{D}^G_{x_0}(\Gamma)(z)}(\alpha,\psi)\| \\
= & \frac{1}{t}\|S_{\Gamma(x_0+t.z)}(\alpha,\psi) - S_{\Gamma(x_0)+t.\mathcal{D}^G_{x_0}(\Gamma)(z)}(\alpha,\psi)\| \\
\leq & \frac{1}{t}D(\Gamma(x_0+t.z),\Gamma(x_0)+t.\mathcal{D}^G_{x_0}(\Gamma)(z))\|(\alpha,\psi)\| \\
= & \frac{o(t)}{t} \to 0 \quad \text{as} \quad t \to +0.
\end{aligned}$$

Thus, a support function $S_{\Gamma(x)}(\alpha\psi)$ is Gâteaux differentiable at x_0 and

$$\mathcal{D}^G_{x_0}(S_{\Gamma(x)}(\alpha,\psi)) = S_{\mathcal{D}^G_{x_0}(\Gamma(x))}(\alpha,\psi).$$

Conversely, suppose that $S_{\Gamma(x)}(\alpha\psi)$ is differentiable at x_0 and $\mathcal{D}^G_{x_0}(S_{\Gamma(x)})(\alpha,\psi) = S_\Lambda(\alpha,\psi)$. Then, for any $z \in X$

$$\begin{aligned}
& D(\Gamma(x_0+t.z),\Gamma(x_0)+t.\Lambda) \\
= & \max_{\|(\alpha,\psi)\|=1}\|S_{\Gamma(x_0+t.z)}(\alpha,\psi) - S_{\Gamma(x_0)+t.\Lambda}(\alpha,\psi)\| \\
= & \max_{\|(\alpha,\psi)\|=1}\|S_{\Gamma(x_0+t.z)}(\alpha,\psi) - S_{\Gamma(x_0)}(\alpha,\psi) - t.S_\Lambda(\alpha,\psi)\| \\
= & o(t)
\end{aligned}$$

as $t \to +0$ and the Theorem is proved. ■

5 Stability of Fuzzy Differential Inclusion

Let $\Gamma : X \to \mathcal{F}(X)$ be a fuzzy-valued mapping. Let $\alpha : X \to [0,1]$ be a function and J a interval in $\mathbb{R}$. The problem [see [20]]: find $x \in C(J,X)$ such that

$$x^{'}(t) \in L_{\alpha(x(t))}\Gamma(x(t)) \tag{3}$$

is said a fuzzy differential inclusion.

A quasilinear differential inclusion is defined by a differential inclusion of form

$$x^{'} \in F(x),$$

where $F : X \to \mathcal{K}_C(X)$ is a quasilinear operator.

Consider the fuzzy differential inclusion (3), assuming the condition $\Gamma(\theta) = \chi_{\{\theta\}}$. We say that the equilibrium position $x = \theta$ of (3) is Lyapunov-stable if the following conditions hold;

(a) There is a $\delta_0 > 0$ such that if $\|x(t_0)\| < \delta_0$, then there exists a solution $x(t)$ such that $\|x(t)\| < \delta_0$ for any $t \geq t_0$.

(b) For any $\epsilon > 0$ there exists a $0 < \delta_1 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \epsilon$ for any $t \geq t_0$.

A Lyapunov-stable equilibrium position $x = \theta$ is said to be asymptotically stable is there exists a positive number $\delta_2 \leq \delta_0$ such that if $\|x(t_0)\| < \delta_2$, then $\lim_{t\to\infty} \|x(t)\| = 0$.

The next result was proved in [12]

Theorem 12 *Suppose that the set-valued mapping $F : X \to K_C(X)$ is positive-homogeneous and upper semicontinuous. Assume that any solution $x(t)$ of the differential inclusions $x^{'} \in F(x)$ tends to θ as $t \to \infty$. Let $G : X \to K_C(X)$ be an upper semicontinuous set-valued mapping, with $\|G(x)\| = o(\|x\|)$ as $\|x\| \to 0$. Then there exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of the differential inclusion $x^{'} \in F(x) + G(x)$ with $\|x(0)\| < \delta$ satisfies the inequality*

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$.

Theorem 13 *Suppose that the point θ is an equilibrium position of the fuzzy differential inclusion (3). Moreover, suppose that the fuzzy-valued mapping $\Gamma : X \to \mathcal{F}(X)$ is differentiable at θ and that there exists a number $\delta_0 > 0$ such that any solution $x(t)$ of (3) exists on the whole* *interval* $[0,+\infty)$ *if* $\|x(0)\| \leq \delta_0$. *If for some* $\alpha \in [0,1]$ *the equilibrium position* $x=\theta$ *of the quasilinear differential inclusion*

$$x^{'} \in L_\alpha \mathcal{D}_\theta^F(\Gamma)(x) \tag{4}$$

is asymptotically stable, then this point is an asymptotically stable equilibrium position of the fuzzy differential inclusion (3), that is, there exist $\sigma > 0$, $k > 0$ *and* $\delta > 0$ *such that any solution* $x(t)$ *of (3) satisfies the inequality*

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$ *if* $\|x(0)\| < \delta$.

Proof: Since Γ is differentiable at θ, then the application $\widetilde{\mathcal{D}_\theta^F(\Gamma)} : X \to K_C(X)$ defined by

$$\widetilde{\mathcal{D}_\theta^F(\Gamma)}(x) = L_\alpha \mathcal{D}_\theta^F(\Gamma)(x),$$

exists for all $x \in X$, is homogeneous and uniformly continuous. Also, since the equilibrium position $x=\theta$ of the quasilinear differential inclusion 4 is asymptotically stable, then any solution $x(t)$ of 4 tends to θ as $t \to \infty$.

Now,

$$\begin{aligned}
\|\bar{\Gamma}(x)\| &= H(L_{\alpha(x)}\Gamma(x), \theta) \\
&\leq D(\Gamma(x), \chi_{\{0\}}) \\
&\leq D(\Gamma(x), \mathcal{D}_\theta^F(\Gamma)(x)) + D(\mathcal{D}_\theta^F(\Gamma)(x), \chi_{\{0\}}) \\
&\leq o(\|x\|) + \|\mathcal{D}_\theta^F(\Gamma)\|_{\mathcal{F}}\|x\| \\
&= o(\|x\|) \ \text{ as } \|x\| \to 0.
\end{aligned}$$

Thus, due to Theorem 12, exist $\sigma > 0$, $k > 0$ and $\delta > 0$ such that any solution $x(t)$ of (3) satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\| \exp(-\sigma t)$$

for all $t \geq 0$ if $\|x(0)\| < \delta$ and the theorem is proved. ■

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