# A rank-three condition for invariant $(1,2)$-symplectic almost Hermitian structures on flag manifolds

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**Abstract**

This paper considers invariant $(1,2)$-symplectic almost Hermitian structures on the maximal flag manifod associated to a complex semi-simple Lie group $G$. The concept of cone-free invariant almost complex structure is introduced. It involves the rank-three subgroups of $G$, and generalizes the cone-free property for tournaments related to $\mathrm{Sl}(n,\mathbb{C})$ case. It is proved that the cone-free property is necessary for an invariant almost-complex structure to take part in an invariant

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$(1,2)$-symplectic almost Hermitian structure. It is also sufficient if the Lie group is not $B_l$, $l \geq 3$, $G_2$ or $F_4$. For $B_l$ and $F_4$ a close condition turns out to be sufficient.

# 1 Introduction

The subject matter of this paper is the invariant almost Hermitian structures on the generalized flag manifolds associated to semi-simple complex Lie algebras and groups. Let $G$ be a complex semi-simple Lie group and denote by $\mathbb{F} = G/P$ the maximal flag manifold of $G$, where $P$ is a Borel (minimal parabolic) subgroup of $G$. Alternatively, $\mathbb{F} = U/T$ where $U$ is a compact real form of $G$ and $T = P \cap U$ a maximal torus.

The $U$-invariant almost Hermitian structures on $\mathbb{F}$ have been studied recently in [2] and [10] with different methods. First, in [2] the group $G$ is specialized to be $\mathrm{Sl}(n, \mathbb{C})$, so that $U = \mathrm{SU}(n)$ and $\mathbb{F}$ is identified with the manifold of complete flags of subspaces of $\mathbb{C}^n$. In this case there exists a natural bijection between the set of $U$-invariant almost complex structures on $\mathbb{F}$ and $n$-player tournaments. Taking advantage of this bijection in [2] the invariant structures were studied with the aid of the combinatorics of tournaments (see also [3]).

On the other hand, [10] adopts the general set up, and studies invariant structures on the flag manifold associated to an arbitrary semi-simple complex group $G$. The methods of [10] are intrinsic in the sense that the combinatorial questions are resolved within the framework of root systems and Weyl groups.

In both papers the basic issue is the description of the $(1,2)$-symplectic Hermitian structures. One of the main results is the derivation of a standard form for the corresponding invariant almost-complex structures. In [2] the standard form is given in terms of stair-shaped incidence matrices of tournaments, while in the general setting of [10] it is proved that the $(1,2)$-symplectic Hermitian structures can be put in correspondence to the abelian ideals of a Borel subalgebra. Although the results of [10] extend those of [2] the proofs are completely independent. In particular, the notion of cone-free tournament – which plays a central role in [2] as a necessary and sufficient condition – does not appear in [10], leaving a gap in the development of the theory.

The purpose of this paper is to fill this gap, by extending the cone-free concept to the context of semi-simple Lie algebras, and analyzing its relation to the $(1,2)$-symplectic structures. The cone-free property for the $A_l$ series can be translated into a condition involving quadruples of roots, and thus makes sense in general (see Definition 3.1). We maintain the name of cone-free for the property stated in terms of roots. It is related to the $(1,2)$-symplectic structures as follows: An invariant Hermitian structure is a pair $(J, \Lambda)$ with $J$ a $U$-invariant almost complex structure and $\Lambda$ an invariant Riemannian metric. The cone-free property refers to the invariant almost complex structures. Such a structure is said to be $(1,2)$-admissible if there exists $\Lambda$ such that $(J, \Lambda)$ is $(1,2)$-symplectic. We prove in Theorem 3.3 that the cone-free property is necessary for $J$ to be $(1,2)$-admissible. It is also sufficient if the semi-simple Lie algebra does not contain components of the types $B_l$, $l \geq 3$, $G_2$ or $F_4$. The point is that for the Lie algebras with rank $\geq 3$, the cone-free property concerns the restriction of $J$ to the rank-three subalgebras, and is equivalent to $(1,2)$-admissibility in $A_3$ and $C_3$ but not in $B_3$. For this reason the correct condition for the Lie algebras $B_l$ and $F_4$ (which are the only ones which contain $B_3$) is that the restriction of $J$ to any rank-three subalgebra is $(1,2)$-admissible.

We regard our approach here as an application of the affine Weyl group characterization of the $(1,2)$-symplectic structures, proved in [10]. Indeed we check that a certain $J$ is $(1,2)$-admissible by showing that it belongs to the class of affine invariant almost complex structures, which are defined by means of alcoves of the affine Weyl group (see Definition 6.3 below). It was proved in [10] that affine structures are $(1,2)$-admissible and conversely. Through the affine structures we have access to the algebra of integer alcove coordinates developed by Shi [11]. This algebra is used to solve the combinatorial problems arising in the study of invariant structures.

The relation of cone-free tournaments with $(1,2)$-symplectic structures on the classical flag manifolds is discussed in Mo and Negreiros [7] and Paredes [9]. The necessity of the cone-free property for $(1,2)$-admissibility was first stated and proved in [7], with the aid of the moving frame method, while evidence for sufficiency was provided in [9], by checking small-sized tournaments. A general proof of sufficiency for tournaments of arbitrary size was given in [2].

Our attempt to understand the $(1,2)$-symplectic structures was motivated by the study of harmonic maps into flag manifolds. However, after studying them in [10] it became clear that among the invariant almost Hermitian structures on the flag manifolds the $(1,2)$-symplectic ones form an outstanding class, allowing the classification of the invariant structures given in [10].

# 2 Preliminaries

Let $\mathfrak{g}$ and $\mathfrak{h}$ be a simple complex Lie algebra and a Cartan subalgebra. Denote by $\Pi$ the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$, and let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h},\ [H, X] = \alpha(H) X\}$$

be the one-dimensional root space corresponding to $\alpha \in \Pi$. Given $\alpha \in \mathfrak{h}^*$ we let $H_\alpha$ be defined by $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the Cartan-Killing form of $\mathfrak{g}$ and define $\mathfrak{h}_\mathbb{R}$ to be the subspace spanned over $\mathbb{R}$ by $H_\alpha$, $\alpha \in \Pi$. We fix once and for all a Weyl basis of $\mathfrak{g}$ which amounts to choosing for each $\alpha \in \Pi$ an element $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha} \rangle = 1$, and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$ and $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root (see Helgason [4], Chapter IX).

Given a choice of positive roots $\Pi^+ \subset \Pi$, denote by $\Sigma$ the corresponding simple system of roots and let $\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ be the Borel subalgebra generated by $\Pi^+$. Let $\mathbb{F} = G/P$ be the associated maximal flag manifold, where $G$ is any connected complex Lie group with Lie algebra $\mathfrak{g}$ and $P$ is the normalizer of $\mathfrak{p}$ in $G$. Let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}$ spanned by $i\mathfrak{h}_\mathbb{R}$ and $A_\alpha$, $iS_\alpha$, $\alpha \in \Pi$, where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$. Denote by $U$ the corresponding compact real form of $G$. By the transitive action of $U$ on $\mathbb{F}$ we can write $\mathbb{F} = U/T$ where $T = P \cap U$ is a maximal torus of $U$.

If $b_0$ stands for the origin of $\mathbb{F}$, the tangent space at $b_0$ identifies naturally with the subspace $\mathfrak{q} \subset \mathfrak{u}$ spanned by $A_\alpha$, $iS_\alpha$, $\alpha \in \Pi$. Analogously, the complex tangent space of $\mathbb{F}$ is identified with $\mathfrak{q}_\mathbb{C} = \mathfrak{g} \ominus \mathfrak{h} \subset \mathfrak{u}$, spanned by the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Pi$. Clearly, the adjoint action of $T$ on $\mathfrak{g}$ leaves $\mathfrak{q}$ invariant.

## 2.1 Invariant metrics

A $U$-invariant Riemannian metric on $\mathbb{F}$ is completely determined by its value at $b_0$, that is, by an inner product $(\cdot, \cdot)$ in $\mathfrak{q}$, which is invariant under the adjoint action of $T$. Such an inner product has the form $(X, Y)_\Lambda = -\langle \Lambda(X), Y \rangle$ with $\Lambda : \mathfrak{q} \to \mathfrak{q}$ positive-definite with respect to the Cartan-Killing form. The inner product $(\cdot, \cdot)_\Lambda$ admits a natural extension to a symmetric bilinear form on the complexification $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{q}$. These complexified objects are denoted the same way as the real ones. The $T$-invariance of $(\cdot,\cdot)_{\Lambda}$ amounts to the elements of the standard basis $A_{\alpha}$, $iS_{\alpha}$, $\alpha \in \Pi$, being eigenvectors of $\Lambda$, for the same eigenvalue. Thus, in the complex tangent space we have $\Lambda(X_{\alpha}) = \lambda_{\alpha} X_{\alpha}$ with $\lambda_{\alpha} = \lambda_{-\alpha} > 0$. We denote by $ds^2_{\Lambda}$ the invariant metric associated with $\Lambda$. In the sequel we allow abuse of notation and write simply $\Lambda$ instead of $ds^2_{\Lambda}$.

## 2.2 Invariant almost complex structures

In the sequel we use the abbreviation *iacs* for $U$-invariant almost complex structure on $\mathbb{F}$. An *iacs* is completely determined by its value $J : \mathfrak{q} \to \mathfrak{q}$ in the tangent space at the origin. The map $J$ satisfies $J^2 = -1$ and commutes with the adjoint action of $T$ on $\mathfrak{q}$. We denote by the same letter the real valued structure $J$ and its complexification to $\mathfrak{q}_{\mathbb{C}}$. The invariance of $J$ entails that $J(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{\alpha}$ for all $\alpha \in \Pi$. The eigenvalues of $J$ are $\pm i$ and the eigenvectors in $\mathfrak{q}_{\mathbb{C}}$ are $X_{\alpha}$, $\alpha \in \Pi$. Hence $J(X_{\alpha}) = i\varepsilon_{\alpha} X_{\alpha}$ with $\varepsilon_{\alpha} = \pm 1$ satisfying $\varepsilon_{\alpha} = -\varepsilon_{-\alpha}$. As usual the eigenvectors associated to $+i$ are said to be of type $(1,0)$ while $-i$-eigenvectors are of type $(0,1)$. Thus the $(1,0)$ vectors are linear combinations of $X_{\alpha}$, $\varepsilon_{\alpha} = +1$, and the $(0,1)$ vectors are spanned by $X_{\alpha}$, $\varepsilon_{\alpha} = -1$.

An *iacs* on $\mathbb{F}$ is completely prescribed by a set of signs $\{\varepsilon_{\alpha}\}_{\alpha \in \Pi}$ with $\varepsilon_{-\alpha} = -\varepsilon_{-\alpha}$. In the sequel we allow some abuse of notation and identify the invariant structure on $\mathbb{F}$ with $J = \{\varepsilon_{\alpha}\}$.

Since $\mathbb{F}$ is a homogeneous space of a complex Lie group it has a natural structure of a complex manifold. The associated integrable *iacs* $J_c$ is given by $\varepsilon_{\alpha} = +1$ if $\alpha < 0$. The conjugate structure $-J_c$ is also integrable. These are called the standard *iacs*.

## 2.3 Kähler form

It is easy to see that any invariant metric $ds^2_{\Lambda}$ is almost Hermitian with respect to any *iacs* $J$, that is, $ds^2_{\Lambda}(JX, JY) = ds^2_{\Lambda}(X, Y)$ (cf. [13], Section 8). Let $\Omega = \Omega_{J,\Lambda}$ stand for the corresponding Kähler form

$$\Omega(X, Y) = ds^2_{\Lambda}(X, JY) = -\langle \Lambda X, JY \rangle.$$

This form extends naturally to a $U$-invariant 2-form defined on the complexification $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{q}$, which we also denote by $\Omega$. Its values on the basic vectors are:

$$\Omega\left(X_\alpha, X_\beta\right) = -i\lambda_\alpha\varepsilon_\beta\langle X_\alpha, X_\beta\rangle.$$

Since $\langle X_\alpha, X_\beta\rangle = 0$ unless $\beta = -\alpha$, $\Omega$ is not zero only on the pairs $(X_\alpha, X_{-\alpha})$, at which $\Omega$ takes the value $i\lambda_\alpha\varepsilon_\alpha$.

The following formula is well known (see [6]).

**Lemma 2.1** *Let $\omega$ be an invariant $k$-differential form on the homogeneous space $L/H$. Then*

$$d\omega\left(X_1, \ldots, X_{k+1}\right) = (k+1)\sum_{i<j}(-1)^{i+j}\,\omega\left([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots \widehat{X}_j, \ldots, X_{k+1}\right).$$

*for $X_1, \ldots, X_{k+1}$ in the Lie algebra $\mathfrak{l}$ of $L$.*

Specializing this lemma to the form $\Omega$ we get

$$-\frac{1}{3}d\Omega\left(X, Y, Z\right) = -\Omega\left([X, Y], Z\right) + \Omega\left([X, Z], Y\right) - \Omega\left([Y, Z], X\right) \qquad (1)$$

From (1) an easy computation yields that $d\Omega\left(X_\alpha, X_\beta, X_\gamma\right)$ is zero unless $\alpha + \beta + \gamma = 0$. In this case

$$d\Omega\left(X_\alpha, X_\beta, X_\gamma\right) = -i3m_{\alpha,\beta}\left(\varepsilon_\alpha\lambda_\alpha + \varepsilon_\beta\lambda_\beta + \varepsilon_\gamma\lambda_\gamma\right) \qquad (2)$$

with $m_{\alpha,\beta}$ as in Section 2 (cf. [10], Proposition 2.1).

Taking into account (2) we make the following distinction between two types of roots triples.

**Definition 2.2** *Let $J = \{\varepsilon_\alpha\}$ be an iacs. A triple of roots $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma = 0$ is said to be a $\{0, 3\}$-triple if $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon_\gamma$, and a $\{1, 2\}$-triple otherwise.*

Recall that an almost Hermitian manifold is said to be $(1, 2)$-symplectic (or quasi-Kähler) if

$$d\Omega\left(X, Y, Z\right) = 0$$

when one of the vectors $X, Y, Z$ is of type $(1, 0)$ and the other two are of type $(0, 1)$.The structure is $(2, 1)$-symplectic if the roles of $(1, 0)$ and $(0, 1)$ are interchanged. Accordingly, the structure is $(i, j)$-symplectic if the $(i, j)$ component $d\Omega^{(i,j)}$ of $d\Omega$ is zero.

In our invariant setting we have the following criterion for an invariant pair $(J, \Lambda)$ to be $(1, 2)$-symplectic, which follows immediately from formula (2), and the fact that $X_\alpha$ has type $(1, 0)$ if $\varepsilon_\alpha = +1$ and $(0, 1)$ if $\varepsilon_\alpha = -1$ (see [10], Proposition 2.3 and [13], Theorem 9.15).

**Proposition 2.3** *The invariant pair* $(J = \{\varepsilon_\alpha\}, \Lambda = \{\lambda_\alpha\})$ *is* $(1,2)$*-symplectic if and only if*

$$\varepsilon_\alpha \lambda_\alpha + \varepsilon_\beta \lambda_\beta + \varepsilon_\gamma \lambda_\gamma = 0$$

*for every* $\{1,2\}$*-triple* $\{\alpha, \beta, \gamma\}$.

In the sequel $J$ is said to be $(1,2)$-admissible if there exists $\Lambda$ such that the pair $(J, \Lambda)$ is invariant and $(1,2)$-symplectic.

# 3 The cone-free property

Given a set of four roots $q = \{\alpha, \beta, \gamma, \delta\}$ with $\alpha + \beta + \gamma + \delta = 0$ we say that a triple of roots $\{(u+v), w_1, w_2\}$ is extracted from $q$ by $u$ and $v$ if $\{u, v, w_1, w_2\} = \{\alpha, \beta, \gamma, \delta\}$. Of course, any such triple satisfies $(u+v) + w_1 + w_2 = 0$. The cone-free condition is stated in terms of such triples.

**Definition 3.1** *Let* $J = \{\varepsilon_\alpha\}$ *be an* iacs. *We say that* $J$ *is* cone-free *if the following condition is satisfied:*

- *If* $q = \{\alpha, \beta, \gamma, \delta\}$ *contains no pairs of opposite roots and* $\alpha + \beta + \gamma + \delta = 0$ *then the number of* $\{0,3\}$*-triples extracted from* $q$ *is different from* 1.

In this definition the hypothesis that the quadruples do not have opposite roots is redundant and is included only for emphasis sake. Indeed, suppose, for instance, that $\beta = -\alpha$. Then $\delta = -\gamma$, and the possible triples extracted from the quadruple are $(\alpha + \gamma, -\alpha, -\gamma)$, $(\alpha - \gamma, -\alpha, \gamma)$, $(-\alpha + \gamma, \alpha, -\gamma)$ and $(-\alpha - \gamma, \alpha, \gamma)$. It is easy to see that in this set $\{0,3\}$-triples appear in pairs, independently of $J$.

Except when the root system is $G_2$ the cone-free property is a condition on the rank-three subsystems of the root system. In fact, since there are no opposite roots in $\{\alpha, \beta, \gamma, \delta\}$ the subspace $V$ spanned by these roots is either two or three dimensional. However, it is easy to see that in the rank-two root systems $A_1 \oplus A_1$, $A_2$ and $B_2$, which are different from $G_2$, there are no such sets of roots. Hence, the intersection of $\Pi \cap V$ is a rank-three root system if we are not in $G_2$ (see Section 5 below for a discussion of $G_2$).

The explanation for the term cone in the above definition comes from the relation between *iacs* in the flag manifolds of the $A_l$ series (the Lie algebras $\mathfrak{sl}(n, \mathbb{C})$, $n = l + 1$) and tournaments. Recall that an $n$-player tournament is a complete directed graph $T = (N, E)$ where $N$ is an ordered set, $|N| = n$, and $E$ stands for the arrows of $T$. With each tournament $T$ there is assigned its incidence matrix $\varepsilon = \varepsilon_T$, which is a real skew-symmetric matrix with all off-diagonal entries $\pm 1$. If $(a, b) \in E$ we say that $a$ wins against $b$ and set $\varepsilon_{ab} = 1$ and $\varepsilon_{ba} = -1$.

On the other hand, in the standard realization, the roots of $A_l$ are $\alpha_{jk}$, $1 \le j \neq k \le l+1$, with $\alpha_{kj} = -\alpha_{jk}$. Thus an *iacs* on the corresponding flag manifold is given by the signs $\varepsilon_{jk} = \varepsilon_{\alpha_{jk}} = \pm 1$, $j \neq k$. These numbers are assembled to form the incidence matrix $\varepsilon$ of some tournament, establishing a one-to-one correspondence between the *iacs* on the maximal flag manifold of $A_l$ and $n$-players tournaments.

A 3-cycle in a tournament is a 3-players subtournament $\{i, j, k\}$ which forms the loop $i \to j \to k \to i$. When $T$ is the tournament associated to the *iacs* $J$, a 3-cycle $\{i, j, k\}$ corresponds to the $\{0, 3\}$-triple $\{\alpha_{ij}, \alpha_{jk}, \alpha_{ki}\}$ (see [2]).

Now, up to isomorphism, there are four distinct 4-player tournaments. The two of them which contain a single 3-cycle are called *cones*. Each of them contains a cycle and a winner or a loser. The other equivalence classes of 4-player tournaments contain an even number of cycles (zero or two).

**Proposition 3.2** *In the maximal flag manifold associated to $A_{n-1} = \mathfrak{sl}(n, \mathbb{C})$, an* iacs *is cone-free in the sense of Definition 3.1 if and only if no 4-player subtournament of the associated tournament is a cone.*

**Proof:** Assume first that an *iacs* $J$ with corresponding tournament $T$ is cone-free in the sense of Definition 3.1. Let $\{i, j, k, l\}$ be a 4-player subtournament, and consider the corresponding set of four roots $\{\alpha_{ij}, \alpha_{jk}, \alpha_{kl}, \alpha_{li}\}$ which satisfies

$$\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} = 0.$$

From this set we extract the four triples $\{\alpha_{ik}, \alpha_{kl}, \alpha_{li}\}$, $\{\alpha_{jl}, \alpha_{li}, \alpha_{ij}\}$, $\{\alpha_{ki}, \alpha_{ij}, \alpha_{jk}\}$ and $\{\alpha_{lj}, \alpha_{jk}, \alpha_{kl}\}$. Each one of these triples corresponds to a 3-player subtournament (e.g. $\{\alpha_{ik}, \alpha_{kl}, \alpha_{li}\}$ is associated to $\{i, k, l\}$), in such a way that $\{0, 3\}$-triples correspond to 3-cycles. Hence, by our generalized cone-free condition $\{i, j, k, l\}$ is not a cone.

For the converse, note that a set of four roots $\{\alpha, \beta, \gamma, \delta\}$ with $\alpha + \beta + \gamma + \delta = 0$ which do not contain opposite roots spans a rank-three root subsystem, and hence the set has the form $\{\alpha_{ij}, \alpha_{jk}, \alpha_{kl}, \alpha_{li}\}$ for $1 \le i, j, k, l \le n$. Repeating the above argument we get the generalized cone-free condition if the tournament has no cones. $\square$

We proceed now to prove that the cone-free condition is necessary for an *iacs* to be $(1,2)$-admissible. Write $d\Omega^{\{0,3\}} = d\Omega^{(0,3)} + d\Omega^{(3,0)}$ and $d\Omega^{\{1,2\}} = d\Omega^{(1,2)} + d\Omega^{(2,1)}$, so that

$$d\Omega = d\Omega^{\{0,3\}} + d\Omega^{\{1,2\}}.$$

We get a necessary condition for $d\Omega^{\{1,2\}} = 0$ by exploiting the fact that $d^2 = 0$, computing formally $d^2\Omega\,(X_\alpha, X_\beta, X_\gamma, X_\delta)$. Analogous to the case of $d\Omega$ the only quadruples $\{\alpha, \beta, \gamma, \delta\}$ of interest are those satisfying $\alpha + \beta + \gamma + \delta = 0$. Using the exterior derivative formula of Lemma 2.1, we get for these quadruples, that $d^2\Omega$ is the sum of the following six terms:

1. $+m_{\alpha,\beta}m_{\gamma,\delta}\left(\varepsilon_{\alpha+\beta}\lambda_{\alpha+\beta} + \varepsilon_\gamma\lambda_\gamma + \varepsilon_\delta\lambda_\delta\right)$
2. $-m_{\alpha,\gamma}m_{\beta,\delta}\left(\varepsilon_{\alpha+\gamma}\lambda_{\alpha+\gamma} + \varepsilon_\beta\lambda_\beta + \varepsilon_\delta\lambda_\delta\right)$
3. $+m_{\alpha,\delta}m_{\beta,\gamma}\left(\varepsilon_{\alpha+\delta}\lambda_{\alpha+\delta} + \varepsilon_\beta\lambda_\beta + \varepsilon_\gamma\lambda_\gamma\right)$
4. $+m_{\beta,\gamma}m_{\alpha,\delta}\left(\varepsilon_{\beta+\gamma}\lambda_{\beta+\gamma} + \varepsilon_\alpha\lambda_\alpha + \varepsilon_\delta\lambda_\delta\right)$
5. $-m_{\beta,\delta}m_{\alpha,\gamma}\left(\varepsilon_{\beta+\delta}\lambda_{\beta+\delta} + \varepsilon_\alpha\lambda_\alpha + \varepsilon_\gamma\lambda_\gamma\right)$
6. $+m_{\gamma,\delta}m_{\alpha,\beta}\left(\varepsilon_{\gamma+\delta}\lambda_{\gamma+\delta} + \varepsilon_\alpha\lambda_\alpha + \varepsilon_\beta\lambda_\beta\right)$

These terms cancel mutually (e.g. the coefficient of $\varepsilon_\alpha\lambda_\alpha$ is $m_{\alpha,\beta}m_{\gamma,\delta} + m_{\beta,\gamma}m_{\alpha,\delta} + m_{\gamma,\alpha}m_{\beta,\delta}$ which is known to be zero, see [4], Lemma III 5.3). In order to look at them closer let us take, for instance, the first one. The coefficient $m_{\alpha,\beta}$ is not zero if and only if $\alpha + \beta$ is a root. But $\alpha + \beta = -(\gamma + \delta)$, so that both coefficients $m_{\alpha,\beta}$ and $m_{\gamma,\delta}$ are simultaneously zero or not. The same remark is true for the other terms. Next, in each term the sum appearing in braces has the form $d\Omega\,(X_\xi, X_\eta, X_\theta)$ with $(\xi, \eta, \theta)$ a triple extracted from $\{\alpha, \beta, \gamma, \delta\}$ if the coefficients $m_{*,*}$ are not zero.

These comments yield an alternative proof of the following result of [7].

**Theorem 3.3** *A necessary condition for $(J, \Lambda)$ to be $(1,2)$-symplectic is that $J$ is cone-free in the sense of Definition 3.1.*

**Proof:** Let $q = \{\alpha, \beta, \gamma, \delta\}$ be a root quadruple such that $\alpha+\beta+\gamma+\delta = 0$. Among the six terms above, those corresponding to $\{1,2\}$-triples extracted from $q$ are zero if $d\Omega^{\{1,2\}} = 0$. On the other hand a term corresponding to an extracted $\{0,3\}$-triple is not zero. Hence, for $d^2\Omega$ to be zero it is not possible to extract just one $\{0,3\}$-triple. □

# 4 Rank-three Lie algebras

The cone-free condition involves sets of four roots whose sum is zero in such a way that no two roots are opposite to each other. This has the consequence that the subspace spanned by the roots is three dimensional if the root system is not $G_2$. Hence, excluding $G_2$ the cone-free condition refers to the rank-three subsystems of roots. The purpose of this preparatory section is to look at those rank-three root systems (mainly the irreducible ones $A_3$, $B_3$ and $C_3$) required to study the cone-free condition in general root systems.

Note first that the rank-three reducible root systems are $A_1 \oplus A_1 \oplus A_1$, $A_1 \oplus A_2$ and $A_1 \oplus B_2$. It is easy to check that any *iacs* in these root systems are $(1,2)$-admissible, and thus satisfy the cone-free condition.

Concerning $A_3 = \mathfrak{sl}(4,\mathbb{C})$, an *iacs* $J$ on the maximal flag manifold corresponds to a 4-tournament $T$. By Proposition 3.2, $J$ satisfies our cone-free condition if and only if $T$ does not contain a cone. We know that such *iacs* are $(1,2)$-admissible (see [2], [3]). Actually, the set of cone-free *iacs* has two equivalence classes, which are represented by the incidence matrices

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}. \tag{3}$$

The class represented by the first matrix consists of the standard *iacs*.

Now, we look at the more delicate $B_3$. In its standard realization the positive root system is $L \cup S$ where $L = \{e_i \pm e_j : 1 \le i < j \le 3\}$ and $S = \{e_i : 1 \le i \le 3\}$ are the sets of long and short roots, respectively.

The set $L$ is isomorphic to the positive root system $L_3 = \{\alpha_{ij} : 1 \le i < j \le 3\}$ of $A_3$ via the bijection:

- Simple roots: $\alpha_{12} \leftrightarrow e_2 - e_3$; $\alpha_{23} \leftrightarrow e_1 - e_2$; $\alpha_{34} \leftrightarrow e_2 + e_3$.

- Height 2: $\alpha_{13} \leftrightarrow e_1 - e_3$; $\alpha_{34} \leftrightarrow e_1 + e_3$.
- Height 3: $\alpha_{14} \leftrightarrow e_1 + e_2$.

Now, let $J = \{\varepsilon_\alpha\}$ be a cone-free *iacs* in $B_3$. Its restriction $J^l$ to $L$ is also cone-free so that we can assume that it is represented by one of the two matrices in (3). It remains to see what happens at the short roots $e_1$, $e_2$ and $e_3$. Regarding $e_3$, we can assume without loss of generality that $\varepsilon_{e_3} = +1$. In fact, the reflection $r_3$ with respect to $e_3$ leaves $L_3$ invariant fixes the highest root $e_1 + e_2$. Hence, we can replace $J$ by $r_3 \cdot J$ without affecting its values in $L_3$ if $J^l$ is represented by one of the matrices in (3). As to $e_1$ and $e_2$ we have

**Lemma 4.1** $\varepsilon_{e_1} = \varepsilon_{e_2}$.

**Proof:** Consider the quadruple $(-e_1) + (e_1 - e_2) + (e_2 - e_3) + e_3 = 0$. The triples extracted from it are $\{-e_2, e_2 - e_3, e_3\}$, $\{-e_1 + e_3, e_1 - e_2, e_2 - e_3\}$, $\{e_2, -e_1, e_1 - e_2\}$ and $\{e_1 - e_3, -e_1, e_3\}$.

Note that $\{-e_1 + e_3, e_1 - e_2, e_2 - e_3\}$ is a $\{1,2\}$-triple. Suppose that $\varepsilon_{e_2} = -1$. Then $\{-e_2, e_2 - e_3, e_3\}$ is a $\{0,3\}$-triple, and $\{e_2, -e_1, e_1 - e_2\}$ is a $\{1,2\}$-triple, forcing the last triple to be $\{0,3\}$, which implies $\varepsilon_{e_1} = -1$. The root $e_1 + e_2$ does not appear in the extracted triples, ensuring that our arguments are independent of the choice of $J^l$.

On the other hand from the quadruple $(e_1 - e_2) + (e_2 + e_3) + (-e_1) + (-e_3) = 0$, the only extracted triple which is not automatically of type $\{0,3\}$ is $\{e_2, e_1 - e_2, -e_1\}$. Hence, this set must be a $\{1,2\}$-triple, so that $\varepsilon_{e_2} = +1$ implies $\varepsilon_{e_1} = +1$. Again the extracted triples do not involve $e_1 + e_2$, hence it is immaterial which of the $J^l$'s we consider. □

We arrive at the following description of the cone-free *iacs* on $B_3$.

**Proposition 4.2** *Denote by $M(J)$ the set of positive roots $\alpha$ of $B_3$ such that $\varepsilon_\alpha = -1$. Fixing the choices of $J^l$ given by (3) and $\varepsilon_{e_3} = +1$, the possible* iacs *satisfying the cone-free condition are:*

1. $M(J_1) = \emptyset$.
2. $M(J_2) = \{e_1 + e_2\}$.
3. $M(J_3) = \{e_1, e_2\}$.

*4.* $M(J_4) = \{e_1, e_2, e_1 + e_2\}$.

*Among them the only* $(1,2)$*-admissible* iacs *are* $J_1$ *and* $J_2$*.*

**Proof:** The $(1,2)$-admissibility of $J_1$ and $J_2$ is a consequence of the abelian ideal shape of [10]. On the other hand, $J_3$ and $J_4$ are not $(1,2)$-admissible. To see this consider the triples $\{e_1, e_1 + e_3, -e_3\}$ and $\{e_1, e_3, -e_1 - e_3\}$. They are $\{1,2\}$-triples for both $J_3$ and $J_4$. Now, assume that $\Lambda = \{\lambda_\alpha\}$ is $(1,2)$-symplectic with respect to $J_3$ or $J_4$. Then $\lambda_{e_1+e_3} = \lambda_{e_1} + \lambda_{e_3}$ and $\lambda_{e_3} = \lambda_{e_1} + \lambda_{e_1+e_3}$, forcing $\lambda_{e_1} = 0$, a contradiction.

Finally, it is straighforward but cumbersome to verify that $J_3$ and $J_4$ indeed satisfy the cone-free condition. One must write down the quadruples of roots of $B_3$ summing up zero, and their extracted triples, and check that the $\{0,3\}$-triples do not appear isolated. □

The discussion of $C_3$ follows the same pattern as that of $B_3$. In the standard realization of $C_3$, its short roots coincide with the long roots of $B_3$, whereas the long roots are given by $\pm 2e_i$, $i = 1, 2, 3$. Again we can assume that the restriction $J^s$ of a cone-free *iacs* $J$ to the short roots has one of the incidence matrices (3). Also, after applying the reflection with respect to $e_3$ we can assume that $\varepsilon_{2e_3} = +1$. With the aid of these choices we can check the quadruples of $C_3$ and prove the

**Proposition 4.3** *Denote, as before, by* $M(J)$ *the set of positive roots* $\alpha$ *of* $C_3$ *such that* $\varepsilon_\alpha = -1$*. Fixing the above choices of* $J^s$ *and* $\varepsilon_{e_3} = +1$*, the possible* iacs *satisfying the cone-free condition are:*

*1.* $M(J_1) = \emptyset$.

*2.* $M(J_2) = \{e_1 + e_2, 2e_1\}$.

*3.* $M(J_3) = \{2e_2, e_1 + e_2, 2e_1\}$.

*Each* $M(J_i)$*,* $i = 1, 2, 3$*, is an abelian ideal of the set of positive roots, so that the cone-free* iacs *are* $(1,2)$*-admissible.*

**Proof:** The proposition is a consequence of the following implications:

$$\varepsilon_{e_1+e_2} = +1 \Rightarrow \varepsilon_{2e_2} = \varepsilon_{2e_1} = +1, \qquad \varepsilon_{e_1+e_2} = -1 \Rightarrow \varepsilon_{2e_1} = -1.$$

which are easy consequences of the cone-free property applied to the quadruples $\{e_1 - e_2, 2e_2, -e_2 + e_3, -e_1 - e_3\}$, $\{e_1 - e_2, e_1 - e_3, e_2 + e_3, -2e_1\}$ and $\{e_1 - e_2, e_2 - e_3, e_1 + e_3, -2e_1\}$, respectively. $\square$

# 5 $G_2$

As mentioned above, $G_2$ is the only rank-two root system where the cone-free condition is not vacuous. For the sake of completeness we analyze here the *iacs* on $G_2$ which satisfy this condition. We write the positive roots as

$$\begin{matrix}\alpha_1\\ \alpha_2\end{matrix} \quad \alpha_1 + \alpha_2 \quad \alpha_1 + 2\alpha_2 \quad \alpha_1 + 3\alpha_2 \quad 2\alpha_1 + 3\alpha_2.$$

The set of short roots $\{\pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\}$ is an $A_2$-root system. Let $J$ be an *iacs* on $G_2$ and denote by $J^s$ its restriction to the set of short roots. In $A_2$ there are two equivalence classes of *iacs*, so that we can assume without loss of generality that $J^s$ is one of the following two *iacs*:

1. $J_1^s = \{\varepsilon_{\alpha_2} = +1, \varepsilon_{\alpha_1+\alpha_2} = +1, \varepsilon_{\alpha_1+2\alpha_2} = +1\}$.
2. $J_2^s = \{\varepsilon_{\alpha_2} = +1, \varepsilon_{\alpha_1+\alpha_2} = +1, \varepsilon_{\alpha_1+2\alpha_2} = -1\}$.

Denote by $r$ the reflection with respect to $\alpha_1$. It satisfies $r\alpha_2 = \alpha_1 + \alpha_2$ and $r(\alpha_1 + 2\alpha_2) = \alpha_1 + 2\alpha_2$. This implies that $r$ leaves $J^s$ invariant. Hence, we may assume that $\varepsilon_{\alpha_1} = +1$.

Now, assuming that $J$ satisfies the cone-free condition, it remains to determine the values of $\varepsilon_{\alpha_1+3\alpha_2}$ and $\varepsilon_{2\alpha_1+3\alpha_2}$. Up to change of signs there are the following three zero-sum root quadruples:

1. $q_1$: $(\alpha_1) + (\alpha_2) + (\alpha_1 + 2\alpha_2) + (-2\alpha_1 - 3\alpha_2) = 0$.
2. $q_2$: $(\alpha_2) + (\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2) + (-2\alpha_1 - 3\alpha_2) = 0$.
3. $q_3$: $(\alpha_1) + (\alpha_1 + 3\alpha_2) + (-\alpha_1 - \alpha_2) + (-\alpha_1 - 2\alpha_2) = 0$.

First suppose that $J^s = J_1^s$. Writing down the triples extracted from $q_3$, it is straighforward to check that $\varepsilon_{\alpha_1+3\varepsilon_2} = -1$ implies that $\varepsilon_{2\alpha_1+3\varepsilon_2} = -1$. Hence, the possible cone-free *iacs* are $\begin{smallmatrix}+\\+\end{smallmatrix}\,{+}{+}{+}{+}$, $\begin{smallmatrix}+\\+\end{smallmatrix}\,{+}{+}{+}{-}$ and $\begin{smallmatrix}+\\+\end{smallmatrix}\,{+}{+}{-}{-}$.

By the abelian ideal property stated in [10], these *iacs* are $(1,2)$-admissible, and hence they are indeed cone-free.

Suppose now that $J^s = J^s_2$. Looking at the triples extracted from $q_1$ it is easy to see that $\varepsilon_{\alpha_1+3\alpha_2} = +1$ implies $\varepsilon_{2\alpha_1+3\alpha_2} = +1$. Since there are no other restrictions, the cone-free *iacs* are $\begin{matrix}+\\+\end{matrix}\ + - + +$, $\begin{matrix}+\\+\end{matrix}\ + - - +$ and $\begin{matrix}+\\+\end{matrix}\ + - - -$. The last one is $(1,2)$-admissible, whereas, similar to the $B_3$ case, one can check that the first two are not $(1,2)$-admissible. (We remark that in checking the cone-free property the quadruple $q_2$ is irrelevant, since in it each extracted triple appears twice.)

# 6 The affine Weyl group

In this section we recall the definition of the *affine iacs* introduced in [10]. These structures are constructed by counting hyperplanes separating a given alcove and the basic one. We refer to Humphreys [5] as a basic source for the affine Weyl group. Consider the subspace $\mathfrak{h}_{\mathbb{R}}$ introduced in Section 2. To conform with the usual notation we often identify $\mathfrak{h}_{\mathbb{R}}$ with its dual $\mathfrak{h}^*_{\mathbb{R}}$ and write $\langle x, \alpha\rangle$ instead of $\alpha(x)$, $x \in \mathfrak{h}_{\mathbb{R}}$, $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$. Given $\alpha \in \Pi$ and $k \in \mathbb{Z}$ define the affine hyperplane

$$H(\alpha, k) = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha\rangle = k\}.$$

The complement $\mathcal{A}$ of the set of hyperplanes $H(\alpha, k)$, $\alpha \in \Pi$, $k \in \mathbb{Z}$, is the disjoint union of connected open simpleces called *alcoves*. Given an alcove $A$ and a root $\alpha$, by definition there exists an integer $k_\alpha = k_\alpha(A)$ such that

$$k_\alpha < \langle x, \alpha\rangle < k_\alpha + 1 \qquad x \in A.$$

Of course, $k_\alpha = [\alpha(x)]$ for any $x \in A$ where $[a]$ denotes the integer part of the real number $a$. According to Shi [11], the integers $k_\alpha(A)$ are called the *coordinates* of the alcove $A$. An alcove is completely determined by its coordinates. A necessary and sufficient condition for $k_\alpha$, $\alpha \in \Pi$, to be the coordinates of an alcove are given by the inequalities below. In writing down these inequalities we must look $\Pi$ as the set of co-roots of another root system $\widetilde{\Pi}$:

$$\Pi = \{\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha\rangle} : \alpha \in \widetilde{\Pi}\}.$$

The root system is normalized so that $|\alpha| = 1$ if $\alpha$ is a short root.

**Proposition 6.1** *A set of integers $k_\alpha$, $\alpha \in \widetilde{\Pi}^+$, form the coordinates of an alcove if and only if for every pair of roots $\alpha, \beta \in \widetilde{\Pi}$ such that $\alpha + \beta \in \widetilde{\Pi}$, the following inequalities hold:*

$$\begin{array}{rcl} |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 & \leq & |\alpha+\beta|^2 \left(k_{\alpha+\beta}+1\right) \\ & \leq & |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1. \end{array} \tag{4}$$

**Proof:** See [11], Lemma 1.2 and Proposition 5.1. □

**Remark:** It is easy to see that the inequalities in this proposition are equivalent to

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\gamma|^2 k_\gamma \leq 1.$$

For later reference we note also the following easy necessary condition.

**Lemma 6.2** *A necessary condition for the integers $k_\alpha \in \mathbb{Z}$, $\alpha \in \Pi$, to be the coordinates of an alcove is that $k_{\alpha+\beta}$ is either $k_\alpha + k_\beta$ or $k_\alpha + k_\beta + 1$ whenever $\alpha$, $\beta$ and $\alpha + \beta$ are roots.*

**Proof:** We have, for all $x \in A$, $k_\alpha < \langle x, \alpha \rangle < k_\alpha + 1$ and $k_\beta < \langle x, \beta \rangle < k_\beta + 1$, so that

$$k_\alpha + k_\beta < \langle x, \alpha + \beta \rangle < k_\alpha + k_\beta + 2.$$

Hence, the integer part of $\langle x, \alpha + \beta \rangle$ is either $k_{\alpha+\beta} = k_\alpha + k_\beta$ or $k_{\alpha+\beta} = k_\alpha + k_\beta + 1$. □

**Definition 6.3** *Given an alcove $A$ with coordinates $\{k_\alpha : \alpha \in \Pi\}$, the* iacs *$J(A) = \{\varepsilon_\alpha(A)\}$ is defined by $\varepsilon_\alpha(A) = (-1)^{k_\alpha}$. We say that $J$ is an affine* iacs *if it has the form $J = J(A)$ for some alcove $A$.*

Note that $J(A)$ is indeed an *iacs*, since $k_{-\alpha} = -k_\alpha - 1$, so that $\varepsilon_{-\alpha}(A) = -\varepsilon_\alpha(A)$. The following theorem is one of the main results in [10]. It provides the criterion which will be used in the sequel for ensuring that *iacs* are $(1,2)$-admissible.

**Theorem 6.4** *An* iacs *$J$ is $(1,2)$-admissible if and only if it is affine.*

## 7 Simply-laced root systems

In this section we prove that the cone-free condition is sufficient for an *iacs* to be $(1, 2)$-admissible, in case the algebra $\mathfrak{g}$ has a simply-laced Dynkin diagram, i.e. $\Pi = A_l$, $D_l$, $E_6$, $E_7$ or $E_8$. The doubly-laced case will be treated in Section 8. We use the equivalence between the affine and $(1, 2)$-admissible *iacs*, as stated in Theorem 6.4, and construct an alcove $A$ such that $J = J(A)$ if $J$ satisfies the cone-free condition. Thus the purpose of this section is to prove the following statement.

**Theorem 7.1** *Let $\Pi$ be a simply-laced root system, and suppose that $J = \{\varepsilon_\alpha\}$ is a cone-free* iacs *on $\mathbb{F}$. Then $J$ is affine.*

The proof will consist of several steps. By definition of affine *iacs* we must find a set of integers $\{k_\alpha : \alpha \in \Pi\}$ satisfying the inequalities of Shi (4) such that $\varepsilon_\alpha = (-1)^{k_\alpha}$, $\alpha \in \Pi$. In a simply-laced root system the roots have the same length, simplifying these inequalities. In fact, we have the following equivalent condition for a set $k_\alpha$ to be the coordinates of an alcove.

**Lemma 7.2** *Let $\Pi$ be simply-laced. Then the integers $k_\alpha \in \mathbb{Z}$, $\alpha \in \Pi$, form the coordinates of an alcove if and only if either $k_{\alpha+\beta} = k_\alpha + k_\beta$ or $k_{\alpha+\beta} = k_\alpha + k_\beta + 1$ when $\alpha$, $\beta$ and $\alpha + \beta$ are roots.*

**Proof:** The condition is necessary by Lemma 6.2. Conversely, if $\Pi$ is simply-laced, the $|\cdot|^2$ appearing in inequalities (4) are equal to 1, hence they reduce to

$$k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1.$$

Therefore, these inequalities are satisfied by $k_\alpha$, $\alpha \in \Pi$, if they are under the conditions of the statement. □

Before proceeding we prove some lemmas.

**Lemma 7.3** *Let $\Pi_* \subset \Pi$ be a root subsystem of $\Pi$. Then $\Pi_*^+ = \Pi_* \cap \Pi^+$ is a choice of positive roots in $\Pi_*$.*

**Proof:** There exists $\gamma \in \mathfrak{h}_{\mathbb{R}}^*$ such that

$$\Pi^+ = \{\alpha \in \Pi : \langle \alpha, \gamma \rangle > 0\}.$$

Of course, $\langle \alpha, \gamma \rangle \neq 0$ for all $\alpha \in \Pi$. Let $\gamma_1$ be the orthogonal projection of $\gamma$ onto the subspace of $\mathfrak{h}_{\mathbb{R}}^*$ spanned by $\Pi_*$. For $\beta \in \Pi_*$ we have $\langle \beta, \gamma \rangle = \langle \beta, \gamma_1 \rangle$, so that $\langle \beta, \gamma_1 \rangle \neq 0$ for all $\beta \in \Pi_*$. Hence, $\gamma_1$ is regular for $\Pi_*$, implying that

$$\Pi_*^+ = \{\beta \in \Pi_* : \langle \beta, \gamma_1 \rangle > 0\}$$

is a choice of positive roots in $\Pi_*$. Using again $\langle \beta, \gamma \rangle = \langle \beta, \gamma_1 \rangle$, $\beta \in \Pi_*$, it follows that $\Pi_*^+ = \Pi_* \cap \Pi^+$, proving the lemma. □

**Lemma 7.4** *Fix a simple system of roots $\Sigma$, and let $J = \{\varepsilon_\alpha\}$ be an affine* iacs. *Suppose that a set of integers $m_\alpha \in \mathbb{Z}$, $\alpha \in \Sigma$, satisfies $\varepsilon_\alpha = (-1)^{m_\alpha}$. Then there exists an alcove $A$ such that $J = J(A)$ and $k_\alpha(A) = m_\alpha$, $\alpha \in \Sigma$.*

**Proof:** Put $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ and define $\{\omega_1, \ldots, \omega_l\}$ by $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$. Also, let $A^1$ be an alcove such that $J = J(A^1)$, that is, $\varepsilon_\alpha = (-1)^{k_\alpha(A^1)}$. Since $\varepsilon_\alpha = (-1)^{m_\alpha}$, the integers $m_{\alpha_i} - k_{\alpha_i}(A^1)$ are even. Now, a translation $t_\lambda$ with $\lambda$ spanned over $\mathbb{Z}$ by $\omega_i$, $i = 1, \ldots, l$, maps alcoves into alcoves, and the coordinates are changed according to

$$k_\alpha(t_\lambda A) = \langle \alpha, \lambda \rangle + k_\alpha(A). \tag{5}$$

Take $\lambda = d_{\alpha_1}\omega_1 + \cdots + d_{\alpha_l}\omega_l$, with $d_{\alpha_i} = m_{\alpha_i} - k_{\alpha_i}(A^1)$. Then the coordinates of $A = t_\lambda A^1$ are $k_\alpha(A) = \langle \alpha, \lambda \rangle + k_\alpha(A^1)$, and since $\langle \alpha, \lambda \rangle$ is even for all $\alpha$, we conclude that $J = J(A)$. Furthermore, for a simple roots $\alpha_i$ we have $k_{\alpha_i}(A) = d_{\alpha_1} + k_\alpha(A^1) = m_{\alpha_i}$, proving the lemma. □

Now, for proving Theorem 7.1 we construct $k_\alpha$, $\alpha \in \Pi$, by induction on the height of $\alpha$. Thus let us fix once and for all a simple system of roots $\Sigma$ with $\Pi^+$ the corresponding set of positive roots. Then given $J = \{\varepsilon_\alpha\}$ define:

1. Let $\alpha \in \Sigma$. Then $k_\alpha = (1 - \varepsilon_\alpha)/2$.
2. Let $\alpha, \beta, \gamma \in \Pi^+$ be such that $\alpha = \beta + \gamma$. Then

$$k_\alpha = k_\beta + k_\gamma + \frac{1 - (-1)^{k_\beta + k_\gamma}\varepsilon_\alpha}{2}. \tag{6}$$

3. Let $\alpha \in -\Pi^+$. Then $k_\alpha = -k_{-\alpha} - 1$.

A case by case analysis shows easily that the coordinates $\{k_\alpha\}$ so defined satisfy $\varepsilon_\alpha = (-1)^{k_\alpha}$. Also, the condition of Lemma 7.2 is readily satisfied. The point is to show that $k_\alpha$ is independent of the decomposition $\alpha = \beta + \gamma$ used in (6). We prove this by induction on the height $h(\alpha)$ of $\alpha \in \Pi^+$. If $h(\alpha) = 1$, the root is simple, and no decomposition $\alpha = \beta + \gamma$, $\beta, \gamma \in \Pi^+$ exists, hence $k_\alpha$ is well defined.

Now take $\alpha \in \Pi^+$ such that $\alpha = \beta_1 + \gamma_1 = \beta_2 + \gamma_2$, $\beta_i$, $i = 1, 2$, positive roots, and hence having height smaller than $h(\alpha)$. By the inductive hypothesis $k_{\beta_i}$, $k_{\gamma_i}$, $i = 1, 2$ are well defined. We must show that $k_{\beta_1} + k_{\gamma_1} = k_{\beta_2} + k_{\gamma_2}$.

Denote by $V \subset \mathfrak{h}^*_{\mathbb{R}}$ the subspace spanned by $\beta_1$, $\gamma_1$, $\beta_2$ and $\gamma_2$. We have $\dim V = 2$ or 3.

In case $\dim V = 2$, the subset $V \cap \Pi$ is a rank-two system of roots, containing two roots ($\beta_1$ and $\gamma_1$) whose sum is a root. Hence $V \cap \Pi$ is irreducible, and since our original root system is simply-laced, it follows that $V \cap \Pi$ is an $A_2$ system. Now, in $A_2$ a root is written uniquely as a sum of two roots, hence there is nothing to prove.

Suppose then that $\dim V = 3$, and let $\Pi_* = V \cap \Pi$ be the corresponding rank-three system. Since the roots in $\Pi$ have the same length, either $\Pi_* = A_1 \oplus A_2$ or $\Pi_* = A_3$. Again, there is nothing to prove in the $A_1 \oplus A_2$ case.

Assuming that $\Pi_* = A_3$, let $J_*$ be the restriction of $J$ to $\Pi_*$. Then $J_*$ is $(1, 2)$-admissible and hence affine.

Now, by Lemma 7.3, $\Pi_*^+ = \Pi_* \cap \Pi^+$ is a positive root system. Let $\Sigma_* \subset \Pi^+$ the corresponding set of simple roots.

**Lemma 7.5** *$\alpha$ is the highest root in $\Pi_*^+$.*

**Proof:** Write the positive roots of $A_3$ as $\alpha_{ij}$, $1 \leq i < j \leq 4$, so that $\alpha$ is one of these roots. It is not a simple root, since $\alpha = \beta_1 + \gamma_1$ with $\beta_1, \gamma_1 \in \Pi_*^+$. Also, a root of height 2 in $A_3$ is written uniquely as a sum of two positive roots. Hence the height of $\alpha$ in $\Pi_*^+$ is not 2, since $\dim V = 3$ and $\alpha = \beta_1 + \gamma_1 = \beta_2 + \gamma_2$. Therefore, the height of $\alpha$ in $\Pi_*^+$ is three, that is, $\alpha$ is the highest root. $\square$

By this lemma and the equality $\Pi_*^+ = \Pi_* \cap \Pi^+$ we conclude that the height of $\alpha$ in $\Pi^+$ is bigger than the height (in $\Pi^+$) of any $\gamma \in \Sigma_*$. Hence, the inductive hypothesis ensures that $k_\gamma$ is well defined for $\gamma \in \Sigma_*$.

Now, by the cone-free assumption, there exists an alcove $A^*$ in the affine system of $\Pi_*$ such that $J_* = J(A^*)$. By Lemma 7.4 we can choose $A^*$ so that $k_\gamma(A^*) = k_\gamma$ for all $\gamma \in \Sigma_*$. The integers $k_\delta(A^*)$, $\delta \in \Pi_*$, satisfy the conditions of Lemma 7.2. Also, $J_* = J(A^*)$ is the restriction of $J$ to $\Pi_*$. Hence starting with $k_\gamma(A^*) = k_\gamma$, $\gamma \in \Sigma_*$, the values of $k_\delta(A^*)$, $\delta \in \Pi_*^+$, are determined according to the rules used to define $k_\alpha$. This means that within $\Pi_*^+$, $k_\alpha$ is well defined. However, the decompositions $\alpha = \beta_1 + \gamma_1 = \beta_2 + \gamma_2$ are inside $\Pi_*^+$, so that the value of $k_\alpha$ does not depend upon one of these decompositions, concluding the proof of Theorem 7.1.

**Corollary 7.6** *In a simply-laced situation let $A^1$ and $A^2$ be alcoves such that $J(A^1) = J(A^2)$. Then there exists $\lambda$ with $\langle\lambda, \alpha\rangle \in 2\mathbb{Z}$ for every root $\alpha$ such that $A^2 = t_\lambda A^1$.*

**Proof:** As in the proof of Lemma 7.4 let $\{\alpha_1, \ldots, \alpha_l\}$ be a simple system of roots and $\{\omega_1, \ldots, \omega_l\}$ its dual basis, and put

$$\lambda = d_{\alpha_1}\omega_1 + \cdots + d_{\alpha_l}\omega_l$$

with $d_{\alpha_i} = k_{\alpha_i}(A^2) - k_{\alpha_i}(A^1)$. The assumption $J(A^1) = J(A^2)$ implies that $d_{\alpha_i}$, $i = 1, \ldots, l$, are even integers, so that $\langle\lambda, \alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in \Pi$. According to the change of coordinates formula (5), to see that $A^2 = t_\lambda A^1$ we must check that $k_\alpha(A^2) = \langle\lambda, \alpha\rangle + k_\alpha(A^1)$ for every positive root $\alpha$. This is done by induction on the height of $\alpha$: If $\alpha$ is simple, the equality holds by definition of $\lambda$. On the other hand if $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Pi^+$, we assume by induction that the equality is true for $\beta$ and $\gamma$. In particular, $k_\beta(A^1) + k_\gamma(A^1) \equiv k_\beta(A^2) + k_\gamma(A^2) \bmod 2$. Now, from the construction performed in the proof of Theorem 7.1, it follows that formula (6) holds for both sets of integers $k_\delta(A^1)$ and $k_\delta(A^2)$, $\delta \in \Pi$, with the same $\varepsilon_\alpha$. Therefore, $k_\alpha(A^i) - k_\beta(A^i) - k_\gamma(A^i)$ is independent of $i = 1, 2$. Thus applying the inductive hypothesis we get

$$k_\alpha\left(A^2\right) = k_\alpha\left(A^1\right) + \langle\lambda, \beta\rangle + \langle\lambda, \gamma\rangle = k_\alpha\left(A^1\right) + \langle\lambda, \alpha\rangle,$$

concluding the proof. □

**Remark:** It is worth mentioning that with Theorem 7.1 we get an indirect proof of a result in tournament theory, namely Theorem 3.5 of [2] which asserts that the vertices of a tournament $T$ can be rearranged so that its incidence matrix becomes stair-shaped in case $T$ has no cones. In fact, this result follows by piecing together Proposition 3.2, Theorem 7.1 and the results of [10] on invariant almost complex structures (see [10], Theorem 4.12).

## 8 Doubly-laced root systems

In this section we look at the cone-free property for the doubly-laced diagrams ($B_l$, $C_l$ and $F_4$). The final result for $C_l$ differs from $B_l$ and $F_4$.

**Theorem 8.1** *Let* $\Pi$ *be a root system and* $J$ *an* iacs *on the corresponding maximal flag manifold.*

1. *Suppose that* $\Pi$ *is* $C_l$. *Then* $J$ *is affine (and hence* $(1,2)$*-admissible) if and only if* $J$ *satisfies the cone-free property.*
2. *Suppose that* $\Pi$ *is* $B_l$ *or* $F_4$, *and that the restriction of* $J$ *to any rank-three subsystem is affine. Then* $J$ *is affine, and hence* $(1,2)$*-admissible.*

**Remark:** The rank-three condition for $B_l$ and $F_4$ is equivalent to $J$ being cone-free together with the additional assumption that the restriction of $J$ to any $B_3$-subsystem is affine. This assumption is not required for $C_l$ because it does not contain $B_3$-subsystems.

The proof of Theorem 8.1 uses the corresponding result for simply-laced diagrams (Theorem 7.1), applied to the set of short roots of $\Pi$. Let $\Pi^s$ and $\Pi^l$ denote the sets of short roots and long roots, respectively. We have the disjoint union $\Pi = \Pi^s \cup \Pi^l$. Both sets $\Pi^s$ and $\Pi^l$ are simply-laced root systems (for example, in $\Pi = B_l$, $\Pi^l$ is a $D_l$ while $\Pi^s$ is reducible with $l$ orthogonal components).

Let $J^s$ stand for the restriction of $J$ to $\Pi^s$. Clearly, under the conditions of Theorem 8.1, $J^s$ satisfies the cone-free assumption of Theorem 7.1, so that $J^s$ is affine in $\Pi^s$. Thus there are integers $k_\alpha$, $\alpha \in \Pi^s$, with $\varepsilon_\alpha = (-1)^{k_\alpha}$ such that $k_\alpha$ form the coordinates of an alcove in $\Pi^s$. We shall prove Theorem 8.1 by extending these coordinates to $\Pi^l$.

For the doubly-laced root systems, we have the following characterization of the coordinates of alcoves, which is obtained from the inequalities of Shi after a case-by-case analysis.

**Proposition 8.2** *In a doubly-laced root system a set of integers* $k_\alpha$, $\alpha \in \mathbb{Z}$, *are the coordinates of an alcove if and only if the following inequalities are satisfied. Each inequality is satisfied by a triple of roots as indicated, where* $s$ *means short root and* $l$ *long root.*

1. $(\alpha, \beta, \alpha+\beta) = (l, l, l)$*:* $k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1$

2. $(\alpha,\beta,\alpha+\beta)=(s,s,s)$: $2k_\alpha+2k_\beta+1\le 2k_{\alpha+\beta}+2\le 2k_\alpha+2k_\beta+5$
3. $(\alpha,\beta,\alpha+2\beta)=(l,s,l)$: $k_\alpha+2k_\beta\le k_{\alpha+2\beta}\le k_\alpha+2k_\beta+2$
4. $(\alpha,\beta,(\alpha+\beta)/2)=(l,l,s)$: $k_\alpha+k_\beta\le 2k_{(\alpha+\beta)/2}+1\le k_\alpha+k_\beta+2$

**Proof:** See [10], Proposition 5.4. □

**Lemma 8.3** *Suppose the doubly-laced root system* $\Pi$ *is irreducible, and let* $\alpha$ *be a long root. Then there exists a short root* $\beta$ *such that* $\langle\alpha,\beta\rangle\neq 0$.

**Proof:** There are a long root $\alpha_1$ and a short root $\beta_1$ such that $\langle\alpha_1,\beta_1\rangle\neq 0$ (look e.g. at the Dynkin diagram). The Weyl group $\mathcal{W}$ leaves invariant both $\Pi^l$ and $\Pi^s$, and since $\Pi$ is irreducible, these subsets are orbits of $\mathcal{W}$. Hence, for a long root $\alpha$ there exists $w\in\mathcal{W}$ with $\alpha=w\alpha_1$. Thus, $\langle\alpha,w\beta_1\rangle\neq 0$. □

**Lemma 8.4** *Let* $\alpha$ *be a long root. Then there are short roots* $\beta$ *and* $\gamma$ *such that*

$$\alpha=\beta+\gamma.$$

**Proof:** By the previous lemma there exists a short root $\beta$ with $\langle\alpha,\beta\rangle\neq 0$. Let $\Pi_2$ be the intersection of $\Pi$ with the subspace spanned by $\alpha$ and $\beta$. It has rank-two and contains two roots $\alpha$ and $\beta$ of different length with $\langle\alpha,\beta\rangle\neq 0$. Hence $\Pi_2$ is a $B_2$. The lemma follows then by looking at the roots of $B_2$. □

Now, we write down the conditions for a set of integers to be the coordinates of an alcove in terms of the short and long roots.

**Lemma 8.5** *In a doubly-laced root system* $\Pi$ *the following conditions are necessary and sufficient for a set of integers* $k_\alpha$, $\alpha\in\Pi$, *to be the coordinates of an alcove:*

1. *The integers* $k_\alpha$, $\alpha\in\Pi^l$, *are the coordinates of an alcove in the root system of the long roots.*
2. *The integers* $k_\alpha$, $\alpha\in\Pi^s$, *are the coordinates of an alcove in the root system of the short roots.*

3. *Take a long root* $\alpha = \beta + \gamma$ *with* $\beta$ *and* $\gamma$ *short roots. Then either* $k_\alpha = k_\beta + k_\gamma$ *or* $k_\alpha = k_\beta + k_\gamma + 1$.

**Proof:** Suppose first that $k_\alpha$, $\alpha \in \Pi$, are the coordinates of an alcove. Then the first and second sets of inequalities in Proposition 8.2 together with the corresponding inequalities in the simply-laced case show that the restriction of $k_\alpha$ to the long roots as well as to the short roots are coordinates of alcoves. Furthermore, the last condition is necessary by Lemma 6.2.

We prove sufficiency by showing that the three conditions of the lemma imply the inequalities of Proposition 8.2. The first two sets of those inequalities are equivalent to our conditions on the sets of long and short roots, respectively. For the other two we make a case by case analysis. As before $l$ means long root and $s$ short root.

- $(\alpha, \beta, \alpha + 2\beta) = (l, s, l)$: $k_\alpha + 2k_\beta \leq k_{\alpha+2\beta} \leq k_\alpha + 2k_\beta + 2$. Put $\gamma = \alpha + \beta$ and $\delta = \alpha + 2\beta$. Note that $\gamma$ is a short root (look at the roots of $B_2$) $\delta$ is a long root, and $\delta = \beta + \gamma$. Hence by the third condition either $k_\delta = k_\beta + k_\gamma$ or $k_\delta = k_\beta + k_\gamma + 1$. On the other hand, $\alpha = -\beta + \gamma$ is a sum of short roots giving rise to a long root. So that either $k_\alpha = -k_\beta + k_\gamma - 1$ or $k_\alpha = -k_\beta + k_\gamma$. Now, we plug these possibilities into $2k_\beta \leq k_\delta - k_\alpha \leq 2k_\beta + 2$. We list below the inequalities that arise:

$$\begin{array}{ll} k_\delta - k_\alpha = (k_\beta + k_\gamma) - (-k_\beta + k_\gamma - 1) & 2k_\beta \leq 2k_\beta + 1 \leq 2k_\beta + 2 \\ k_\delta - k_\alpha = (k_\beta + k_\gamma) - (-k_\beta + k_\gamma) & 2k_\beta \leq 2k_\beta \leq 2k_\beta + 2 \\ k_\delta - k_\alpha = (k_\beta + k_\gamma + 1) - (-k_\beta + k_\gamma - 1) & 2k_\beta \leq 2k_\beta + 2 \leq 2k_\beta + 2 \\ k_\delta - k_\alpha = (k_\beta + k_\gamma + 1) - (-k_\beta + k_\gamma) & 2k_\beta \leq 2k_\beta + 1 \leq 2k_\beta + 2 \end{array}$$

  Hence the third set of inequalities of Proposition 8.2 holds under the conditions of the lemma.

- $(\alpha, \beta, (\alpha + \beta)/2) = (l, l, s)$: $k_\alpha + k_\beta \leq 2k_{(\alpha+\beta)/2} + 1 \leq k_\alpha + k_\beta + 2$. Put $\gamma = (\alpha + \beta)/2$ and $\delta = (\beta - \alpha)/2$. Both $\gamma$ and $\delta$ are short roots (again look at $B_2$). We have $\beta = \gamma + \delta$ and $\alpha = \gamma - \delta$, so that $k_\beta = k_\gamma + k_\delta$ or $k_\beta = k_\gamma + k_\delta + 1$ and $k_\alpha = k_\gamma - k_\delta - 1$ or $k_\alpha = k_\gamma - k_\delta$. Plugging these choices into $k_\alpha + k_\beta \leq 2k_\gamma + 1 \leq k_\alpha + k_\beta + 2$ we get:

$$\begin{array}{ll} k_\alpha + k_\beta = (k_\gamma - k_\delta - 1) + (k_\gamma + k_\delta) & 2k_\gamma - 1 \leq 2k_\gamma + 1 \leq 2k_\gamma + 1 \\ k_\alpha + k_\beta = (k_\gamma - k_\delta - 1) + (k_\gamma + k_\delta + 1) & 2k_\gamma \leq 2k_\gamma + 1 \leq 2k_\gamma + 2 \\ k_\alpha + k_\beta = (k_\gamma - k_\delta) + (k_\gamma + k_\delta) & 2k_\gamma \leq 2k_\gamma + 1 \leq 2k_\gamma + 2 \\ k_\alpha + k_\beta = (k_\gamma - k_\delta) + (k_\gamma + k_\delta + 1) & 2k_\gamma + 1 \leq 2k_\gamma + 1 \leq 2k_\gamma + 3 \end{array}$$

Concluding the proof of the lemma. □

We return now to Theorem 8.1. Let $J = \{\varepsilon_\alpha\}$ be an *iacs* in the doubly-laced root system $\Pi$, which satisfies the cone-free property. Then the restriction $J^s$ of $J$ to the short roots $\Pi^s$ is cone-free. Hence, by Theorem 7.1, $J^s$ is affine, so that there are integers $k_\beta$, $\beta \in \Pi^s$, forming the coordinates of an alcove in $\Pi^s$, such that $\varepsilon_\beta = (-1)^{k_\beta}$ for all $\beta \in \Pi^s$.

Maintaining this choice of alcove in $\Pi^s$ we intend to extend the integers $k_\alpha$ to the long roots. Taking into account the third condition of Lemma 8.5, we must define $k_\alpha$, $\alpha \in \Pi^l$, by the expression (6) already used in the simply-laced case, but now with $\alpha$ a long root and $\beta$ and $\gamma$ short roots, such that $\alpha = \beta + \gamma$. Again, the very expression for $k_\alpha$ ensures that $\varepsilon_\alpha = (-1)^{k_\alpha}$. Hence, in order to proceed we must prove that the integers $k_\alpha$, $\alpha \in \Pi^l$, are well defined, and form the coordinates of an alcove.

**Lemma 8.6** *Let $k_\beta$, $\beta \in \Pi^s$, be the coordinates of an alcove in $\Pi^s$, representing $J^s$. Let $\alpha > 0$ be a long root with $\alpha = \beta + \gamma$, $\beta$ and $\gamma$ short roots, and put*

$$k_\alpha = k_\beta + k_\gamma + \frac{1 - (-1)^{k_\beta + k_\gamma}\,\varepsilon_\alpha}{2}. \tag{7}$$

*Then $k_\alpha$ is independent of the short roots $\beta$ and $\gamma$.*

**Proof:** Let $\alpha = \beta_1 + \gamma_1$ be another sum with $\beta_1$ and $\gamma_1$ short roots. Denote by $V$ the subspace spanned by the roots $\beta$, $\gamma$, $\beta_1$ and $\gamma_1$, and let $\Pi_* = V \cap \Pi$ be the corresponding subsystem. The possible dimensions of $V$ are 2 or 3. If $\dim V = 2$, $\Pi_*$ is a $B_2$ system, so that the components in the two sums are equal. Similarly, in $B_3$ there is only one way of writing a long root as a sum of two short roots. Hence we can assume that $\Pi_*$ is $C_3$.

By Proposition 4.3 any cone-free *iacs* in $C_3$ is affine. Of course, the restriction $J_*$ of $J$ to $\Pi_*$ is cone-free. Hence, there are integers, say $m_\delta$, $\delta \in \Pi_*$, which are the coordinates of an alcove in $C_3$, such that $\varepsilon_\delta = (-1)^\delta$ for all $\delta \in \Pi_*$. In particular $k_\delta \equiv m_\delta \mathrm{mod} 2$ for every short root $\delta \in \Pi_*$. The set of short roots in $C_3$ forms an $A_3$-root system, so that we can apply Corollary 7.6, to get $\lambda$ such that for every short root $\delta \in \Pi_*$ we have $k_\delta = m_\delta + \langle\lambda, \delta\rangle$ and $\langle\lambda, \delta\rangle \in 2\mathbb{Z}$. Since the long roots in $C_3$ are linear combinations of short roots with integer coefficients, it follows that $\langle\lambda, \delta\rangle \in 2\mathbb{Z}$ for the long roots in $\Pi_*$ as well. Therefore, $m_\delta + \langle\lambda, \delta\rangle$, $\delta \in \Pi_*$, are the coordinates of an alcove $A^*$ such that $J_* = J(A^*)$.

Now, let $\alpha = \beta + \gamma$ be as in the statement. By the third condition in Lemma 8.5 we have

$$m_\alpha + \langle \lambda, \alpha \rangle = k_\beta + k_\gamma + \frac{1 - (-1)^{k_\beta + k_\gamma} \varepsilon_\alpha}{2}.$$

Since the left hand side is independent of the way $\alpha$ is written as a sum of short roots, the lemma follows. □

It remains to prove that the extension of $k_\alpha$ to the long roots given by (7) form the coordinates of an alcove. For this we use Lemma 8.5, and verify that the three conditions of that lemma are satisfied. Firstly, the integers $k_\alpha$ were chosen so that they form the coordinates of an alcove on the short roots. Also, the compatibility condition (3) follows immediately from the definition of $k_\alpha$ in (7). Hence, the point is to show that the integers $k_\alpha$ are coordinates of an alcove on the long roots. At this point we consider $C_l$ separately. In fact, the set $\Pi^l$ of long roots of $C_l$ is a decomposable root system with $l$ orthogonal positive roots. In such a root system there are no restrictions on the integers to be the coordinates of an alcove. In particular, for $C_l$ the condition of Lemma 8.5 regarding the long roots is vacuous. Therefore, Theorem 8.1 is true in the $C_l$ case.

To consider $B_l$ and $F_4$ we prove first the following easy lemma.

**Lemma 8.7** *Let $J = \{\varepsilon_\alpha\}$ be an affine* iacs *in the root system*

$$B_l = \{\pm (e_i \pm e_j) : 1 \le i < j \le l\} \cup \{\pm e_i : 1 \le i \le l\}.$$

*Suppose we are given integers $k_i$, $i = 1, \ldots, l$ such that $\varepsilon_{e_i} = (-1)^{k_i}$. Then there exists an alcove $A$ with coordinates $k_\alpha(A)$ satisfying $k_{e_i} = k_i$, $i = 1, \ldots, l$ and such that $J = J(A)$.*

**Proof:** Is similar to the proof of Lemma 7.4, after taking into account that the short roots $e_i$, $i = 1, \ldots, l$, span $B_l$ over $\mathbb{Z}$. □

Finally, we can conclude the proof of Theorem 8.1 for $B_l$ and $F_4$, by showing that the extension defined in Lemma 8.6 are indeed the coordinates of an alcove on the set of long roots.

**Lemma 8.8** *Given $k_\beta$, $\beta \in \Pi^s$, define $k_\alpha$, $\alpha \in \Pi^l$ as in Lemma 8.6. Then $\{k_\alpha\}$ are the coordinates of an alcove in $\Pi^l$.*

**Proof:** By Lemma 7.2 we must show that if $\alpha$, $\beta$ and $\alpha+\beta$ are long roots then either $k_{\alpha+\beta}=k_\alpha+k_\beta$ or $k_{\alpha+\beta}=k_\alpha+k_\beta+1$.

Write $\beta=\gamma_1+\gamma_2$ as a sum of short roots and denote by $V$ the subspace spanned by $\{\alpha,\gamma_1,\gamma_2\}$. Let $\Pi_*$ be the root system $V\cap\Pi$. We claim that $\Pi_*$ is a $B_3$-subsystem. In fact, $\dim V>1$ because $\beta\neq\pm\alpha$ and we cannot have $\dim V=2$, since this would imply that $\Pi_*$ is a $B_2$-system, because it contains short and long roots and a pair of roots ($\gamma_1$ and $\gamma_2$) whose sum is a root. But in $B_2$ the sum of two long roots is not a root. Hence, $\dim V=3$. Analogous arguments show that $\Pi_*$ is irreducible. Now, $\Pi_*$ has roots of different length, so that either $\Pi_*=B_3$ or $C_3$. However, in $C_3$ no sum of two long roots is a root. Therefore, $\Pi_*=B_3$, as claimed.

By looking at the roots of $B_3$ we can ensure that, since $\alpha+\beta$ is a root, one of the roots in the decomposition of $\beta$, say $\gamma_1$, is such that there exists a short root $\delta$ with $\alpha=-\gamma_1+\delta$. Hence, $\alpha+\beta=\gamma_2+\delta$, and we have

- $k_{\alpha+\beta}=k_{\gamma_2}+k_\delta+\left(1-(-1)^{k_{\gamma_2}+k_\delta}\,\varepsilon_{\alpha+\beta}\right)$,
- $k_\alpha=-k_{\gamma_1}-1+k_\delta+\left(1+(-1)^{k_{\gamma_1}+k_\delta}\,\varepsilon_\alpha\right)$, and
- $k_\beta=k_{\gamma_1}+k_{\gamma_2}+\left(1-(-1)^{k_{\gamma_1}+k_{\gamma_2}}\,\varepsilon_\beta\right)$.

These formulae imply that the dependence of $k_{\alpha+\beta}-(k_\alpha+k_\beta)$ on the integers $k_\gamma$, $\gamma\in\Pi$, is only mod2.

Now, we use the cone-free condition to get an alcove $A^*$ in $V$ such that $J_*=J(A^*)$, where $J_*$ is the restriction of $J$ to $\Pi_*$. For a root $\gamma\in\Pi_*$, $\varepsilon_\gamma=(-1)^{k_\gamma(A^*)}=(-1)^{k_\gamma}$, so that $k_\gamma(A^*)\equiv k_\gamma\,(\mathrm{mod}2)$. The formulae above are true with $k_\gamma(A^*)$, $\gamma\in\Pi$, in place of $k_\gamma$. But we know that for the coordinates of an alcove either $k_{\alpha+\beta}(A^*)-(k_\alpha(A^*)+k_\beta(A^*))=0$ or $1$. Hence, either $k_{\alpha+\beta}-(k_\alpha+k_\beta)=0$ or 1, concluding the proof of the lemma. □

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