Counting Domains in $\{p, q\}$ Tessellations

Eduardo Brandani Silva
Departamento de Matemática
Universidade Estadual de Maringá 87020-900 - Maringá - PR, Brazil
Marcelo Firer
Universidade Estadual de Campinas
Instituto de Matemática, IMECC, 13083-970 - Campinas - SP, Brazil
Reginaldo Palazzo Jr.
Universidade Estadual de Campinas
Faculdade de Engenharia Elétrica, FEECC, 13083-970 - Campinas - SP, Brazil

Abstract

For any given regular $\{p, q\}$ tessellation in the hyperbolic plane, we compute the number of vertices and tiles to be found as we distance from a given point, enabling a complete characterization of the asymptotic behavior.

1 Introduction

In the designs of communications systems the choice of signal constellation to be used play a fundamental role, mainly because the performance of the system is dependent of such signal constellation. In order to became a real system, is necessary to have strong instruments to manipulate the signals, generally a suitable algebraic structure. Forney [1] introduced the possibility of considering uniform geometric codes, build up from lattices Λ in $\mathbb{R}^n$ that become a finite set of signal points, a constellation, just after taking a convenient quotient by a sublattice $\Lambda' \subset \Lambda$. Many other possibilities for constellations and associated codes arise if we realize that the same kind of construction may be done in other ambient metric spaces X ([2]) taking the care to consider a properly discontinuous group of isometries Γ and a normal subgroup $\Gamma' \subset \Gamma$.
One of the most important such possibilities is to consider constellations of points in the hyperbolic plane ([3], [4], [5]). The main potential for coding in the hyperbolic plane is the infinitude of essentially distinct tessellations, in contrast with the Euclidean case. Not only we can find infinite constellations, we also can find infinitely many properly discontinuous groups of isometries, not isomorphic (as abstract subgroups) one to the other. Moreover, rigidity (in the sense of Mostow) does not hold in the (2-dimensional) hyperbolic plane, so, for each co-compact properly discontinuous group of isometries Γ, there are uncountable many subgroups isomorphic to Γ but not conjugated to it. In other words, to every such subgroup there is a situation similar to the, essentially unique, situation found in $\mathbb{R}^n$.
Suppose we have such a signal constellation in the hyperbolic plane and let z be a point in the constellation. If there is random hyperbolic perturbation of this point, a noise, we get a point w that distance, let us say r from the original point. Since any practical approach to signal constellations faces the need of making efficient algorithms, the question of finding out how many maximum-likelyhood regions may be found at a distance r from z becomes a relevant one. This what we do in this work, where we compute this quantity for every regular ($\{p, q\}$) tessellation and essentially to every distance r.

2 Counting Tiles and Vertex

Let $\mathbb{X}$ be either the Euclidean plane $\mathbb{E}^2$, the hyperbolic plane $\mathbb{H}^2$ or the sphere S^2. A *polygon* in $\mathbb{X}$ is a closed set D, with non empty interior, bounded by a finite number of geodesic arcs. Each of such arc is called an *edge* of D and a point in the intersection of two edges a *vertex.* A polygon is said *regular* if all its edges, as well as the angles between the edges at any vertex are congruent. A p, q *regular tessellation* of $\mathbb{X}$ is a family ${D_i}_{i\in\mathbb{N}}$ of isometric regular polygons (called *tiles of the tessellation*) with p edges such that:

i) $\cup_{i\in\mathbb{N}} = \mathbb{X}$.

ii) $\mathrm{int}\,(\mathrm{D_i}) \cap \mathrm{int}\,(\mathrm{D_j}) =$ if $i \neq j$.

iii) If $D_i \cap D_j \neq$ then it is either a common edge or a common vertex.

iv) If z is a vertex of D_{i_1}, then it is a vertex of exactly q polygons $D_{i_1}, \cdots, D_{i_q}$.

It is a well known result that a p, q tessellation occurs in S^2 if $0 < (p-2)(q-2) < 4$ (corresponding to the five Platonic solids), in $\mathbb{R}^2$ if $(p-2)(q-2) = 4$ or in $\mathbb{H}^2$ if $4 < (p-2)(q-2)$. Those possibilities are mutually excluding. These multiplicity of tessellations represent the opportunities offered by the hyperbolic case.
We assume here that $\{D_i\}_{i\in\mathbb{N}}$ is a regular $\{p, q\}$ tessellation of the hyperbolic plane. Let p_0 be the barycentre of the tile D_0, which has p vertices and denote by C_0 the family consisting $\{D_0\}$ containing only this tile. In our first level, we consider a ball B_1, centered at p_0, that contains D_0 but no other tile. We denote by C_1 the collection of all tiles that intersect B_1. This is the family of all tiles which has either an edge or a vertex in common with D_0. We choose now a ball B_2, centered at p_0, containing every tile in C_1 but no other tile. We denote by C_2 the family of all tiles that intersects B_2. In this way, given C_j, we choose a ball B_{j+1} centered at p_0 ant containing all tiles of C_j but no other and define C_{j+1} to be the family of all tiles that intersects B_{j+1}. In this way we get a family $\{C_j\}_{j\in\mathbb{Z}}$ such that $C_j \subsetneq C_{j+1}$ and $\mathbb{H}^2 = \cup_{j\in\mathbb{N}} C_j$. The set of tiles added in the passage from one level to the next one is denoted by $L_{j+1} := C_{j+1} \backslash C_j$.
Our goal is to give an explicit formula for $P_j = \mathrm{card}\,(C_j)$. We will also consider the number V_j of vertices of polygons contained in C_j. Note that V_j is not just a multiple of P_j, since there are vertices joined by different number of polygons. The amount of new polygons and edges added at the level j will be denoted by $NP_j = P_j - P_{j-1}$ and $NV_j = V_j - V_{j-1}$ respectively. Note that all the quantities depend on p and q. Note that

$$V_{j+1} = \sum_{i=0}^{j+1} NV_i \quad P_{j+1} = \sum_{i=0}^{j+1} NP_i$$

Similar counting could be performed if we had started with a ball centered not at the barycentre of a polygon D_0 but at one of its vertices, asking for the quantities $\widetilde{P}_j$ and $\widetilde{V}_j$ defined in an analogous way. We do not need to perform explicit computations in this case since, by a standard duality argument, we find that $\widetilde{P}_j\,(p, q) = V_j\,(q, p)$ and $\widetilde{V}_j\,(p, q) = P_j\,(q, p)$.
We denote by $v_{j,1}, v_{j,2}, ..., v_{j,l}$ the vertices that belongs to L_j but not to L_{j-1}. A vertex $v_{j,i}$ is said to be of *type* $t\,(v_{j,i}) = k$ if it is a vertex of k edges in L_j. Note that $2 \leq t\,(v_{j,i}) \leq q$. Whenever we are counting either vertices $v_{j,i}$ or edges $e^{j,i}$ with the index $0 \leq i \leq m$, we consider the indices modulo $m+1$, that is, the pair $v_{j,i}, v_{j,i+1}$ is the pair $v_{j,k}, v_{j,0}$ when $i = m$. Generally, m will be understood from the context.
The counting process will be done in three different cases, namely, when $p, q \geq 4$, $p = 3$ and $q = 3$. The first case will be done in greater details, since it illustrate the type of argument used in the next two cases. To make it easier for the reader, intermediate steps in the counting process are typed using bold fonts: they are all gathered in the proposition that follows.

2.1 First Case: $p \geq 4$ and $q \geq 4$

We start counting the number of vertices in the level L_j.
In the level C_0 there are p vertices, each one of type 2. When considered as vertices of the next level, C_1, each one of those vertices become a vertex of type q. So when constructing the next level, we must add $q-2$

edges to each of this vertices, summing up $p\,(q-2)$ edges. We may count them in an clockwise order and denote by $e_1^{0,i}, e_2^{0,i}, ..., e_{q-2}^{0,i}$ $(i = 1, ..., p)$, the edges added to the vertex $v_{0,i}$. Each of this edge contribute with one vertex in the L_1 level, so, by now, we counted $\mathbf{p\,(q-2)}$ vertices in L_1. Each pair of consecutive edges $e_j^{0,i}, e_{j+1}^{0,i}$ are edges of the same polygon in L_1 so, this polygon has $p-3$ more vertices that must be counted at the level L_1. Since there are $p\,(q-3)$ pairs of such edges, we counted $\mathbf{p(q-3)(p-3)}$ more vertices in level L_1. To conclude the counting of this level, we note that $e_{q-2}^{0,i}, e_1^{0,i+1}$ and the edge joining $v_{0,i}$ to $v_{0,i+1}$ are three edges of a polygon in L_1 not counted yet. They determine 4 vertices of this polygon, consequently we must add $p-4$ more edges, a total amount of $\mathbf{p(p-4)}$. We sum all this edges and conclude that

$$NV_1 = p\,(q-2) + p\,(p-3)\,(q-3) + p\,(p-4)\,.$$

We note that all these vertices are of type 2, except the $p\,(p-2)$ vertices of the edges $e_j^{0,i}$, that are all of type 3, since we are considering $p \geq 4$.
We can calculate now NV_2. In level L_1, there are $NV_{1,3} = p\,(q-2)$ vertices of type 3 and $NV_{1,2} = p\,[(p-3)\,(q-3) + (p-4)]$ vertices of type 2. As we did before, we add $q-2$ edges $e_1^{1,i}, e_2^{1,i}, ..., e_{q-2}^{1,i}$ to each one of the vertices $v_{1,1}^2, v_{1,2}^2, ..., v_{1,NV_{1,2}}^3$ of type 2 and $q-3$ edges to each of the vertices $v_{1,1}^3, v_{1,2}^3, ..., v_{1,NV_{1,3}}^3$of type 3 and they give rise to $\mathbf{(q-2)\,NV_{1,2} + (q-3)\,NV_{1,2}}$ new vertices, all them of type 3. Let $v_{1,i}^2$ be a vertex of type 2 of L_1 and $e_j^{1,i}, e_{j+1}^{1,i}$ be two consecutive edges added to $v_{1,i}$: each of this pair give rise to a new polygon, with $p-3$ vertices not yet counted. Since there are $q-3$ pairs of such edges at each of the $NV_{1,2}$ vertices of L_1, they sum up $\mathbf{(p-3)\,(q-3)\,NV_{1,2}}$ new vertices. The same happens to the vertices $v_{1,i}^3$ of type 3 of L_1, but now, to each $NV_{1,3}$ of them, we can find $q-4$ pairs of consecutive edges, and they sum $\mathbf{(p-3)\,(q-4)NV_{1,3}}$ new vertices. Finally, the edges $e_{q-2}^{2,i}, e_1^{2,1+1}$ and the edge of L_1 joining $v_{2,i}$ to $v_{2,i+1}$ determine another polygon in L_2 and $p-4$ new vertices, all of type 2and we added $\mathbf{(p-4)NV_{1,2}}$ new vertices of type 2. We can now sum up all of them and get:

$$\begin{aligned} NV_{1,3} &= (q-2)\,NV_{1,2} + (q-3)\,NV_{1,3} \\ NV_{1,2} &= [(p-4) + (p-3)\,(q-3)]\,NV_{1,2} + (p-3)\,(q-4)NV_{1,3}. \end{aligned}$$

The following levels work in the same way and we get the recursive formula:

Proposition 1 *Given a regular $\{p, q\}$ tessellation, with $p, q > 3$, the number of vertices in the level j is given by the recursive formula:*

$$\begin{aligned} NV_{1,2} &= p\,[(p-3)\,(q-3) + (p-4)]\,, \quad NV_{1,3} = p\,(q-2) \\ NV_{j+1,2} &= [(p-4) + (p-3)\,(q-3)]\,NV_{j,2} + (p-3)\,(q-4)NV_{j,3} \\ NV_{j+1,3} &= (q-2)\,NV_{1,2} + (q-3)\,NV_{1,3} \end{aligned}$$

Now we compute the number NP_k of polygons at the level L_k. As before, level C_0 has only type 2 vertices and to each vertex $v_{0,i}^2$ we add $q-2$ edges, $e_1^{0,i}, ..., e_{q-2}^{0,i}$. Each pair of consecutive edges $e_j^{0,i}, e_{j+1}^{0,i}$ $(i = 1, ..., p; j = 1, ..., q-3)$ corresponds to a polygon in L_1, so for the moment we have $\mathbf{p\,(q-3)}$ new polygons. But each of the edges joining $v_{0,i}^2$ and $v_{0,i+1}^2$, together with edges $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$ determines another polygon (a total of $\mathbf{p}$) and we find

$$NP_1 = p + p\,(q-3)\,.$$

The inductive step is again given by the determination of NP_2. As we did before, we add $q-2$ edges to each type-2 vertices of L_1 and $q-3$ to type-3 vertices. If $v_{1,i}^2$ is a type-2 vertex and $e_1^{1,i}, ..., e_{q-2}^{1,i}$ are the new edges attached to it, each pair of the $q-3$ consecutive edges $e_j^{1,i}, e_{j+1}^{1,i}$ corresponds to a new polygon in L_2. The same happens at the type-3 vertices, but now, at each one we added $q-4$ polygons. By the moment, we added $\mathbf{(q-3)NV_{1,2}}$+$\mathbf{(q-4)NV_{1,3}}$ polygons. But the edges $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$ together with the one joining $v_{1,i}$ to $v_{1,i+1}$ determines another polygon in L_1, resuming in $\mathbf{NV_1}$ more polygons. This can be sumarised in the following:

Proposition 2 *Given a regular $\{p,q\}$ tessellation with $p,q>3$, the number of polygons in the level j is given by the recursive formula:*

$$\begin{aligned} NP_0 &= 1, \quad NP_1 = p + p\,(q-3) \\ NP_{k+1} &= (q-3)NV_{k,2} + (q-4)\,NV_{k,3} + NV_k = (q-2)NV_{k,2} + (q-3)\,NV_{k,3} \end{aligned}$$

2.2 Case 2: $p=3$ and $q\geq 6$

In this case our fundamental domains are triangle. We begin with a triangle, with vertices $v_{0,1}, v_{0,2}$ and $v_{0,3}$. To each of these vertices, all of type 2, we add $q-2$ edges, determining $p\,(q-2) = 3\,(q-2)$ new vertices. If $e_j^{0,i}$ and $e_{j+1}^{0,i}$ are consecutive edges added to vertex $v_{0,i}$, they determine a new triangle in L_1, to which we must add another edge, but not a new, yet uncounted vertex. In a similar way, the edges $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$ must determine a triangle in L_1, so they have a common vertex, counted twice in the previous calculation, so that at level L_1 we wave $3\,(q-2) - 3 = 3\,(q-3)$ vertices. The vertices corresponding to the edges $e_j^{0,i}$, for $j = 2,...,q-3$ are all vertices of type 3: each one is joined to a vertex in the previous level, C_0, and to vertices of the previous and next edge, $e_{j-1}^{0,i}$ and $e_{j+1}^{0,i}$ respectively. But, since the vertices corresponding to $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$ coincide, they are type-4 vertices: they are joined to two vertices in the C_0 (determined by the edges $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$), just like the vertices determined by edges $e_{q-3}^{0,i}$ and $e_2^{0,i+1}$. So, the total amount of vertices in L_1 is given by

$$NV_1 = NV_{1,3} + NV_{1,4} = p\,(q-4) + p.$$

To compute the number of vertices in the next level, we add $q-3$ new edges $e_1^{1,i},...,e_{q-3}^{1,i}$ to each of the $NV_{1,3}$ type-3 vertices of L_1. Each pair of consecutive edges, excluding the extreme ones, gives rise to a new triangle to which an edge must be added, so, the $\mathbf{q-5}$ edges $e_2^{1,i},...,e_{q-4}^{1,i}$ added to the $\mathbf{NV_{1,3}}$ vertices of type 3 of L_1 contribute with a type-3 vertex in L_2. The same happens with the $\mathbf{NV_{1,4}}$ type-4 vertices of L_1, but now, excluding the extreme edges $e_1^{1,i}$ and $e_{q-4}^{1,i}$, there are only $q-6$ edges, hence $\mathbf{q-6}$ type-3 new vertices in L_2. As we did before, $e_{q-\varepsilon}^{1,i}$, the last edge added to each of the $\mathbf{NV_1}$ vertex $v_{1,i}$ (ε may be 3 of 4, according to the type of the vertex to $v_{1,i}$), and $e_1^{1,i+1}$, the first edge added to vertex $v_{1,i+1}$ join a common vertex, a type-4 vertex in L_2. The passage from level L_1 to level L_2 gives us the inductive step and we conclude:

Proposition 3 *Given a regular $\{p,q\}$ tessellation with $p=3, q\geq 6$, the number of vertices in the level j is given by the recursive formula:*

$$\begin{aligned} V_0 &= p, \quad NV_{1,3} = p\,(q-4)\,, \quad NV_{1,4} = p = 3 \\ NV_{j+1,3} &= (q-5)\,NV_{j,3} + (q-6)\,N_{j,4} \quad NV_{j+1,4} = NV_{j,3} + NV_{j,4} = NV_j \end{aligned}$$

We compute now the number NP_j of new polygons at level L_j. At the first level, at each of the 3 vertices of C_0, we introduced a $q-2$ edges, each pair of consecutive ones gives rise to a new triangle, a total of $\mathbf{3(q-3)}$ new triangles. Beside those, the edges $e_{q-2}^{0,i}$ and $e_1^{0,i+1}$ determine a new one and we find $NP_1 = 3 + 3\,(q-3)$. To each vertex $v_{1,i}$ of type 3 in L_1 we must add $q-3$ edges $e_1^{1,i},...,e_{q-3}^{1,i}$. Each pair of consecutive of this edges determine a new triangle, a total of $q-4$ for each one of those vertices. The same happens to type-4 vertices, but now we introduce only $q-4$ edges, resulting in $q-5$ pairs of consecutive edges an hence $q-5$ new triangles. By the moment, we have computed $(q-4)\,NV_{1,3} + (q-5)\,NV_{1,4}$ new polygons. It remains to consider the pairs of edges $e_{q-\varepsilon}^{1,i}$ and $e_1^{1,i+1}$ added to vertex $v_{1,i}$ ($\varepsilon = t\,(v_{1,i})$), which determines a new triangle for each $i = 1,...,NP_1$. This is the inductive step and we get the following:

Proposition 4 *Given a regular $\{p,q\}$ tessellation with $p=3, q\leq 6$, the number of polygons in the level j is given by the recursive formula:*

$$\begin{aligned} NP_0 &= 1, \quad NP_1 = p + p\,(q-3) \\ NP_{k+1} &= V_k + (q-4)\,NV_{k,3} + (q-5)\,NV_{k,4} = (q-3)\,NV_{k,3} + (q-5)\,NV_{k,4} \end{aligned}$$

2.3 Case 3: $p \geq 6, q = 3$

Let us consider a $\{p,q\}$ tessellation $\{D_i\}_{i\in\mathbb{N}}$. We start from a p-polygon and, to each one of this vertices we must add one single edge, each one contributing to a type-3 vertex in L_1. If $v_{0,i}$ is a vertex of C_0 with added edge $e^{0,i}$, then $e^{0,i}, e^{0,i+1}$ and the edge joined $v_{0,i}$ to $v_{0,i+1}$ are edges of the same polygon, that are determined by $p-4$ vertices, they are all of type 2. So, $\mathbf{NV_{1,2}= p\,(p-4)}$ and $\mathbf{NV_{1,3}= p}$. The type-2 and type-3 vertices of L_1 are uniformly distributed, in a way, there are $p-4$ consecutive type-2 vertices between each pair of type-3 vertices.
To each type-2 vertex of L_1 we must add one single edge that, as before, will be of type 3 in L_2. If $v_{1,i}$ and v_{i+1} are consecutive type-2 vertices of L_1, the edges $e^{1,i}$ and $e^{1,i+1}$ are edges of one polygon, that has $p-4$ yet uncounted vertices, all of type-2. Since there are $p = \mathbf{NV_{1,3}}$ sequences of $p-4$ consecutive such vertices, we get $p\,(p-5)$ such polygons, contributing with $\mathbf{(p-4)\,(p-5)\,NV_{1,3}}$ type-2 vertices in L_2. There are also p type-3 vertices. If $v_{1,i}$ is such a vertices, it belongs to a same polygon as $v_{1,i-1}$ and $v_{1,i+1}$, just like the edges $e^{1,i-1}$ and $e^{1,i+1}$. We have already counted 5 of the vertices of this polygon, so there are more $(p-5)$ for each of the type-2 vertices of L_1 associated to this vertex, a total amount of $\mathbf{(p-5)\,NV_{1,3}}$. This is the inductive step and we found the following:

Proposition 5 *Given a regular $\{p,q\}$ tessellation with $p \geq 6, q = 3$, the number of vertices in the level j is given by the recursive formula:*

$$\begin{aligned} NV_0 &= p, \quad NV_{1,2} = p\,(p-4)\,, \quad NV_{1,3} = p \\ NV_{j+1,2} &= (p-4)\,(p-5)\,NV_{1,3} + (p-5)\,NV_{1,3}, \quad NV_{j+1,3} = NV_{j,2} \\ NV_{j+1} &= NV_{j+1,2} + NV_{j+1,3} \end{aligned}$$

We count now the polygons added at each level. Obviously, $NP_0 = 1$ and $NP_1 = p$. To compute the next level, that allows us to make the inductive step, we notice that we added one edge for each type-2 vertex of L_1. Two consecutive such edges contribute with a new polygon, so we find that there are $\mathbf{NV_{1,2}}$ new polygons. We generalize to obtain:

Proposition 6 *Given a regular $\{p,q\}$ tessellation with $p \geq 6, q = 3$, the number of polygons in the level j is given by the recursive formula:*

$$NP_0 = 1, \quad NP_1 = p, \quad NP_{k+1} = NV_{k,2}.$$

3 Asymptotic Behavior

It is a well known fact that the fundamental group Γ of compact surface of negative curvature has exponential growth, in the sense that, given any finite set of generators of $\Gamma = \langle \gamma_1, ..., \gamma_n \rangle$, the number B_k of elements of Γ that can be represented as words of length at most k in the generators $\gamma_1, ..., \gamma_n$ is an exponential function on k, that is, there are constants $c, \lambda > 0, b > 1$ such that $\lim_{k\to\infty} \frac{cb^{\lambda k}}{B_k} = 1$.
In all the cases where a $\{p,q\}$ tessellation is associated with a discrete group of isometries of the hyperbolic plane - all $\{4g,4g\}$ and $\{4g+2, 2g+1\}$ tessellations are so ([2]) - we can describe B_k in a similar way we described P_k. We start with one single polygon and, at each step, instead of adding every polygon that has either an edge or an vertex contained in the previous one, we add only the polygons with a face contained in the previous stage. So, we find that for such tessellations, $B_k < P_k < V_k$ and it will not be surprising that P_k and V_k behave exponentially in all the cases. This is indeed what we find. Moreover, we find explicitly the coefficients c, λ and b, depending on the pair $\{p,q\}$.
For each pair (p,q) the functions $V_{k+1}\,(p,q)$ and $P_{k+1}\,(p,q)$ are attained as a linear combination of the previous $NV_{k,i}\,(p,q)$, the number of vertices of type i (either 2, 3 or 4, depending on the case) added in the previous level, with coefficients being polynomials in p and q. Since we are concerned with the asymptotic behavior, we look only at the higher degree at each level k. Looking at the recursive formulas found in

propositions $1-6$, it is easy to see that the higher exponents of p and q in the relevant cases are as given in the table bellow:

	$p \geq 4, q \geq 4$	$p = 3, q \geq 6$	$p \geq 6, q = 3$
Higher coefficient in $V_k(p,q)$	$p^{k+1}q^k$	pq^k	p^{3k}
Higher coefficient in $P_k(p,q)$	$p^k q^k$	pq^k	$p^{3(k-1)}$

.

With this table in mind, it is immediate to verify that the asymptotic behavior of the functions is the same as exponential functions $c(b)^{\lambda k}$ with c, b and λ as follows:

	$p \geq 4, q \geq 4$	$p = 3, q \geq 6$	$p \geq 6, q = 3$
$V_k(p,q) \sim$	$c = p, b = pq, \lambda = 1$	$c = p, b = q, \lambda = 1$	$c = 1, b = p, \lambda = 3$
$P_k(p,q) \sim$	$c = 1, b = pq, \lambda = 1$	$c = p, b = q, \lambda = 1$	$c = \frac{1}{p^3}, b = p, \lambda = 3$

.

References

[1] G. D. Forney Jr., Geometrically uniform codes, IEEE Trams. Inform. Theory, vol. 37, pp. 1241-1260, Sept. 1991.

[2] A. F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, 1982.

[3] E. B. da Silva, R. Palazzo Jr., and S. R. Costa, Improving the performance of asymmetric M-PAM signal constellations in Euclidean space by embedding then in hyperbolic space, Proceedings 1998 IEEE Information Theory Workshop, Killarney, Ireland, June 22-26, pp. 98-99.

[4] E. B. da Silva, and R. Palazzo Jr., M-PSK signal constellations in hyperbolic space achieving better performance than the M-PSK signal constellations in Euclidean space, 1999 IEEE Information Theory Workshop, Metsovo, Greece, June 27 - July 1.

[5] E. B. da Silva, R. Palazzo Jr., and M. Firer, Performance analisys of QAM-like constellations in hyperbolic space, 2000 International Symposium on Information Theory and Its Applications, Honolulu, USA, November 5-8, pp. 568-571.

[6] P. A. Firby and C. F. Gardiner, Surface Topology, Second edition, Ellis Horwood, 1991.